

COMPUTATION OF ASYMMETRIC PERIODIC CONFIGURATIONS IN THE THREE-BODY PROBLEM

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Abstract: In this paper, we present the algorithm and the methodology used for locating periodic configurations for the planetary problem of three bodies. The repeated relative configurations of the planets can be associated with periodic orbits in an appropriate rotating frame of reference. In this frame, the general planar three-body problem, which is the basic model, is reduced to three degrees of freedom. Up to recent years, only symmetric periodic orbits have been presented in the literature. Recently, however, asymmetric periodic orbits have been computed. In the present work we describe the algorithm used for computing asymmetric configurations, and discuss the efficiency and the implementation of the method. An example of a 2/1 resonant family of asymmetric periodic orbits is given as an application of the algorithm.

1 INTRODUCTION

The three-body problem (TBP) is a well-known Hamiltonian dynamical system with a number of few degrees of freedom that depends on particular assumptions made. The system consists of three bodies, of masses $m_i, i = 1, 2, 3$, which move under their mutual Newtonian gravitational forces in a plane or in space. Even as a mathematical model, the TBP is of great interest because of its complicated dynamics. A special case is the TBP of *planetary type*, which describes the motion of two planets around a star. Such a system has been proved a good model for describing the dynamics of various extra-solar systems, discovered in the last decade [1].

Although in the TBP of planetary type the two planets have very small masses, their mutual interaction may be sufficient for destabilizing the system. An important objective of a dynamical analysis of the system is to determine the regions in phase space where regular motion or chaos exists. Regular motion guarantees the long-term evolution of the planetary system, while chaos usually leads the planets to collision or to escape. One method for determining the regular regions in phase space is to run a large number of trajectories and to use an indicator that classifies the trajectories as chaotic or regular [2]. However, since the phase space is of many dimensions, only two-dimensional sections of phase space can be mapped. Another approach is the construction of an averaged system of two degrees of freedom, the computation of its equilibrium points and the study of their stability [3]. A third method, which is the subject of the present work, is based on the determination of the periodic orbits of the system in an appropriate frame of reference. The periodic orbits constitute the backbone of the phase space. Stable periodic orbits are surrounded by regular trajectories in their neighborhood, while unstable ones are usually associated with chaotic motion. Thus, the determination of all periodic orbits, which form monoparametric curves in phase space, reveals regions of stable and chaotic motion.

The computation of periodic orbits in the restricted three-body problem (RTBP) goes back in the 1950's [4,5]. Computations of periodic orbits in the general three-body problem have been performed mainly by Hadjidemetriou (see [6] and references therein). However, only symmetric periodic orbits have been computed. In some recent studies by Voyatzis and Hadjidemetriou [7,8], results on asymmetric periodic orbits of the planar TBP have been presented, including applications in the dynamics of extra-solar planetary systems. In the present work we describe the algorithm for the determination of periodic orbits in the planar TBP in a rotating frame of reference.

In the next section we describe the model under consideration. In section 3 we define the various

periodic configurations which may appear. We discuss their properties and give the periodicity conditions, which must be satisfied. We also describe the various computational aspects for their numerical evaluation. Finally in section 4 we present some examples, by considering a random search in phase space for symmetric and asymmetric configurations.

2 THE MODEL AND FRAMES OF REFERENCE

2.1. The inertial frame of reference and basic definitions.

In the present work we consider the planar case, i.e. all bodies move on a plane [9]. By introducing a Cartesian coordinate system OXY (*inertial frame of reference*) the positions of the bodies are given by the vectors $\vec{r}_i, i = 1, 2, 3$ and the differential equations of motion have the form:

$$\ddot{\vec{r}}_i = - \sum_{j=1, j \neq i}^3 \frac{Gm_j(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^3}, \quad i = 1, 2, 3 \quad (1)$$

Without loss of generality, the center of mass is considered at rest, located at $O(0,0)$, i.e. we can use only two of the three equations in (1), e.g. those with $i=1, 2$. Then, the position of the third body can be derived from the equation of the center of mass:

$$\vec{r}_3 = -(m_1\vec{r}_1 + m_2\vec{r}_2) / m_3 \quad (2)$$

A special case of the TBP is the *three-body problem of planetary type*. In this case the two bodies have very small (but not negligible) masses compared with that of the third one, namely

$$m_1 \ll m_3, \quad m_2 \ll m_3, \quad O(m_1) = O(m_2).$$

The two bodies with the small masses are called Planets and are denoted by P_1 and P_2 , while the heavy body is the Star.

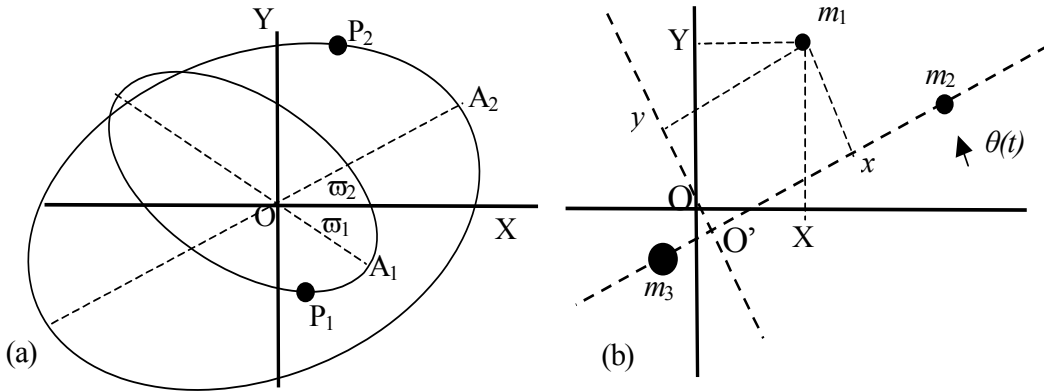


Figure 1: (a) The unperturbed Keplerian orbits of the planets P_1 and P_2 in the inertial frame. (b) The rotating frame of reference $O'xy$.

By setting $m_1=m_2=0$, we get the *unperturbed problem*, where the planets P_1 and P_2 move in a central force field provided by the Star and their bounded motion is described by Keplerian ellipses of semimajor axes a_i , eccentricities e_i and longitude of pericenter ϖ_i , where the subscript i refers to the planet P_i (see Fig.1a). The position of each planet on the corresponding ellipses is given by an ‘‘anomaly’’, e.g. the mean anomaly M_i . The difference $\Delta\varpi = \varpi_2 - \varpi_1$ is called apsidal difference. The period of planet’s motion obeys the Kepler’s law $T_i \sim a_i^{3/2}$. The planet P_1 is referred as the ‘‘inner’’ planet of the system, and P_2 is referred to as the ‘‘outer’’ one, in the sense that their periods obey the relation $T_1 < T_2$, although those orbits may intersect each other. The system is in *resonance* ‘‘ p/q ’’, with p, q prime integers, when

$$\frac{T_1}{T_2} = \left(\frac{a_1}{a_2} \right)^{3/2} = \frac{p}{q}, \quad p, q \in \mathbb{Z}^+ \setminus 0 \quad (3)$$

The resonance is related to a periodic orbit of the system with period $T = qT_1 = pT_2$.

If we set $m_1=0$ and $m_2 \neq 0$ (or $m_1 \neq 0$ and $m_2=0$), we get the so called restricted three body problem (RTBP), which is a non-integrable system with complicated phase space structure and various types of periodic orbits [4,5]. We remark that the RTBP has asymmetric periodic orbits only in the case where $m_1 \neq 0$ and $m_2=0$, namely when the outer planet is the massless one [10].

2.2. The rotating frame of reference and reduction of degrees of freedom.

The system of the planar TBP as it is described in the inertial frame of reference is a dynamical system of four degrees of freedom. We can reduce the degrees of freedom by introducing a *rotating frame of reference* $O'xy$, whose x -axis is the line $Star-P_1$, the origin $O'(X',Y')$ is the center of mass of these two bodies, and the y -axis is perpendicular to the x -axis (Fig. 1b). In this rotating frame, P_1 moves on the x -axis and P_2 in the $O'xy$ plane. The coordinates defining the system are the position x_1 of P_1 , the position (x_2, y_2) of P_2 and the angle θ between the x -axis and a fixed direction in the inertial frame [11]. We remark that the above frame of reference does not rotate uniformly.

After unit normalization we can set $G=1$. The Lagrangian of the system can be written in the form

$$L = T - V, \quad (4)$$

where

$$T = \frac{1}{2} \bar{\mu}_1 (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} \bar{\mu}_2 (\dot{x}_2^2 + \dot{y}_2^2 + 2\dot{\theta}(x_2 \dot{y}_2 - \dot{x}_2 y_2) + \dot{\theta}^2 (x_2^2 + y_2^2)) + V(x_1, x_2, y_2) \quad (5)$$

$$\mu = \frac{m_1}{m_1 + m_3}, \quad \bar{\mu}_1 = \mu m_3, \quad \bar{\mu}_2 = \frac{m_1 + m_3}{m_1 + m_2 + m_3} m_2,$$

$$V = -\frac{m_1 m_2}{r_{12}} - \frac{m_1 m_3}{r_{13}} - \frac{m_2 m_3}{r_{23}} \quad (6)$$

$$r_{13} = r, \quad r_{12} = \sqrt{(x_1 - x_2)^2 + y_2^2}, \quad r_{23} = \sqrt{(x_2 + \mu r)^2 + y_2^2}$$

and

$$r = x_1 / (1 - \mu) \quad (7)$$

The differential equations of motion are given by the Lagrange equations

$$\ddot{x}_1 - x_1 \dot{\theta}^2 + \frac{(1-\mu)^2}{\bar{\mu}_1} \frac{\partial V}{\partial x_1} = 0 \quad (8)$$

$$\ddot{x}_2 - y_2 \ddot{\theta} - 2\dot{y}_2 \dot{\theta} - x_2 \dot{\theta}^2 + \frac{1}{\bar{\mu}_2} \frac{\partial V}{\partial x_2} = 0, \quad (9)$$

$$\ddot{y}_2 + x_2 \ddot{\theta} + 2\dot{x}_2 \dot{\theta} - y_2 \dot{\theta}^2 + \frac{1}{\bar{\mu}_2} \frac{\partial V}{\partial y_2} = 0 \quad (10)$$

We can obtain that the variable θ is ignorable in the Lagrangian. Consequently, the integral of angular momentum exists:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \bar{\mu}_1 r^2 \dot{\theta} + \bar{\mu}_2 (\dot{\theta}(x_2^2 + y_2^2) + x_2 \dot{y}_2 - \dot{x}_2 y_2) = \text{const.} \quad (11)$$

By using (11), the angular velocity $\dot{\theta}$ can be determined by the variables $x_1, x_2, y_2, \dot{x}_1, \dot{x}_2, \dot{y}_2$ and the

system is reduced to three degrees of freedom. Working with normalized variables (values of order 1), we consider $m_{tot} = m_1 + m_2 + m_3 = 1$ and the p_θ constant is set such that $x_1 \approx 1.0$ and $T_1 \approx 2\pi$. Also it is $\partial L / \partial t = 0$ and thus the system has the energy integral

$$E = T + V \quad (12)$$

2.3. Transformation between reference frames.

By considering the inertial coordinates (X_i, Y_i) , we pass to the rotating coordinates (x_i, y_i) according to the following transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} X - X' \\ Y - Y' \end{pmatrix} \quad (13)$$

and

$$\theta = \text{Arc tan} \frac{Y_2 - Y_1}{X_2 - X_1}.$$

Conversely, we have the transformation from (x_1, x_2, y_2, θ) to (X'_i, Y'_i) , $i=1,2,3$, where primes denote the inertial frame with center $O'(X', Y')$ (the center of mass of the star and the planet P_i) as it follows :

$$\begin{aligned} X'_1 &= x_1 \cos \theta, & X'_2 &= \sqrt{x_2^2 + y_2^2} \cos \theta, & X'_3 &= -\mu X'_1 \\ Y'_1 &= x_1 \sin \theta, & Y'_2 &= \sqrt{x_2^2 + y_2^2} \sin \theta, & Y'_3 &= -\mu Y'_1 \end{aligned} \quad (14)$$

and, finally,

$$X_i = X'_i + X', \quad Y_i = Y'_i + Y'. \quad (15)$$

For the three-body problem of planetary type, the short interval evolution corresponds to almost Keplerian motion, and the equations of motion (1) show singularities only at collisions. In contrast to this, in many cases, in the rotating frame of reference, trajectory solutions of the Lagrange equations (8)-(10) are complicated and they show “cusps”, i.e. points where the orbits are not differentiable. Such a situation leads to a failure of the numerical integration. Therefore, it is computationally efficient to integrate the equations in the inertial frame and, then, to obtain the system variables in the rotating frame using the transformations (13)-(15).

We remark that at $t=0$ the inertial and the rotating frames coincide with respect to the direction of their axes. Also, $\theta=\theta(t)$ is known only after the solution of the system.

3 PERIODIC CONFIGURATIONS

3.1 Basic definitions

In the inertial frame, a periodic orbit of period T is an orbit which satisfies the following condition:

$$X_i(T) = X_i(0), \quad Y_i(T) = Y_i(0), \quad \dot{X}_i(T) = \dot{X}_i(0), \quad \dot{Y}_i(T) = \dot{Y}_i(0), \quad i = 1, 2 \quad (16)$$

Only few periodic solutions (16) are known (e.g. the Lagrange solution [9] and the “figure 8” [13]) and are unstable. However, as we mentioned in the introduction, our aim is to obtain periodic configurations for the system, i.e. solutions that are periodic of period T in the rotating frame of reference. Such a solution denotes that the relative position of the planets is periodic. The system is not necessarily periodic in the inertial frame, i.e. the planets do not return to their initial position after $t=T$.

A periodic orbit of period T in the inertial frame is defined by the relations

$$\begin{aligned} x_1(T) &= x_{10}, & x_2(T) &= x_{20}, & y_2(T) &= y_{20}, \\ \dot{x}_1(T) &= \dot{x}_{10}, & \dot{x}_2(T) &= \dot{x}_{20}, & \dot{y}_2(T) &= \dot{y}_{20}, \end{aligned} \quad (17)$$

where the subscript 0 denotes the value of the variable at $t=0$. We remark that (17) does not imply (16) because is $\theta(T) \neq \theta_0$, in general. Next, we will consider periodic orbits in the rotating frame of reference.

For the classification of periodic orbits we take into account that the system in the rotating frame

possesses the *fundamental symmetry* Σ :

$$x \rightarrow x, \quad y \rightarrow -y, \quad t \rightarrow -t$$

Indeed, for preserving p_0 in (11), we should apply the transformation $\theta \rightarrow -\theta$ and then we can conclude that the equations (8)-(10) remain invariant. For the system in the rotating frame the symmetry Σ is written as

$$(x_1, x_2, y_2, \dot{x}_1, \dot{x}_2, \dot{y}_2) \rightarrow (x_1, x_2, -y_2, -\dot{x}_1, -\dot{x}_2, \dot{y}_2) \quad (18)$$

The fundamental symmetry means that if we take the symmetric orbit of the body P_2 (in the rotating frame) about the axis $O'x$ and we change the direction of the velocity \dot{x}_1 of the body P_1 then we get new valid solution for the system, which is called “mirror solution” (see Fig. 2). In the inertial frame the corresponding “mirror orbits” of the two planets have the same semimajor axes and eccentricities but the longitude of pericenters and the mean anomalies take opposite values.

3.2. Symmetric periodic orbits

A periodic orbit of period T is called *symmetric* if it is invariant under the fundamental symmetry Σ or, equivalently, if we apply the transformation (18) to any point of the periodic orbit we get a point of the same orbit.

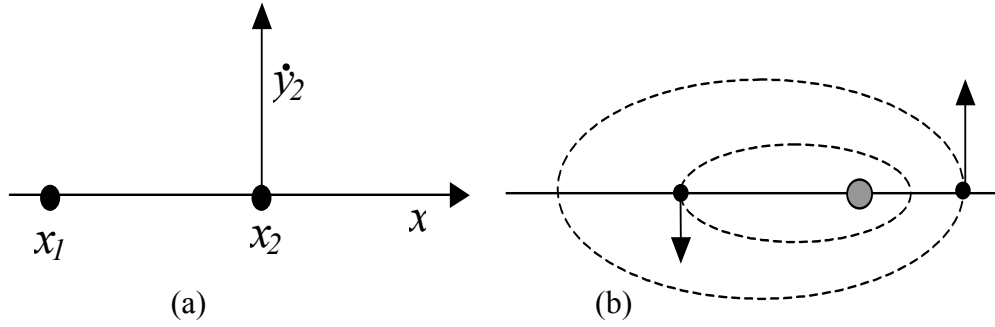


Figure 2. Initial conditions for a symmetric periodic orbit (a) in the rotating frame and (b) in the inertial frame.

Since the part of the orbit at $y_2 > 0$ is identical with that at $y_2 < 0$, their union forms a smooth orbit in phase space only if the orbit of P_2 crosses perpendicularly the axis Ox when the velocity of the body P_1 is zero, and such a situation must occur twice during one period. By interpreting such symmetry in the sense of the Keplerian ellipses of the planetary orbits, we conclude that for symmetric configurations the apsidal difference $\Delta\varpi$ is 0 or 180° (aligned or anti-aligned periastra, respectively). Additionally, there is at least one moment, during one period, when the planets are found either at a periastron or an apoastron, i.e. $M_1=0$ or 180° and $M_2=0$ or 180° .

Taking into account the symmetry, we can select convenient initial conditions for the periodic orbit when it crosses perpendicularly the x -axis. In particular, we assume that the planet P_2 starts from the x -axis perpendicularly ($y_2=0, \dot{x}_2=0$), and at that time $\dot{x}_1=0$, and after some time $t=T/2$, T being the period, the planet P_2 crosses again the x -axis perpendicularly, again with $\dot{x}_1=0$. The hyperplane $y_2=0$ of the six-dimensional phase space is called the *reference plane* of the orbit. During the time span $[0, T)$ the reference plane is intersected by the orbit $2k$ times (including the initial position). The non-zero integer k is called the *multiplicity of the orbit* and the k th intersection of the symmetric periodic orbit with the reference plane takes place at $t=T/2$. So, provided that we start the orbit from the reference section (i.e. $y_2(0)=y_2(T/2)=0$), the periodicity conditions for a symmetric periodic orbit are the boundary conditions:

$$\dot{x}_1(0) = \dot{x}_1(T/2), \quad \dot{x}_2(0) = \dot{x}_2(T/2), \quad (19)$$

This means that, in the rotating frame, the non zero initial conditions of a symmetric periodic orbit

are (x_{10}, x_{20}, y_{20}) and a periodic orbit is represented by a point in the 3D space corresponding to the three non-zero initial conditions.

3.3. Asymmetric periodic orbits

An asymmetric periodic orbit is not invariant under Σ but it is mapped in another orbit, which is also periodic, and is called “mirror image” orbit. The initial and periodicity conditions of an asymmetric periodic orbit can be defined in two different ways:

Case A. The planets start in *conjunction*, namely, all of them are located on the x -axis, the planet P_1 is not in rest and the planet P_2 does not start moving perpendicularly from the x -axis (Fig. 3). After time $t=T$, T being the period, the planets return to the same location in the rotating frame. So, given the reference plane $y_2(0) = y_2(T/2) = 0$, the periodicity conditions are

$$\begin{aligned} x_1(T) &= x_1(0), & \dot{x}_1(T) &= \dot{x}_1(0) \neq 0, \\ x_2(T) &= x_2(0), & \dot{x}_2(T) &= \dot{x}_2(0), & \dot{y}_2(T) &= \dot{y}_2(0) \end{aligned} \quad (20)$$

Thus, all variables, except y_2 , which defines the reference plane, are necessary for the localization of an asymmetric periodic orbit.

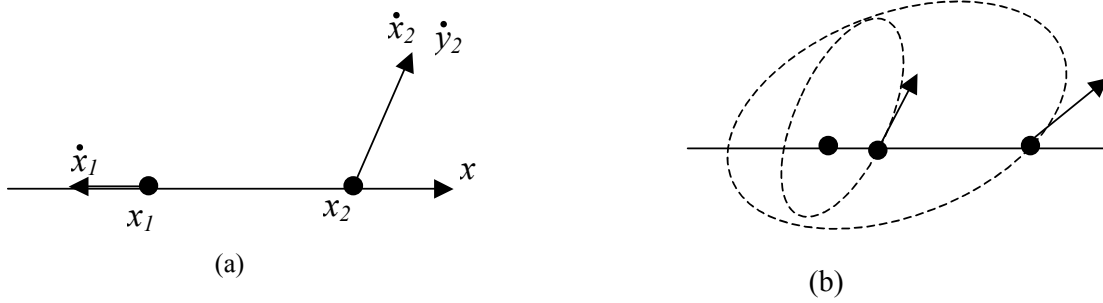


Figure 3. Initial conditions of *case A* for an asymmetric periodic orbit (a) in the rotating frame and (b) in the inertial frame.

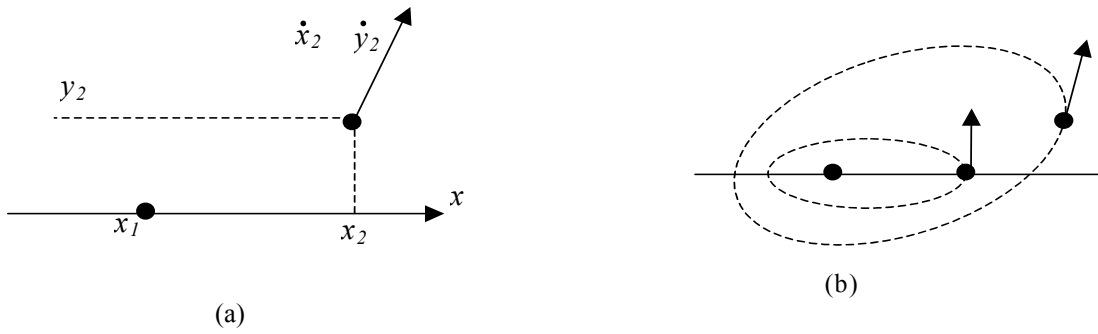


Figure 4. Initial conditions of *case B* for an asymmetric periodic orbit (a) in the rotating frame and (b) in the inertial frame.

Case B. The initial conditions are taken when the planet P_1 passes from its periastron, which, without loss of generality, can be taken on the x -axis. Thus the reference plane is defined by $\dot{x}_1 = 0$ where, at $t=0$ or $t=T$, it is $\varpi_1 = 0$ and $M_1 = 0$. The planet P_2 can be located everywhere in the plane $O'xy$ with any velocity components \dot{x}_2 and \dot{y}_2 (Fig. 4). So, provided that $\dot{x}_1 = 0$, the periodicity conditions are

$$\begin{aligned} x_1(T) &= x_1(0), & x_2(T) &= x_2(0), & y_2(T) &= y_2(0), \\ \dot{x}_2(T) &= \dot{x}_2(0), & \dot{y}_2(T) &= \dot{y}_2(0) \end{aligned} \quad (21)$$

Thus, all variables, except \dot{x}_1 , which defines the reference plane, are necessary for the localization of an asymmetric periodic orbit.

3.4. Parametric continuation and families of periodic orbits

Considering a periodic orbit with initial conditions $(x_{10}, \dot{x}_{10}, x_{20}, y_{20}, \dot{x}_{20}, \dot{y}_{20})$ on a fixed reference plane, we can proceed to a parametric continuation of the periodic orbit by adopting a non-zero initial condition, e.g. x_{10} , as a parameter. When this continuation is possible (see [6], [11]), at the new value $x'_{10}=x_{10}+\Delta x_{10}$ corresponds a new periodic orbit with some initial conditions

$$(x_{10} + \Delta x_{10}, \dot{x}_{10} + \Delta \dot{x}_{10}, x_{20} + \Delta x_{20}, y_{20} + \Delta y_{20}, \dot{x}_{20} + \Delta \dot{x}_{20}, \dot{y}_{20} + \Delta \dot{y}_{20})^*$$

where the star symbol indicates that the new initial conditions are restricted by the fixed reference plane and by the fixed zero-valued components. By changing smoothly the parameter x_{10} , we form a characteristic curve in the space of initial conditions, which is a *monoparametric family* of periodic orbits.

3.5. Continuation via differential corrections

Given a periodic orbit $(x_{10}, \dot{x}_{10}, x_{20}, y_{20}, \dot{x}_{20}, \dot{y}_{20})$ of multiplicity k and a small parameter displacement Δx_{10} , the new periodic orbit is localized by determining the displacements of the other initial conditions, which are supposed to be small too, from the periodicity conditions. The *differential corrections* to the initial conditions are calculated using a *Newton-Raphson Shooting* iterative process [11]. Next, we describe the formulation of the algorithm for each particular case. The evolution of a variable z along an orbit with initial conditions $x_{10}, \dots, \dot{y}_{20}$ will be denoted as:

$$z(t) = z(t; x_{10}, \dot{x}_{10}, x_{20}, y_{20}, \dot{x}_{20}, \dot{y}_{20})$$

3.5.1. Symmetric periodic orbits

Starting with a periodic orbit of initial conditions $(x_{10}, \dot{x}_{10}=0, x_{20}, y_{20}=0, \dot{x}_{20}=0, \dot{y}_{20})$ we are looking for another periodic orbit at $x'_{10}=x_{10}+\Delta x_{10}$ with initial conditions $(x'_{10}, \dot{x}_{10}=0, x_{20}+\Delta x_{20}, y_{20}=0, \dot{x}_{20}=0, \dot{y}_{20}+\Delta \dot{y}_{20})$. The periodicity conditions (19) are written as :

$$\begin{aligned} \dot{x}_1(t^*; x'_{10}, 0, x_{20} + \Delta x_{20}, 0, 0, \dot{y}_{20} + \Delta \dot{y}_{20}) &= 0 \\ \dot{x}_2(t^*; x'_{10}, 0, x_{20} + \Delta x_{20}, 0, 0, \dot{y}_{20} + \Delta \dot{y}_{20}) &= 0 \end{aligned}$$

The time t^* denotes the time that corresponds to the k th section of the orbit with the reference plane. The differential corrections are given by:

$$\begin{pmatrix} \Delta x_{20} \\ \Delta \dot{y}_{20} \end{pmatrix} = \begin{pmatrix} \partial \dot{x}_1 / \partial x_{20} & \partial \dot{x}_1 / \partial \dot{y}_{20} \\ \partial \dot{x}_2 / \partial x_{20} & \partial \dot{x}_2 / \partial \dot{y}_{20} \end{pmatrix}_0^{-1} \begin{pmatrix} \dot{x}_1(t^*; x'_{10}, x_{20}, \dot{y}_{20}) \\ \dot{x}_2(t^*; x'_{10}, x_{20}, \dot{y}_{20}) \end{pmatrix} \quad (22)$$

3.5.2. Asymmetric periodic orbits with periodicity conditions of case A

Starting with a periodic orbit of initial conditions $(x_{10}, \dot{x}_{10}, x_{20}, y_{20}=0, \dot{x}_{20}, \dot{y}_{20})$ we are looking for another periodic orbit at $x'_{10}=x_{10}+\Delta x_{10}$ with initial conditions $(x'_{10}, \dot{x}_{10}+\Delta \dot{x}_{10}, x_{20}+\Delta x_{20}, y_{20}=0, \dot{x}_{20}+\Delta \dot{x}_{20}, \dot{y}_{20}+\Delta \dot{y}_{20})$. The periodicity conditions are

$$\begin{aligned} x_{20}(t^*; x_{10} + \Delta x_{10}, x_{20} + \Delta x_{20}, y_{20} = 0, \dot{x}_{10} + \Delta \dot{x}_{10}, \dot{x}_{20} + \Delta \dot{x}_{20}, \dot{y}_{20} + \Delta \dot{y}_{20}) &= x_{20} + \Delta x_{20} \\ \dot{x}_{10}(t^*; x_{10} + \Delta x_{10}, x_{20} + \Delta x_{20}, y_{20} = 0, \dot{x}_{10} + \Delta \dot{x}_{10}, \dot{x}_{20} + \Delta \dot{x}_{20}, \dot{y}_{20} + \Delta \dot{y}_{20}) &= \dot{x}_{10} + \Delta \dot{x}_{10} \\ \dot{x}_{20}(t^*; x_{10} + \Delta x_{10}, x_{20} + \Delta x_{20}, y_{20} = 0, \dot{x}_{10} + \Delta \dot{x}_{10}, \dot{x}_{20} + \Delta \dot{x}_{20}, \dot{y}_{20} + \Delta \dot{y}_{20}) &= \dot{x}_{20} + \Delta \dot{x}_{20} \\ \dot{y}_{20}(t^*; x_{10} + \Delta x_{10}, x_{20} + \Delta x_{20}, y_{20} = 0, \dot{x}_{10} + \Delta \dot{x}_{10}, \dot{x}_{20} + \Delta \dot{x}_{20}, \dot{y}_{20} + \Delta \dot{y}_{20}) &= \dot{y}_{20} + \Delta \dot{y}_{20} \end{aligned} \quad (23)$$

where, now, t^* is the time of the $(2k-1)$ th section with the reference plane because the conditions (20) refer to $t=T$ and not to $t=T/2$ as in the symmetric cases. The differential corrections are given by:

$$\begin{pmatrix} \Delta x_{20} \\ \Delta \dot{x}_{10} \\ \Delta \dot{x}_{20} \\ \Delta \dot{y}_{20} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_2}{\partial x_{20}} - 1 & \frac{\partial x_2}{\partial \dot{x}_{10}} & \frac{\partial x_2}{\partial \dot{x}_{20}} & \frac{\partial x_2}{\partial \dot{y}_{20}} \\ \frac{\partial \dot{x}_1}{\partial x_{20}} & \frac{\partial \dot{x}_1}{\partial \dot{x}_{10}} - 1 & \frac{\partial \dot{x}_1}{\partial \dot{x}_{20}} & \frac{\partial \dot{x}_1}{\partial \dot{y}_{20}} \\ \frac{\partial \dot{x}_2}{\partial x_{20}} & \frac{\partial \dot{x}_2}{\partial \dot{x}_{10}} & \frac{\partial \dot{x}_2}{\partial \dot{x}_{20}} - 1 & \frac{\partial \dot{x}_2}{\partial \dot{y}_{20}} \\ \frac{\partial \dot{y}_2}{\partial x_{20}} & \frac{\partial \dot{y}_2}{\partial \dot{x}_{10}} & \frac{\partial \dot{y}_2}{\partial \dot{x}_{20}} & \frac{\partial \dot{y}_2}{\partial \dot{y}_{20}} - 1 \end{pmatrix}_0^{-1} \begin{pmatrix} x_2(t^*; x'_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}, \dot{y}_{20}) \\ y_2(t^*; x'_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}, \dot{y}_{20}) \\ \dot{x}_2(t^*; x'_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}, \dot{y}_{20}) \\ \dot{y}_2(t^*; x'_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}, \dot{y}_{20}) \end{pmatrix} \quad (24)$$

3.5.3. Asymmetric periodic orbits with periodicity conditions of case B

Starting with a periodic orbit of initial conditions $(x_{10}, \dot{x}_{10}=0, x_{20}, y_{20}, \dot{x}_{20}, \dot{y}_{20})$ a New periodic orbit with initial conditions $(x'_{10}, \dot{x}_{10} = 0, x_{20} + \Delta x_{20}, y_{20} + \Delta y_{20}, \dot{x}_{20} + \Delta \dot{x}_{20}, \dot{y}_{20} + \Delta \dot{y}_{20})$, where $x'_{10} = x_{10} + \Delta x_{10}$, is seek. The periodicity conditions (21) are written:

$$\begin{aligned} x_{20}(t^*; x_{10} + \Delta x_{10}, x_{20} + \Delta x_{20}, y_{20} + \Delta y_{20}, \dot{x}_{10} = 0, \dot{x}_{20} + \Delta \dot{x}_{20}, \dot{y}_{20} + \Delta \dot{y}_{20}) &= x_{20} + \Delta x_{20} \\ y_{20}(t^*; x_{10} + \Delta x_{10}, x_{20} + \Delta x_{20}, y_{20} + \Delta y_{20}, \dot{x}_{10} = 0, \dot{x}_{20} + \Delta \dot{x}_{20}, \dot{y}_{20} + \Delta \dot{y}_{20}) &= y_{20} + \Delta y_{20} \\ \dot{x}_{20}(t^*; x_{10} + \Delta x_{10}, x_{20} + \Delta x_{20}, y_{20} + \Delta y_{20}, \dot{x}_{10} = 0, \dot{x}_{20} + \Delta \dot{x}_{20}, \dot{y}_{20} + \Delta \dot{y}_{20}) &= \dot{x}_{20} + \Delta \dot{x}_{20} \\ \dot{y}_{20}(t^*; x_{10} + \Delta x_{10}, x_{20} + \Delta x_{20}, y_{20} + \Delta y_{20}, \dot{x}_{10} = 0, \dot{x}_{20} + \Delta \dot{x}_{20}, \dot{y}_{20} + \Delta \dot{y}_{20}) &= \dot{y}_{20} + \Delta \dot{y}_{20} \end{aligned} \quad (25)$$

where t^* is defined as in the previous case and the differential corrections are given by:

$$\begin{pmatrix} \Delta x_{20} \\ \Delta y_{20} \\ \Delta \dot{x}_{20} \\ \Delta \dot{y}_{20} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_2}{\partial x_{20}} - 1 & \frac{\partial x_2}{\partial y_{20}} & \frac{\partial x_2}{\partial \dot{x}_{20}} & \frac{\partial x_2}{\partial \dot{y}_{20}} \\ \frac{\partial y_2}{\partial x_{20}} & \frac{\partial y_2}{\partial y_{20}} - 1 & \frac{\partial y_2}{\partial \dot{x}_{20}} & \frac{\partial y_2}{\partial \dot{y}_{20}} \\ \frac{\partial \dot{x}_2}{\partial x_{20}} & \frac{\partial \dot{x}_2}{\partial y_{20}} & \frac{\partial \dot{x}_2}{\partial \dot{x}_{20}} - 1 & \frac{\partial \dot{x}_2}{\partial \dot{y}_{20}} \\ \frac{\partial \dot{y}_2}{\partial x_{20}} & \frac{\partial \dot{y}_2}{\partial y_{20}} & \frac{\partial \dot{y}_2}{\partial \dot{x}_{20}} & \frac{\partial \dot{y}_2}{\partial \dot{y}_{20}} - 1 \end{pmatrix}_0^{-1} \begin{pmatrix} x_2(T; x'_{10}, x_{20}, y_{20}, \dot{x}_{20}, \dot{y}_{20}) \\ y_2(T; x'_{10}, x_{20}, y_{20}, \dot{x}_{20}, \dot{y}_{20}) \\ \dot{x}_2(T; x'_{10}, x_{20}, y_{20}, \dot{x}_{20}, \dot{y}_{20}) \\ \dot{y}_2(T; x'_{10}, x_{20}, y_{20}, \dot{x}_{20}, \dot{y}_{20}) \end{pmatrix} \quad (26)$$

By obtaining the differential corrections, the calculations are repeated with the new corrected initial conditions. If the algorithm converges, then the periodic orbit is efficiently localized and the time t^* approximates the half period $T/2$ for the symmetric periodic orbits or the period T for the asymmetric ones. The subscript “0” of the matrix of differential corrections denotes that the derivatives (the components of the matrix) are calculated along the orbit, which is the approximation of the periodic orbit obtained in the previous step.

3.6. Computational aspects

For the practical implementation of the overall algorithm we are concerned about the following points:

- The method of numerical integration, which we use, is either the *Taylor Series Expansions* [14] or the *Bulirsch-Stoer algorithm* of variable step [12]. The tolerance in the accuracy is set to 10^{-15} . Since the time interval of integration is very sort (usually the period is about 10 – 60 t.u.), the numerical solutions are found with high accuracy. The second point where we should set a tolerance is the intersection of the numerical solutions with the reference plane ($y(t^*)=0$ or $\dot{x}_1(t^*) = 0$). The accuracy of such an approximation is set to be one order lower than that of the numerical integration and the satisfaction of the condition is based on a bisection-type method. Finally, the periodicity conditions are satisfied within accuracy of 10^{-11} - 10^{-13} .
- The parametric continuation of the families of periodic orbits is optimized when the displacement Δx_{10} in the parameter of the family is followed with a prediction for the next non-zero initial conditions, before performing differential corrections. Suppose that the non-zero initial conditions, denoted by χ_0 , are written as functions of the parameter x_{10} , i.e. $\chi_0 = \chi_0(x_{10})$. Then, the next initial conditions after the displacement Δx_{10} , i.e. $\chi'_0 = \chi_0(x_{10} + \Delta x_{10})$, can be approximated by an extrapolation based on the previous N steps. However, initial condition x_{10} which is assumed to be the parameter of the characteristic curve

does not change monotonically along the family and, subsequently, the variables $\chi_0 = \chi_0(x_{10})$ are not defined uniquely. Thus, we introduce as parameter of the characteristic curve its “length”

$$s(x_{10}, \dot{x}_{10}, x_{20}, y_{20}, \dot{x}_{20}, \dot{y}_{20}) = \int_A^B ds, \quad (27)$$

where A is an arbitrary starting point, and $B(x_{10}, \dot{x}_{10}, x_{20}, y_{20}, \dot{x}_{20}, \dot{y}_{20})$ is the last point of the known segment of the characteristic curve after some steps. Thus, s takes an increasing value along the characteristic curve and the non-zero initial conditions are uniquely defined by the functions $\chi_0 = \chi_0(s)$. The extrapolation procedure is applied to each variable $\chi_0 = \chi_0(s)$ separately. We use polynomial extrapolation of degree N , $N \geq 3$. The first $N+1$ points, which are required for the implementation of the polynomial interpolation, are taken by using very small displacements Δx_{10} in the first steps of continuation procedure and the degree of the polynomial extrapolation increases gradually according to the available points until the maximum value N .

c) Along a characteristic curve, the multiplicity of the periodic orbits may change. For the symmetric orbits this usually happens near collisions of the planets. For the asymmetric periodic orbits the change of multiplicity depends on the form of the orbit in the rotating frame and can change without approaching a collision (see the transition of the orbit (d) to orbit (e) in Fig. 6). We can overcome this difficulty by taking into account the fact that, along the characteristic curve, the period T of the orbits varies smoothly. Therefore, instead of considering a predefined fixed multiplicity, i.e. a number of intersections of the orbit with the reference plane, we consider the intersection which takes place in the time interval defined by $T \pm \Delta T$, where $\Delta T \ll T$.

d) The calculation of the derivatives in the differential correction algorithm is done numerically, and is based on the numerical solutions around the periodic orbit, and up to first order. For example, the derivative $\partial x_2 / \partial x_{20}$ in (26) is calculated in the following way:

$$\begin{aligned} x_2^{(1)}(T) &= x_2(T; x'_{10}, x_{20} - \varepsilon, y_{20}, \dot{x}_{20}, \dot{y}_{20}) \\ x_2^{(2)}(T) &= x_2(T; x'_{10}, x_{20} + \varepsilon, y_{20}, \dot{x}_{20}, \dot{y}_{20}), \quad \varepsilon \ll 1 \end{aligned}$$

and

$$\frac{\partial x_2}{\partial x_{20}} \approx \frac{x_2^{(2)}(T) - x_2^{(1)}(T)}{2\varepsilon} \quad (28)$$

The above approximation is satisfactory for the differential corrections. A better estimation of the derivatives is achieved by using the “Neville’s algorithm” [12].

4 EXAMPLES OF RESONANT SYMMETRIC AND ASYMMETRIC PERIODIC ORBITS

The determination of the families of symmetric and asymmetric periodic orbits for a particular problem is done systematically starting from the unperturbed problem and the circular orbits (for details see [6]-[8]). We can perform also a random search in particular regions of phase space. The algorithm of differential corrections has a domain of convergence and, therefore, if the initial conditions are selected in such a domain, we can obtain a periodic orbit which is used as a starting point for computing the whole family by using parametric continuation. An example is given in the following.

We consider the planets with masses $m_1=0.001$ (as Jupiter) and $m_2=0.0003$ (as Saturn) and the star with $m_3=1-(m_1+m_2)$. We have performed a random search for an asymmetric periodic orbit, considering the initial conditions of case A in the phase space domain defined by

$$\begin{aligned} x_{10} &= 1.00, & x_{20} &= 2 \pm 0.2, & y_{20} &= 0 \\ \dot{x}_{10} &= 0.1 \pm 0.05, & \dot{x}_{20} &= 0.1 \pm 0.05, & \dot{y}_{20} &= 1.5 \pm 0.5 \end{aligned}$$

The first successful convergence was taken for the asymmetric periodic orbit with initial conditions

$$\begin{aligned} x_{10} &= 1.000000000000, & x_{20} &= 2.365218874571, & y_{20} &= 0 \\ \dot{x}_{10} &= -0.261577123782, & \dot{x}_{20} &= -0.134008686585, & \dot{y}_{20} &= -1.745915579242. \end{aligned}$$

The above orbit is a 2/1 resonant periodic orbit with $e_1=0.371$ and $e_2=0.876$, and is stable. Its figure is similar to that of the orbit 4 (see fig. 6). Note that, if we change the sign of the velocity components \dot{x}_{10} and \dot{x}_{20} , we get the mirror orbit due to the symmetry Σ .

Starting from the above asymmetric periodic orbit, we perform the continuation and calculate the corresponding family where the above orbit belongs to. The whole family consists of 2/1 resonant periodic orbits. It is presented in the projection planes e_1 - e_2 and x_1 - E in Fig. 5a and 5b, respectively. At

point A the velocity components \dot{x}_{10} and \dot{x}_{20} become zero. This means that a symmetric periodic orbit is met. Indeed, the point A belongs also to a family of symmetric periodic orbits presented by the dashed curve in figures 5a and 5b. Consequently, A is a bifurcation point or, equivalently, a generating orbit for the family of asymmetric orbits. Note that at point A the stability of the symmetric orbits changes; thus, A is of critical stability, as it should be. The other edge of the family was not found. The continuation breaks after point E, because the orbits become strongly unstable and their neighborhood in phase space consists of strongly chaotic orbits.

Along the family of asymmetric orbits, the type of stability changes. This change takes place at the points A, B, C, D and E. Especially at point E the orbits turn from simple unstable to doubly unstable [6,8]. In the graph of Figure 5b we observe that the change of stability, along the family of asymmetric orbits, takes place at the extrema of the energy.

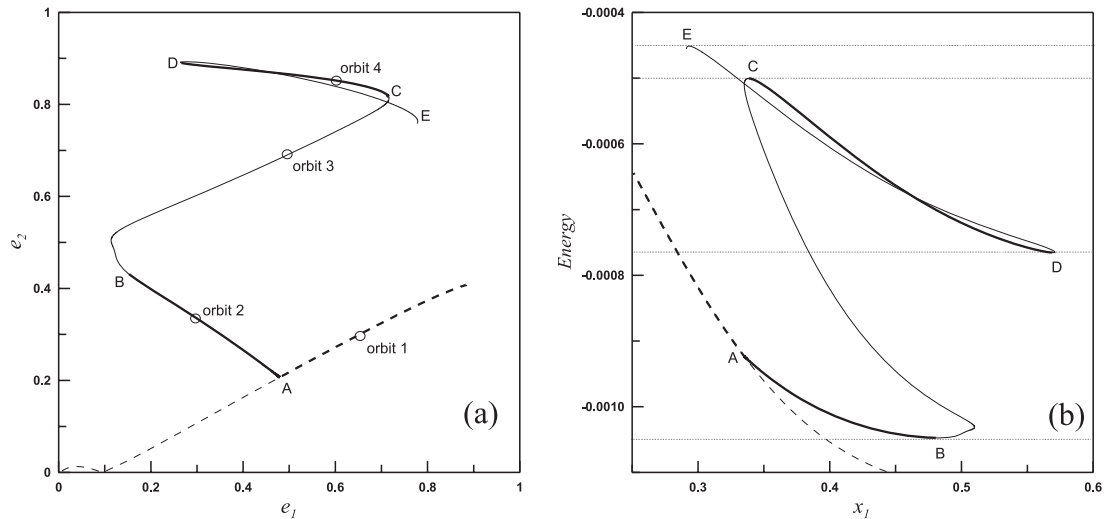


Figure 5. The characteristic curve (solid curve) of the family of asymmetric orbits projected in the planes (a) e_1 - e_2 and (b) x_1 - E . The dashed curve shows a family of symmetric periodic orbits. The bold or thin lines indicate that the corresponding orbits are stable or unstable, respectively. The dotted horizontal lines in (b) indicate the extrema of the energy where the change of stability is observed.

Some typical periodic orbits, indicated as “orbit 1”, “orbit 2” etc. in Fig. 5a, are shown in figure 6. In the panel (a) we present the symmetric “orbit 1” in the projection plane x_2 - y_2 . The symmetry with respect to the axis $y_2=0$ is clear. In panel (b) we present the corresponding planetary orbits in the inertial frame. The orbits are almost Keplerian. The planets start perpendicularly from the axis $Y=0$ which coincides with the line of apsides of the two planets; in particular, both planets start from their periastron and $\Delta\varpi = 0$.

In panels (c)-(e) the asymmetric orbits 2,3 and 4 are represented in the projection plane x_2 - y_2 . We can easily observe the non-symmetric shape of the orbits. Note that from the orbit in (d) to that in (e), the multiplicity changes from $k=1$ to $k=2$. In panels (f)-(h) the same orbits are presented in the inertial frame. The periastron of the inner planet P_1 is located on the $Y=0$ axis, where the planet is located at $t=0$. At this moment (and for any other moment when P_1 passes through the axis $Y=0$), the planet P_2 is neither at its periastron nor at its apostron. Also, the asymmetry is indicated geometrically by the line of apsides of the planet P_2 , which does not coincide with that of planet P_1 , namely, $\Delta\varpi \neq 0$ holds. It is shown that in cases of panels (g) and (h) the planetary orbits intersect. This fact does not affect the stability of the “orbit 4” (case h) which, although it corresponds to highly eccentric motion with intersecting planetary orbits, is nevertheless stable, due to the phase protection provided by the 2/1 resonance [15].

We remark that the periodic orbits refer to the inertial frame of reference. The orbits presented in figure 6 are given for an integration time interval $(0, T)$. The shape of orbits referring to the rotating frame is invariant for any integration time interval because of the periodicity. But the Keplerian ellipses in the inertial frame precess slowly in time.

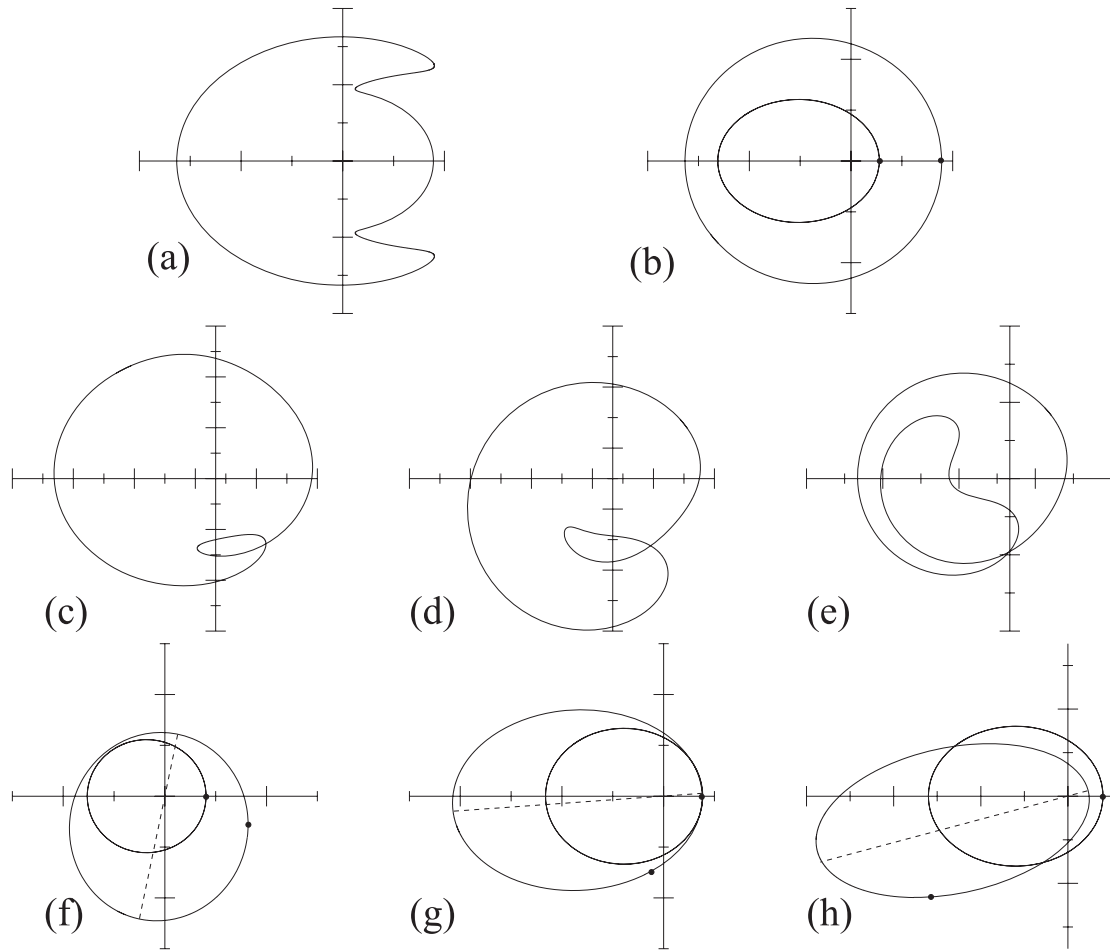


Figure 6. The periodic orbits 1-4 indicated in Figure. 5a. (a)-(b) The symmetric periodic orbit 1 in the rotating projection plane x_2 - y_2 and the corresponding planetary orbits in the inertial frame, respectively. Panels (c)-(d) present the asymmetric orbits 2,3 and 4 in the plane x_2 - y_2 and panels (g)-(h) present the corresponding planetary orbits in the inertial frame.

5 CONCLUSIONS

In this paper we present and discuss about the algorithm for locating periodic orbits in the general planar three body problem of planetary type using a rotating frame of reference. We emphasize the localization of asymmetric periodic orbits, which they have been proved to exist recently. These orbits play an important role in the stability of planetary motion, even for high eccentricities. In our Solar system, asymmetries are not indicated for the planets, since all of them move with almost circular orbits. However, the observations of extra-solar planetary systems show that stability can be obtained for asymmetric configurations and relatively large eccentricities.

The overall algorithm for computing the periodic orbits is numerical, since the system is non-integrable. For the localization of a symmetric periodic orbit, the solution of two periodicity conditions is required. For the asymmetric ones the periodicity conditions are four. Also, for the asymmetric orbits, we can define two different, but equivalent, cases of initial and periodicity conditions.

The method of differential corrections can be applied efficiently in both cases (symmetric or asymmetric), provided that our numerical integration is sufficiently accurate and we start sufficiently close to the target periodic orbit, i.e. we start with initial conditions inside the range of convergence of the Newton-Raphson algorithm. There are indications that such a range of convergence is significantly smaller for asymmetric periodic orbits than that of the symmetric ones.

The continuation of the periodic orbits and the construction of the whole family is achieved computationally by using differential corrections assisted by a polynomial extrapolation scheme of degree ≥ 3 . The continuation becomes computationally costly when the orbits are very unstable and

near them the phase space is filled with strongly chaotic motion. The characteristic curves of the families of asymmetric periodic orbits are, in general, of more complicated shape with respect to the symmetric ones (see Fig. 5, and the results in [7,8]).

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