ΑΡΙΣΤΟΤΕΛΕΙΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΘΕΣΣΑΛΟΝΙΚΗΣ - ΠΑΝΕΠΙΣΤΗΜΙΟ ΒΟΥΡΓΟΥΝΔΙΑΣ
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ΕΥΡΩΠΑΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΑΚΟ ΠΡΟΓΡΑΜΜΑ ERASMUS
PROGRAMME EUROPEEN UNIVERSITAIRE ERASMUS

ETUDES DOCTORALES: MATHEMATIQUES ET APPLICATIONS FONDAMENTALES

1987-1988
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Rapport sur

MODULATED WAVES IN ONE-DIMENSIONAL NONLINEAR PHYSICAL SYSTEMS

par

Georges Voyatzis

de l' Université Aristote de Thessalonique
realisé dans le
Laboratoire d' Optique du Réseau Cristallin
U.F.R. Sciences et Techniques
Université de Bourgogne

Dijon 1988

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Directeur:
Prof. Michel Remoissenet

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The main subject of this work is nonlinear waves which are supported by the Nonlinear Schrodinger Equation (NLS). The most important solutions are given and their properties and characteristics are discussed. The significant role of the NLS equation in Optical fibers and Nonlinear Electrical Transmission Lines is also indicated. At the last part, some generalization forms of the NLS equation are noticed. Since an enormous amount of literature on this field exists, many references are given for a detailed study.

It is my pleasure to express my thanks to Prof. Remoissenet for his constant and valuable supervision during this work. I would also to thank the director of the program Prof. Sp. Pnevmatikos of the University of Thessaloniki for his encouragement and the financial support. Finally I am grateful to B.Michaux, J.M.Bilbault, T.Kofane, H.Desfoutaines, D.Barday and P.Rudel for their essential help and cooperation during my stay in Dijon.

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C O N T E N T S

1. WHAT IS A SOLITON

The first observation of a soliton was made by John Scott Russell in 1834. He was looking at a boat on a narrow channel and when the boat stopped a rounded smooth heap of water was formed. The heap continued to proceed without changing of shape or diminution of speed for a long distance. In 1895, Korteweg and de Vries developed a form of a partial differential equation in order to study shallow water waves. Their equation included dispersion and nonlinear terms and exhibited solutions with properties which were able to describe the J.S. Rusells wave.

Interest in these waves began to grow up in the early of 60's. M.D. Kruskal and N.J. Zabusky studied the Korteweg de Vries equation :

$$U_t + UU_x + k^2 U_{xxx} = 0^* \qquad 1.1$$

using one of the first computers. For a small k and periodic boundary conditions, they observed([10]) that the front of the wave was going more and more steep because of the existence of the nonlinear term UU_x. When the wave was being too steep, dispersion effects increased creating a balance between nonlinearity and dispersion. The final result was that the initial sinusodial wave transformed into a series of 'humps' with remarkable properties which will be discussed later. Many equations have been derived which provide 'humps' as solutions but few of them have properties like the Rusell's wave properties or like the KdV equation's ones.

1.a. Definitions, characteristics and properties of solitons. [2]-[7],[10],[14]-[16]

One dimension wave solutions can be derived by P.D.Eqs and they consist of arbitrary functions of x and t where x,t

* subscripts denote partial differentiation.

are space and time coordinates respectively. Special solutions which depend on x and t only through $X=x-vt$, where v is a fixed constant, are called <u>travelling waves.</u> The qualitative form of a travelling wave is an infinite series of 'humps' which are periodically arranged with spacing d and velocity v (fig.1a).

A <u>solitary wave</u> is obtained by letting $d \rightarrow \infty$. Its transition from one asymptotic state as $x \rightarrow -\infty$ to another as $x \rightarrow \infty$ is essentially localized.

In figure 1b a 'pulse-like' solitary wave is given. The two asymptotic values of the amplitude as $x \rightarrow -\infty$ and as $x \rightarrow +\infty$ are equal and is easy to observe that it can be obtained from the travelling wave of fig.1a setting $d \rightarrow \infty$. In figure 1c a 'kink-like' solitary wave is represented. For this type of wave the asymptotic values are different and it is not so easy to understand which travelling wave generated it.

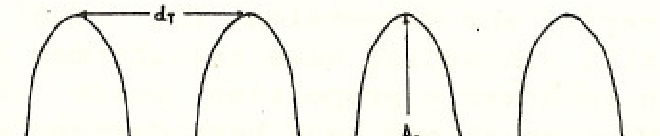
With the above definition of solitary waves we can classify the Rusells wave in them. Also we saw that solitary waves of the KdV equation are caused by the balance between nonlinearity and dispersion. Let's study the effects of these terms separately.

Consider the nonlinear and dispersionless equation of the form:

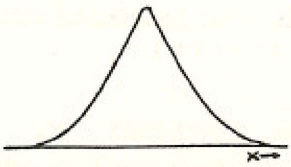
$$u_t + u \cdot u_x = 0 \qquad 1.2$$

It can be checked easily that eq. 1.2 admits the wave solution :

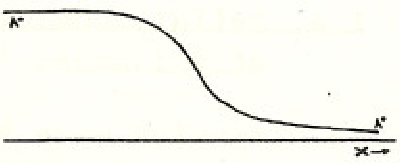
$$u(x,t) = f(x-u \cdot t) \qquad 1.2a$$



<u>fig.1a</u> A travelling wave



<u>fig.1b</u> A 'Pulse-like' solitary wave



<u>fig.1c</u> A 'Kink-like' solitary wave

It is known that for a wave solution the multiplier of t, in the arbitrary function f, coincides with the velocity of the wave. For waves of the form 1.2.a we can note that the velocity of a point of displacement u is equal to that displacement. This means that a point which corresponds to a large amplitude moves faster than an other which corresponds to a small amplitude.

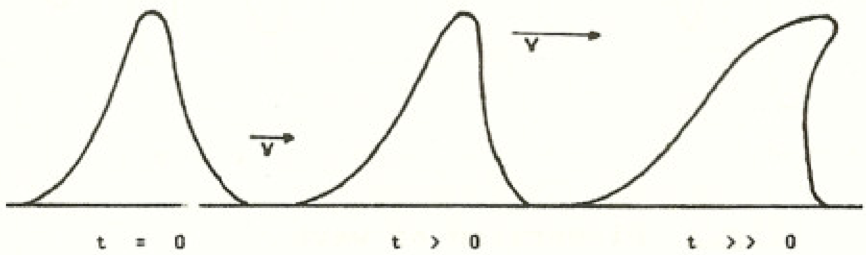


fig. 2: Nonlinear breaking of a wave

So we can conclude that when nonlinear terms appear in a wave equation, the wave steepens due to the continual injection of high frequencies. After finite time, the wave breaks up and discontinuities occur.

Consider now the linear P.D. wave equation :

$$u_t + u_{xxx} = 0 \qquad 1.3$$

The most elementary wave solution of eq. 1.3 is the harmonic solution:

$$u(x,t) = A \exp[i(kx - wt)] \qquad 1.3.a$$

where k is the wavenumber and w is the angular frequency. If we substitute the solution 1.3.a into the original equation 1.3 we find the relation :

$$w = k^3 \qquad 1.4$$

The above 'dispersion relation' is connected with the next two concepts :

<u>Phase velocity</u> $v_p = w/k = k^2$
<u>Group velocity</u> $v_g = dw/dk = 3k^2$

The phase velocity measures how fast a point of a constant phase moves, while the group velocity measures how fast the energy of the wave is transferred. If a wave is composed of a superposition of elementary components with different k's it will disperse.

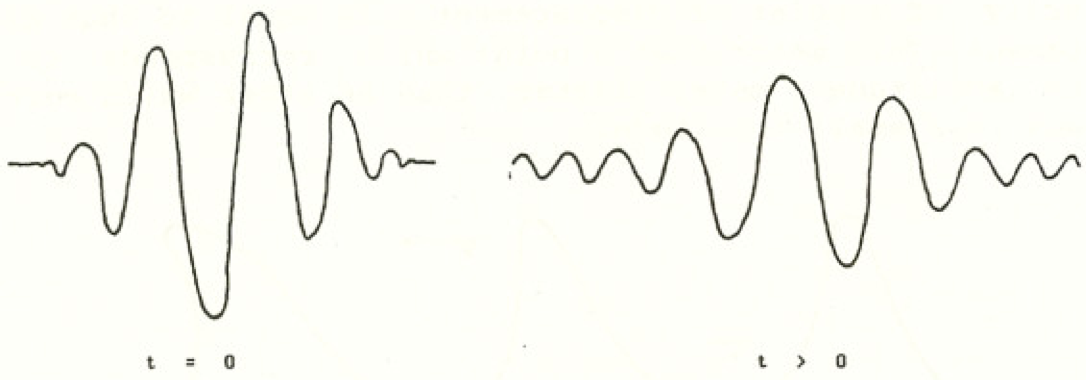


fig.3 Dispersion of wave

So, different components of the wave move at different speeds (higher frequencies move slowly) resulting to the spreading of the wave which is not recognizable after finite time.

After the above discussion we can give the conditions which support a solitary wave. The simplest example of a solitary wave is a pulse-like travelling wave of a dispersionless and linear wave equation e.g.

$$u_{xx}-(1/c^2)u_{tt}= 0 \qquad 1.5$$

The wave solution of eq. 1.5 maintains its form in space and time since all the points of the wave move with the same speed c. The effect of introducing dispersion into eq. 1.5 is to destroy the possibility of solitary waves because the various Fourier components will propagate at different velocities. Introducing nonlinearity without dispersion it removes again the possibility of the existence of a solitary wave because the pulse energy is continuously injected into higher frequenciy modes. If both dispersion and nonlinearity are present in the wave equation, steeping and spreading lead to a solution which is constant in frame moving at speed v_g. So when the effects of nonlinearity and dispersion are balanced then solitary waves appear.

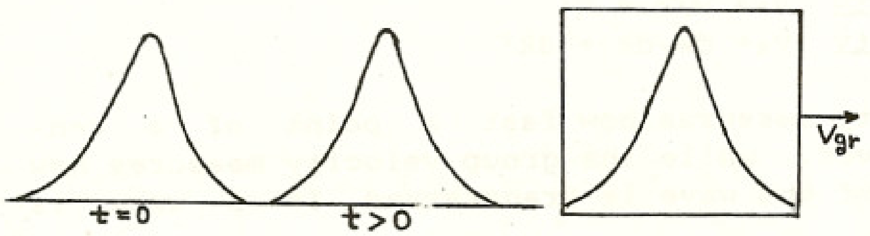
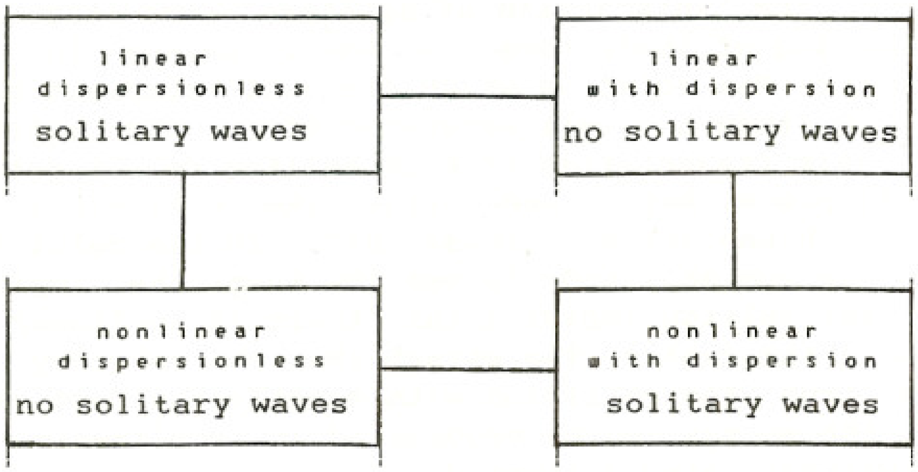


fig.4: Solitary wave in time.



Solitary waves of KdV equation and some other nonlinear evolution equations are characterized by the property that a taller solitary wave moves faster than a shorter one (see Eq. 1.2). Consider now the initial-value problem where two solitary waves of distinct amplitudes are placed with the shorter wave in front of the taller one. After a finite time the taller solitary wave will catch up with the shorter one and they will undergo a nonlinear interaction. 'Particle-like' properties of solitary waves [i.e. i) steady progressing pulse-like solution, ii) preservation of their shapes and speeds after an interaction] have been observed in many cases. Zabusky and Kruskal called these solitary waves solitons.

So, <u>a soliton</u> $u=u(x-vt)$ is a solitary wave solution of a wave equation which preserves asymptotically its shape and velocity upon collisions with other solitary waves.

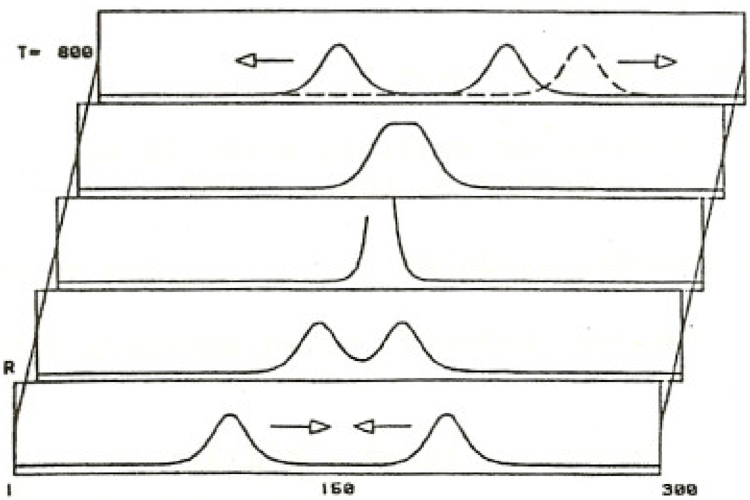


fig. 5 Interaction of pulse-like solitons

Although the interaction of solitary waves takes place without changing of their form, a small change in their phase appears. Two solitons which are well separated before the collision, they are well separated after the collision too but their trajectories of each soliton do not coincide in the (x,t)-plane, before and after the collision; this shows that each one suffer a phase shift. In the collision's region the maximum amplitude is smaller than the amplitude of the largest soliton implying that there is no linear superposition in the centre. The effect that the solitary wave re-emerges from the collision with exactly the same shape, shows that the energy can be propagated in localized stable 'packets' without being dispersed.

1.b Prototype wave equations that exhibit solitons

Recently many equations which possess soliton solutions have been derived. The most important and well studied equations are the Korteweg de Vries (KdV) equation, the sine-Gordon equation and the Cubic (or Nonlinear) Schrodinger (NLS) equation. Many other equations can be reduced in one of these types or in their generalizations. Nowadays, many physical systems, from every branch of physics, are described by these equations. A concise description of them follows. The NLSE will be the main subject in the next parts.

A. The KdV equation

The KdV equation is a useful approximation for many systems where one wishes to include a simple nonlinearity and a simple dispersion effect.

$$u_t + u \cdot u_x + u_{xxx} = 0 \qquad 1.6$$

If we wish to discover solitary waves in eq. 1.6 we must look for solutions depended on the variable $X=x-vt$. Since

$u_t=-v\, du/dX$, $u_x=du/dX$ and so on,

equation 1.6 can be reduced to an ordinary differential equation :

$$-v\, du/dX + u\, du/dX + d^3u/dX^3 = 0 \qquad 1.7$$

After integrations and for $v>0$, we can find the solution in the following form :

$$u = -12v\cdot A\cdot[A\cdot\exp(v^{1/2}/2)-\exp(-v^{1/2}X/2)]^{-2} \qquad 1.8$$

For simplicity we consider the boundary conditions:

$$u(X) \longrightarrow 0$$
$$du/dX \longrightarrow 0$$
$$d^2u/dX^2 \longrightarrow 0$$

as $|X| \longrightarrow \infty$. These boundary conditions mean that the seeked solution has equal asymptotic values as $X \longrightarrow +\infty$ and as $X \longrightarrow -\infty$. Boundary conditions are satisfied for A=-1 and the solution becomes:

$$u = 3v\ \mathrm{sech}^2[v^{1/2}(x-vt)/2]$$

Solitary wave

$$u_o = 3\cdot v,\ w \approx v^{1/2}$$

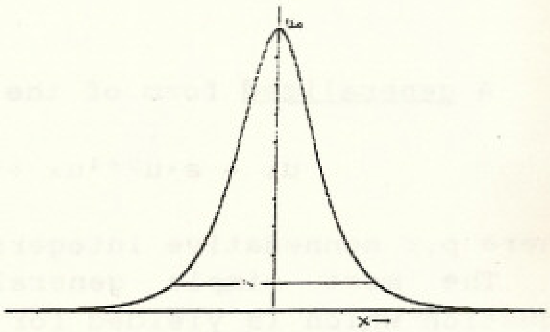


fig. 6. The KdV solitary wave.

Zabusky and Kruskal showed after numerical computions, that the solution 1.9 is a soliton.

The KdV equation was derived by Korteweg and de Vries as :

$$u_t + a\cdot uu_x + u_{xxx} = 0 \qquad 1.10$$

where a is a real parameter. It can be derived from a Langrangian density

$$L=(1/2)f_x f_t+(a/6)f_x^3+f_x Y_x+(1/2)Y^2 \qquad 1.11^*$$

where $f_x=u$ and $Y=f_{xx}$. Eq. 1.11 is completely integrable and is met with in 1) ion-acoustic waves in plasma, 2)magnetohy-

* For a Lagrangian $L=L(f_x, f_t, f)$ the Euler-Langrange equation is :

$$\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial f_x}\right)+\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial f_t}\right)-\frac{\partial L}{\partial f}=0$$

drodynamic waves, 3) non harmonic lattices, 4) Solid state e.t.c. A doublet solution for two interacting solitons has been found by Tappert:

$$u = \frac{72}{a} \cdot \frac{3+4\cosh(2x-8t)+\cosh(4x-64t)}{[\cosh(x-28t)+\cosh(3x-36t)]^2}$$

For large t the above solution approaches the supperposition of two solitons in the form:

$$u=(12k_i^2/a)\mathrm{sech}^2[k_i\,(x-4k_i^2t)+w_i] \qquad i:1,2$$

with $k_1=1, k_2=2$ and w_i constant.

A generalized form of the KdV equation is

$$u_t + a\cdot u^{p+1}u_x + u_{x^{2r+1}} = 0$$

where p,r nonnegative integers and $u_{x^{2r+1}} = \partial^{2r+1}/\partial x^{2r+1}$.

The most simple generalization is the modified KdV equation which is yielded for p=r=1 :

$$u_t + a\cdot u^2 u_x + u_{xxx} = 0$$

The mKdV equation supports, as solutions, solitary waves which are solitons. In figure 5 a soliton collision is demonstrated.

B. The sine-Gordon Equation

In particle and plasma Physics, many models can be represented by the linear Klein-Gordon equation

$$u_{xx} - u_{tt} = m^2 u \qquad 1.12$$

which is derived from the lagrangian density

$$L = 1/2(u_x^2 - u_t^2) + 1/2\, m^2u^2$$

Replacing the term m^2u by a simple periodic extension we get:

$$u_{xx} - u_{tt} = m^2 \sin u \qquad 1.13$$

SG equation is derived from the Langrangian density

$$L = 1/2\ (u_x{}^2 - u_t{}^2) - m^2\cos(u)$$

It is completely integrable and conserves its energy.

Suppose the existence of a travelling wave solution $u=u(x-vt)$ where v is the constant group velocity. Inserting this form into eq.1.13 it yields

$$d^2u/dx^2 = \sin(u)/(1-v^2) \qquad 1.14$$

where the variable x replaced the term x-vt and m=1 has been taken. Eq. 1.14 is the equation of a simple plane pendulum. So SG equation can be model a line of coupled pendulums undergoing large oscillations including complete rotations. We can find two fundamental types of solutions [10]:

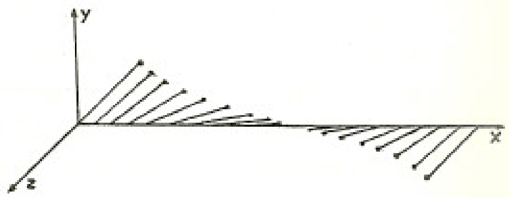


fig. 7.

i) Plasma waves : They correspond to the oscillations of the pendulum and they are supported by the condition $v^2>1$. The solution is given by :

$u=2\cdot\arcsin\{k\cdot sn[(x-x_o)/l\ ;k]\}$

sn: elliptic function with modulus k.

x_o: arbitrary constant

$l = (v^2-1)^{1/2}$

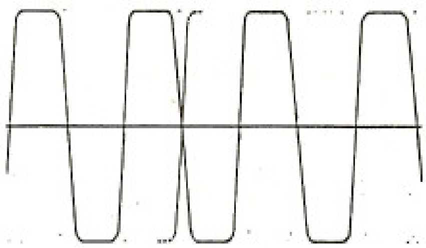


fig.8. A plasma wave

ii) fluxon waves : They correspond to rotational motion of the pendulum and they appear when $v^2<1$. The solution is

$$u = 2\cdot\arcsin\{\pm cn[(x-x_o)/(k\cdot l)\ ;k]\}$$

where parameters are kept as before. For a wave which corresponds to a rotation in u by $2\cdot\pi$ as x goes from $-\infty$ to $+\infty$ the above solution takes the form

$$u = 4\cdot\arctan[\ \exp \pm(x/l)] \qquad 1.15$$

Sign '+' corresponds to a positive sense of rotation (fluxon or kink solution) and sign '-' to a negative sense of rotation (antifluxon or antikink solution).

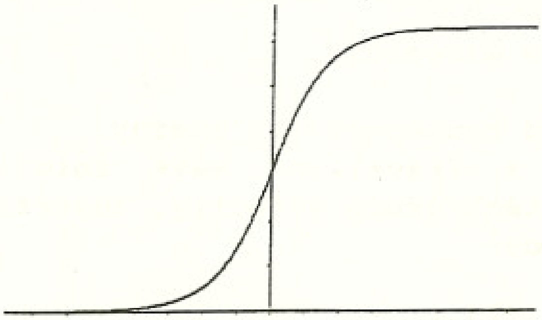


fig.9a: Kink solution

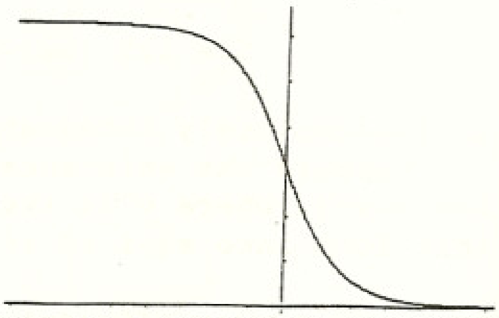


fig.9b : antikink solution

The above solutions are solitary waves and are called solitons and antisolitons respectively. Since the total rotation must be conserved, the difference between the number of solitons and the number of antisolitons must be conserved in any collision. Solitons and antisolitons are created and destroyed in pairs. Perring and Skyrme found (1962) the analytical expressions for the collisions[7]

soliton - soliton collision (fig.10a)

$$u = 4\cdot\arctan[v\cdot\sinh(x/l)/\cosh(vt/l)]$$

soliton - antisoliton collision (fig.10b)

$$u = 4\cdot\arctan[\sinh(vt/l)/(v\cdot\cos(x/l))]$$

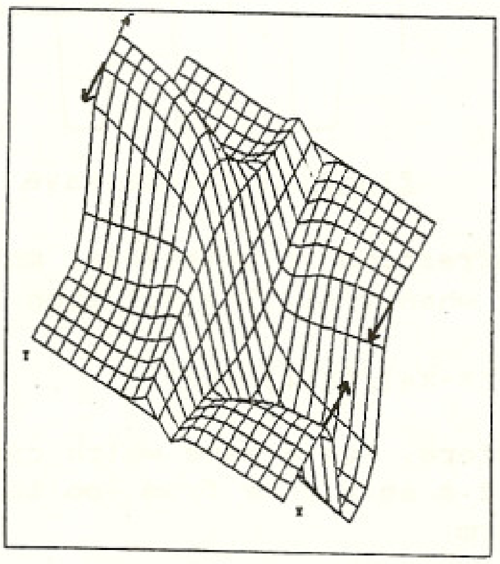


fig. 10.a

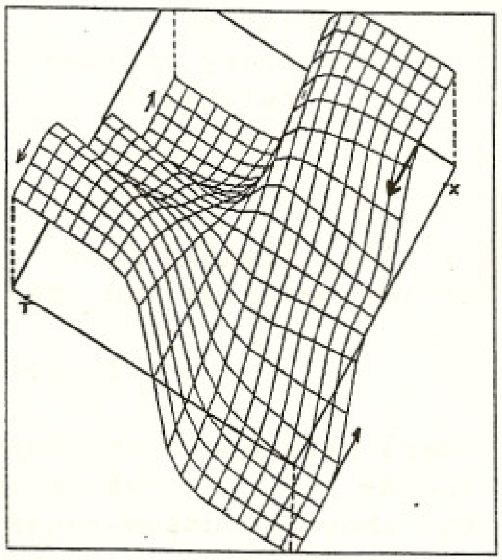


fig. 10.b

The sine-Gordon equation has been used to describe 1) propagation of a crystal dislocation, 2) a unitary theory for elementary particles, 3) propagation of a magnetic flux on a Josephson line, 4) bloch wall motion of magnetic crystals e.t.c. Experiments and theoretical results showed that the generalized Klein-Gordon equation

$$u_{xx}-u_{tt}= F(u)$$

has soliton solutions if and only if $F(u) \equiv \sin u$ ([5])

C. The Nonlinear Schrodinger Equation .

We noticed that the KdV equation includes a balance between dispersion and nonlinearity. This balance can be quite different from system to system. Another important equation is the nonlinear Schrodinger (NLS) equation which plays a significant role in the theory of propagation of wavetrain envelopes. The NLS equation is

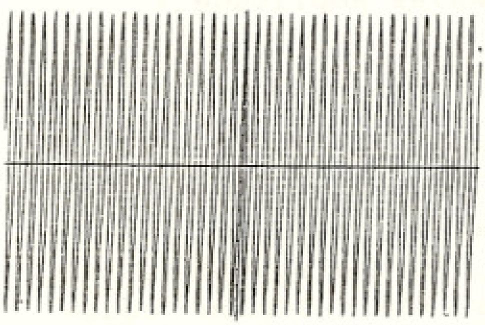
$$i\cdot\frac{\partial u}{\partial t} + P\cdot\frac{\partial^2 u}{\partial x^2} + Q\cdot|u|^2u = 0 \qquad 1.16$$

where P and Q are real parameters.

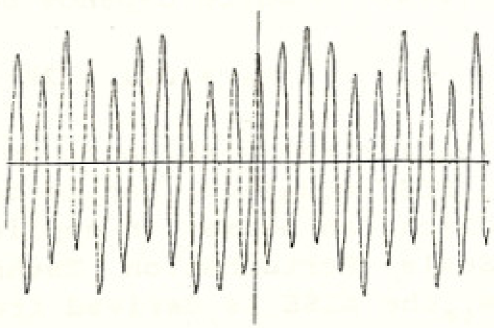
The NLS equation is generic to all conservative systems which are weakly nonlinear and strongly dispersive. It appears in many stable dispersive systems where no dissipation finally occurs. These nonlinear systems usually admit harmonic wavetrain solutions

$$u = \alpha\cdot\exp[i(kx-w(k)t)] \qquad 1.17$$

When the nonlinear terms are small enough to be neglected, the amplitude α appears to remain constant in time (fig. 11.a). Tne effect of nonlinearity on these harmonic oscillations is a variation of the amplitude in both space and time. So nonlinearities create a modulation on the linear wave producing higher harmonics (fig.11.b) The NLS equation describes the slow temporal and spatial envelope evolution of an almost monochromatic wavetrain.



<u>fig.11a</u> Monochromatic wave



<u>fig.11b</u> Modulated wave

In order to give an example, we will show that the NLS equation can represent the evolution of a modulated harmonic wavetrain. Consider a wave with an amplitude dependent dispersion relation :

$$w = w\,(k, |u|^2)$$

where $u=u(x,t)$ is a slow variation of a modulated wave. By expanding the above relation in a Taylor series around w_0 and k_0 we obtain:

$$w-w_0 = w_{k(o)}(k-k_0)+(1/2)w_{kk(o)}(k-k_0)^2+w \cdot |E|^2 + \ldots \qquad 1.18$$

The expansion is only limited to the first order in the nonlinearity but to the second one in the dispersion (the first order term of the dispersion represents undistorted propagation of the wave). The quantities $w-w_0$ and $k-k_0$ act like operators on the amplitude u (eq. 1.18 is the Fourier space equivalent of an operator equation [2][4][20]). So we can transform eq. 1.18 using the operators

$$w-w_0 \dashrightarrow i\partial/\partial t, \quad k-k_0 \dashrightarrow i\,\partial/\partial x$$

If, now, eq. 1.18 operates on u, we get

$$i\cdot\frac{\partial u}{\partial t} + \left.\frac{\partial w}{\partial k}\right|_o \frac{\partial u}{\partial x} + \frac{1}{2}\left.\frac{\partial^2 w}{\partial k^2}\right|_o \frac{\partial^2 u}{\partial x^2} - \left.\frac{\partial w}{\partial u}\right|_o |u|^2 u = 0 \qquad 1.19$$

Now we use the transformation

$$x \longrightarrow x-[\partial w/\partial k]_o t \qquad t \longrightarrow 1/2\ [\partial^2 w/\partial k^2]_o t$$

where the new x is the coordinate moving with group velocity $v_{gr}=[\partial w/\partial k]_o$ and the new t is the normalized time. At last, we find the NLS equation which governs a wavetrain with dispersion relation which depends on the waves amplitude :

$$i\cdot u_t + u_{xx} + 2k\cdot|u|^2 u = 0 \qquad 1.20$$

Its derivation has nothing to do with quantum mechanics.

The NLS equation can be derived from many other equations, which can be met in applied Physics, using the Multiple Scale Perturbation Technique. In the appendix, for instance, the NLSE is derived from the Klein-Gordon equation*.

NLSE has been used to describe 1) one dimensional self modulation of a monochromatic wave, 2) Self trapping phenomena in nonlinear optics, 3) propagation of a heat pulse in a solid, 4) Langmuir waves in plasma, 5) gravity waves in deep waters and (6) is related to the Gindurg-Landau equation of supercontactivity.

In 1971, Zakharov and Shabat found an exact analytical solution of the NLSE of the form $iu_t=-u_{xx}-2|u|^2u$. Further studies showed that the NLSE supports three different kinds of solutions: envelopes and holes, which are solitary waves, and periodic wavetrains. These three kinds are discussed at the next part. Numerical studies of collisions between these solitary solutions indicate that they are solitons. Its form is different from the KdV solitons and they do not possess relationship between amplitude and velocity.

Additional References : [53] [60] [72] [76] [82] [86] [[redacted]] [90] [71]

* By observing the sequence of the calculations, we can make a generalization in order to find a class of equations which yield the NLSE. It can be proved [2] that the class of PDEs of the form

$$L(\partial/\partial t;\partial/\partial x)u=\sum_i(M^iu)(N^iu)+\quad(P^iu)(R^iu)(Q^iu)$$

where L,M,N,P,R,Q are scalar differential operators in $\partial/\partial x$ and $\partial/\partial t$, can lead us to the derivation of the NLSE using the multiple scale perturbation method.

2. SOLITONS OF THE N L S EQUATION

The integrability of a classical dynamical system, which is described by a set of ordinary differential equations, is connected with the number of the invariants of the system. So a classical system with N degrees of freedom is completely integrable if N invariants of motion exist and they satisfy the evolution property (i.e. the Poisson bracket of any pair of them must be zero[134]). Consider now a partial differential equation :

$$u_t = K(u) \qquad 2.1$$

where K is a nonlinear operator from Æ (a linear space) into Æ. A functional I[ ] is to be a <u>constant</u> of motion (or invariant or integral) if

$$dI[u(t)]/dt = 0$$

for all the solutions u(t) in Æ. Since Æ has infinite dimension the integrability of eq. 2.1 is connected with the existence of an infinite number of constants. Consider, now, the operators D[] and F[] (Æ⟶Æ) so that D[u(x,t)] depend on x and F[u(x,t)] depend on t. If

$$\partial D[u(x,t)]/\partial t + \partial F[u(x,t)]/\partial x = 0 \qquad 2.2$$

then 2.2 is called a 'conservation law' and provides the constant of motion [7]:

$$I[u] = \int D[u(x,t)]\, dx$$

Conservation laws and constants of motion provide simple and efficient methods to study quantitative and qualitative properties of solutions. Also, the existence of an infinite number of conservation laws verifies the integrability of eq. 2.1, i.e. eq. 2.1 is exactly solvable.[7] [9] [14]

The Backlund transformation theory can supply the integrals of an equation in many cases. Zakharov and Shabat found an infinite set of conservation laws for the nonlinear Schrodinger equation thus proving its integrability.

2.1. SOLUTIONS OF THE NLSE [7] [97]

The NLS equation can be solved using the Inverse Scattering method which gives analytical forms of travelling wave solutions. Since this method is quite complicated we shall present a simple way which is usful to find the most important solutions.

Consider the NLS equation :

$$iu_t + P\cdot u_{xx} + Q|u|^2u = 0 \qquad 2.3$$

We look for wave solutions of the form

$$u = A(x,t)\, e^{if(x,t)} \qquad 2.4$$

where A and f are real functions. By taking the derivatives of 2.4 with respect to x and t

$$u_t=A_t e^{if}+iAf_t e^{if}$$

$$u_{xx}=A_{xx}e^{if}+iA_xf_xe^{if}+iA_xe^{if}+iA(ie^{if}+f_{xx}e^{if})$$

$$|u|^2= A^2$$

and substituting them into eq. 1.3, since both real and imaginary part must be zero; we obtain the equations :

$$A_t+P\cdot A_xf_x+P\cdot(A\cdot f_x)_x = 0 \qquad 2.5.a$$

$$-A\cdot f_t+P\cdot A_{xx}-P\cdot A\cdot f_x{}^2+Q\cdot A^3 = 0 \qquad 2.5.b$$

Since we seek for a travelling wave solution, A and f must be of the form:

$$f=f(x-v_ct) \text{ and } A=A(x-v_et)$$

where v_c,v_e are the carrier and the envelope velocity respectively. So we can assume only one variable for f and A. Let is $x\longrightarrow x-vt$. Eqs 2.5 can be transformed with the relations:

$$A_x \longrightarrow A_x \qquad\qquad f_x \longrightarrow f_x$$
$$A_t \longrightarrow -v_e A_t \qquad\qquad f_t \longrightarrow -v_c f_t$$

Now, eqs 2.5 can be treated as ordinary differential equations with respect to the new variable x.

$$P\cdot A_x f_x + P\cdot(A\cdot f_x)_x - v_e A_x = 0 \qquad 2.6.a$$

$$P\cdot A_{xx} - P\cdot A\cdot f_x^2 + v_c A\cdot f_x + Q\cdot A^3 = 0 \qquad 2.6.b$$

If we multiply eq.2.6.a with 2A we take :

$$2P[2AA_x f_x + A^2 f_{xx}] - 2AA_x v_e = 0 \implies 2P\cdot d(A^2 f_x)/dx - u_e dA^2/dx = 0$$

Integrating we get

$$A^2\cdot(2P\cdot f_x - v_e) = \text{const.} \qquad 2.7$$

Equation 2.7 yields

$$f_x = [\text{const}/(2A^2) + v_e/2]/(2P) \qquad 2.8$$

and after substitution in eq.2.6b we obtain a differential equation only for A. Since this differential equation will be very complicated we assume const. equal to zero i.e.

$$f_x = v_e/(2P) \quad \text{or} \quad f = (v_e/2P)x + \text{const}(=0)$$

Now, eq. 2.6.b takes the form

$$A_{xx} = D\cdot A - PQ\cdot A^3 \qquad 2.9$$

where $D=(v_e^2-2v_e v_c)/(4P^2)$. Eq. 2.9 is classified in an integrable family of nonlinear equations of the form

$$x'' = a + bx + cx^2 + dx^3$$

with a first integral of the form :

$$(x')^2 = \text{const} + 2ax + bx^2 + 2/3cx^3 + 1/2dx^4$$

So, Eq. 2.9 yields

$$(dA/dt)^2 = -(1/2)PQ\,A^4 + D\cdot A^2 + C = L(A) \qquad 2.10$$

A has been chosen to be real function so L(A) must be always positive. The form of the solution of A depends on the form of the polynomial L(A). We can distinguish the following cases:

CASE A .

We assume that L(A) has two single roots and one double. Since a root A_0 is associated with $-A_0$ which is also a root, L(A) must be of the form

$$L(A) = A^2(A-A_0)\cdot(A+A_0)$$

i.e. $A_d=0$ is a double root of L(A). The above expression is possible only when

$$C = 0 \qquad \text{and} \qquad A_0=[2D/(PQ)]^{1/2}$$

A_0 must be real, so both D and PQ must be positive or negative.

For the case PQ<0 and D<0 the quantity A must satisfy the condition $|A|>A_0$ so that L(A)>0 (fig 12). This solution can not be accepted because it does not describe a wave propagation.

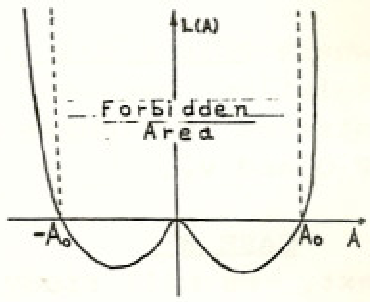


fig.12: PQ<0 & D<0

For PQ>0 and D>0 the L(A) has the form of figure 13. So these conditions can give as a solution a real bounded amplitude bounded amplitude for the seeked wave. The shape of the polynomial L(A) results from the following conditions which must be satisfied as $x \longrightarrow \pm\infty$:

$$A = 0, \qquad dA/dx = 0, \qquad d^2A/dx^2=0$$

i.e. A=0 is an asymptotic value (fig.13).

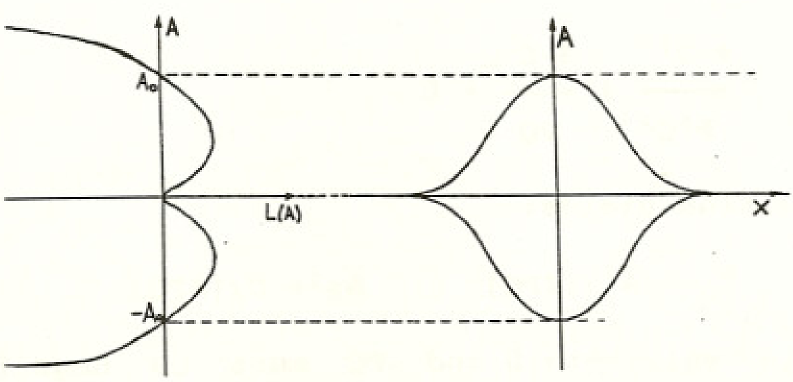


fig.13 PQ>0 & A>0

An analytical form of the solution can be found after integration of eq. 2.10.

$$2.10 \Longrightarrow \int \frac{dA}{A(A^2-A_0^2)^{1/2}} = \int dx \Longrightarrow (A'=A/A_0) \Longrightarrow$$

$$\int \frac{dA'}{A'(A'^2-1)^{1/2}} = A_0 \int dx \Longrightarrow \text{arcsech}(A') = A_0 x + \text{const}(=0) \Longrightarrow$$

$$\Longrightarrow \quad A = A_0 \,\text{sech}(A_0 x).$$

So we achieved to define functions A and f and consequently one solution of the NLS equation can be given by :

$$u=(2D/PQ)^{1/2}\text{sech}[(2D/PQ)^{1/2}(x-v_e t)]\cdot\exp[iv_e(x-v_c t)/(2P)]$$

where PQ>0 & D>0 . This solution is called **'envelope'** solution because of the A(x)s shape and it is demonstrated in figures 14a,14b,14c and 14d for some values of $P\cdot Q$ and v_e.

CASE B .

Next, we will study the case where L(A) appears to have two doublets of roots i.e.

$$L(A) = (-1/2)PQ\cdot(A-A_0)^2(A+A_0)^2$$

Eq. 2.10 can be written:

$$L(A)=(-1/2)PQ \;(A'^2-(2D/PQ)A'-2C/PQ) \qquad 2.12$$

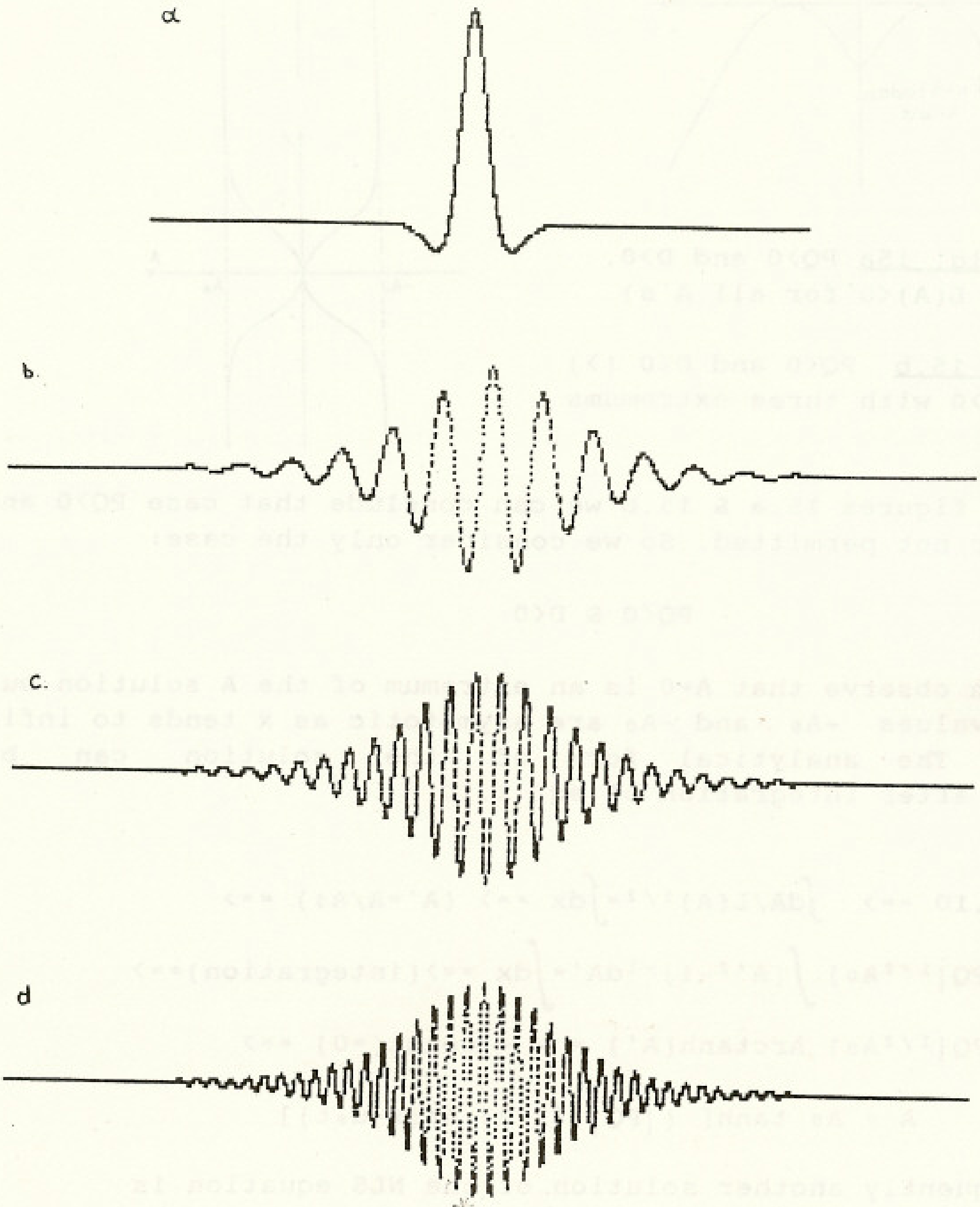
where $A'= A^2$. L(A) supports two doublet roots if :

$$\frac{4\,D^2}{P^2Q^2} + \frac{8\cdot C}{PQ} = 0$$

The above condition yields:

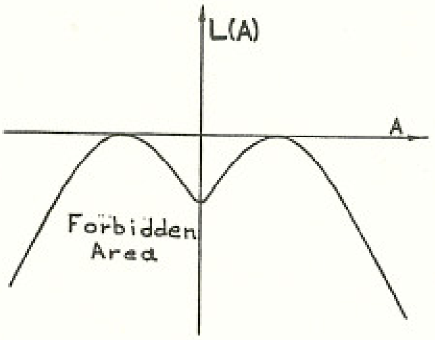
$$C = -A^2/(2PQ) \qquad A_0^2= D/(4PQ)$$

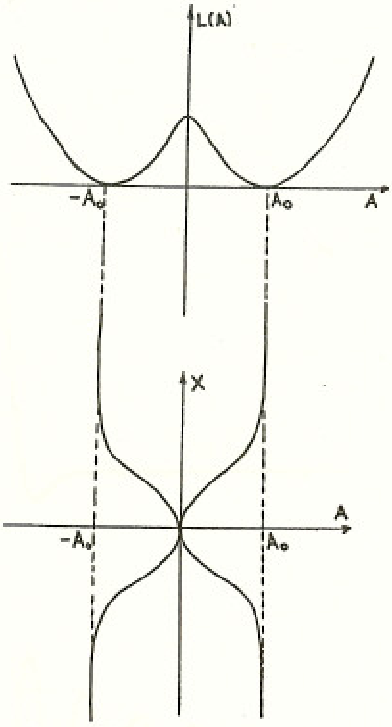
Since A_0 is real, both D and PQ must be negative or



<u>fig. 14</u>: Envelope solitons a) PQ=1 D=2 b) PQ=100 D=10
c) PQ=500 D=50 d) PQ=1000 D=100

positive.





(^)fig. 15a $PQ>0$ and $D>0$.
($L(A)<0$ for all A's)

fig. 15.b $PQ<0$ and $D<0$ (>)
$L(A)>0$ with three extremums

From figures 15.a & 15.b we can conclude that case $PQ>0$ and $D>0$ is not permitted. So we consider only the case:

$$PQ<0 \ \& \ D<0$$

We can observe that $A=0$ is an extremum of the A solution but the values $+A_0$ and $-A_0$ are asymptotic as x tends to infinity. The analytical form of the solution can be found after integration of eq. 2.10

$$2.10 \Longrightarrow \int dA/L(A)^{1/2} = \int dx \Longrightarrow (A' = A/A_0) \Longrightarrow$$

$$2/(|PQ|^{1/2}A_0) \int (A'^2-1)^{-1} dA' = \int dx \Longrightarrow (\text{integration}) \Longrightarrow$$

$$2/(|PQ|^{1/2}A_0)\ \mathrm{Arctanh}(A') = x + \mathrm{const}(=0) \Longrightarrow$$

$$A = A_0 \tanh[\ (|PQ|^{1/2}/2)\ A_0(x-u_e t)]$$

Consequently another solution of the NLS equation is

$$u=[D/2PQ]^{1/2}\tanh[\,|P|(|D|^{1/2}/2)\cdot(x-u_e)]\cdot e^{i(u_e/2P)(x-u_c t)}$$

This solution is called '**shock**'* solution and it is demonstrated in figure 16.

* See App. B for a more general formula of this solution which is called 'hole' solution.

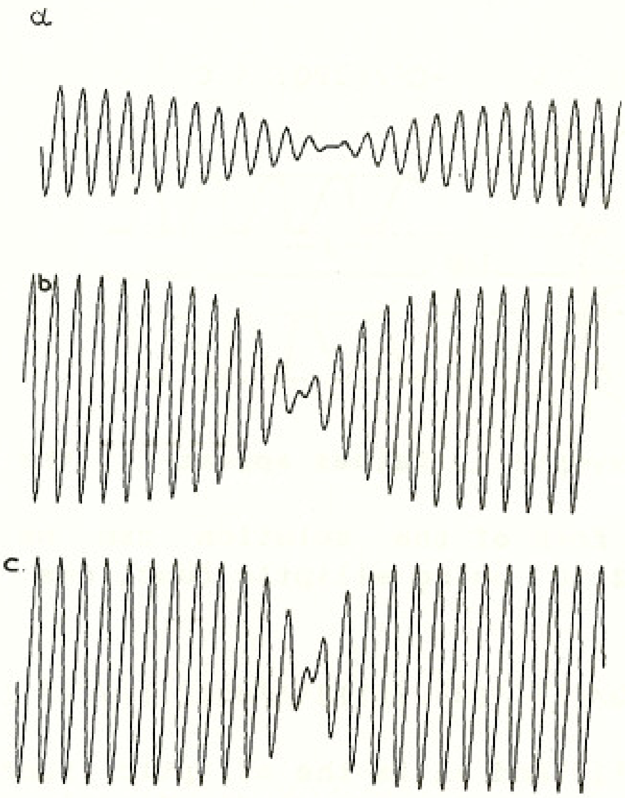


fig. 16 'Shock' solutions.

The last possible case is to assume four different roots for the L(A). Since it appears symmetric, it will have the form :

$$L(A)=(-1/2)PQ\cdot(A\pm A_1)(A\pm A_2)$$

L(A) is also written in the form 2.12. Now, the following condition must be satisfied:

$$4\,D^2/(PQ)^2 + 8C/(PQ) > 0 \qquad 2.14$$

If PQ>0, eq. 2.14 yields $C > -D^2/(2PQ)$. A_1, A_2 must be real so eq. 2.12 must give positive roots.

$$A'_{1,2}=(A^2)_{1,2} = \frac{D}{PQ} \pm(1/2)\left(\left(\frac{2D}{PQ}\right)^2 + \frac{8C}{PQ}\right)^{1/2}$$

We can observe that A_1, which corresponds to the '+'sign, is always positive, but A_2, which corresponds to '-'sign, is positive if and onlv if $C<0$. So we must accept the conditions (fig.17):

$$PQ > 0 \quad \& \quad -D^2/(2PQ) < C < 0$$

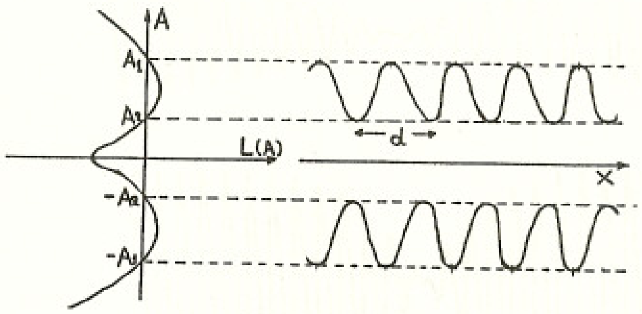


fig. 17. No asymptotic values appear for the A(x)

The analytical form of the solution can be found by integrating eq. 2.10 using elliptic functions. Finally we can take:

$$u = A_1[1-k\cdot sn^2\{(PQ/4)^{1/2}(x-u_e t)\}]^{1/2}\exp[i(u_e/2P)(x-u_c t)]$$

where $k=[1-(A_2{}^2/A_1{}^2)]$ and sn is the elliptic function with modulus k. Because sn is a periodic function, the above solution of the NLSE is a **'periodic'** wave with spatial period $d=(4/A_1)\cdot F(k)$, where F is the Jacobi's elliptic integral of the first kind.

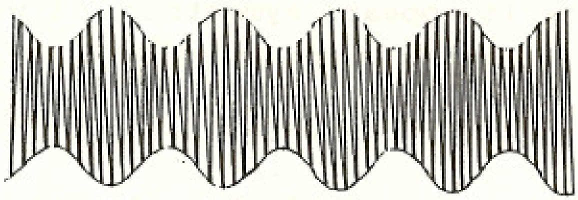


fig.18. A 'Periodic' wave solution

The three cases treaded above give the most interesting and simple solutions of the NLS equation. They have been found setting the con:tant in equation 2.8 equal to zero. If we do not make this ansatz, the equations become complicated but they can still support many other solutions. In the appendix we demonstrate another special solution which is called **'hole'** solution and is the more general form of the shock solution [97].

Envelope and hole solutions, since they are well localized in x and t, are solitary waves. Yuen and Lake (1975) studied experimentally the envelope solutions on a surface of shallow waters [10]. Figure 19 shows the evolution of the envelopes in this experiment. In figure 20 a head of collisions take place between two packets of waves with different carrier frequencies. In both cases, the envelopes behave like solitons. Numerical studies of the solutions of the NLSE, showed that envelope and hole solitary waves are solitons.

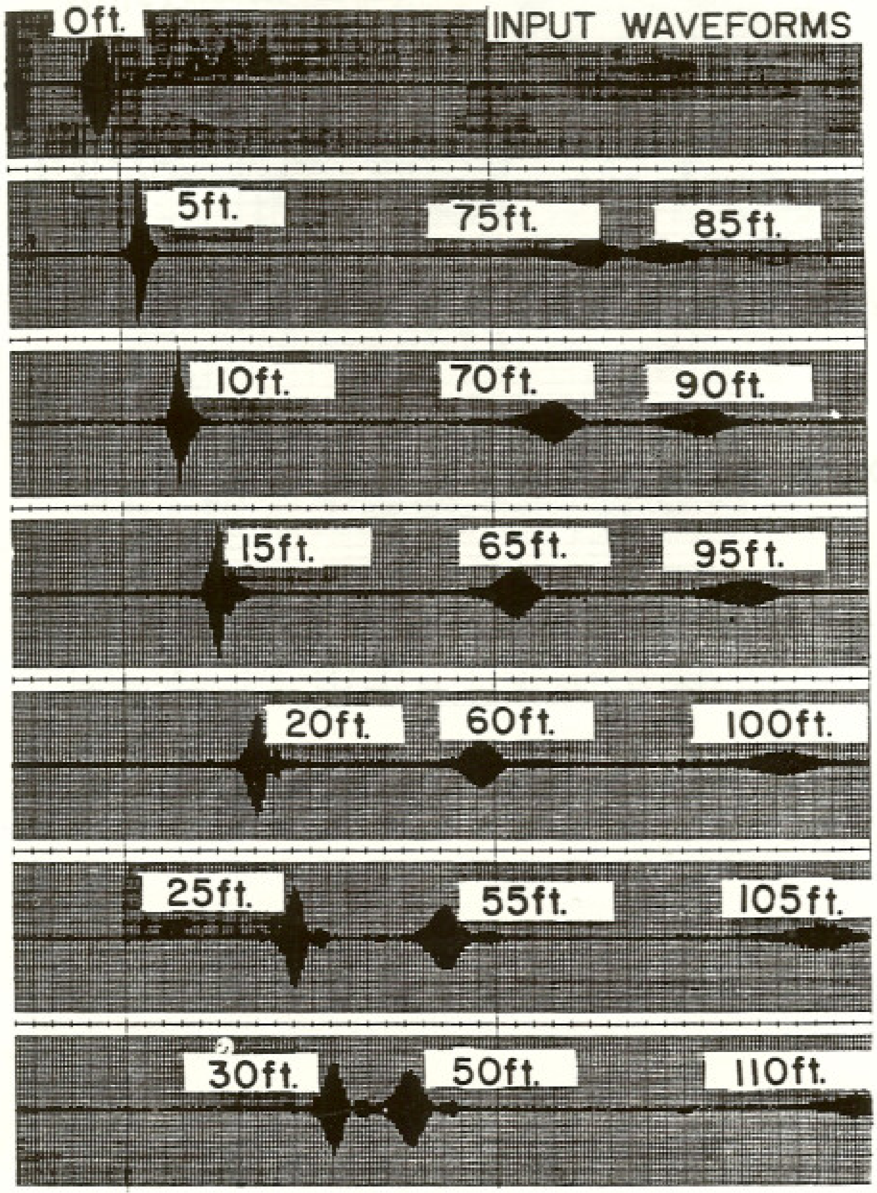


fig. 19 Evolution of envelope solitons.

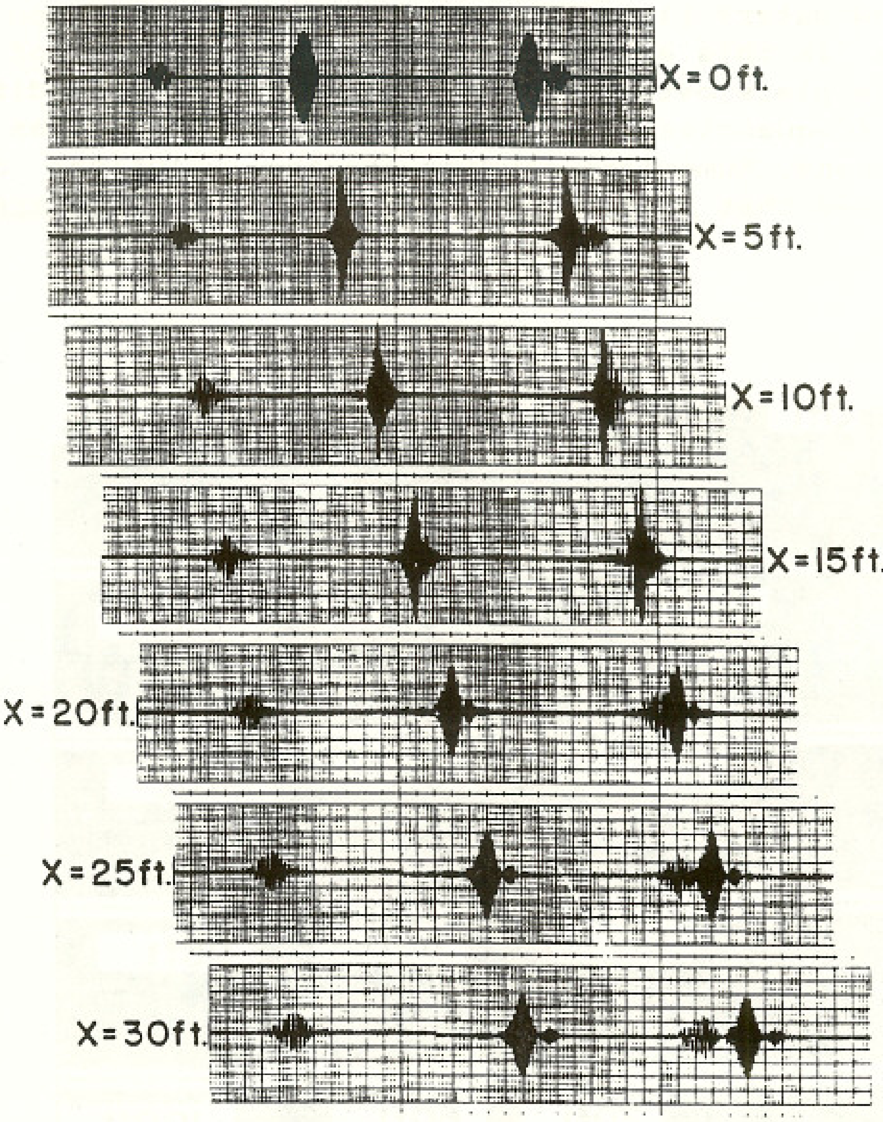


fig. 20 Collisions of envelope solitons

2.2 MODULATIONAL INSTABILITY

In the first part it has been mentioned that a soliton is characterized by its special stability upon collisions. Therefore, the stability theory for travelling waves should play a major role in the general theory of solitons. There is no standard definition of the term 'stability'[7] and a systematic method to investigate it. The most common method to study the stability of a special solution is to assume a 'small linear perturbation' on this solution. If this initial perturbation grows up with the time then instability occurs. The modulational instability is referred to perturbations in the wave amplitude a [11].

$$a' = a + \varepsilon a_1$$

where $\varepsilon \ll 1$ and $a_1 \approx \exp[i(kx-st)]$. a' is assumed to be a solution of the original nonlinear equation and its substitution in the nonlinear equation yields equations which can determine the evolution of a_1. Since $\varepsilon \ll 1$, our equations can be linearized. This linearization is an approximation and its results are referred only to small time intervals. Finally the instability can be checked by the dispersion relation of k and s.

Consider a wavetrain with wavenumber k_0 and angular frequency w_0.

$$u(X,T) = a \exp[i(k_0X - w_0T)] \qquad 2.15$$

where a is a slow varying function of $t=\varepsilon^2 T$ and $x=\varepsilon(X-w'_0T)$ with $w'_0 = dw_0/dk_0$. The quantity a satisfies the NLS equation:

$$a_t = iP \cdot a_{xx} + iQ \cdot |a|^2 a \qquad 2.16$$

We have already studied solutions of the form $a=A(x,t)\exp[if(x,t)]$ for eq. 2.16. In this case the wave propagates as

$$u(X,T) = A(x,t) \exp[i\int(X,T)]$$

where $\int = k_0X - w_0T + f(x,t)$. The local wavenumber and frequency can be found using corresponding derivatives of the total phase $\int$ [12]:

$$k = df/dX = k_0 + \varepsilon f_x = k_0 + \varepsilon K \qquad 2.17a$$

$$w = df/dT = w_0 + \varepsilon w'_0 f_x - \varepsilon^2 f_t \qquad 2.17b$$

At this point note that the conservation of the number of the waves is expressed by $k_T + w_X = 0$

Inserting $a = A \cdot \exp(if)$ into eq. 2.16 and taking into account the derivatives

$$a_t = A_t e^{if} + iAe^{if} f_t$$

$$a_{xx} = A_{xx} e^{if} + iA_x f_x e^{if} + iAf_{xx} e^{if} + iA_x f_x e^{if} - Af_x{}^2 e^{if}$$

and the relation:

$$|a|^2 a = A^2 A e^{if}$$

equation 2.16 becomes:

$$A_t + iAf_t = iPA_{xx} - PA_x f_x - PAf_{xx} - PA_x f_x - iPAf_x + iQA^2 \qquad 2.18$$

The imaginary and the real part of the above equation give

$$f_t = P(A_{xx}/A - K^2) + iQA^2 \qquad 2.19$$

where $K = f_x$

$$(A^2)_t + 2P(A^2 K)_x = 0^* \qquad 2.20$$

Since we can not find the general form of the solution of eqs. 2.19 & 2.20 we should look for special solutions.

We select the monochromatic solution

$$A = A_0 = \text{const.} \qquad f = QA_0{}^2 t + \text{const.} \qquad 2.21$$

Now, we are going to study the modulational instability of this wavetrain under a linear sense. So we set:

$$A = A_0 + A' \quad , \quad K = K' \qquad 2.21$$

where A', K' are assumed small enough so that higher order

* The derivation of eq. 2.19 is succeeded after a multiplication of the real part of 2.18 by 2A

terms of A' and K' will be neglected. Differentiating eq. 2.19 with respect to x we have*

$$K_t + 2P\,KK_x = P(A_{xx}/A)_x + Q(A^2)_x \qquad 2.23$$

and by substituting 2.22 in 2.23 we can take the equation

$$K'_t = 2QA_0A'_x + PA'_{xxx}/A_0 \qquad 2.24$$

In the same way, equation 2.20 gives

$$A'_t = -PA_0K'_x \qquad 2.25$$

Since we are interested in the modulational instability, it is convenient to have one equation which consists only of terms of A'. So

$$2.25 \Longrightarrow A'_{tt}=-PA_0K'_{tx} \Longrightarrow (2.24) \Longrightarrow A'_{tt}=-PA_0(2QA_0A'_{xx}+$$

$$+PA'_{xxxx}/A_0) \Longrightarrow A'_{tt}=-2PQA_0{}^2A'_{xx}- PA'_{xxxx}/2 \qquad 2.26$$

Eq. 2.26 provides a great number of special solutions for A', so there are many different perturbations for the wave-train 2.21. Usually we consider perturbations of the form:

$$A' = m\,e^{ilx+st}\ (^{**})\ ,\quad m{:}\text{constant} \qquad 2.27$$

By taking into account the solution 2.27, eq.2.26 yields the dispersion relation between l and s :

$$s^2=2PQA_0{}^2l^2-P^2l^4 \quad \text{or} \quad s = \pm l[2PQA_0{}^2-(Pl)^2]^{1/2} \qquad 2.28$$

From the dispersion relation 2.28 we can conclude that if s is real and positive then A' grows up with the time t and the wavetrain appears unstable. <u>So instability occurs when :</u>

$$PQ > 0 \qquad \text{and} \qquad 0<l^2<2A_0{}^2(Q/P)$$

---

*,** Upon linearization we neglected high order terms of A_x and K_x too. This is permitted only when the perturbation is of a suitable form. The perturbation 2.27 supports the above linearization for small l i.e. 2.27 is a longwave perturbation.

The maximum growth of instability corresponds to

$$ds/d(l^2) = 0 \implies l^2 = (P/Q)A_0^2$$

for

$$s_{max} = \text{(eq. 2.28)} = Q^2 A_0^2$$

s_{max} expresses the maximum rate of growth because $dA'/dt = m \cdot s \cdot e^{i(lx-st)}$.

Conclusion: A modulated wavetrain is unstable for small longwave perturbations if the free parameters of the NLSE satisfy the relation $P \cdot Q > 0$ [12].

In appendix (C) a different procedure to find the final dispersion relation is given (see, also, [17]).

Benjamin and Feir worked out the first numerical experiments on instability. They showed that when the waves have a finite nonlinear amplitude, in many cases, the uniform wavetrain appears unstable under infinitesimal modulational perturbations. The effects of modulational instability are demonstrated in the following figures. Initially we have a wavetrain which transfers all the energy (fig. 21a) and it propagates in a nonlinear medium. The interaction with this medium causes the appearance of sidebands.[11] The energy is gradually transferred to the sidebands increasing their amplitude. When the amplitude of the sidebands approaches the carrier amplitude, the wave breaks up into envelope solitons which are separated and connected by small tails which do not transfer energy. The envelope solitons which are formed, break up giving secondary sidebands.

As shown, an analysis of the Benjamin-Feir modulational instability gives conditions which support solitons. These solitons are obtained after a finite time when a sufficient increase of the sidebands occurs. They appears unstable and break up into secondary solitons[10]. Numerical analysis [97] shows reccurence in modulation instability : solitons disappear and sidebands are formed again. Finally the wavetrain takes the form of its initial form. This process is repeated in the time and is called Fermi -Pasta -Ulam recurrence (figure 22).

Additional references: [49] [50] [51] [58] [69] [70] [71] [73] [74] [79] [80] [81] [83] [85] [90] [91] [93] [94] [95]

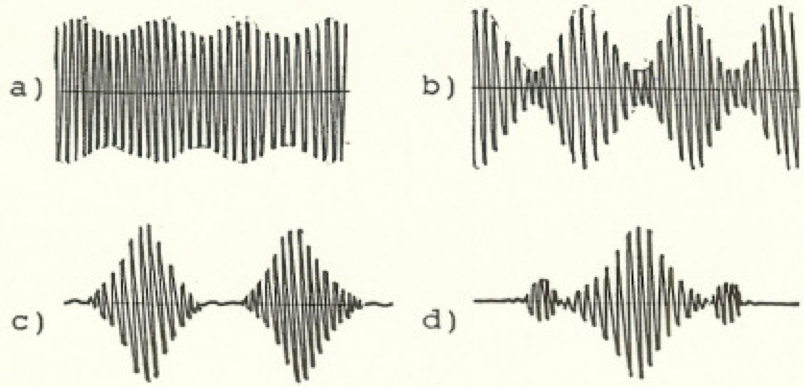


fig.21:The evolution of a modulated wave

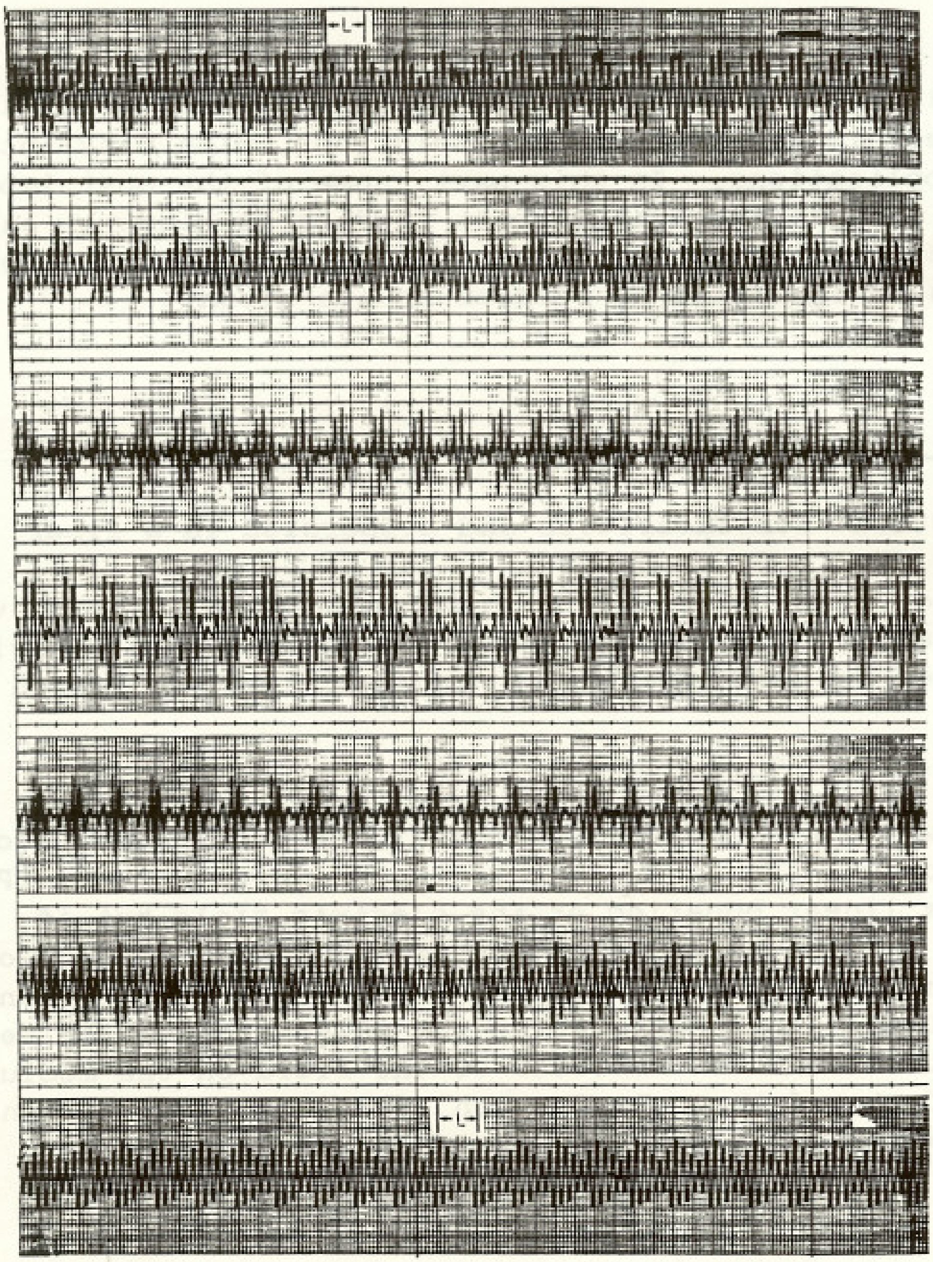


fig. 22 : The evolution of a nonlinear wavetrain with large initial amplitude modulation

3. EXAMPLES OF 1D SYSTEMS MODELED BY THE NLSE.

In the part 1 it has been mentioned that the NLS equation is used to describe many physical models and, in most of cases, it provides soliton solutions, with many interesting applications. In this part, some of the most important models are presented though not in details since an enormous ammount of literature on this field already exists. For further investigation many references are given.

3.1 OPTICAL FIBERS.[20]-[30]

3.1.a Nonlinearities and dispersion in a fiber.

When a pulsed signal with an electrical field variation $\tilde{E}(z,t)$ propagates through a medium, its behavior is governed by the Maxwell electromagnetic wave equation

$$\nabla^2\tilde{E} - (1/c^2)\tilde{E}_{tt} = (4\pi/c^2)P_{tt} \qquad 3.1$$

We assume a homogeneous, neutrally charged medium, so the only source for the light is the polarization term P. Eq. 3.1 is a linear equation of the electric field but if $\tilde{E}$ is strong enough the polarization of the medium depends on the field and produces significant nonlinear effects on the pulse signal. In order to find the pulse propagation we must take into account the effects of the field on the medium and the effect of the polarization back on the pulses. In this case the polarization can be expanded as

$$P = x_1\varepsilon_0\tilde{E} + x_2\tilde{E}^2 + x_3\tilde{E}^3 + \dots \qquad 3.2$$

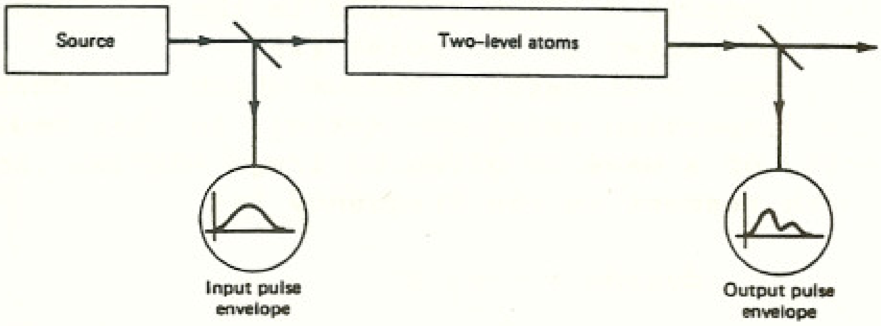


fig. 23 :A typical change of a pulse propagating in a fiber.

where the first term stands for the linear suspectibility and the rest terms represent weak higher order nonlinearities. If we maintain only the three first terms of 3.2 the dielectric constant of the medium and the index of refraction can be written as [20] :

$$\varepsilon = \varepsilon_1 + \varepsilon_2 E^2$$
$$n = n_0 + n_2 E^2$$

The nonlinearity, for sinusodial waves, can be described most conveniently as:

$$n = n_0 + n_2 |E|^2$$

This change of the index, which is produced by the optical signal itself, is referred as the optical Kerr effect. We can observe an important effect on the pulse field. Since in an optical fiber we have $n_2>0$, intensity is higher in the center of the beam, and causes an increase in the index of the ref-raction as compared to the wings. So the initial pulse becomes steeper. This effect is called self focusing [20],[33], [34],[37].

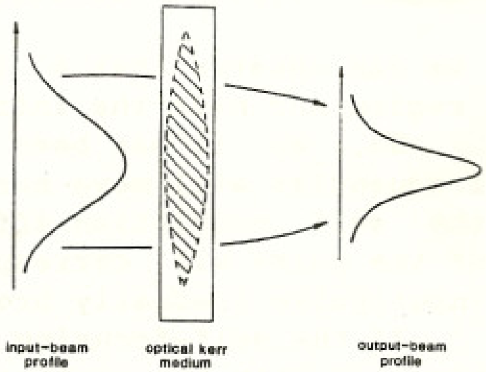


fig.24: Self-focusing

Another important effect is the self-phase modulation effect [20],[24]. The index of refraction changes in time as $\Delta n(t)=n_2|E(t)|^2$ followed by a change in the phase of the pulsed field :

$$f(t)=2\pi\cdot\Delta n\cdot L/\lambda$$

where λ is the wavenumber and L the length of the medium. Usually n_2 is positive which results to the increase of the frequency at the center of the pulse [20].

Consider, now, a dispersive medium which is characterized by the dispersion relation $k=k(w)$. In this medium the group velocity of a wave is given by $1/v_g = dk/dw_0$ and its variation with respect to the frequency is:

$$dv_g/dw = - v_g{}^2 k''$$

where $k''=d^2k/dw^2$. Figure 25 shows a typical variation of the index of refraction and the variation of k". The dispersion parameter k" changes from positive at shorter wave lengths to negative at longer ones.

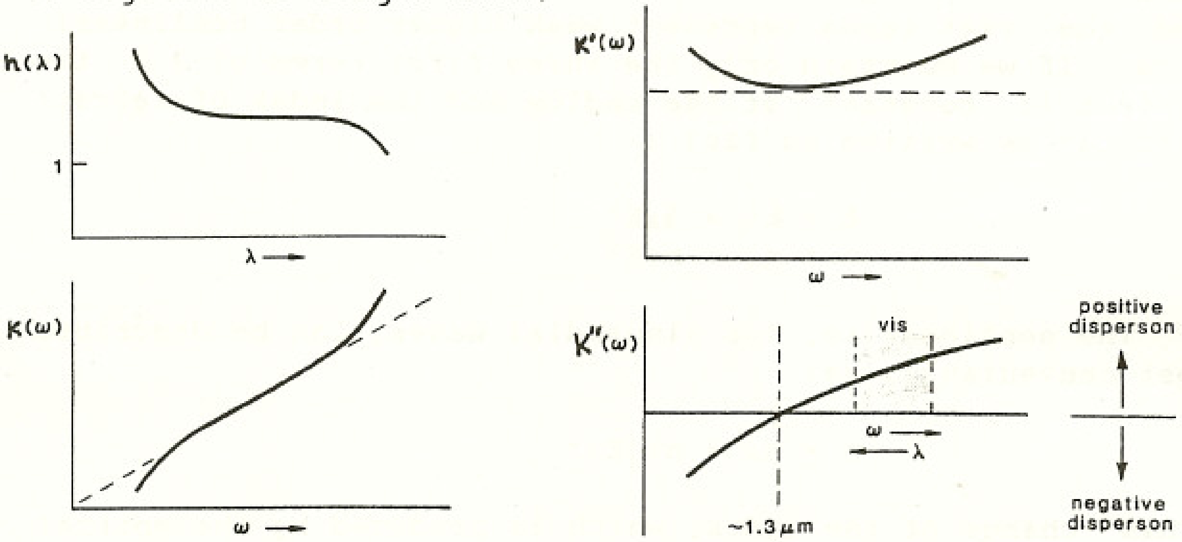


fig.25: Typical variations of n,k,k' and k" in a fiber

We can observe that k" is usually positive in the visible region and near the infrare region.

So, as it has been mentioned in the first part, higher frequencies will move more slowly. By taking into account the self modulation effect we can conclude that the center of the pulse will corresponds to low group velocity i.e. the input pulse gradually broadens.

If the self-focusing effect and the pulse broadening are compared with the nonlinear and dispersion effects which have been discussed in the first part, we can conclude that solitary waves can be formed under the balance of the above effects. This balance can be obtained using a suitable value for the power of the wave, which causes a change of the index of the refraction [24]. Next we derive the equations which describes the propagation of a pulse in a nonlinear medium and also we will verify the existence of solitary waves.

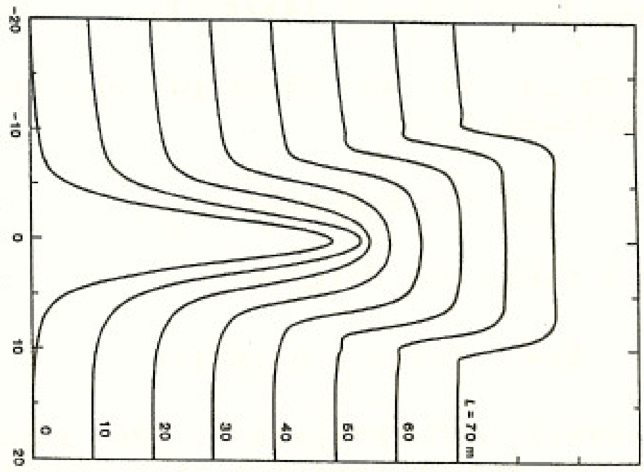


fig.26: Pulse broadening

3.1.b. The equation of propagation.

Consider an optical signal of the form $\tilde{E}(z,t)=E(z,t)\cdot \exp[i(w_0 t-k(w_0)z)]$ which travels on the +z direction. E(z,t) is assumed to be a slow varying amplitude and using the Fourier transformation in a close plane, it can be given by [20]:

$$E(z+dz,t)= 1/2\pi\int_{-\infty}^{\infty} dz\, \Delta w \int_{-\infty}^{\infty} dt' E(z,t')\cdot e^{i[\Delta w \Delta t - \Delta k \Delta z]}$$

where $\Delta w=w-w_0$ and $k=k(w)-k(w_0)$. According to the optical Kerr effect we suppose that the k=k(w) is of the form

$$k(w)=k(w_0)+\beta_2 |E^2|+k'(w-w_0)+1/2\cdot k''(w_0)(w-w_0)^2+ \ldots$$

where $\beta_2=2\pi n_2/c$. The general form of above relation has been already mentioned in the first part as $w=w(k,|E^2|)$. It has been showed that a dispersion relation of this kind leads to the Nonlinear Schrodinger equation. In this case we have :

$$i\frac{\partial E}{\partial z} + k'\frac{\partial E}{\partial t} + \frac{k''}{2}\cdot\frac{\partial^2 E}{\partial t^2} - \beta_2/2\cdot|E|^2 E = 0$$

If we compare this equation with the eq. 1.20 we see that the space and time variables changed roles.

A more formal derivation of the NLSE can be obtained by starting from the Maxwell equation 3.1. Since we are familiar with the 1D NLSE we reduce eq. 3.1 to one dimension :

$$E_{xx} - (1/c^2)\cdot E_{tt} = (4\pi/c^2)P_{tt} \qquad 3.3$$

The polarization P in an isotropic and centro-symmetric medium can be written as

$$P = a_1 E + a_3 E^3 + \dots.$$

Now equation 3.3 takes the form :

$$E_{xx} - \beta\cdot E_{tt} = h\cdot(E^3)_{tt} \qquad 3.4$$

where $\beta = c^{-2}(1+4\pi a_1)$ and $h = 4\pi a_3 c^{-2}$. We assume that the polarization of the medium causes a slow amplitude variation on the wave propagated to the x axis. So according to the multiple scale perturbation theory (see Appendix) we can expand E as

$$E = \varepsilon E_1 + \varepsilon^2 E_2 + \varepsilon^3 E_3 + \dots\dots \qquad 3.5$$

and we introduce a new set of slow space and time variables

$$X_n = \varepsilon^n x \quad ; \; T_n = \varepsilon^n t$$

with the derivative operators :

$$\frac{\partial}{\partial x} \longrightarrow \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2} + \dots$$

$$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + \dots.$$

$$\frac{\partial^2}{\partial x^2} \longrightarrow \frac{\partial^2}{\partial x^2} + 2\varepsilon\frac{\partial^2}{\partial x\partial X_1} + \varepsilon^2\frac{\partial^2}{\partial X_1^2} + \varepsilon\frac{\partial^2}{\partial x\partial X_2} + \dots$$

$$\frac{\partial^2}{\partial t^2} \longrightarrow \frac{\partial^2}{\partial t^2} + 2\varepsilon\frac{\partial^2}{\partial t\partial T_1} + \varepsilon^2\frac{\partial^2}{\partial T_1^2} + 2\varepsilon\frac{\partial^2}{\partial t\partial T_2} + \dots$$

By using the above expansion and the new slow variables we derive from eq. 3.4 a set of equations by setting each terms of the same order of ε equal to zero.

$$O(\varepsilon) : \quad L\,E_1 = 0 \qquad 3.6.a$$

$$O(\varepsilon^2) : \quad L\,E_2 = -2\cdot\left(\frac{\partial^2}{\partial x\partial X_1} - \beta\cdot\frac{\partial^2}{\partial t\partial T_1}\right) E_1 \qquad 3.6.b$$

$$O(\varepsilon^3) : \quad L\,E_3 = -\left(\frac{\partial^2}{\partial X_1{}^2} - \beta\cdot\frac{\partial^2}{\partial T_1{}^2}\right)E_1 - 2\cdot\left(\frac{\partial^2}{\partial x\partial X_1} - \frac{\partial^2}{\partial t\partial T_1}\right) E_1$$
$$-2\cdot\left(\frac{\partial^2}{\partial x\partial X_1} - \beta\cdot\frac{\partial^2}{\partial t\partial T_1}\right) E_2 + h(E_{1\,3})_{tt} \qquad 3.6.c$$

. . . .

where $L\equiv \partial^2/\partial x^2-\beta\cdot\partial^2/\partial t^2$.

From eq. 3.6.a we can have

$$E_1 = E(X_n,T_n)\cdot e^{i(kx-wt+f)} + c.c \qquad 3.7$$

with $w^2=k^2/\beta$, f:constant. Substituting eq.3.7 into 3.6.b we obtain

$$L\,E_2 = -2i\cdot[k\cdot E_{X1} + w\beta\cdot E_{T1}]e^{if} \qquad 3.8$$

The term on the right hand side of 3.8 is secular and it grows exponentialy with time t. It can be removed by defining a new variable X which travels with the group velocity $v_g = dw/dk = \beta^{-1/2}$ of the wave 3.8. So

$$X = X_1 - \beta^{-1/2}T_1 \qquad 3.9$$

and now the amplitude E can be written as a function of X,X_2 and T_2 (X_n,T_n with $n>2$ are assumed to be neglected).

The third in the righ hand side of eq. 3.6.c can be removed in the same way. Also we have

$$\frac{\partial^2 E_1}{\partial X_1{}^2} = \frac{\partial^2 E_1}{\partial X^2} \quad \& \quad \frac{\partial^2 E_1}{\partial T_1{}^2} = 1/\beta\cdot\frac{\partial^2 E}{\partial X^2}\cdot e^{if} + c.c.$$

$$\frac{\partial^2 E_1}{\partial x \partial X_2} = ik \frac{\partial E}{\partial X_2} e^{if} + c.c.$$

$$\frac{\partial^2 E_1}{\partial t \partial T_2} = -iw \frac{\partial E}{\partial T_2} e^{if} + c.c.$$

$$(E_1{}^3)_{tt} = [Ee^{if} + E^* e^{-if}]^3{}_{tt} = hw^2[9E^3 e^{3if} - 3E^2 E^* e^{if}] + c.c.$$

Now eq. 3.6.c can be written :

$$L\ E_3 = -(1 - 1/\beta)\cdot\frac{\partial^2 E}{\partial X^2} e^{if} - 2i\cdot k\cdot\frac{\partial E}{\partial X_2} - w\beta\cdot\frac{\partial E}{\partial T_2} e^{if} +$$

$$+ hw^2[9E^3 e^{3if} - 3E^2 E^* e^{if}] + c.c.$$

The terms multiplied by e^{if} are secular and can be removed by setting :

$$(1-\beta^{-1})\frac{\partial^2 E}{\partial X^2} + 2ki\cdot\frac{\partial E}{\partial X_2} - \beta^{1/2}\frac{\partial E}{\partial T_2} + 3hw^2|E|^2 E = 0 \qquad 3.10$$

The variables X_2, T_2 are dependent. Transforming to a frame of reference moving with group velocity $\beta^{-1/2}$ we can introduce the variable

$$Z = X_2 - \beta^{-1/2} T_2$$

and eq. 3.10 takes the form of the NLS equation :

$$i\frac{\partial E}{\partial Z} + P\cdot\frac{\partial^2 E}{\partial X^2} + Q\cdot|E|^2 E = 0 \qquad 3.11$$

where $P=(1-\beta^{-1/2})/2k$ and $Q=3hw^2/2k$.

If we would not low the dimension in the Maxwell equation we would be ended up, following the same route, with the 2-D NLSE which, in a simple form, that is

$$E_{xx}+E_{yy} - i\cdot E_z + 2\cdot|E|^2 E = 0$$

This equation refers to a wave which propagates to the z direction ([2], [21], [23], [31]).

3.1.c Numerical simulations and experimental results.

In the previous parts we have seen the most interesting features of solitons. If there is no loss, soliton travels in the form of a pulse which does not change its shape. For optical, fibers it represents a pulse of just the right amplitude such that the pulse broadening effects of dispersion are exactly balanced by the pulse narrowing effects of nonlinearity. In 1973 Hasegawa and Tappert [22] showed that the nonlinear change in the refractive index could be used to balance the dispersion. They derived the NLS equation and after numerical analysis showed that pulse solitons occur when the power is sufficient for the nonlinearity to balance the dispersion. They examined the envelope solution :

$$u(x,t) = E_s \cdot \mathrm{sech}[(t-x/v_g)/\tau_o] \cdot e^{i(kx-\omega t)}$$

where τ_o is he half-width of the pulse. The critical value of power is $P_s=3/4 \cdot E_s(\varepsilon_o/\mu_o)^{1/2}$. For $P \ll P_s$ dispersive spreading takes place (fig.27a) but for $P=P_s$ stationary nonlinear pulses are observed (fig.27b).

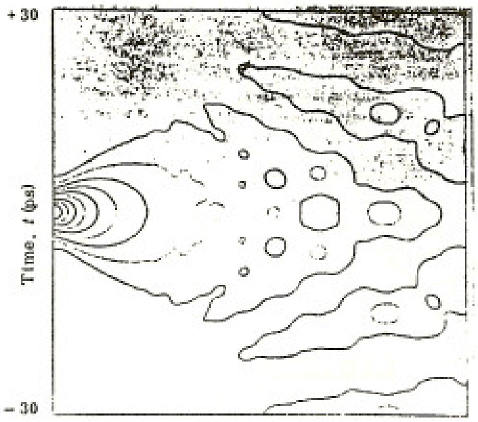


fig. 27a

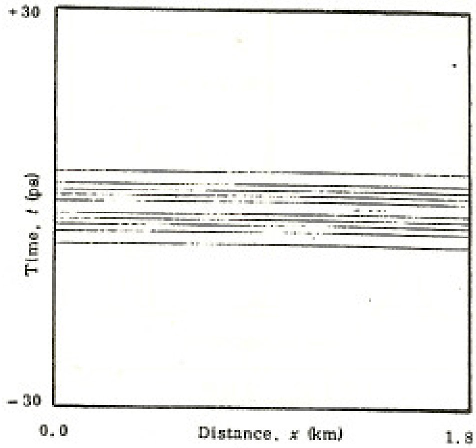


fig.27b

If $P > P_s$ higher order solitons appear exhibiting more complex behavior. Mollenauer et al [27][28] studied numerically the behavior in this case. They observed that there is no change of the period z_o of solitons but they become more compressed with an increase in their amplitude (figure 28b). Later on, multisoliton solutions were studied theoretically by Lamb [35] and J.C. Eilbeck et al [38]. They found formulas which yield the broadening of these pulses. They observed that the pulses undergo oscillations in shape becoming more structured (fig.29). K Konno and H. Suzuki [37] presented numerical studies for three dimensional waves.

In 1980, Mollenauer et al [24] [26] made the first experimental observation for the above theoretical plans . They observed the compression and the splitting of picosecond pulses in the negative dispersion region of a single-mode optical fiber having low loss for wavelength at about 1.3 µm. Figure 30 shows the results of the experiment. For P < 1.2 W the dispersion causes a broadening of the pulse. For P=1.2 W the output pulse is similar to the input one while for P>1.2 W a narrowing of the pulse begins. The behavior of the pulse was in close agreement with the predictions based on the NLS equation. In 1986 F.Salin et al [29] experimented on Femtosecond pulses ($\lambda \approx$ 620 nm). By using a CPM laser they observed high-order solitons with properties which could be explained if the NLS equation was taking into account.

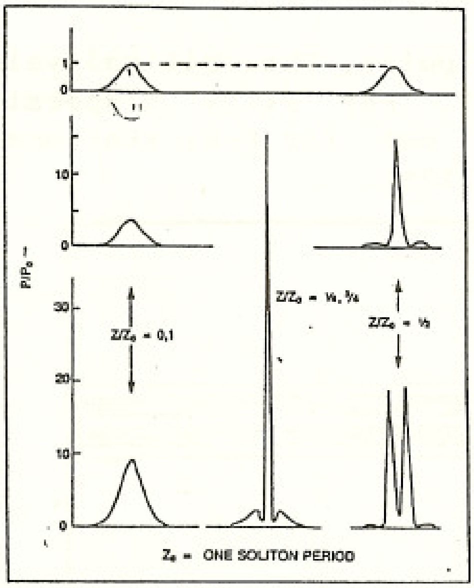


fig. 28a

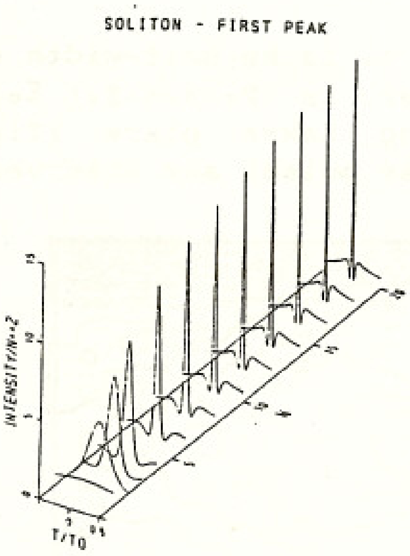


fig.28b

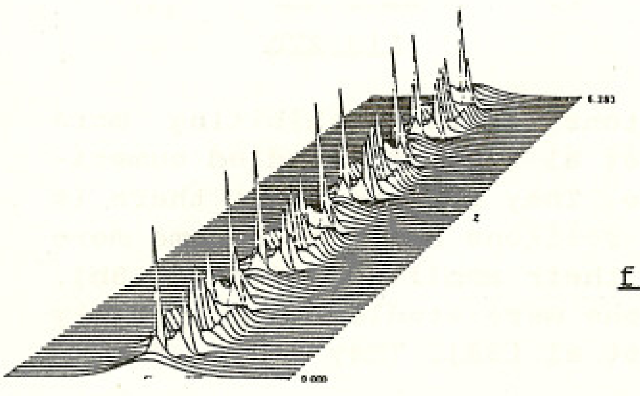


fig. 29: The propagation of a 4-soliton solution

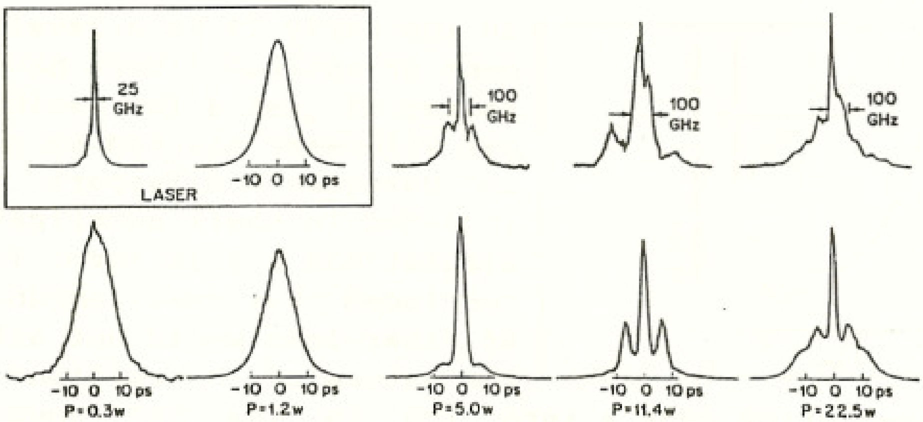


fig.30 : The input pulse and the spectra of the output pulses (below).

In 1985 K.Tai, A Hasegawa and A. Tomita [32] studied experimentally the instability of the pulses (in 2nd part the conditions which cause instability have been mentioned). They observed [32] the appearance of sidebands in agreement with the theoretical predictions.

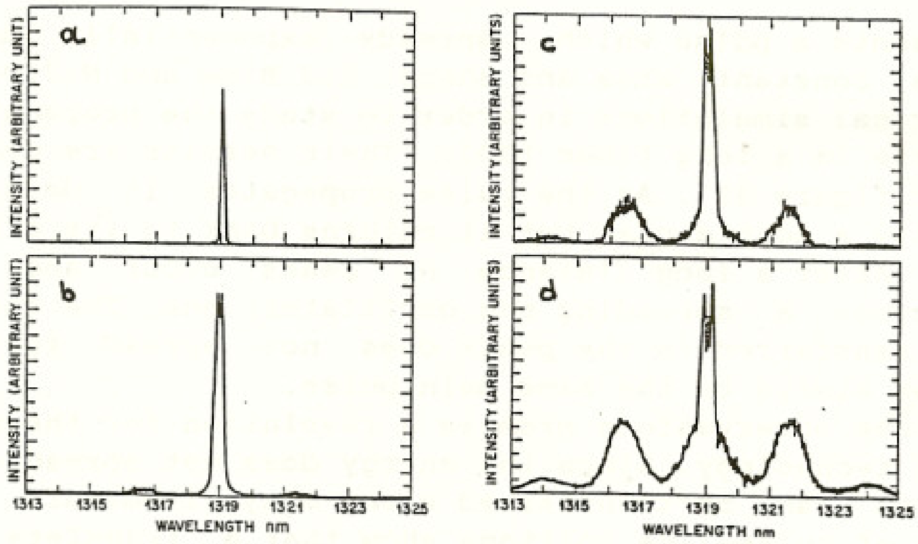
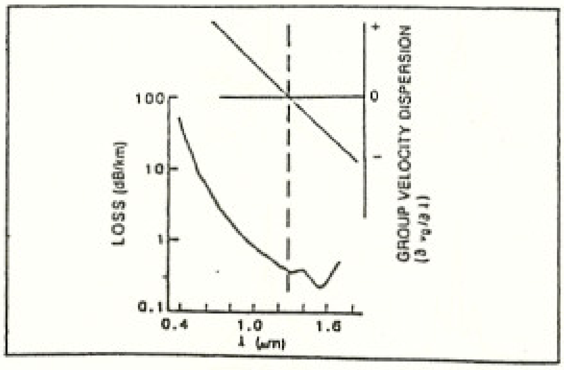


fig.31 : Instability of a narrow pulse.

3.1.d Optical Fibers with loss.



In the early 1970s a development of optical fibers having losses of just a few decibels per kilometer started to grow up. Hasegawa and Tappert included in their numerical simulations a loss term. They concluded that small amounts of fiber-loss would not affect the nature of the solitons.
A linear absorbtion adds a term $i\sigma u$ in the NLS equation which can be written in this case as

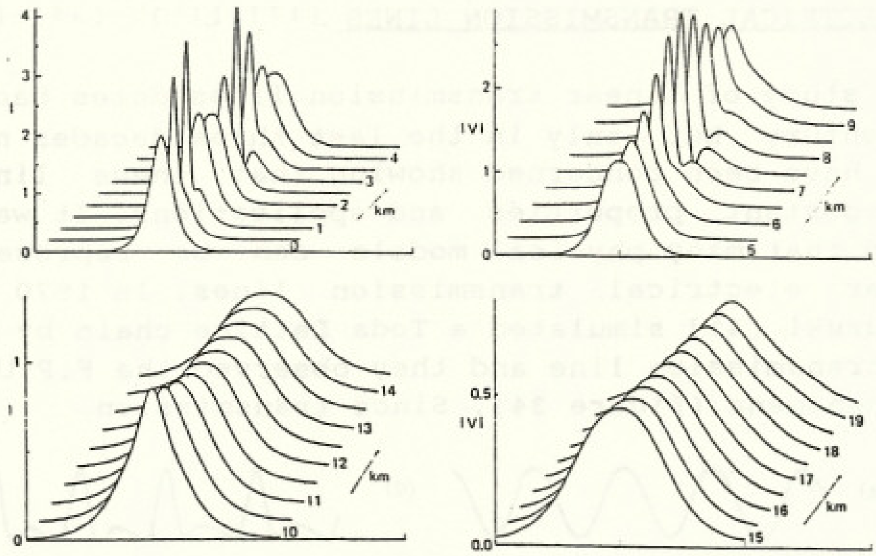
$$i \cdot u_z i\Gamma u + \frac{1}{2} u_{tt} + 2 \cdot |u|^2 u = 0$$

where $\Gamma = \sigma\tau^2/|k_2|$. In this case the pulse is given by ([2],[14]) :

$$u = \exp\{[i(1-e^{-z/\sigma})]-2\sigma z\} \cdot \mathrm{sech}(t \cdot e^{-2\sigma z})$$

and represents a pulse which spreads exponentially maintaining a constant area and shape. K.J.Blow and N.J.Doran made numerical simulations in order to study the propagation of the pulse in a long fiber [30]. Their results are demonstrated in figure 33 . As the pulse propagates it develops periodically a double peak but it returns back to its original shape. After a long distance no peaks occur and the pulse becomes a spreading non oscillatory one. The energy which is transferred to the peaks does not spread further but returns always to the same main pulse.

The above observations promise a revolution for the communication technology. Since the energy does not spread out, informations can be transferred along many kilometers. The properties of pulse-like solitons show that a transfere of many gigabits per second is possible. There is much work yet to be done, but it is clear that communication links based on soliton propagation are eminently of much practical importance.



<u>fig.33</u> : The propagation of a pulse in a long fiber with loss. The absorption is not able to destroy the pulses.

Additional references : [52] [64] [65] [68] [75] [77] [78] [84] 87] [88] [92]

ELECTRICAL TRANSMISSION LINES [17]-[19], [39]-[48]

The study of linear transmission lines dates back in the 19^{th} century but only in the last three decades nonlinear effects have been concerned showing that these lines have many important properties and applications. It was, also, realized that many physical models can be represented by nonlinear electrical transmission lines. In 1970 R.Hirota and K.Suzuki [41] simulated a Toda Lattice chain by an electrical transmission line and they observed the F.P.U. recurrence phenomena (figure 34). Since transmission

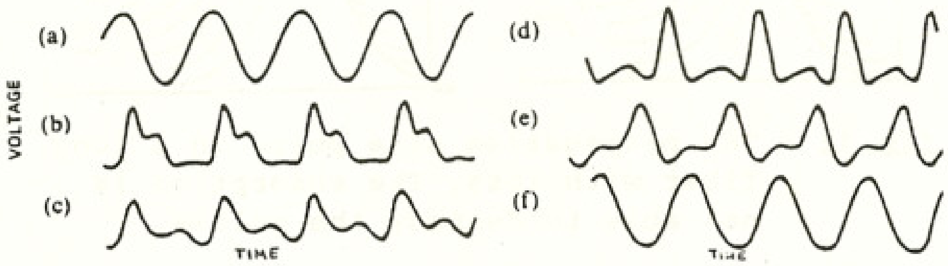


fig.34 : Oscillograms showing recurrence phenomena.

lines are relatively inexpensive and convenient devices, they are used to study the propagation and the properties of nonlinear excitations like solitons.

A transmission line consists of a large number of identical sections with capacitors and inductors. Many different forms of sections are used. Nonlinearities appear by using capacitors (or inductors) with capacitance (or inductance) which is a function of voltage. In most cases the propagation of a signal in the line is described by the KdV or the NLS equation.

3.2.a The NLS equation.

A monoinductance electrical transmission line is illustrated in figure 35. The inductance L is constant but the capacitance C depends on the voltage V by the relation :

$$C = C_o V_o \log(1+V_n/V_o)/ V_n \qquad 3.12$$

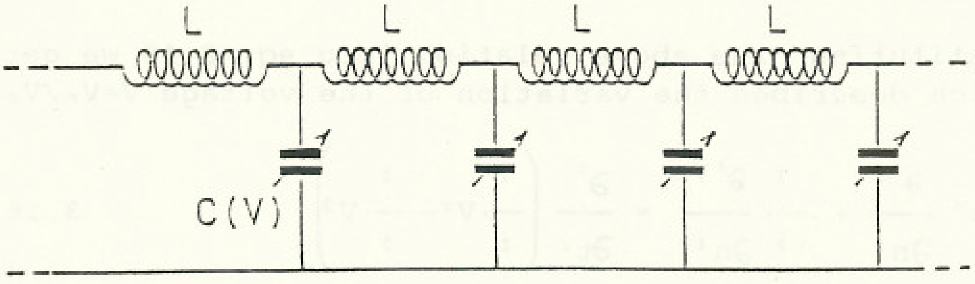


fig. 35.a

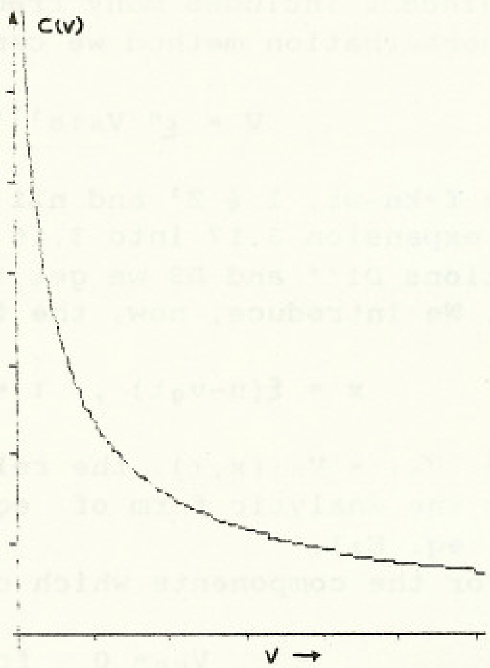


fig.35.b

The fundamental equations which describe the above circuit are:

$L(\partial I_n/\partial t)=V_{n-1}-V_n$ 3.13a

$\partial Q/\partial t = I_n-I_{n+1}$ 3.13b

$Q_n= C(V_n)\cdot V_n$ 3.13c

where the index refers to the correspondence shell in the line. If we derivate eq. 3.13b in respect to t and by taking into account eq. 3.13a we obtain:

$$L\frac{\partial^2 Q_n}{\partial t^2} = V_{n+1}+ V_{n-1}- 2V_n \qquad 3.14.$$

We also have :

$$\mathrm{Log}(1+V) = V - (1/2)V^2+ (1/3)V^3 + \ldots.$$

and equation 3.14 is written :

$$V_o\frac{\partial^2}{\partial t^2}\left(\frac{V_n}{V_o} - \frac{1}{2}\left(\frac{V_n}{V_o}\right)^2+\frac{1}{3}\left(\frac{V_n}{V_o}\right)^3\right) = \frac{1}{L\cdot C_o}\, V_{n+1}-2V_n + V_{n-1}$$

For sufficiently long wave length signals ($k\longrightarrow 0$) we can use the continuum limit approximation :

$$V_{n\pm 1}= V_n \pm \frac{\partial V_n}{\partial n} + \frac{1}{2!}\frac{\partial^2 V_n}{\partial n^2} \pm \frac{1}{3!}\frac{\partial^3 V_n}{\partial n^3} + \ldots.$$

After substituting the above relation into eq. 3.15 we get the PDE which describes the variation of the voltage $V=V_n/V_o$

$$\frac{\partial^2 V}{\partial t^2} - u^2 \frac{\partial^2 V}{\partial n^2} + \frac{1}{12}\frac{\partial^4 V}{\partial n^4} = \frac{\partial^2}{\partial t^2}\left(\frac{1}{2}\cdot V^2 - \frac{1}{3}\cdot V^3\right) \qquad 3.16$$

where $u^2 = 1/LC_o$.

Since V includes many frequency components, according to the perturbation method we can expand $V=V(t)$ as :

$$V = \varepsilon^n V_{nl} e^{ilf} \quad (*) \qquad 3.17$$

where $f=kn-wt$, $l \in Z^{\pm}$ and $n \geq 1$ (integer). If we substitute the expansion 3.17 into 3.16 and by taking into account the relations D1** and D3 we get the expanded form E1 of eq. 3.16. We introduce, now, the following slow variables :

$$x = \varepsilon(n-v_g t) \;,\quad \tau = \varepsilon^2 t \;,\quad v_g = dw/dk$$

Since $V_{nl} = V_{nl}(x,\tau)$, the relations D2 are satisfied and E_1 gives the analytic form of eq.3.16 in the new variables (see. eq. E_2).

For the components which correspond to **l=0** we can obtain that:

$$V_{no} = 0 \quad \text{for every } n \qquad 3.18$$

If we distinguish the components with **l=1** we have:

$\underline{\varepsilon(0).}$: $\quad -w^2V_{11} + u^2k^2V_{11} - (u^2/12)k^4V_{11} = 0$

Now the dispersion relation is:

$$w^2 = k^2u^2(1-k^2/12) \qquad 3.19$$

We suppose that components of high harmonics are negligible for $|l|>2$. Now the components of the second order of ε satisfy the equation :

$\underline{\varepsilon^2(0)}$: $\quad (2wv_g - 2u^2k + u^2k^3/3)\cdot V_{11} + O.O(0)^* = -w^2ll'V_{1l}V_{1l'} -$

$w^2l^2V_{1l}V_{1l'} = (3.18) = 0(l=0 \text{ or } l'=0) + (4w^2V_{12}\cdot$

$V_{1(-1)} - 4w^2V_{12}V_{1(-1)} = 0 \quad ==>$

*We use the Einstein convention for repeated indices
** Eqs D1,D2,D,E1,E2 and E3 are given in the appendix (D)

$$v_g = u^2(k/w)\cdot(1-k^2/6) \qquad 3.20$$

$\varepsilon^3(0)$: In this case we can obtain equation E3. By taking into account the relations 3.18, 3.19 and 3.20 we find:

$$-2iw(V_{11})_\tau + (v_g^2 - u^2 + u^2k^2/2)\cdot(V_{11})_{xx} = -w^2V_{1(-1)}V_{22}$$

$$+ w^2|V_{1(\pm1)}|^2V_{11} \qquad 3.21$$

For the second harmonics of V we can obtain from E2 the following relations :

$\varepsilon(0)$: $[w^2 - u^2k^2(1-k^2/3)]\cdot V_{12} = 0 \implies$ (by 3.19) $\implies$

$$V_{12} = 0 \qquad 3.22$$

$\varepsilon^2(0)$: $(-w^2-u^2k^2-u^2k^2/3)\cdot V_{22} = -(w^2/2)V_{11}^2 \implies$

$$\implies V_{22} = \tfrac{1}{2}w^2/[w^2-u^2k^2(1-k^2/3)]\cdot V_{11}^2 \qquad 3.23$$

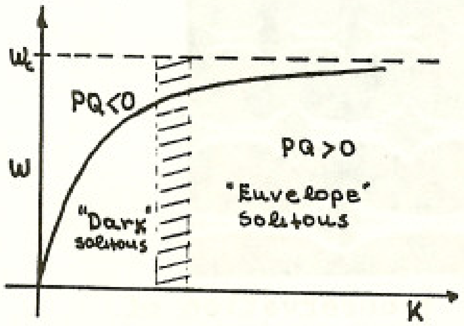
If we substitute V_{22} from 3.23 into 3.21 we obtain the NLS equation :

$$i(V_{11})_\tau + P\cdot(V_{11})_{xx} + Q\cdot|V_{11}|^2V_{11} = 0$$

$$P = (-1/2w)\cdot[v_g^2 - u^2(1-k^2/12)]$$

$$Q = (w/2)\cdot\{1-(w^2/2)[w^2-u^2k^2(1-k^2/3)]\}$$

Since our calculations are exact only for k-->0 it must be P>0 and Q<0. For this case (PQ<0) the above NLS equation gives hole solutions. If we do not use the continuum limit approximation and study the discrete case in the same way, we find [67]:



$$P = -(u/4)\cdot\sin(k/2)$$

$$Q = w_o^2/(4\cdot w) - (7/16)w$$

where $w_o = 2/(LC_o)^{-}$. With the above coefficients we have information for all ks. The P·Q changes from negative to positive. So we can find regions which provide envelope solutions.

3.2.b Numerical simulations and experiments.

K.Muroya, N.Saitoh and S.Watanabe [19] made analytical calculations and experimental observations of the transmission line of figure 35.a. They observed dark solitons and they found a good agreement between the experiments and the results of the NLS equation (figure 37). They, also, veri-

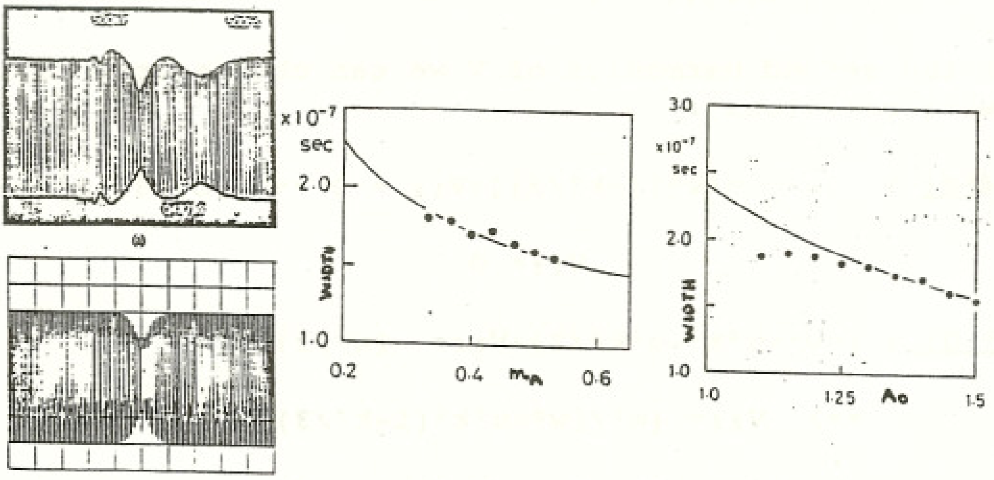
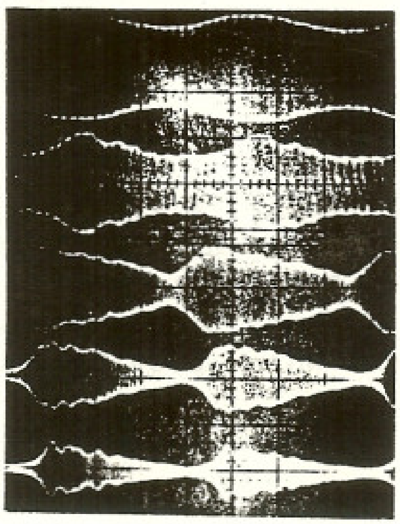


fig. 37 : Experimental and theoretical results of dark solitons in a transmission line.

fied the relations between the width, the modulational depth, the amplitude and the velocity of the dark solitons. The velocity decreases with the modulational depth and it has a little increase with the increase of the carrier amplitude.

Several authors have shown that the nonlinear dispersive transmission lines allow the formation of modulational instability and eventual formation of envelope solitons. Figure 38 shows the experimental observations of Yagi and Noguchi [46].



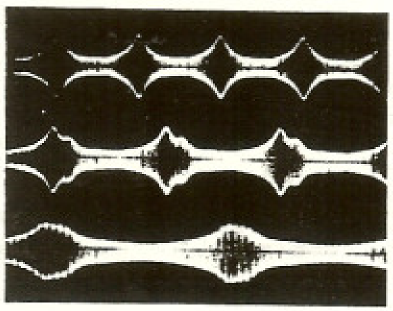


fig. 38 : ↑ observation of envelope solitons
⟵ The spatial modulation of a gently modulated initial wave.

From their numerical studies, Mizumura and Noguchi [40] concluded that there was a little harmonic generation during the wave propagation and the modulational instability appeared as it had been predicted by the theory. After certain time, the side bands which had appeared, diminished and recurrence phenomena took place.

Also, many experiments have been carried out in the O.R.C. laboratory of Dijon using monoinductance and biinductance LC transmission lines and a pulse voltage generator controlled by a microcomputer [17][18][47][48]. In figure 39 there is a representation of the experimental set up using monoinductance transmission line composed of 96 sections. The linear inductors are about 300 μH and the nonlinear capacitors are biased by $V_0=1$ Volt with $C_0\approx500$ pF. The diodes are used to give better nonlinear behavior to the capacitors. The resistance R_L is adjusted to prevent reflection effects.

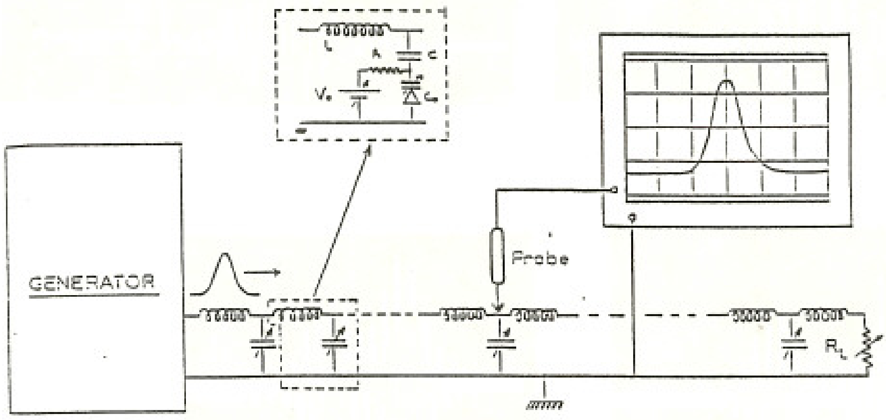


fig. 39 : Experimental representation

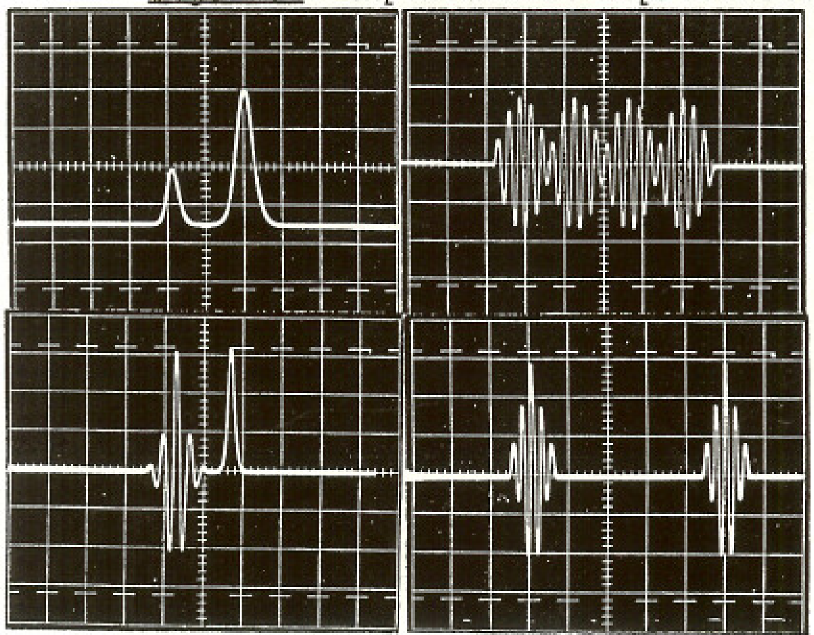
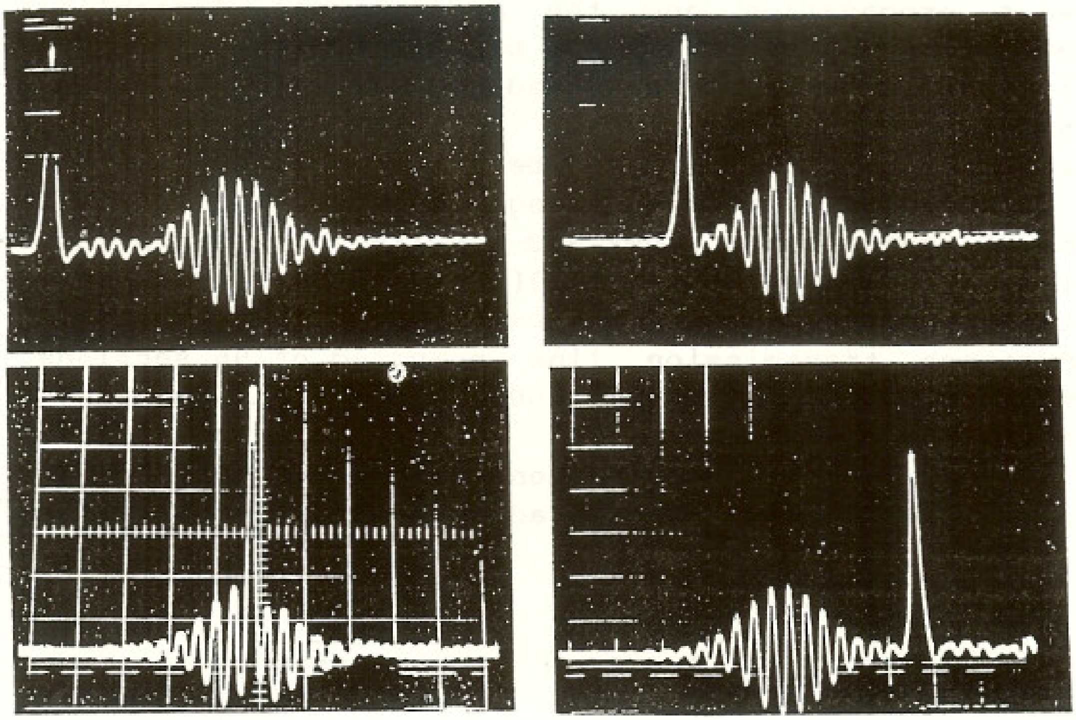
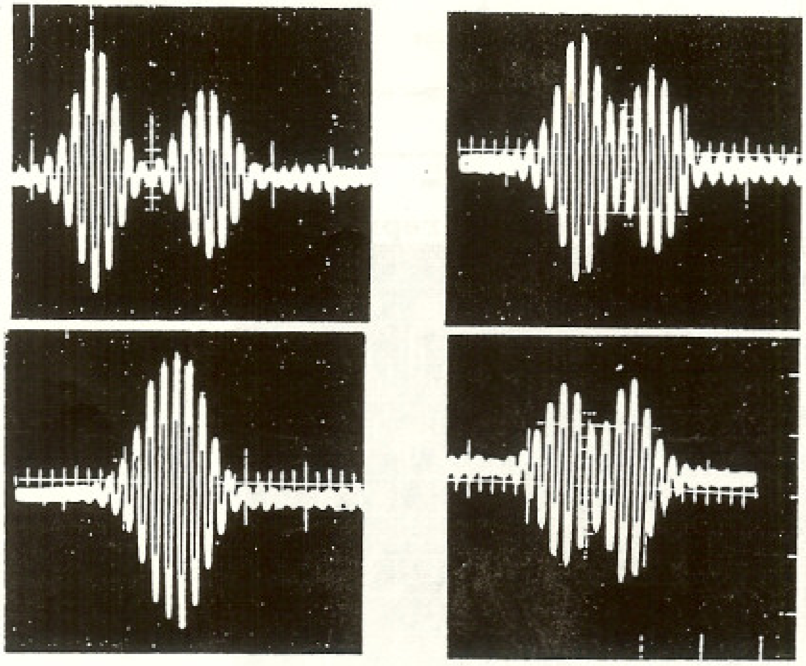


fig. 40. Output voltage signals by the generator



(a)



(b)

fig. 41 : a) pulse - envelope interaction
b) envelope - envelope interaction

The experimental results are in a good agreement with the theoretical ones when the height of the solitons is not too large. Figure 41.a illustrates an interaction of one pulse and one envelope soliton where only a weak and negligible emission of radiation had been observed. In figure 41.b a collision between envelope solitons is represented. The phase shift was too small to be observed. In a recent work [48], J.M. Bilbault and M. Remoissenet studied the 'two-soliton' envelope solution of the line in figure 35. In figure 42.a the propagation of such solitons is demonstrated by numerical simulation. They also studied the modulational instability of the line and observed the form of side bands.

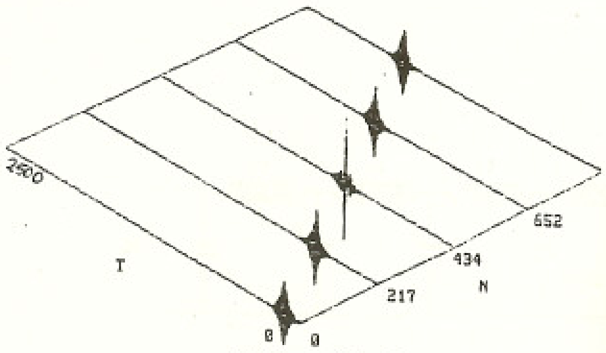


fig. 42.a

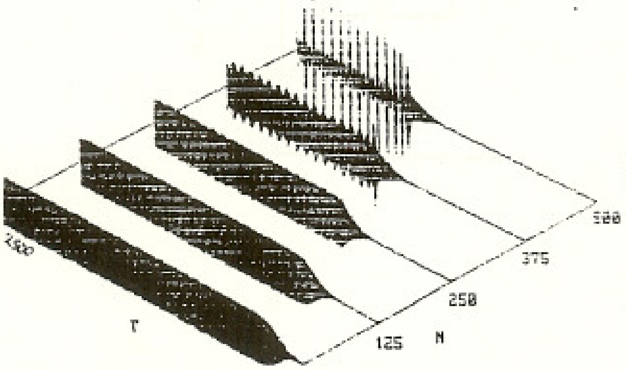


fig. 42.b The modulational instability of a carrier wave in the line.

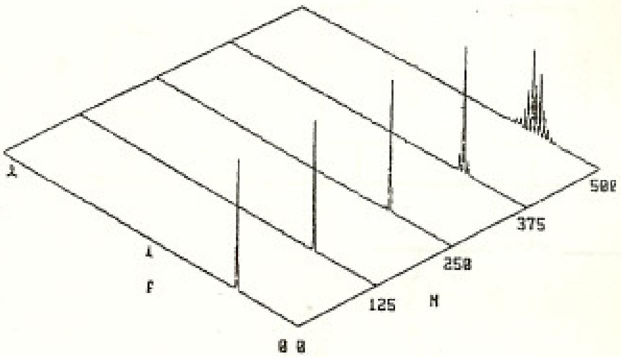


fig.42.c: The Fourier spectrum.

3.2.c Other transmission lines

Besides the monoinductance LC transmission line there are a variety of other electrical lines where solitons can propagate (figure 43). In fig. 43.a a biinductance line is drawn. Experiments in the laboratory showed that this line can support low frequency (acoustic) pulse solitons and low frequency envelope solitons [17]. For the E.T.L.(e) , Y.Nejoh [39] derived the NLS equation using the continuum limit approximation and he observed dark solitons. He, also, showed that electron plasma waves can be simulated by this line. The E.T.L.(c) was studied by Fukushima et al

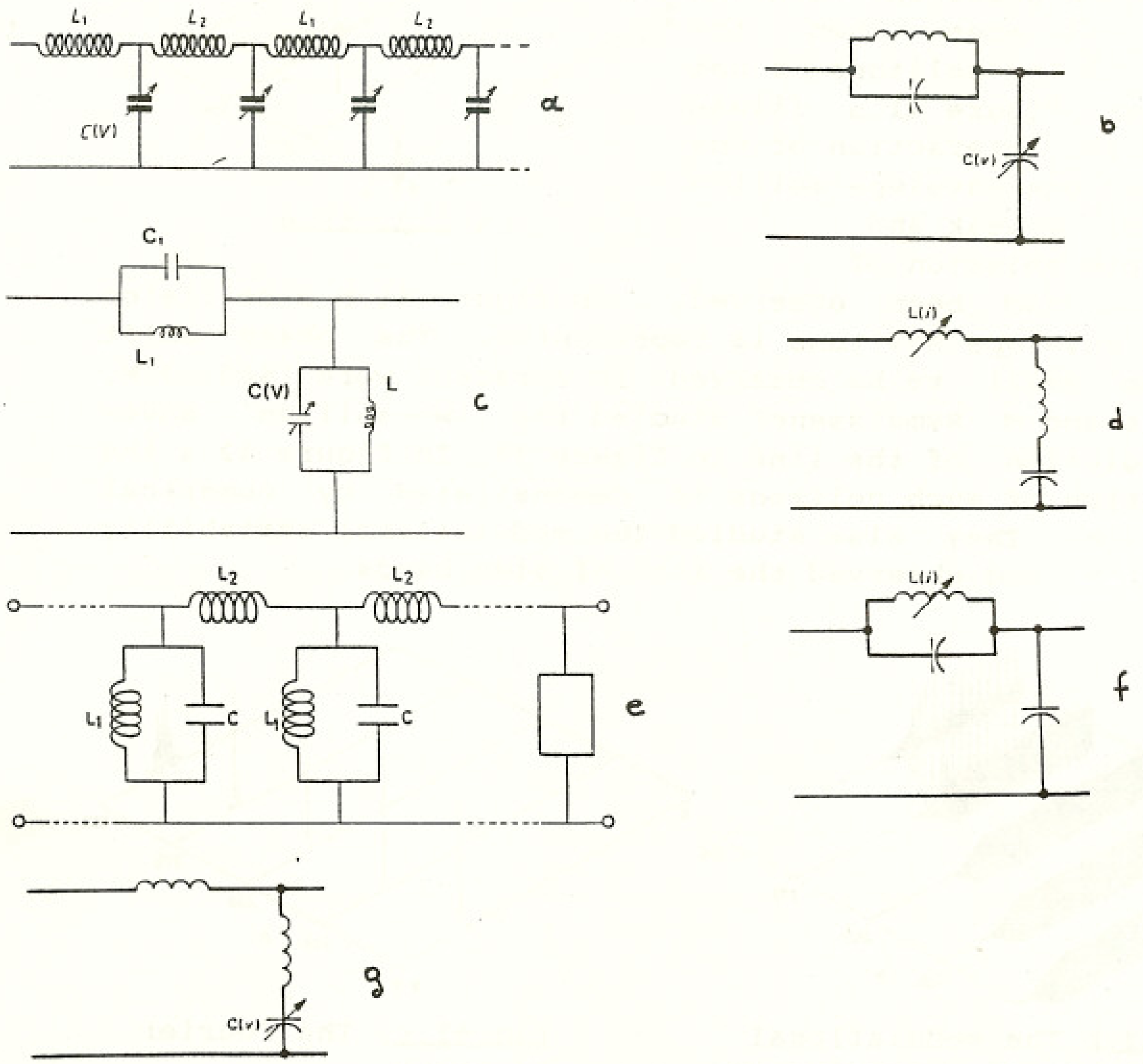


fig. 43: Cells of transmission lines

[42],[43]. They described the line with the modified NLS equation :

$$i \cdot u_t - \beta \cdot u_{xx} - \alpha \cdot |u|^{1/2} u = 0$$

with envelope soliton solutions of the form :

$$u(n,t) = (4/\alpha^2) f_o^2 \mathrm{sech}^4[(f_o/5)^{1/2}/2(\Gamma - uQ_o t)] \cdot e^{ik\Gamma - i\omega o \Omega}$$

$$\Gamma = (2/\beta)^{1/2} \cdot n \qquad f_o = (5/n) \cdot (\Omega + K^2) \qquad u = -2 \cdot K$$

After experiments, they observed the envelope solitons predicted by the theory and they studied the modulational instability of wavetrains in the line. E.T.Ls.(d) & (f) consist of nonlinear inductors and they were used to simulate electron waves in semiconductors.

3.2.d Applications of E.T.L. to communication systems[44][46]

Nowadays, communication systems are based on linear channels where the dispersion causes a distortion of the information pulses. Since a soliton can propagate along a

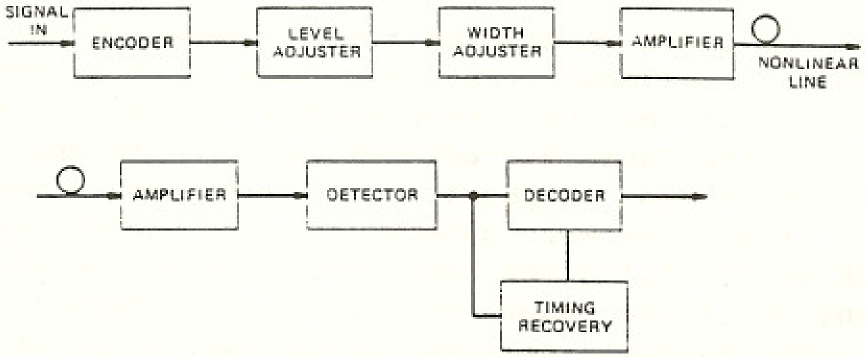


fig.44 : A p.c.m system.

transmission line without changing of its velocity and with no distortion, it seems to be advantageous. P.L. Chue and T.Whitbread carried out experiments using a pulse-code modulation communication system (figure 44). Their nonlinear line was a monoinductance line of 90 LC sections. They observed a lack of distortion and less bit error rate than in a linear line. They, also, concluded that the jitter, which appeared, was caused by the tails of solitons and could be removed. Consequently, since we can achieve better performance, using nonlinear lines, it seems more practical to use solitons as signal carriers in a P.C.M. communication system but much work is to be done on this matter.

Additional references : [62] [89] [98]-[101]

4. GENERALIZATIONS OF THE N L S EQUATION

In the previous parts we studied the NLS equation which includes a quadratic nonlinear term. This form is the most famous and its solutions have been studied in detail. We also mentioned that the Cubic Schrodinger equation is met in many applicable branches of physics. In recent years many authors began to study some more general forms adding new terms in the CSE or modifying the nonlinear term or the dispersion one. Many of these equations present interesting features and they support soliton solutions. Since they describe, in many cases, physical models, their investigation is necessary. Many generalizations have been suggested; in this part we will give some characteristics of the most important forms.

4.1 THE HIGHLY NLS EQUATION[102]-[105]

The highly NLS equation is characterized by a nonlinear term which is an arbitrary function of the intensity $I=|E|^2$. So the highly NLSE describes systems with stronger nonlinearities than the cubic one. It has been suggested by A.E. Kaplan [102] in order to study further nonlinear optical effects. Its form is :

$$2i\cdot E_z + E_{xx} + E\cdot f(|E|^2) = 0 \qquad 4.1$$

where f is an arbitrary function which satisfies the condition $f(0)=0$. For $f=\alpha\cdot|E|^2$ we take the CSE. The first three invariants of the eq. 4.1 are :

$$J_1 = \int_{-\infty}^{\infty} I\cdot dx \qquad : \text{total power}$$

$$J_2 = i\int_{-\infty}^{\infty}(EE_x{}^* - cc)\cdot dx \quad : \text{'transverse' momentum}$$

$$J_3 = \int_{-\infty}^{\infty}\left(|E_x|^2 - \int_0^I f(s)ds\right)\cdot dx \quad : \text{'transverse' energy}$$

Stationary solutions, with a nonvarying intensity profile (i.e. $\partial|E|^2/\partial z = 0$), are of the form :

$$E(x,z) = u(x-v\cdot z)\cdot\exp(i\delta z/2 - ivx) \qquad 4.2$$

where $u=|E|$ is a real function which satisfies the condition $u \longrightarrow 0$ as $|x| \longrightarrow \infty$. δ is a real constant and v plays the role of the velocity.

For the solution 4.2, eq. 4.1 gives :

$$d^2u/dx^2 + u\cdot[f(u^2)-\delta] = 0 \qquad 4.3$$

which yields:

$$x = \int\left(\int_0^{u^2}[\delta - f(u^2)]\cdot d(u^2)\right)^{-1/2} du$$

The above integral gives the amplitude profile of the solitary wave and it can be analytically evaluated only for some particular class of functions $f(u^2)$. The total power is given by :

$$P(\delta) = \int_0^{Im(\delta)} dI \;/\; [\delta - F(I)]^{1/2} \qquad 4.4.a$$

where

$$F(I) = I^{-1}\int_0^I f(I)\cdot dI \quad , \; F(0)=0 \qquad 4.4.b$$

$Im(\delta)$ is the peak intensity of the solitary wave. The multistability of a singular soliton is realized when the function $\delta(P)$ becomes multivalued (or the curve $\delta(P)$ is "S-shaped").

In order to verify that the solitary waves of 4.1 are solitons we have to study their behavior under interactions. By taking into account some special examples of functions f, Kaplan stated that solitary waves of eq. 4.1 remain stable after collisions, if $d\delta/dP>0$ [102]. This condition holds if the nonlinearity's dependence on the intensity has a sufficiently sharply increasing range.

R.H.Enns and S.S. Rangnekar carried out numerical simulations in order to verify the conclusions of Kaplan. They examined [103] the step function :

$$f(I) = \begin{cases} 0 & I<I_o \\ \alpha(1-I_o^2/I^2) & I>I_o \end{cases}$$

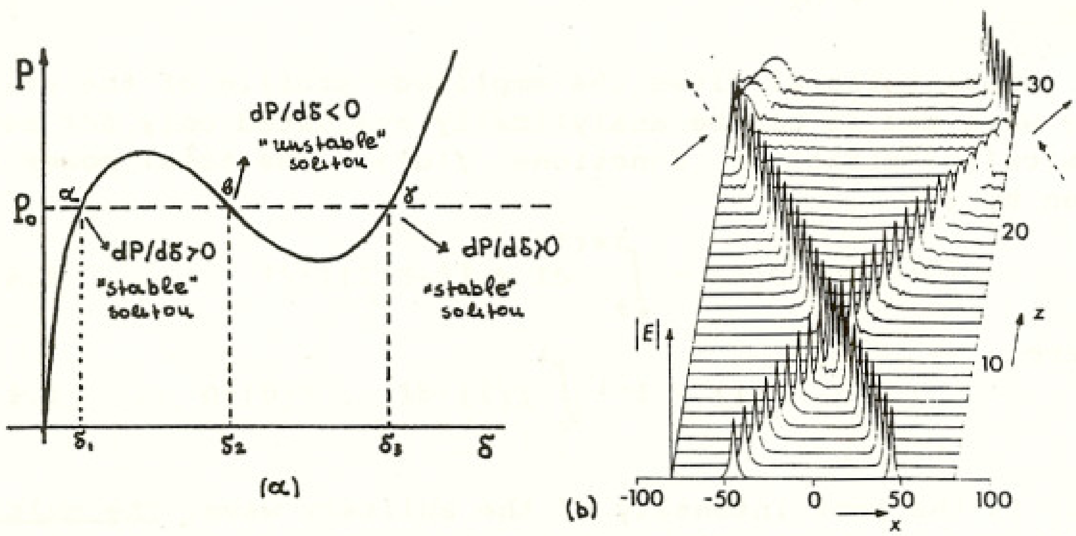
which yield :

$$u(x,0)=\begin{cases} I_o^{1/2}\exp[(\alpha\beta)^{1/2}(x_o-|x|)] & |x|\geq x_o \\ I_o^{1/2}[1+\beta^{1/2}\cos\{2[\alpha(1-\beta)]^{1/2}x\}]/(1-\beta)^{1/2} & |x|\leq x_o \end{cases}$$

where α is real parameter, $\beta\equiv(\delta+v^2)/\alpha$ and

$$x_o = \arccos(-\beta^{1/2})/2[\alpha(1-\beta)]^{1/2}.$$

Their experiments showed that solitary waves with $d\delta/dP>0$ were stable under collisions but solitary waves with $d\delta/dP<0$ were dispersing after the collision (figure 45).



<u>fig. 45</u> : a) δ-P diagram which gives the conditions for the existence of solitons. b) Collision of a stable solitary wave (soliton) with an unstable one.

The above authors examined many models using polynomial or step forms of function f [104][105]. They observed solitons (**bistable**) solutions in agreement with the Kaplans theory. They, also, mentioned the existence of "robust" bistable solitons. These are solitons which survive under large perturbations. They noted that solitary waves of eq. 4.1 are **robust** solitons if a) $d\delta/dP>0$ b) $f(I)/I^2=0$ as $I\longrightarrow\infty$ c) $f(I)$ is a nonnegative nondecreasing function for $I>0$.

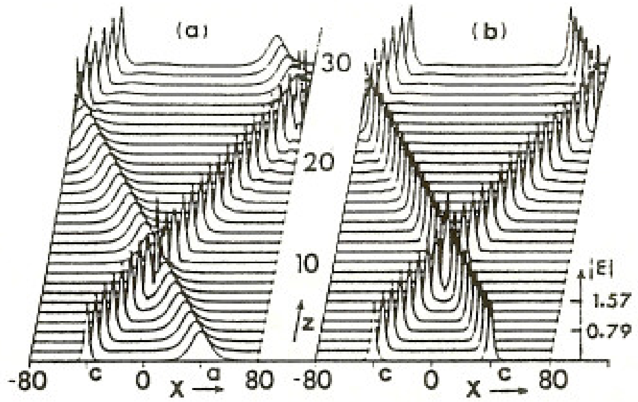


fig. 46 : Interaction of robust solitons
a) solitons with different values of δ
b) solitons with the same δ

THE EXPONENTIAL N L S EQUATION [110]-[112]

In the third part we showed that the CSE is a very useful model for wave propagation in a nonlinear medium. However, when the CSE is generalized to two or three dimensions with the same quadratic nonlinearity, localized solutions are found to be unstable in general. The CSE does not take care of the vector character of the electric field **E** and it refers to weak nonlinearities of the medium. N.Tzoar and J.Gerten [110] showed that the solitons which propagate in a nonlinear medium, shows a saturation tendency to collapse. They, also noticed that the above phenomenon is caused because the quadratic nonlinear term is not enough to balance the dispersion. P.K.Kaw et al [111] assumed higher order nonlinearities in order to admit localized stationary solutions. They found that stationary solutions can be supported by a nonlinear susceptibility of the form $x_{NL}\approx\exp(-|u|^2)-1$. E.W. Laedke and K.H.Spatschek [112] replaced the quadratic nonlinear term in the CSE with the above nonlinearity and they suggested the **scalar exponential equation**.

$$i\cdot u_t + \nabla^2 u + [1-\exp(-|u|^2)]\cdot u = 0 \qquad 4.5$$

We introduce now a nonlinear frequency shift $\tilde{n}^2$ and assume that eq. 4.8 supports solutions of the form :

$$u = G(r)\cdot \exp(i\cdot \tilde{n}^2 t) \qquad 4.5.a$$

A necessary and sufficient criterion for stability of the solutions 4.5.a is [112] :

$$\partial_{\tilde{n}^2} \int dr\cdot r^2 G^2 > 0 \qquad 4.6$$

The relation 4.9 is satisfied for $0 < \tilde{n}^2 < 2/3$. This shows the existence of stable 3-D envelope solutions for sufficiently large $\tilde{n}^2$ values (figure 47.a).

Also, Laedke and Spatschek studied the **vectorial exponential equation** which has the form :

$$i\cdot E_t + \nabla^2 E - q\nabla x \nabla x E + [1-\exp(-E\cdot E^*)]E = 0$$

They found that its solutions show similar behavior with the scalar expotential equation.

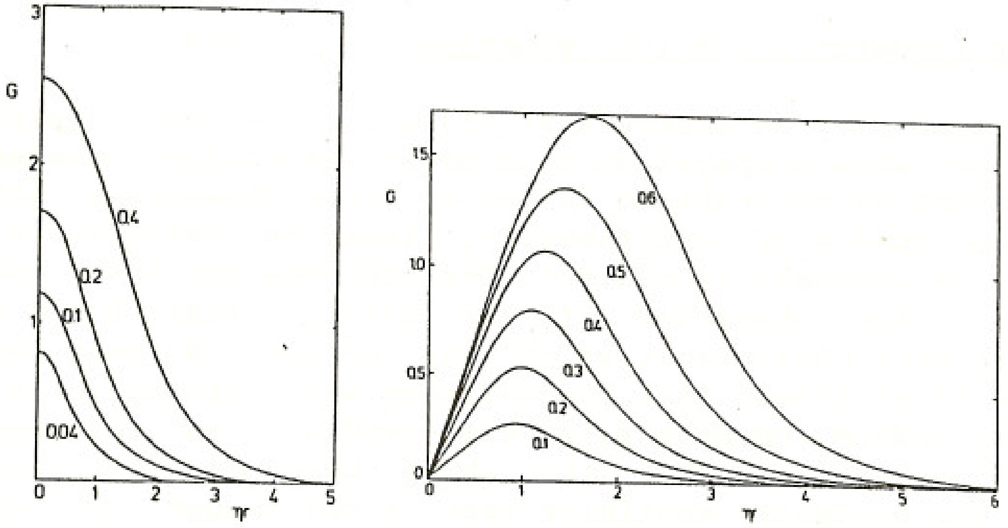


fig. 47. a) Stationary solutions of the S.E. NLS
b) Stationary solutions of the V.E. NLS

The exponential NLS equation plays an important role in plasma physics where it can explain the existence of stable spherically symmetric Langmuir solitons.

THE DERIVATIVE N L S EQUATION [106]-[108]

The NLS equation can describe modulated waves only when the characteristic length of the variation of the envelopes is much larger than the wave length of the carrier. The DNLS equation is available even in the case that both lengths are comparable to each other. Its form is

$$i \cdot q_t + q_{xx} + m \cdot i \cdot (|q|^2 q)_x = 0 \quad m = \pm 1 \qquad 4.7$$

The first derivation of eq. 4.7 was by K.Mio et al. Using the multiple scale perturbation method, they found that eq. 4.7 can describe Alfven waves in plasma [106]. After numerical computations, they showed the stability of the solutions and they observed envelope-like and dark-like solitons.

D.J.Kaup and A.C.Newell using the inverse scattering method, obtained, under the vanishing condition :

$$|q(x,t)| \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \pm \infty$$

the one-soliton solution [107] :

$$|q(x,t)|^2 = \frac{8|f|\sin^2\theta}{\cosh(4|f|z \cdot \sin\theta + A_0) - m \cdot \cos\theta} \qquad 4.8$$

$$f = a + i \cdot b$$
$$\theta = b \cdot (x - x_0)$$
$$z = x + 4 \cdot a \cdot t + z_0$$

The solution 4.8 becomes similar to that of NLS equation for $\cos\theta = 0$. For $m \cdot \cos\theta \geq 0$ the solution is called "anomalous" and for $m \cdot \cos\theta \leq 0$ "normal". As $m \cdot \cos\theta \to 1$ then "algebraic" solitons occur which are:

$$|q(x,t)|^2 = 16|f| \;/\; (1 - 16|f|^2 Z^2)$$

where $Z = z + A_0/(4|f|\sin\theta)$. These solitons are unstable i.e. if the initial data are changed by a finite amount then these solitons either disperse away or become very broad sech-like solitons.

T.Kawata et al studied the collisions of solitons of the DNLS equation numerically [109]. They observed that solitons behaved like particles except for interchanging their roles. The intermediate region between solitons was growing gradually when peaks approached each other and they exchanged their heights rapidly. In figure 49a an anomalous-normal soliton collision is demonstrated where a small dispersion has been found in the collision region.

The same authors studied the solutions of the DNLS under the nonvanishing condition :

$$|q(x,t)| \longrightarrow q^{+} > 0 \quad \text{as} \quad x \longrightarrow \pm \infty$$

and they found soliton solutions which are given by:

$$|q(x,t)|^{2} = (q^{+})^{2} - 2im\,(\log[\equiv^{*}/\equiv])_{x}\,^{(*)}$$

They named the above solution "paired" soliton. These solitons

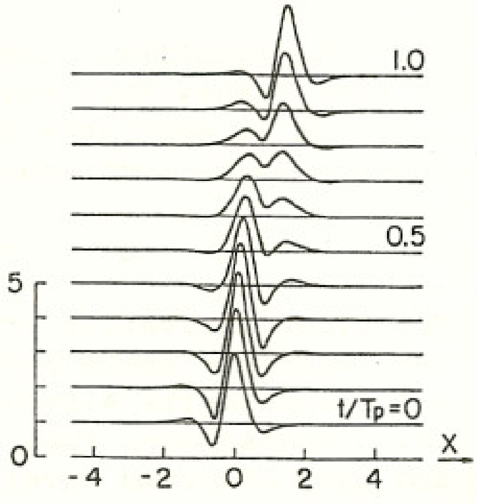


fig.48: 'Paired' soliton.

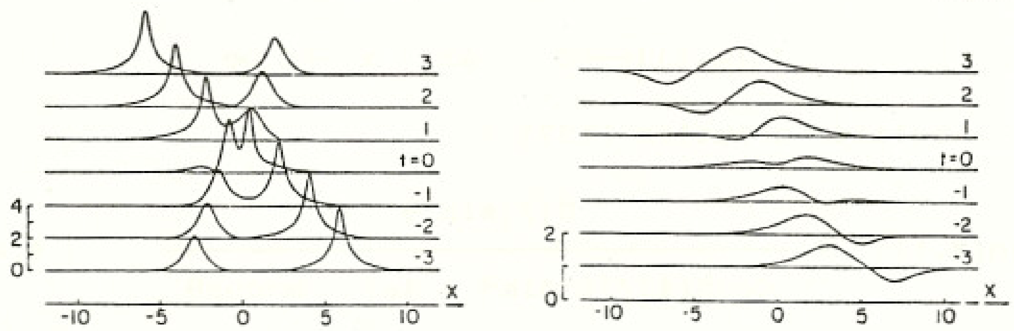


fig.4 :a) "anomalous-normal" soliton collision b) dark-bright soliton collision.

do not keep the same shape as propagate but they appear to change it periodically in time (figure 48). The above solution includes two kinds of solitons i)**bright** and ii)**dark** solitons. They showed that their behavior is like the solitons one for the 'vanishing condition' case [109].

Other generalizations

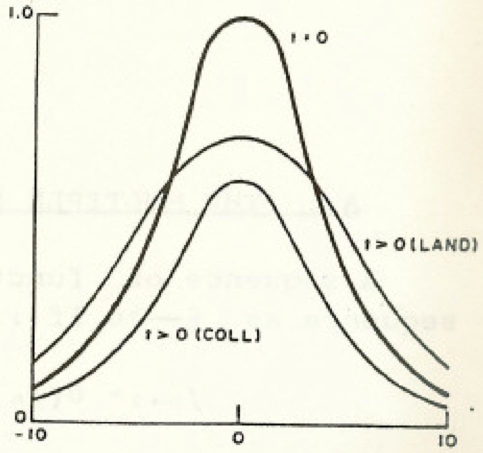
In the third part we mentioned the modified NLS equation which is used to describe the propagation of envelope solitons in some electrical transmission lines. We also noticed the NLS equation with loss which provides stable but reduced solitons. Next we notice some other generalizations which are used to describe nonlinear wave propagation in many physical models.

*for details see eq. 6.14 in ref.[108]

<u>i) Damped NLSE</u> : It has been proposed by D.R. Niccholson et all [113] and it has the form :

$$i \cdot E_t + E_{xx} + |E|^2 E + i \cdot L[E] = 0$$

where L is a real Hermitian operator. The same authors derived the Landau damping operator and the collisional damping one which can explain the propagation of nonlinear waves in plasma. They found solitons with reduced amplitude (figure 50). For the Landau operator the damping increases with the soliton velocity.



<u>fig. 50</u>

<u>ii) Logarithmic NLSE</u>: It has the form :

$$iu_t + u_{xx} + |u|^2 u - b \cdot \ln(\alpha |u|^2) = 0$$

where α and b are real coefficients. It has been suggested to describe nuclear phenomena [114].

<u>iii) NLSE with variable coefficients</u> : It is given by

$$i \varepsilon q_t + \varepsilon^2 q_{xx} + \tilde{n}(x,t) |q|^2 q = 0$$

where $\tilde{n}$ is a real function and ε is a small parameter ($0<\varepsilon<1$). V. Subochev and V. Tsupin found conditions which support soliton-like solutions [115].

additional references : [116] - [133]

A P P E N D I X

A . THE MULTIPLE SCALE PERTURBATION TECHNIQUE

A sequence of functions $f_n(\varepsilon)$ is called asymptotic sequence as $\varepsilon \longrightarrow 0$ if :

$$f_{n+1} = O(f_n) \quad \text{for every } n \geq 0$$

Consider a function f(x) which is approximated by series

$$f(x; \ ) = \sum_{n=0}^{N} f_n(\varepsilon) \cdot f_n(x) + O(f_N(\varepsilon)) \qquad \varepsilon \longrightarrow 0$$

This expression is called an <u>asymptotic approximation</u> to N+1 terms of the function f(x) with respect to the asymptotic sequence $\{f_n\}$. If the above expression holds for every $N \geq 0$ it is called **asymptotic expansion.**

Convergent series expansions are always asymptotic but asymptotic ones are not necessarily convergent. In asymptotic expansions, no infinite series are needed, instead only few terms of the asymptotic expansion are required when is small enough. The most common asymptotic expansion used is the :

$$f_n(\varepsilon) = \varepsilon^n$$

So a function f can be expanded as :

$$f(x; \ ) = \sum_{n=0}^{N} \varepsilon^n f_n(x) + O(\varepsilon^{n+1})$$

Multiplication and differentiation of such asymptotic expansions are always permitted.

In the multiple scale perturbation technique we introduce slow varying variables which are defined by:

$$X^n{}_i \equiv \varepsilon^n \cdot x_i$$
$$T^n \equiv \varepsilon^n \cdot t$$

Using the above slow variables we can describe the asymptotic behavior of oscillations. Now, the derivatives should be replaced by:

$$d / dx_i = \sum_{n=0}^{\infty} \varepsilon^n \partial/\partial X^n{}_i$$

A function $f(x_i,t)$ can be expanded by series of the form

$$f(x_i,t;\ ) = \sum_{n=0}^{\infty} \varepsilon^n f_n(T^n,X^n) \qquad n>0$$

By substituting the above formulas into the equation which describes the system, we can distinguish the terms in different powers of . So we obtain a number of equations which can be studied separately. The equations which are going to describe the asymptotic behavior are obtained if we remove the secular terms which grow up in time making the multiple scale method not useful.

We will apply the above method to the cubically nonlinear Klein-Gordon equation :

$$u_{xx} + u_{tt} = a\cdot u - b\cdot u^3 \qquad 1$$

which comes from the Hamiltonian density :

$$H = 1/2\ (u_x{}^2+u_t{}^2) + 1/2\cdot a\cdot u^2 - 1/4\cdot b\cdot u^3 \qquad 2$$

We define a set of slow space and time variables as :

$$X_n = \varepsilon^n\cdot x \qquad T_n = \varepsilon^n\cdot t \qquad (\varepsilon<<1) \qquad 3$$

The above variables describe the envelope motion and they are considered independently.

If we plot the potential of the Hamiltonian (2) we will see that u=0 is a minimum, so we expand the variable u around this point :

$$u = \varepsilon^p u^1 + \varepsilon^{2p} u^2 + \varepsilon^{3p} u^3 + \ldots. \qquad 4$$

where p is, as yet, an unknown positive number. If we substitute eqs. 3 & 4 in 1 we obtain :

$$\{(\partial/\partial x + \varepsilon\partial/\partial X_1 + \ldots)^2-(\partial/\partial t + \varepsilon\partial/\partial T_1 + \ldots)^2-a\}\cdot(\varepsilon^p u^1+$$

$$+\varepsilon^{2p} u^2+ \ldots) + b\cdot(\varepsilon^p u^1+\varepsilon^{2p} u^2+\ldots)^3 = 0 \qquad 5$$

Distinguishing the terms at the same order of ε we obtain:

$O(\varepsilon^{p})$: $(\partial^2/\partial x^2 - \partial^2/\partial t^2)\cdot u^1$ 6.a

$O(\varepsilon^{p+1})$: $2(\partial^2/\partial x\partial X_1 - \partial^2/\partial t\partial T_1)\cdot u^1$ 6.b

$O(\varepsilon^{2p})$: $(\partial^2/\partial x^2 - \partial^2/\partial t^2 - a)\cdot u^2$ 6.c

$O(\varepsilon^{2p+1})$: $2\cdot(\partial^2/\partial x\partial X_1 - \partial^2/\partial t\partial T_1)\cdot u^2$ 6.d

$O(\varepsilon^{p+2})$: $2\cdot(\partial^2/\partial x\partial X_2 - \partial^2/\partial t\partial T_2)\cdot u^1 +$

$+ (\partial^2/\partial X_1^2 - \partial^2/\partial T_1^2)\cdot u^1$ 6.e

$O(\varepsilon^{3p})$: $b\cdot(u^1)^3 + (\partial^2/\partial x^2 - \partial^2/\partial t^2 - a)\cdot u^3$ 6.f

The determination of p is achived under the requirement that function u^1 should has a "sensible" form. For the above lower orders we have two choices i) $p+1=2p$ ii) $p+1=3p$. From (ii) we have $p=1/2$ and we can discover that it leads to a non sensible conclusion for the form of u^1. Case (i) gives **p=1** and eqs.(6) become :

$$O(\varepsilon): \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - a\right) u^1 = 0 \qquad 7.a$$

$$O(\varepsilon^2): \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - a\right) u^2 = -2\left(\frac{\partial^2}{\partial x\partial X_1} - \frac{\partial^2}{\partial t\partial T_1}\right) u^1 \qquad 7.b$$

$$O(\varepsilon^3): \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - a\right) u^3 = -2\left(\frac{\partial^2}{\partial x\partial X_1} - \frac{\partial^2}{\partial t\partial T_1}\right) u^2 -$$

$$-2\left(\frac{\partial^2}{\partial x\partial X_2} - \frac{\partial^2}{\partial t\partial T_2}\right) u^1 - b\cdot(u^1)^3 - \left(\frac{\partial^2}{\partial X_1^2} - \frac{\partial^2}{\partial T_1^2}\right) u^1 \qquad 7.c$$

7.a ==> $u^1 = A(X_1, X_2, \ldots, T_1, T_2, \ldots)\, e^{i\theta} + cc$ 8 where $\theta=kx-wt+\delta$ and $w^2=k^2+a$ $(a>0)$. So function A is an arbitrary complex amplitude function of the scales. It varies take place slowly and expresses the amplitude of an envelope wave.

Now we are going to discover more details for the function A i.e. we must determine the function $A(X_i,T_i)$. From eqs. 8 and 7.b we have:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - a\right) u^2 = -2i \; k\cdot\frac{\partial A}{\partial X_1} + w\cdot\frac{\partial A}{\partial T_1} \; e^{i\theta} + cc \qquad 9$$

We observe that the solution of the homogeneous part of eq.9 is like (8). So, since the term $e^{i\theta}$ appears in the nonhomogeneous part and in the solution of the homogeneous one, the particular solution of u^2 contains secular terms as e.g. $\theta\cdot e^{i\theta}$. As $t\longrightarrow\infty$, secular terms will blow up and the perturbation theory will become invalid for $t>{}^{-1}$. In order to prevent the above behavior we must set A=constant or we must assume the following, more general, condition :

$$k\cdot\frac{\partial A}{\partial X_1} + w\cdot\frac{\partial A}{\partial T_1} = 0 \qquad 10$$

From eq. 10 we can conclude that :

$$A = A\ (X,X_2,X_3,\ldots,T_2,T_3,\ldots)$$

where $X=X_1-c_gT_1$ and $c_g=dw/dk$ is the group velocity of the wavetrain 10. Consequently, the secular terms vanish in a frame travelling with the group velocity.

Concerning the $O(\ {}^3)$ terms, we obtain :

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - a\right) u^3 - 2i\left(k\cdot\frac{\partial A}{\partial X_2} + w\cdot\frac{\partial A}{\partial T_2}\right)e^{i\theta} -$$

$$- [(1-c_g)\ {}^2/\ X^2 + 3\cdot b\cdot A^2A^*]\cdot e^{i\theta} - b\cdot A^3 e^{3i\theta} \qquad 11$$

The terms which include u^2 have been neglected because of (10). We observe that the homogeneous part of (11) will include a $e^{i\theta}$ term which appears on the right part of (11). So secular terms are going to created. However $e^{3i\theta}$ can not create secular terms because it will not appear in the solution of the homogeneous part of (11). So we form the equation :

$$(1-c_g{}^2)\frac{\partial^2 A}{\partial X^2} + 3\cdot b\cdot A^2 A^* + 2\cdot i\cdot\left(k\cdot\frac{\partial A}{\partial X_2} + w\cdot\frac{\partial A}{\partial T_2}\right) = 0 \qquad 12$$

Since we can always transfer into a frame of reference with velocity dw/dk, X_2 and T_2 can be assumed dependent; so we need one of them and eq. 12 takes the form :

$$A_{XX} + 2ikw^2a^{-1}\cdot A_{T2} + 3bw^2a^{-1}A|A|^2 = 0 \qquad 13$$

Eq. 13 is the NLS equation and the behavior of A is given by its solutions.

B. THE HOLE SOLUTION - DARK SOLITON

In the second part we found three solutions of the NLS equation. In the third part, we also mentioned the dark soliton. Next, another solution is found by the same way as before.

Consider the NLSE with the form :

$$i\cdot u_t + P\cdot u_{xx} + Q\cdot |u|^2 u = 0$$

We seek solutions of the form :

$$u = r^{1/2}e^{i\theta}, \quad r=r(x), \quad \theta=\theta(x,t)$$

If we substitute the above solution in the NLSE, the imaginary and real part give the equations :

$$(r\cdot\theta_x)_x=0 \Longrightarrow r\cdot\theta_x= c(t) \qquad 1$$

$$P\cdot(r_{xx}/2r)-P(r_x/2r)^2+Q\cdot r = \theta_t+P\cdot\theta_x{}^2 \qquad 2$$

From equation (2) we can observe that its left hand side is a function of x only. So we have :

$$\theta_t + P\cdot(\theta_x)^2= f(x)$$

or

$$\theta_t + P\cdot(c(t)/r)^2= f(x)$$

If we derivate the above equation with respect to x and t we find the following relation for c(t) :

$$d^2c/dt^2-(P/2r^2)\cdot(dr/dx)\cdot dc^2/dt = 0$$

So function c(t) must be a constant and equation (1) becomes

$$r\cdot\theta_x = c(\text{constant}) \quad \Longrightarrow \quad \theta= \int(c/r)dx + F(t)$$

We will look for travelling wave solutions so $\theta=\theta(x-v_{gr}t)$ which means that function F(t) must be $\Omega\cdot t$ and which

$$\theta = \int(c/r)dx + \Omega t \qquad 3$$

where Ω and c are constants.

Equation (2) takes the form :

$$(1/4)\cdot d[P(r_x{}^2/r)+2Q\cdot r^2]/dr = \theta_t+P\cdot\theta_x{}^2 \qquad 4$$

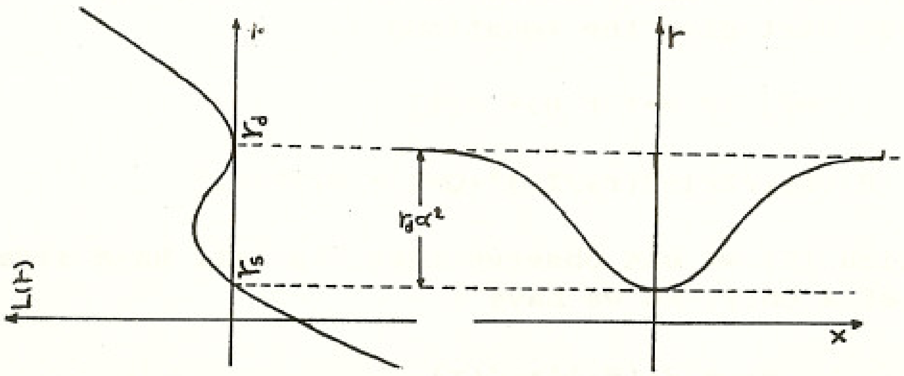
Substituting (3) into (4) and integrating we take the final polynomial which leads to the solutions:

$$r_x^2 = -2(Q/P)\cdot r^3 + (4\Omega/P)\cdot r^2 + (c'/P)\cdot r + 4c^2$$

If we examine the above polynomial for $Q/P>0$ (or $P\cdot Q>0$) we will observe the envelope solution. Here we examine the case:

$$Q/P < 0 \quad \text{or} \quad P\cdot Q < 0.$$

For this case, the polynomial has one positive double root (r_d) and one positive single root (r_s) with $r_d>r_s$. So we can observe a localized shape for the amplitude r with asymptotic value r_d as $x \longrightarrow \infty$.



This polynomial can be written as :

$$(r_x)^2 = |Q/P|/2\cdot(r-r_d)^2(r-r_s) \qquad 5$$

where the r_d and r_s depend on the parameters Ω,c,c',P and Q. After integration of (5) we have :

$$r = r_d\cdot\{1 - a^2\text{sech}^2[(\Gamma\cdot r_d)^{1/2}\cdot a\cdot x]\}$$

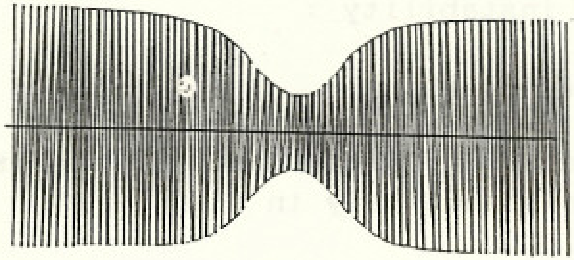
$$\theta=\sin^{-1}\{a\cdot\tanh[(\Gamma r_d)^{1/2}ax]/[1-a^2\text{sech}^2((\Gamma r_d)^{1/2}ax]\}^{1/2}+\Omega t$$

where:

$$a^2 = (r_d-r_s)/r_d$$

$$\Gamma = |Q/P|/2$$

This solution is called **hole** solution and is soliton **(dark)** with the same properties as the envelope soliton. The parameter a is connected with the depth of the hole and for a=1 we take the shock solution.



The dark soliton

C. MODULATIONAL INSTABILITY : A different way

Consider the NLS equation :

$$i \cdot u_t + u_{xx} + 2|u|^2 u = 0 \quad (1)$$

An envelope solution which is constant in x has the form :

$$u(x,t) = a \cdot \exp(2ia^2 t) \quad (2)$$

where a is a real constant. We perturb the solution (2) adding a small amplitude modulation.

$$u' = u + \varepsilon u \cdot (A_1\sigma + A_2\sigma^*) \quad (3)$$

where $\varepsilon \ll 1$, A_1, A_2 are real constants and $\sigma = \exp(ikx - i\Omega t)$. The superscript $*$ denotes the complex conjugate. If we substitute (3) into (1) we have :

$$[-2a^2A_1 + \Omega A_1 - k^2A_1 + 4a^2 A_1 + 2a^2 A_2] \cdot \sigma +$$

$$+ [-2a^2A_2 - \Omega A_2 - k^2A_2 + 4a^2 A_2 + 2a^2 A_1] \cdot \sigma^* = 0$$

The above relation yields:

$$(2a^2 + \Omega - k^2)A_1 + 2a^2A_2 = 0 \quad (4.a)$$

$$2a^2A_1 + (2a^2 - \Omega - k^2)A_2 = 0 \quad (4.b)$$

System (4) supports nontrivial solutions for A_1 and A_2 if

$$\det \begin{bmatrix} 2a^2 - k^2 + \Omega & 2a^2 \\ 2a^2 & 2a^2 - k^2 - \Omega \end{bmatrix} = 0$$

So we can take the final dispersion relation $\Omega = \Omega(k)$ which gives the condition for instability :

$$\Omega = \pm\, k(k^2-4a^2)^{1/2} \qquad 5$$

For $|k|<2\cdot a$ (long wavelength) the modulation appears unstable; u and u' diverge exponentially in time.

maximum growth $d\Omega/d(k^2)=0 \implies k^2=2a^2$

maximum rate growth $-i\cdot\Omega_{max}= 2a^2$

D. Formulas for the derivation of NLSE in the E.T.L

D1.

$$V_t = \varepsilon^n \cdot [(V_{n1})_t - iwV_{n1}] \cdot e^{il\theta}$$

$$V_{tt} = \varepsilon^n \cdot [(V_{n1})_{tt} - w^2 l^2 V_{n1} - 2iwl(V_{n1})_t] \cdot e^{il\theta}$$

$$V_{nn} = \varepsilon^n \cdot [(V_{n1})_{tt} - w^2 l^2 V_{n1} - 2iwl(V_{n1})_n] \cdot e^{il\theta}$$

$$V_{nnnn} = \varepsilon^n \cdot [(V_{n1})_{nnnn} + 4ikl(V_{n1})_{nnn} - 6k^2 l^2 (V_{n1})_{nn}$$

$$-4ik^3 l^3 (V_{n1})_n + k^4 l^4 V_{n1}] \cdot e^{iw\theta}$$

D2

$$(V_{n1})_t = -\varepsilon v_g (V_{n1})_x + \varepsilon^2 (V_{n1})_\tau$$

$$(V_{n1})_{tt} = \varepsilon^2 v_g^2 (V_{n1})_{xx} + \varepsilon^4 (V_{n1})_{\tau\tau} - 2\varepsilon^2 v_g (V_{n1})_{x\tau}$$

$$(V_{n1})_{nn} = \varepsilon^2 (V_{n1})_{xx} \qquad (V_{n1})_{nnnn} = \varepsilon^4 (V_{n1})_{xxxx}$$

D3

$$[(1/2)V^2 - (1/3)V^3]_{tt} = V_t^2 + V \cdot V_{tt} - 2V \cdot V_t^2 - V^2 V_{tt}$$

E1

$$\varepsilon^n e^{il\theta}[(V_{n1})_{tt} - w^2 l^2 V_{n1} - 2iwl(V_{n1})_t - u^2 (V_{n1})_{nn} + u^2 k^2 l^2 V_{n1}$$

$$-2u^2 ikl(V_{n1})_n - (u^2/12)(V_{n1})_{nnnn} - (u^2/3)ikl(V_{n1})_{nnn} +$$

$$(u^2/2)k^2 l^2 (V_{n1})_{nn} + (u^2/3)ik^3 l^3 (V_{n1})_n - (u^2/12)k^4 l^4 V_{n1}] =$$

$$= \varepsilon^{n+n'} e^{i(l+l')}[(V_{nl})_t (V_{n'l'})_t - iwl'(V_{nl})_t V_{n'l'} -$$

$$-iwl(V_{n'l'})V_{nl} - w^2 ll' V_{nl}V_{n'l'} + (V_{nl})_{tt}V_{n'l'} - w^2 l^2 V_{nl}V_{n'l'} -$$

$$-2iwl(V_{nl})_t V_{n'l'} - 2\varepsilon^{n''} e^{il''\theta}[(V_{nl})_t (V_{n'l'})_t V_{n''l''} - iwl'(V_{nl})_t$$

$$\cdot V_{n'l'}V_{n''l''} - iwl(V_{n'l'})_t V_{nl}V_{n''l''} - w^2 ll' V_{nl}V_{n'l'}V_{n''l''}] -$$

$$n'' e^{il''\theta}[(V_{nl})_{tt}V_{n'l'}V_{n''l''} - w^2l^2 V_{nl}V_{n'l'}V_{n''l''} - 2iwl(V_{nl})_t$$

$$\cdot V_{n'l'}V_{n''l''}]].$$

E2.

$$\varepsilon^n e^{il\theta}\cdot[\varepsilon^2 v_g^2 (V_{nl})_{xx} + \varepsilon^4 (V_{nl})_{\tau\tau} - 2\varepsilon^3 v_g (V_{nl})_{x\tau} - w^2 l^2 V_{nl}$$

$$+2\varepsilon iwlv_g(V_{nl})_x - 2\varepsilon^2 iwl(V_{nl})_\tau - \varepsilon^2 u^2 (V_{nl})_{xx} + u^2k^2l^2V_{nl} - 2\varepsilon iu^2kl$$

$$\cdot(V_{nl})_x - \varepsilon^4(u^2/12)(V_{nl})_{xxxx} - \varepsilon^3 i(u^2/3)kl(V_{nl})_{xxx} + \varepsilon^2(u^2/2)$$

$$\cdot k^2l^2(V_{nl})_{xx} + i(u^2/3)k^3l^3(V_{nl})_x - (u^2/12)k^4l^4V_{nl}] =$$

$$\varepsilon^{n+n'}e^{i(l+l')}[\varepsilon^2 v_g (V_{nl})_x (V_{n'l'})_x - \varepsilon^3 v_g (V_{nl})_x (V_{n'l'})_\tau - \varepsilon^3 v_g$$

$$\cdot(V_{nl})_\tau (V_{n'l'})_x + \varepsilon^4 (V_{nl})_\tau (V_{n'l'})_\tau + iwl'v_g(V_{nl})_x V_{n'l'} - \varepsilon^2 iwl'$$

$$(V_{nl})_\tau V_{n'l'} + iwlv_g (V_{n'l'})_x V_{nl} - \varepsilon^2 iwl(V_{n'l'})_\tau V_{nl} - w^2 ll' V_{nl}$$

$$\cdot V_{n'l'} - w^2l^2V_{nl}V_{n'l'} + \varepsilon^2 V_g^2 (V_{nl})_{xx}V_{n'l'} + \varepsilon^4 (V_{nl})_{\tau\tau}V_{n'l'} -$$

$$-2\varepsilon^3 V_g (V_{nl})_{x\tau}V_{n'l'} + 2\varepsilon iwlv_g(V_{nl})_x V_{n'l'} - 2\varepsilon^2 wl(V_{nl})_\tau -$$

$$-2\varepsilon^{n''}e^{il''\theta}[\varepsilon^2 v_g (V_{nl})_x (V_{n'l'})_x V_{n''l''} - \varepsilon^3 v_g (V_{nl})_x (V_{n'l'})_\tau\cdot$$

$$\cdot V_{n''l''} - \varepsilon^3 v_g (V_{nl})_\tau (V_{n'l'})_x V_{n''l''} + \varepsilon^4 (V_{nl})_\tau (V_{n'l'})_\tau V_{n''l''} +$$

$$+ iwl'v_g (V_{nl})_x V_{n'l'}V_{n''l''} - \varepsilon^2 iwl'(V_{nl})_\tau V_{n'l'}V_{n''l''} + iwlv_g\cdot$$

$$\cdot(V_{n'l'})_x V_{nl}V_{n''l''} - \varepsilon^2 iwl(V_{n'l'})_\tau V_{nl}V_{n''l''} - w^2ll'V_{nl}V_{n'l'}\cdot$$

$$\cdot V^{n''l''}] - \varepsilon^{n''}e^{il''\theta}[\varepsilon^2 V_g^2 (V_{nl})_{xx}V_{n'l'}V_{n''l''} + \varepsilon^4(V_{nl})_{\tau\tau}V_{n'l'}\cdot$$

$$\cdot V_{n''l''} - 2\varepsilon^3 V_g (V_{nl})_{x\tau}V_{n'l'}V_{n''l''} - w^2l^2V_{nl}V_{n'l'}V_{n''l''} + 2\varepsilon iwl\cdot$$

$$\cdot(V_{n1})_x V_{n'1'} \cdot V_{n''1''} - \varepsilon^2 2iwl(V_{n1})_\tau V_{n'1'} \cdot V_{n''1''}]]$$

E3 ε^3 & $e^{i\theta}$

$$V_g{}^2(V_{11})_{xx} + 2iw(V_{11})_t - u^2(V_{11})_{xx} + (u^2/2)k^2(V_{11})_{xx} + (n=2)0.0^*$$

$$+(n=3)0.0^* = iwv_g(V_{10})_x V_{11} - iwv_g(V_{12})_x V_{10} + 2iwv_g(V_{1(-1)})V_{12}$$

$$+iwv_g(V_{10})_x V_{11} + 2iwv_g(V_{1(-1)})_x V_{12} - iwv_g(V_{12})_x V_{1(-1)}) + 2wV_{22}$$

$$\cdot V_{1(-1)} + 2w^2 V_{12} V_{2(-1)}) + 2w^2 V_{2(-1)} V_{12} + 2w^2 V_{2(-1)}) V_{22} + 2w^2.$$

$$\cdot V_{1(-1)} V_{22} - w^2 V_{11} V_{20} - w^2 V_{21} V_{10} - 4w^2 V_{12} V_{2(-1)} - 4w^2 V_{22} V_{1(-1)} -$$

$$-w^2 V_{1(-1)} V_{22} - w^2 V_{2(-1)} V_{12} + 2iwv_g(V_{11})_x V_{10} + 4iwv_g(V_{12})_x V_{1(-1)}$$

$$+w^2 V_{11} V_{10} V_{10} + 2w^2 V_{11} V_{11} V_{1(-1)} - 2w^2 V_{11} V_{1(-1)} V_{11} + 4w^2 V_{12} V_{1(-1)}$$

$$\cdot V_{10} + 0.0[V_{10}]^{**} + w^2 V_{11} V_{11} V_{1(-1)} + 0.0[V_{10}].$$

* terms which are neglected because of the dispersion relations

** terms which are neglected because of $V_{no}=0$

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