Dynamics of KBOs near the 4:5 mean motion resonance with Neptune

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Abstract. We study the dynamics at the 4:5 exterior mean motion resonance using the model of the restricted three-body problem with the Sun and Neptune as primaries. The position and the stability character of the periodic orbits determine critically the structure of the phase space and particularly the qualitative characteristics of the long-term evolution of trans-Neptunian objects (TNOs). Using the circular planar model as the basic model, we extend our study to other models by considering the ellipticity of Neptune's orbit and the inclination of TNO's orbit. Families of symmetric periodic orbits of the planar elliptic and the three-dimensional problem (circular or elliptic) are found. The stability of all orbits is also examined. Finally, we show the evolution of a real object, i.e. the 2003 FC128, which is located near the 4:5 resonance.

Keywords: Restricted 3-body problem, Periodic orbits, Kuiper Belt, methods:numerical
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INTRODUCTION

The interest in studying exterior resonances is continuously growing after the discovery of many objects at the edge of the Solar System, beyond the orbit of Neptune, called Egdeworth-Kuiper (E-K) belt objects or trans-Neptunian objects (TNOs). Several studies of the dynamics of the E-K belt have been made up to now ([9], [8], [2], [11], [10]).

The restricted three-body problem (RTBP) is an efficient model for understanding the underlying dynamics of the resonant orbits in Kuiper Belt. In this model the primary bodies are the Sun and Neptune and the small body (of negligible mass) represents a potential TNO. As far as the Kuiper belt dynamics is concerned, the families of symmetric periodic orbits at the 1:2, 2:3 and 3:4 resonances have been studied previously ([5]). A complete study of periodic orbits for higher-order resonances was made in [14].

For the study of inclined orbits, the three-dimensional (3-D) RTBP should be considered as the basic model. Three-dimensional symmetric periodic orbits of the circular RTBP have been studied by several authors in the past ([1], [12]). These studies refer to families of 3D periodic orbits associated to the dynamics near the Lagrangian equilibrium points. As far as the dynamics in Kuiper belt is concerned, families of 3-D symmetric periodic orbits were computed in [5], [6], [7].

The study of the dynamics at the 4:5 mean motion resonance with Neptune will be done by computing the position and the stability of periodic orbits. Other known methods are: averaged Hamiltonians and mapping models. The periodic orbits are of particular interest since they define the structure of the associated resonance. In this paper we shall consider a hierarchy of models: (i) planar circular, (ii) planar elliptic, (iii) 3-D circular and (iv) 3-D elliptic RTBP. We will compute families of symmetric periodic orbits, discuss about the topology of the phase-space and present the long-term evolution of real TNOs near the 4:5 resonance.

THE PLANAR CIRCULAR RTBP

We assume that the orbit of Neptune is circular and we consider a rotating frame of reference *Oxyz* whose *x*-axis is the line joining the Sun (*S*), with Neptune (*N*), the positive direction being from *S* to *N*, its origin is at their centre of mass, the *y*-axis is in the orbital plane of Neptune and the *z*-axis is perpendicular to the *xy* plane. In the usual normalized units where the radius $SN = 1$, the gravitational constant is $G = 1$ and the total mass $m_S + m_N = 1$, the differential equations of motion are the equations of the circular restricted 3-body problem (e.g. [13]).

The symmetric periodic orbits of the planar circular RTBP correspond to the initial conditions:

$$x(0) = x_0,\ y(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = \dot{y}_0. \tag{1}$$

At half the period T of the orbit we must have:

$$y(x_0,\ \dot{y}_0;\ T/2) = 0,\ \dot{x}(x_0,\ \dot{y}_0;\ T/2) = 0. \tag{2}$$

Equations (2) give the periodicity conditions for a symmetric periodic orbit in the planar circular problem. A symmetric periodic orbit of multiplicity p starts perpendicularly from the x-axis and again crosses the x-axis perpendicularly after p intersections in the same direction.

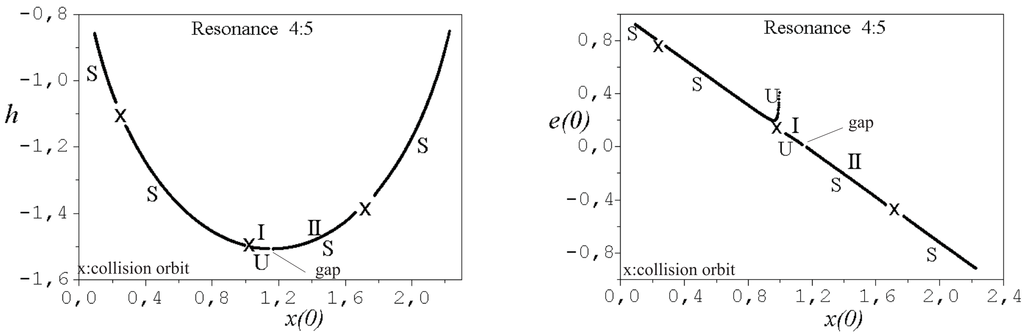

FIGURE 1. The families *I* and *II* of 4:5 resonant symmetric periodic orbits at the plane: (a) $x(0) - h$ and (b) $x(0) - e(0)$. The gap at the resonance is due to the non-continuation of the circular orbits. The symbols "*S*" and "*U*" denote stability and instability respectively. The symbol "*x*" stands for collision orbit. The small body is initially at perihelion ($e(0)$>0) or at aphelion ($e(0)$<0).

There are two families of periodic orbits for the 4:5 resonance, symmetric with respect to the *x*-axis, one of them corresponding to the small body at perihelion at t =0 (Family *I*) and the other to the small body at aphelion (Family *II*). A monoparametric family

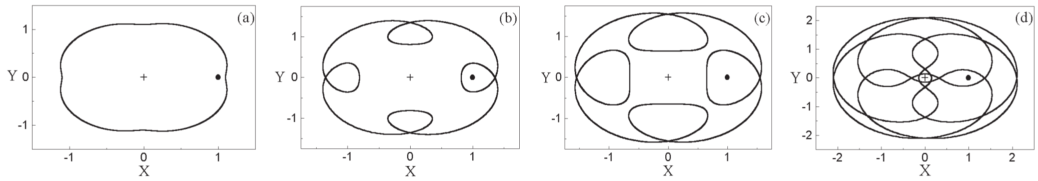

FIGURE 2. Periodic orbits along the family $I_{4:5}$: (a) e=0.05, U (b) e=0.30, S (c) e=0.44, S and (d) e=0.87, S. The development of a non-perpendicular collision orbit and the change of the multiplicity from 1 to 3 and from 3 to 7 is clearly seen. The symbols "+" and "•" denote the positions of Sun and Neptune respectively.

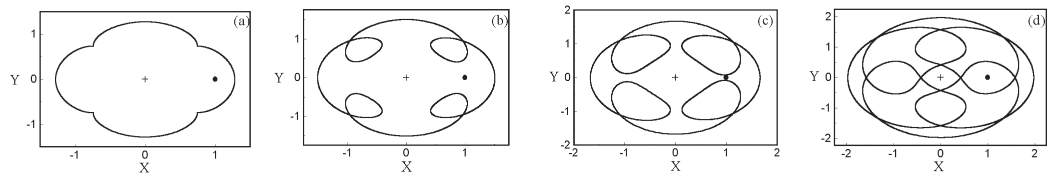

FIGURE 3. Periodic orbits along the family $II_{4:5}$: (a) e=0.10, S (b) e=0.30, S (c) e=0.44, U (close to collision) and (d) e=0.70, S. The development of a non-perpendicular collision orbit and the change of the multiplicity from 1 to 5 is shown.

of symmetric periodic orbits is defined by a smooth curve in the plane $x_0 - h$ of initial conditions, which is called the "*characteristic curve*". Along each family the eccentricity of the small body increases starting from zero.

In Fig. 1 we present the resonant families *I* and *II* of periodic orbits. The stability type is also indicated. In Figs. 2, 3 we show some samples of symmetric periodic orbits of family *I* and *II* respectively. The evolution of the orbits and the change of multiplicity is shown. In Fig. 4 we present two surfaces of section at the energy levels $h_1 = -1.50$ (Fig. 4a) and $h_2 = -1.46$ (Fig. 4b). We note that apart from the 4:5 resonance, several other first order resonances appear. As the h increases, the width of the resonant island decreases (see also [8]).

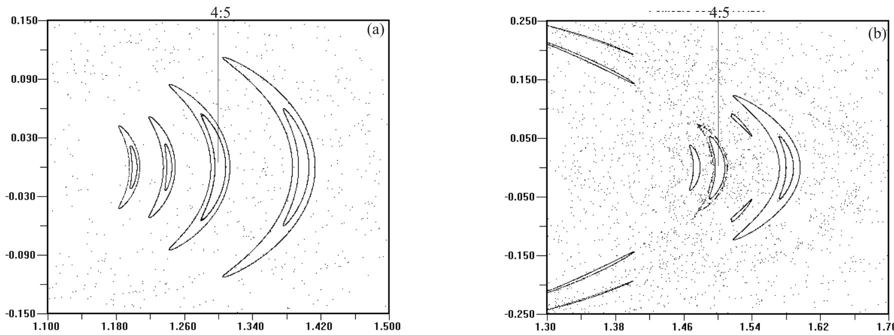

FIGURE 4. Poincaré maps taken at two characteristic energy levels: (a) $h_1 = -1.50$ and (b) $h_2 = -1.46$.

THE PLANAR ELLIPTIC RTBP

In this section we study the resonant dynamics of the elliptic restricted three body problem. We have two degrees of freedom, as in the circular problem, but now the system is *non*-autonomous. The distance between the Sun and Neptune is not constant and the xOy frame is not a uniformly rotating system. The initial conditions also involve the position and velocity of the Sun and Neptune at t=0. For a symmetric periodic orbit of the small body we get:

$$y(0) = 0,\ \dot{x}(0) = 0,\ \dot{r}(0) = 0,\ \theta(0) = 0,\ \pi, \qquad (3)$$

with an indication that Neptune is at perihelion (p) or aphelion (a) at t=0. The nonzero initial conditions are then: $x(0)$=x_0, $\dot{y}(0) = \dot{y}_0$ and also $r(0)$, $\dot{\theta}(0)$, which, however, are determined from Keplerian theory in terms of the eccentricity of Neptune. Thus, the eccentricity of Neptune (e_N) appears as a parameter of the problem. By varying e_N we have a family of periodic orbits of the elliptic problem. For $\dot{r}$(0)=0 and θ(0)=0 or π the periodicity conditions are now:

$$y(x_0,\ \dot{y}_0;\ e_N;\ T/2) = 0,\ \dot{x}(x_0,\ \dot{y}_0;\ e_N;\ T/2) = 0, \qquad (4)$$

where T is the period of the periodic orbit in the rotating frame. Due to the fact that at $t = T/2$ we must have $\dot{r}$=0 and θ=0, π too, it arises that the period T must necessarily be a multiple of T_{Nep} where T_{Nep} is the period of Neptune. This means that the periodic orbits in the elliptic problem in the rotating frame are also periodic in the inertial frame.

There is one pair of families of periodic orbits bifurcating from the planar orbits of the family $II_{4:5}$, at e_1 =0.25, and another pair of such families bifurcating from the point e_2=0.87. The first pair is: $E_{1p}^{4/5}$, $E_{1a}^{4/5}$ and the second one is: $E_{2p}^{4/5}$, $E_{2a}^{4/5}$. These families are shown in Fig. 5. The curves on the x_0-$\dot{y}_0$ plane are the two families of the circular problem that are given in Fig. 1.

THE 3-D CIRCULAR RTBP

We present in this section the 4:5 resonant families of periodic orbits of 3-D circular RTBP. These families bifurcate from the families of type II of the planar circular problem at those orbits whose stability is critical with respect to perturbations along the z-axis. For the computation of the vertical stability we used the method developed by Hénon (1973). There are five vertical critical orbits (VCOs), which are starting points for the continuation of families of 3D periodic orbits. They belong to the family $II_{4:5}$ of the planar problem and are the following: B_1(e_0 =0.22), B_2(e_0 =0.23), B_3(e_0 =0.62), B_4(e_0 =0.73) and B_5(e_0 =0.83). No such VCOs exist on the family $I_{4:5}$.

The families of three-dimensional symmetric periodic orbits have the following symmetries:

Families F: These periodic orbits are symmetric with respect to xz-plane. In this case a periodic orbit can be defined by the initial conditions $x(0) = x_0$, $y(0) = 0$, $z(0) = z_0$, $\dot{x}(0) = 0$, $\dot{y}(0) = \dot{y}_0$, $\dot{z}(0) = 0$ and the periodicity conditions are:

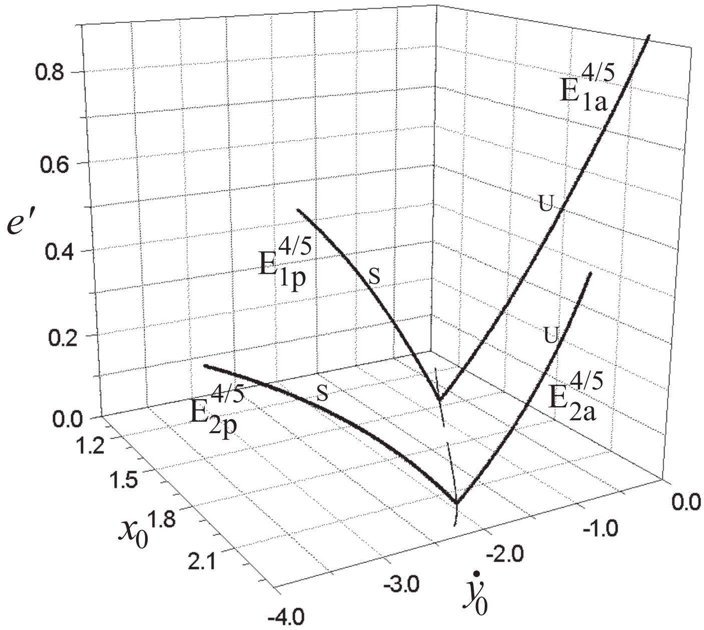

FIGURE 5. Families of periodic orbits of the elliptic restricted three-body problem at 4:5 resonance. Here e' is the eccentricity of Neptune (e_N).The symbols *S, U* denote for stability and instability respectively (from [14]).

$$y(0) = y(T/2) = 0,\ \dot{x}(0) = \dot{x}(T/2) = 0,\ \dot{z}(0) = \dot{z}(T/2) = 0, \tag{5}$$

where T is the period of the orbit. Such a periodic orbit can be represented as a point in 3-D space $x_0 - z_0 - h$, where h is the corresponding Jacobi constant. By varying the value of h we get a monoparametric family of periodic orbits.

Families G: These periodic orbits are symmetric with respect to x-axis. In this case a periodic orbit can be defined by the initial conditions $x(0) = x_0$, $y(0) = 0$, $z(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = \dot{y}_0$, $\dot{z}(0) = \dot{z}_0$ and the periodicity conditions are:

$$y(0) = y(T/2) = 0,\ \dot{x}(0) = \dot{x}(T/2) = 0,\ z(0) = z(T/2) = 0, \tag{6}$$

where T is the period of the orbit. Such a periodic orbit can be represented as a point in 3-D space $x_0 - \dot{z}_0 - h$, where h is the corresponding Jacobi constant. By varying the value of h we get a monoparametric family of periodic orbits.

In Fig. 6 we present the families of 3-D resonant periodic orbits at the 4:5 exterior mean motion resonance with Neptune. The eccentricity and the inclination vary along a family, but the semimajor axis is almost constant, $a_{4:5} \approx 34.88$ A.U. (in non-dimensional units). Sample of periodic orbits are given in Fig.7. Finally, at Fig. 8, we show the projections of these families on the plane $e_0 - i_0$. The stability type is also indicated.

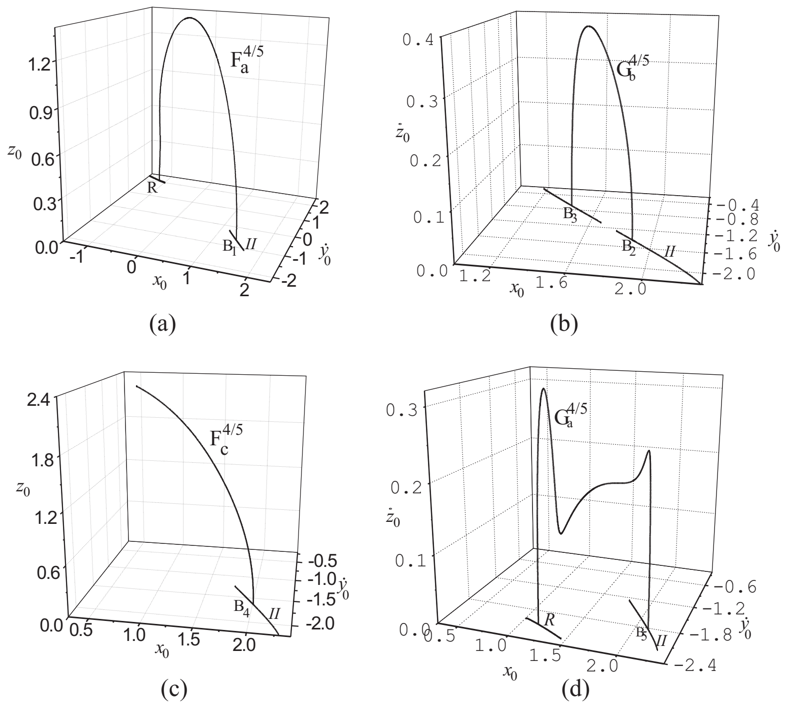

FIGURE 6. Families of periodic orbits of the 3-D circular restricted three-body problem at 4:5 resonance. The starting and the terminating points of each family are also indicated. The axes correspond to the initial conditions of periodic orbits. The symbol *R* stands for retrograde orbits.

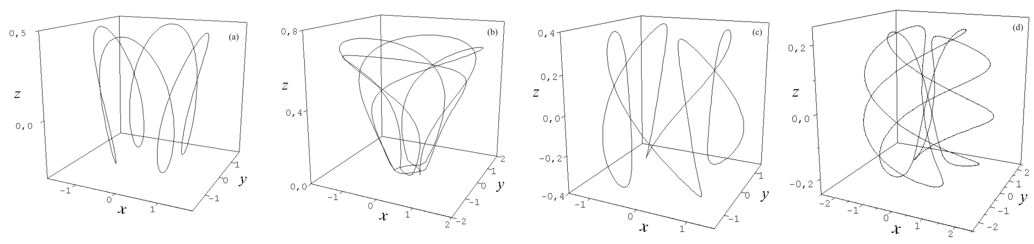

FIGURE 7. Samples of three-dimensional periodic orbits at 4:5 resonance. Each one of them belongs to the corresponding family presented in figure 6.

THE 3-D ELLIPTIC RTBP (ER3BP)

The 3-D elliptic RTBP is a non-autonomous Hamiltonian system with three degrees of freedom. This model is the most realistic one in the framework of the restricted three-body problem, since it describes the motion of the small body in space and the perturbing planet is on an elliptic orbit.

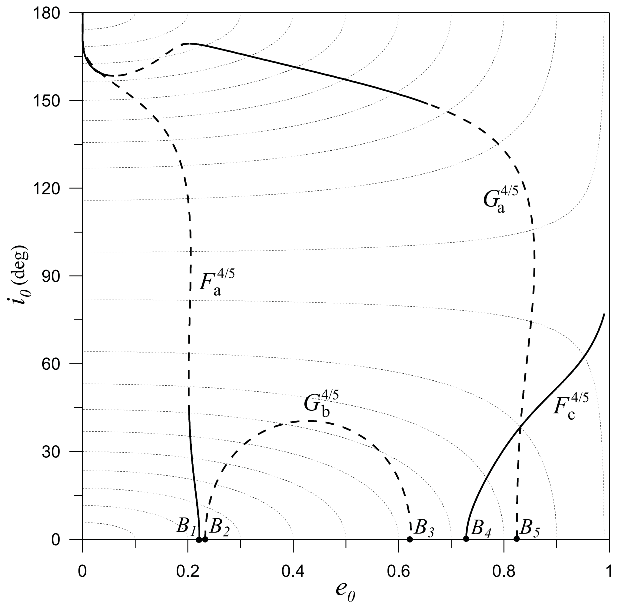

FIGURE 8. Projection of the families of periodic orbits of the 3-D circular restricted three-body problem at 4:5 resonance. Solid segments denote linear stability and dashed lines stand for linear instability. The thin dotted lines denote the curves of z-argument of the angular momentum related to the *Kozai* dynamics (from [7]).

There are two approximations for obtaining periodic orbits in the 3-D ER3BP:

(a) Families of periodic orbits of the 3-D ER3BP, with the eccentricity of Neptune (e_N) as a parameter, bifurcate from the families of the 3-D CR3BP mentioned above at those orbits whose period is a multiple of T_{Nep}. The results are summarized in Table 1.

TABLE 1
Bifurcation points from 3-D CR3BP to 3-D ER3BP

Resonance	Period	Family	Bifurcation Points
$4:5$	10π	F_a	e=0.204, i=80.9°
$4:5$	10π	G_a	e=0.865, i=86.8°

(b) The second way to search for periodic orbits that bifurcate to the 3-D ER3BP is to check the vertical stability of periodic orbits of the corresponding planar elliptic problem. From our results there are not any such periodic orbits which continue from the planar elliptic problem to the three-dimensional one.

A monoparametric family of symmetric periodic orbits is formed by analytic continuation varying the eccentricity of the planet (e') which is the parameter of the family in this case. Such a family can be represented by a curve in the 4D space $x_0 - \dot{y}_0 - z_0 - e'$ or $x_0 - \dot{y}_0 - \dot{z}_0 - e'$ accordingly to the symmetry of the periodic orbits. When $e' \neq 0$ we get two possible configurations of the system at $t = 0$. In the first configuration the planet is located at pericenter and in the second one at apocenter. Thus, from each generating orbit (or bifurcation point) two families bifurcate.

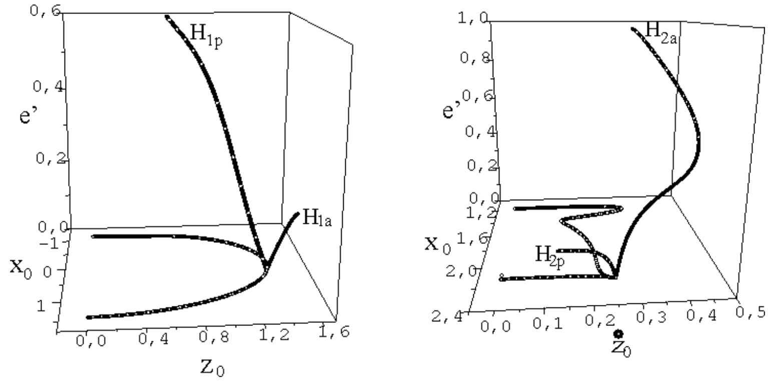

FIGURE 9. Projection of families of 3-D symmetric periodic orbits at the three-dimensional elliptic RTBP in (a) $x_0 - z_0 - e'$ and (b) $x_0 - \dot{z}_0 - e'$ 3-D space of initial conditions. The symbol e' stands for the eccentricity of the orbit of Neptune.

In Fig. 9 we present the families of periodic orbits in the 3-D space. From the first bifurcation point, given in Table 1, we obtain the families H_{1p} and H_{1a}, where p or a indicates the initial position of Neptune (pericenter or apocenter, respectively). Correspondingly, we obtain the families H_{2p} and H_{2a} bifurcating from the second bifurcation point. These families are linearly unstable.

DYNAMICAL EVOLUTION OF TNOS NEAR THE 4:5 NMMR

In this section we study the long-term evolution of the real object *2003 FC128* which is near the 4:5 mean motion resonance with Neptune. Its orbital elements are: a=35.127 A.U., e=0.088, i=2.4°, ω=41.0°, Ω=78.6°, M=41.4° (*Minor Planets Center*). This KBO is on a low eccentric orbit, almost circular. We have chosen a sufficient numerical method to study the long-term evolution of this KBO; the LIE integration method with an adaptive step size (Hanslmeier and Dvorak, 1984). This method uses direct integrations of the equations of motion. The accuracy was preserved at 10^{-11}. The time evolution of the orbital elements is shown in Fig. 10 and indicates a regular orbit. However, the evolution of the angle $\Omega - \Omega_8$ switches from libration to rotation which indicates that the orbit is rather weakly chaotic.

CONCLUSIONS

In this paper we presented families of periodic orbits at the 4:5 resonance in the model of the 3-D elliptic RTBP. The main results are summarized below:

1) There exist two families of elliptic orbits in the planar circular problem at the 4:5 resonance; families $I_{4:5}$ and $II_{4:5}$. The first one is unstable until $e \approx 0.10$ where a perpendicular collision with Neptune occurs; then it becomes stable. The other one ($II_{4:5}$) is all stable except for a small collision area around $e \approx 0.47$. The location of periodic orbits is important for obtaining the structure of the phase space.

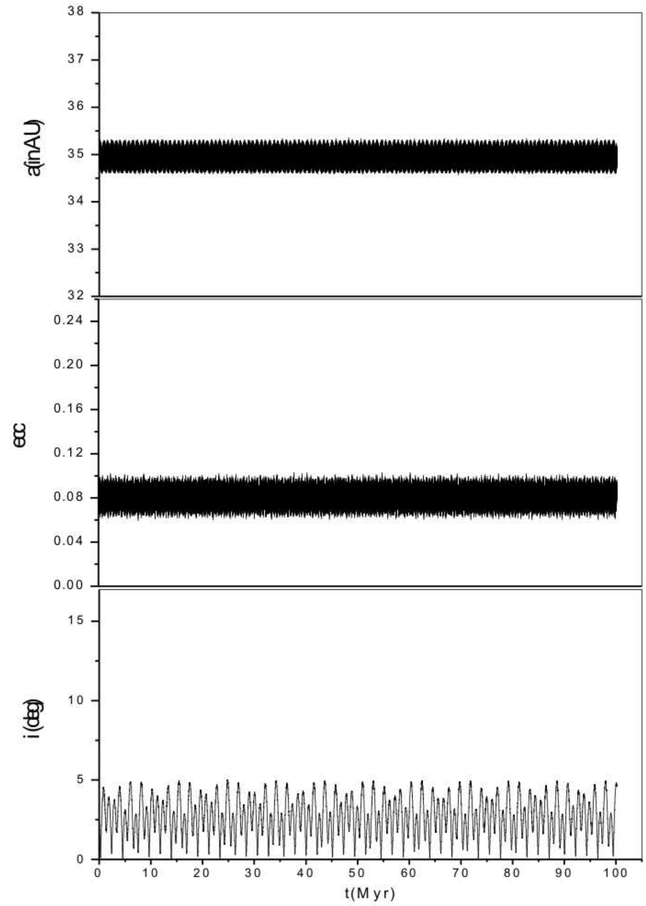

FIGURE 10. Time evolution of the orbital elements a, e i of the test body for 100 Myr. A quasi-periodically behaviour of the orbital elements is observed.

2) The families of periodic orbits of the planar elliptic RTBP bifurcate from the known families of the planar circular problem. We found totally four families of symmetric periodic orbits; two of them are linearly stable and the other ones are unstable.

3) As far as the 3-D circular RTBP is concerned, vertical critical orbits (VCOs) exist only in the family $II_{4:5}$. Families of three-dimensional periodic orbits which bifurcate from vertical critical orbits of the planar problem are symmetric with respect to the xz-plane (Type F) or to the x-axis (Type G). They continue up to high inclination values; stable and unstable orbits coexist in this model.

4) Families of periodic orbits of the 3-D ER3BP were also computed. These families bifurcate from the known families of the 3-D circular problem. They have the same symmetry as the families of periodic orbits in the 3-D circular model and are generated by varying the eccentricity of Neptune (e_N). All of them are linearly unstable.

5) The evolution of the test body (2003 FC128) seems to be weakly chaotic but practically stable.

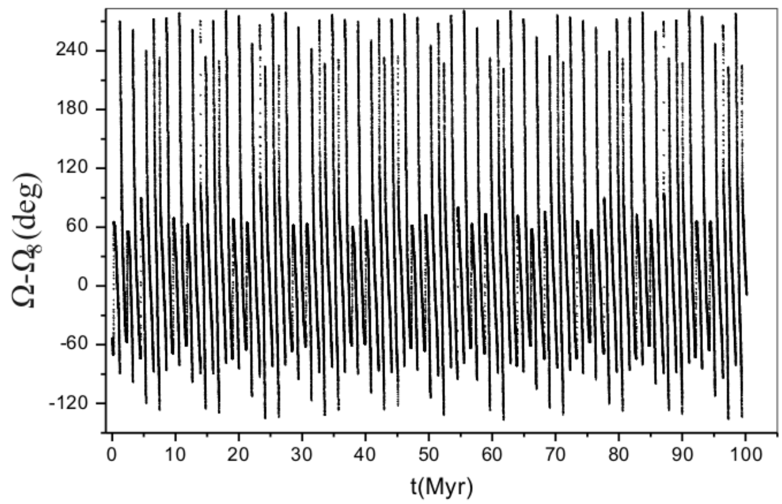

FIGURE 11. Time evolution of $\Omega - \Omega_8$ for 100 Myr. The orbit leaves the ν_{18} resonance; the argument $\Omega - \Omega_8$ switches between libration and circulation.

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REFERENCES

1. Broucke, R.A., 2001, *Cel. Mech. and Dyn. Ast.* **81**, 321
2. Gallardo T. and Ferraz-Mello. S:1998, *Planet Space Sci.*, Vol. 46, No.8, 945-965.
3. Hanslmeier A. and Dvorak R. 1984, A & A **132**, 203
4. Hénon, M., 1973, A & A, 28, 415
5. Kotoulas, T., & Hadjidemetriou, J.D., 2002, Earth, Moon and Planets, **91**, 63
6. Kotoulas, T., 2005, A & A, 429, 1107
7. Kotoulas, T., & Voyatzis, G., 2005, A & A 441, 807-814
8. Malhotra, R.:1996, *Astron. J.* **111 (1)**, 1996, 504-516
9. Morbidelli, A., Thomas, F. and Moons, M:1995, *Icarus* **118**, 322-340.
10. Morbidelli, A., Brown, M.E. & Levison H.F., 2003, Earth, Moon and Planets 92, 1-27
11. Nesvorný, D. & Roig, F.,II 2001, Icarus, 150, 104
12. Papadakis, K.E., 2004, A & A, 425, 1133
13. Roy, A.E.:1982, *Orbital Motion*, Adam Hilger, (2^{nd} ed.).
14. Voyatzis, G., & Kotoulas T.A., 2005, Planetary and Space Science, **53**, 1189-1199