THE 2/1 AND 3/2 RESONANT ASTEROID MOTION
A SYMPLECTIC MAPPING APPROACH

JOHN HADJIDEMETRIOU and GEORGE VOYATZIS
Department of Physics, University of Thessaloniki,
540 06 Thessaloniki, Greece

Abstract.
A comparative study is made between the 2/1 and the 3/2 resonant asteroid motion, with the aim to understand their different behaviour (gap in the 2/1 resonance, group in the 3/2 resonance). A symplectic mapping model is used, for each of these two resonances, assuming the asteroid is moving in the three dimensional space under the gravitational perturbation of Jupiter. It is found that these resonances differ in several points, and although there is, in general, more chaos in the phase space close to the 3/2 resonance, even in the model of circular orbit of Jupiter, there are regions, close to the secondary resonances, which are less chaotic in the 3/2 resonance compared to the 2/1 resonance, and consequently trapping can take place.

Keywords: asteroids, resonance, secondary resonance

1. Introduction

The 2/1 and the 3/2 resonances with Jupiter in the asteroid belt have been studied extensively and there are many papers devoted to the dynamical evolution of an asteroid starting inside these resonances (Ferraz-Mello 1997, Hadjidemetriou and Lemaitre 1997, Hadjidemetriou 1999, Lemaitre and Henrard 1990, Morbidelli 1996). These resonances are both of the same order, but the dynamical behaviour of an asteroid is different. We have a gap at the 2/1 resonance and a group in the 3/2 resonance. This has been explained on the basis that the diffusion rate of an asteroid at the 3/2 resonance is much slower than the diffusion rate at the 2/1 resonance, and for this reason the region near the 3/2 resonance is not yet depleted (Michchenko and Ferraz-Mello 1996, Ferraz-Mello et al. 1998, Morbidelli and Moons 1993, Newsony and Ferraz-Mello 1997, Roig and Ferraz-Mello 1999). The aim of this work is to make a comparative study of the 2/1 and the 3/2 resonant asteroid motion, considering a hierarchy of models in the restricted three body problem: circular planar model, elliptic planar model, three dimensional elliptic model, by making use of a symplectic mapping model which has the basic features of the topology of the phase space of the Poincaré map of the real system. Since the mapping is simpler than the real system, it is suitable to explore the basic dynamics and discover

Celestial Mechanics and Dynamical Astronomy 00: PLEASE RUN AGAIN,
2002
the differences between these two cases of resonant motion, without being obscured from minor details which are not essential in the evolution of the system. In this way we can illustrate the dynamics at these two resonances, their similarities and their differences and explain the results obtained by other investigators by semianalytic, numerical, or frequency map analysis methods. We remark that the present study is meant to be a complement to similar studies on this problem, the symplectic map being used as a tool to illustrate the mechanisms involved in generating chaotic motion in the region of these two resonances. Moreover, since the map is simple and very fast, it allows us to "play" with different models, starting with the simplest and going to the more realistic, in the hierarchy we consider, and in this way we can understand better the factors that are responsible for the long term evolution of the system. In the present study we assume that an asteroid moves under the gravitational perturbation of Jupiter. Although the perturbations from other planets play also a role in the long term evolution, we believe that the consideration of the perturbation of Jupiter only will give us important information on the basic dynamics of the system.

The mapping model is based on the averaged Hamiltonian at each resonance and it is proved (Hadjidemetriou 1993) that it has the same fixed points, with the same stability, as the averaged Hamiltonian. From the theory of averaging it is known that the fixed points of the averaged Hamiltonian correspond to the periodic orbits of the original, non averaged, system. These periodic orbits, in our case, represent the 2/1, or the 3/2 resonances. We remark also that a periodic orbit of the non averaged system appears as a fixed point of the Poincaré map. If the fixed points of the averaged Hamiltonian correspond (both with respect to their position and stability) to all the periodic orbits of the non averaged system, or, equivalently, to the fixed points of the Poincaré map, then the topology of the phase space of the symplectic map is similar to that of the Poincaré map of the non averaged system. If, on the other hand, a fixed point is missing from the averaged Hamiltonian, then the corresponding resonance is also missing. Consequently, a necessary condition for an averaged Hamiltonian and the corresponding symplectic map that is based on it, to be realistic, is to have the same fixed points, with the correct stability properties, as the original model (the elliptic restricted three body problem in this case).

It is a simple matter to compute the fixed points of the averaged Hamiltonian. We found, for both resonances, that the family of fixed points of the simplest model in the hierarchy we are considering, the circular planar case, is quite close to the correct position, but the stability may be very different (there is an interchange between stability and instability as we add terms of increasing order in the eccentricity, (as is shown in Lemaître and Henrard, 1990, for the 2/1 case). If we go, next, to the more realistic model, the elliptic planar case, the situation changes completely. No high
eccentricity resonances are present, for both resonances, because there is no
any bifurcation point on the corresponding family of the fixed points of the
circular case, from which families of periodic (resonant) orbits of the elliptic
problem start, as the eccentricity of Jupiter increases, starting from zero
value. In the real model the bifurcation point is at an eccentricity \( e = 0.73 \)
for the 2/1 resonance and at \( e = 0.46 \) for the 3/2 case (see figure 1 of section
2). This means that the high eccentricity resonances are missing from the
averaged Hamiltonian, both for the 2/1 and the 3/2 resonances. This may be
due to the fact that the perturbation series that give the successive terms
in the averaged Hamiltonian do not converge (because the system is not
integrable). To solve this problem, we introduced in our model the missing
resonances by changing the numerical value of some coefficients and adding
correction terms, by making use of a semianalytic method (Hadjidemetriou
1993, 1999). In this way, the corrected mapping model becomes equivalent
to the Poincaré map of the original system, in the sense that the topology of
the phase space of the mapping model has the same structure as that of the
Poincaré map of the real system. We remark at this point that the correction
to the averaged model does not have the meaning that we improve the
convergence of the perturbation series. We just use the corrected averaged
Hamiltonian as a tool, in order to introduce to our mapping model the
correct resonances, so that it becomes equivalent to the Poincaré map of the
non averaged system.

We start our study with the simplest model. This is the model where
Jupiter moves in a circular orbit and the asteroid moves in the orbital plane
of Jupiter (circular restricted three body problem). This is not a realistic
model, but the topology of its phase space is the basic framework which will
help us to understand the effect of all other perturbations added on this
model, that is, the effect of the eccentricity of Jupiter’s orbit, and motion
in the three dimensional space.

The phase space of the mapping model representing circular motion of
Jupiter, for planar motion, is two dimensional, but becomes four dimensional
when the eccentricity of Jupiter’s orbit is introduced. Since it is not possible
to have a geometrical view of this four dimensional phase space, we must
take projections in two dimensional surfaces. This however will not provide
much insight on the topology of the phase space if these surfaces are not
properly selected. In order to understand the dynamics of the system, we
found it much better to consider the secondary resonances, which in fact cor-
respond to periodic motion of the averaged Hamiltonian, or, equivalently, to
quasiperiodic motion in the four dimensional phase space. These quasiperi-
odic orbits appear as fixed points or closed islands in a two dimensional
"mapping of the mapping", as we shall explain in the following. This two
dimensional subspace of the four dimensional phase space contains much
more information on the dynamical behavior of the system, compared to a
Figure 1. The families of resonant periodic orbits at the 2/1 and 3/2 resonances in the space of initial conditions $x_0$, $h$. The bifurcations point to resonant orbits of the elliptic model and the three dimensional model are indicated.

projection on a coordinate plane. Using these secondary resonances in the 2/1 and the 3/2 resonance we found that the space around them is more chaotic in the 2/1 resonance compared to the 3/2 resonance. This means that trapping can take place in the latter case, as we shall explain in the following.

2. The restricted three body problem

As we mentioned before, the basic model that we consider is the restricted three body problem (e.g. Roy, 1982), with the Sun and Jupiter as primaries and the asteroid as the small body.

The resonances in the restricted circular problem (simplest model) appear as periodic orbits, in the synodic system $xOy$, whose $x$-axis is the line Sun-Jupiter. In particular, there are two families of periodic orbits for the 2/1 and 3/2 resonances, symmetric with respect to the $x$-axis, one of them corresponding to the asteroid at perihelion when $x = 0$, and the other at aphelion. Each member of the family is defined by its initial conditions $x_0, \dot{y}_0$ (and $y_0 = 0, \dot{x}_0 = 0$), or, equivalently, by $x_0, h$, where $h$ is the Jacobi energy constant. These families are shown in figure 1. Along each family the eccentricity of the asteroid increases. One family in each resonance is stable and the other is unstable or stable, as shown in figure 1. On the stable family, in both resonances, there are bifurcation points to families of periodic orbits when the eccentricity of Jupiter is non-zero, and also to three dimensional periodic orbits. The bifurcation for elliptic periodic orbits is at the point where the period is exactly equal to $T = 2\pi/(q/p - 1)$, where $q/p$ is equal to 2/1 or 3/2, i.e. $T = 2\pi$ for the 2/1 resonance and $T = 4\pi$ for the 3/2 resonance. Also, the bifurcation to three dimensional orbits is at the point where the stability with respect to perturbations perpendicular to the $xOy$
Figure 2. The Poincaré surface of section at the 2/1 resonance (upper row) and the 3/2 resonance (lower row), for different values of the eccentricity of the resonant fixed point. The axes are $e \cos(\sigma)$, $e \sin(\sigma)$.

plane is critical. The region of these bifurcation points is at much higher eccentricities for the 2/1 resonance ($0.67 < e < 0.80$) compared to the 3/2 resonance ($0.39 < e < 0.46$), as shown in figure 1. It is these bifurcation points that are missing from the averaged Hamiltonian, as mentioned in the previous section, and consequently the corresponding high eccentricity resonances are missing from the averaged model, before the correction is introduced.

In figure 2 we present the Poincaré mapping of the circular planar problem, on the surface of section $h =$ constant, $y = 0$, where $h$ is the Jacobi energy constant, at the region of the phase space close to the 2/1 and to the 3/2 resonances. For reasons of comparison with the mapping of figure 3, the diagrams are expressed in the axes $e \cos(\sigma)$, $e \sin(\sigma)$, where $e$ is the eccentricity and $\sigma$ the critical angle, defined in the next section. We present four mappings for each resonance, corresponding to different Jacobi energy levels, or equivalently, to different values of the eccentricity of the exact resonance (fixed point, i.e. periodic orbit). Note that the phase space near the 3/2 resonance appears more chaotic than that of the 2/1 resonance, for the same value of the eccentricity.
3. The averaged Hamiltonian

The averaged Hamiltonian at the 2/1 and the 3/2 resonances is expressed in action-angle variables which are given by

\[
S = L - G, \quad S_z = G - H, \quad N = \frac{p+q}{q}L - H,
\]

\[
\sigma = (p+q)\lambda - q\lambda - \varpi, \quad \sigma_z = (p+q)\lambda' - q\lambda - \Omega, \tag{1}
\]

\[
\nu = -(p+q)\lambda' + q\lambda + \varpi'.
\]

where \(L = \sqrt{(1-\mu)a^3}, \quad G = L\sqrt{\mu(1-e^2)}, \quad H = G \cos i, \quad a: \) semimajor axis of the asteroid, \(e: \) eccentricity, \(i: \) inclination, \(\lambda: \) mean longitude, \(\varpi: \) longitude of perihelion, \(\Omega: \) longitude of the node and the primed angles refer to Jupiter. The 2/1 resonance corresponds to \(p = 1, \quad q = 1\) and the 3/2 resonance to \(p = 1, \quad q = 2.\)

The averaged Hamiltonian in action angle variables is given by Šidlichovský (1991) and we kept the following terms in the expansion, for both resonances \((H_c\) is an additional correction term):

\[
H = H_0(S, S_z, N) + \mu eH_1(\sigma) + \mu e^2H_2(\sigma) + \mu(H_3(\sigma) + H_c)
\]

\[
+ \mu e_j H_j(e(S, S_z, N), \sigma, \sigma_z, \nu)
\]

\[
+ \mu \sin^2 \frac{i}{2} H_{ij}(e(S, S_z, N), \sigma, \sigma_z) \tag{2}
\]

\[
+ \mu e_j \sin^2 \frac{i}{2} H_{ij}(e(S, S_z, N), \sigma, \sigma_z, \nu)
\]

\[
+ \mu \sin \frac{i}{2} \sin \frac{j}{2} H_{ij}(e(S, S_z, N), \sigma, \sigma_z, \nu, \phi)
\]

\[
+ \mu e_j \sin \frac{i}{2} \sin \frac{j}{2} H_{ij}(e(S, S_z, N), \sigma, \sigma_z, \nu, \phi).
\]

The above terms have the form

\[
H_0 = -\frac{(1-\mu)^2}{2L^2} - \frac{p+q}{q}L, \quad H_1 = A \cos \sigma, \quad H_2 = C + D' \cos 2\sigma,
\]

\[
H_3 = f e^7(P \cos \sigma + Q \cos 3\sigma + R \cos 5\sigma + T \cos 7\sigma), \quad H_c = f e^7T' \cos 8\sigma,
\]

for the 2/1 resonance, or

\[
H_3 = e^3(P \cos \sigma + Q \cos 3\sigma),
\]

\[
H_c = e^3T' \cos 4\sigma + (\alpha_1 N^3 + \alpha_2 N^2 + \alpha_3 N) + S_z(\beta_1 N^2 + \beta_2 N + \beta_3),
\]
for the 3/2 resonance,

\[
H_j = J \cos \nu + e \left( F \cos(\sigma + \nu) + G \cos(\sigma - \nu) \right) \\
+ e^2 \left( F_1 \cos(2\sigma + \nu) + G_1 \cos(2\sigma - \nu) + J_1 \cos \nu \right) \\
+ e^3 \left[ F_{11} \cos(\sigma + \nu) + G_{11} \cos(\sigma - \nu) \right] \\
+ F_{12} \cos(3\sigma + \nu) + G_{12} \cos(3\sigma - \nu)],
\]

\[
H_{ii} = C_1 + D_1 \cos 2\sigma_z \\
+ e(A_1 \cos \sigma + A_2 \cos(\sigma - 2\sigma_z) + A_3 \cos(\sigma + 2\sigma_z)),
\]

\[
H_{iij} = J_2 \cos \nu + F_2 \cos(2\sigma_z + \nu) + G_2 \cos(2\sigma_z - \nu) \\
+ e(F_5 \cos(\sigma + \nu) + G_3 \cos(\sigma - \nu) \\
+ B_{21} \cos(2\sigma_z + \sigma + \nu) + B_2 \cos(2\sigma_z + \sigma - \nu) \\
+ B_3 \cos(2\sigma_z - \sigma + \nu) + B_4 \cos(2\sigma_z - \sigma - \nu)),
\]

\[
H_{i'j} = F_4 \cos(\sigma_z + \nu - \phi) + G_4 \cos(\sigma_z - \nu + \phi) \\
+ e(B_{11} \cos(\sigma + \sigma_z + \nu - \phi) + B_{21} \cos(\sigma + \sigma_z - \nu + \phi) \\
+ B_{31} \cos(\sigma - \sigma_z + \nu - \phi) + B_{41} \cos(\sigma - \sigma_z - \nu + \phi)),
\]

\[
H_{ii'j} = C_2 \cos(\sigma_z + \phi) + D_2 \cos(\sigma_z - \phi) \\
+ F_5 \cos(\sigma_z + 2\nu - \phi) + G_5 \cos(\sigma_z - 2\nu + \phi) \\
+ e(F_{51} \cos(\sigma + \sigma_z - \phi) + G_{51} \cos(\sigma - \sigma_z + \phi) \\
+ B_{13} \cos(\sigma + \sigma_z + 2\nu - \phi) + B_{23} \cos(\sigma + \sigma_z - 2\nu + \phi) \\
+ B_{33} \cos(\sigma - \sigma_z + 2\nu - \phi) + B_{43} \cos(\sigma - \sigma_z - 2\nu + \phi)).
\]

where \( \phi = \omega' - \Omega' \). This angle is constant for a fixed orbit of Jupiter.

The numerical values of the coefficients are given in Šidlichovský (1991). However, as we mentioned in the introduction, the above expansion of the averaged Hamiltonian, without the corrections, does not contain the correct resonances and we solved this problem by changing the numerical value of a few coefficients, given below, and introduced the correction term \( H_c \).

For the 2/1 resonance the coefficient in the correction term \( H_c \) is \( T' = 21.0 \) and the corrected coefficients are \( D' = -0.238, G_1 = -3.5, C_1 = -0.1517, D_1 = 3.40176, A_1 = -11.0194 \). The value of \( f \) was taken equal to \( f = 0.02 \), in order to have in the mapping model the same chaotic regions as in the circular planar model.

For the 3/2 resonance the coefficients in the correction term \( H_c \) are \( T' = 9.1, \alpha_1 = 0.196833, \alpha_2 = -0.1466, \alpha_3 = -0.0208, \beta_1 = -0.3031, \beta_2 = 0.3774, \beta_3 = -0.100296 \) and the corrected coefficients are \( D' = 2.18, G_1 = 11.3497 \).

Note that the same coefficients \( D', G_1 \) are corrected in both resonances. It is these coefficients that affect the stability of the resonant orbits. The correction of the other coefficients and the addition of the correction terms
was necessary in order to have the correct bifurcation points to the elliptic problem and the three dimensional problem.

For the fixed orbit of Jupiter we used the values $\mu = 0.00095387535$, $e_j = 0.048$ and also $\omega' = \Omega' = 0$, $\sin \frac{\phi}{2} = 0.003151$.

4. The mapping equations

It can be proved (Hadjidemetriou, 1993) that the mapping which is generated from the generating function $W$, given by

\begin{equation}
W = \sigma_n S_{n+1} + \sigma_{z,n} S_{z,n+1} + \nu_n N_{n+1} \\
+ T \left[ H_0(S, N) + \mu W_1(e, \sigma) + \mu e J H_j(e, \sigma, \nu) \right] \\
+ \mu T \left[ \sin^2 \frac{i}{2} H_{\bar{w}}(e, \sigma, \sigma_z) + e_j \sin^2 \frac{i}{2} H_{ij}(e, \sigma, \sigma_z, \nu) \right] \\
+ \mu T \left[ \sin \frac{i}{2} \sin \frac{j'}{2} H_{\bar{w}w}(e, \sigma, \sigma_z, \nu, \phi) + e_j \sin \frac{i}{2} \sin \frac{j'}{2} H_{ijw}(e, \sigma, \sigma_z, \nu, \phi) \right]
\end{equation}

is a good model for the Poincaré map of the original system. In equations (3)

\begin{equation}
W_1 = \mu e H_1(\sigma) + \mu e^2 H_2(\sigma) + (\mu H_3(\sigma) + \mu H_e)
\end{equation}

and $T$ is the period of the resonant periodic orbit of the elliptic restricted three body problem, provided that the averaged Hamiltonian on which it is based has all the resonances of the real system. In the normalized units
we are using it is $T = 2\pi$ for the 2/1 resonance and $T = 4\pi$ for the 3/2 resonance.

The mapping is obtained from the relations

$$
\sigma_{n+1} = \partial W/\partial S_{n+1}, \quad \sigma_{z,n+1} = \partial W/\partial S_{z,n+1}, \quad \nu_{n+1} = \partial W/\partial N_{n+1},
$$

$$
S_n = \partial W/\partial \sigma_n, \quad S_{z,n} = \partial W/\partial \sigma_{z,n}, \quad N_n = \partial W/\partial \nu_n,
$$

which are in implicit form. Note that the generating function $W$ is a function of the old angles and the new actions. Evidently, this mapping is symplectic by construction, according to Hamiltonian theory. For $i = 0$ we have planar motion, and the mapping is four dimensional and corresponds to the planar elliptic restricted three body problem, and for $e_j = 0$ and $i = 0$ the mapping is two dimensional and corresponds to the planar circular restricted 3-body problem.

For reasons of comparison, and to check the validity of the model, we present in figure 3 the mapping for the case of the restricted circular three body problem, for both resonances 2/1 and 3/2, for the same values of the resonant eccentricity as in figure 2. We note that the similarity is good, because the regions of ordered and of chaotic motion in the Poincaré map and in the symplectic mapping model compare very well.

Note that the non averaged circular planar three body problem is an autonomous system with two degrees of freedom, and consequently chaotic motion may appear in some regions. On going to the averaged circular planar problem, we lose one degree of freedom, so the system becomes integrable and consequently it is not realistic for the chaotic regions of the real system. However, going from the averaged Hamiltonian to the mapping model, we gain again the lost degree of freedom. Indeed, the two dimensional mapping model is the equivalent to the two dimensional Poincaré map of the non averaged system, so chaos may appear in the mapping model, if it appears in the Poincaré map. This is indeed what happens in our case.

5. The evolution of the system at the 2/1 and the 3/2 resonances near the secondary resonances

In order to understand better the dynamics of the system, we start with the planar elliptic problem and then we add the third dimension and study the changes. However, even the mapping of the planar model is four dimensional, and in order to understand the topology of the phase space, we must take projections on a two dimensional space. The projection on a coordinate plane is not the best choice one can make, and we decided, instead, to use the secondary resonances (Henrard et al. 1995, Wisdom 1985) to obtain a better picture of the system.
We start with the averaged Hamiltonian (2) for \( i = 0 \) (planar elliptic problem) in the action-angle variables \( \sigma, \nu, S, N \). The unperturbed motion can be considered to take place on a 2-torus, with radii \( S \) and \( N \) and corresponding angles \( \sigma \) and \( \nu \). Let us go now to a new set of canonical action-angle variables \( \bar{\sigma}, \bar{\nu}, \bar{S}, \bar{N} \) by making use of the generating function \( F = \bar{S}(\sigma + \nu) + \bar{N}\nu \). The new variables are \( \bar{\sigma} = \sigma + \nu, \bar{S} = S, \bar{\nu} = \nu, \bar{N} = N - S \). Note that \( \bar{\sigma} = -\omega \) is the longitude of perihelion and in many cases this is a slow angle compared to the angle \( \bar{\nu} \). So, it was proposed (Henrard and Lemaître 1987, Wisdom 1985) to perform a second averaging to the already averaged Hamiltonian and end up with an integrable system in the variables \( \bar{\sigma}, \bar{S} \) only (i.e. in the variables \( S, \sigma + \nu \)). The periodic orbits in his reduced one degree of freedom Hamiltonian system are the secondary resonances (for planar motion). The secondary resonances correspond to quasiperiodic motion of the initial system. The transformation however of a nonintegrable dynamical system to an integrable one is useful only in those regions of phase space where the system behaves, for all practical reasons, as integrable (i.e. to have smooth invariant curves).

Taking advantage of the above idea for the secondary resonances, we consider the mapping to be expressed in the above new variables and take a mapping of the mapping, at the surface of section \( \bar{\sigma} - \bar{\nu} = 0 \) (i.e. \( \sigma = 0 \)) and \( \bar{\sigma} > 0 \) or \( < 0 \). (The angle \( \bar{\sigma} \) is also a "fast" angle compared to \( \omega \)). The results are expressed in the two dimensional space \( e \cos(\sigma + \nu), e \sin(\sigma + \nu) \). This is, in a sense, similar to the above mentioned second averaging, but at the same time we do not lose degrees of freedom and consequently we keep a basic property of the system, i.e. its chaoticity (if it exists at this region of the phase space).

The above technique of the "mapping of the mapping" is, in fact, another way to take a projection of the four dimensional phase space in a two dimensional space, which gives more insight on the dynamics of the system, instead of a projection on a coordinate plane. This is so, because the region of the secondary resonances plays a special role in the structure of the phase space of the whole system, because it is shaped from the stability properties of a quasiperiodic motion. (We remark that although much work has been done for the study of the vicinity of a periodic motion, almost no theory exists for the study of the stability of quasiperiodic motion). In figure 4 we present two examples of secondary resonances in the 2/1 resonance (planar model). The computation of the perturbed motion in the initial conditions of the periodic motion in panel (a) (figure 4c) is for a small time interval, because otherwise the plot would be black. In fact, there is much chaos in this region, as it becomes clear from figure 5. In figure 5a we present the evolution of the system, starting with initial conditions in the vicinity of the 3/1 secondary resonance and also the evolution starting in the vicinity of the 6/1 secondary resonance between the angles \( \sigma \) and \( \omega \). Several secondary
resonances overlap and in this way a path exists for the diffusion of the eccentricity of the asteroid from small to larger values. However, at this level of the model (planar elliptic), two chaotic regions appear which are not connected. In addition, there is a region of rather ordered motion for higher values of the eccentricity.

Let us introduce now one more feature to our model in order to make it more realistic, namely the third dimension. In figure 5b we start with the same initial conditions as with the inner chaotic zone in figure 5a, but now the initial inclination is $i = 1^\circ$. All the chaotic regions overlap and the chaos extends also to regions of ordered motion. In this way the eccentricity can travel to quite large values. This is shown in figure 5c, where the evolution of the eccentricity and the inclination corresponding to 5b is presented.

We come now to the 3/2 resonance and we explore the region of phase space at the secondary resonances. In figures 6a and b, upper row, we present the mapping of the mapping, for planar motion, corresponding to the 3/1 and 4/1 secondary resonances and in figure 6c we present the evolution when the above secondary resonances are perturbed with respect to the initial conditions (in the plane). Compared to the 2/1 resonance (figure 5a), we note that the region of the phase space of the secondary resonances appears more stable and in fact there is no overlap of secondary resonances. Consequently, the excursion of the eccentricity of an asteroid in this region is very small. If the system is perturbed by introducing motion in three dimensions, we found that this behavior does not change (a typical example

\( Figure 4. \) The 3/1 (first row) and the 6/1 (second row) secondary resonances (planar model) in the 2/1 resonance in the axes $e \cos(\sigma + \nu), e \sin(\sigma + \nu)$: (a) The periodic orbit, (b) the islands in the close vicinity of the secondary resonance and (c) the motion on a torus in the vicinity of the secondary resonance (perturbed initial conditions with respect to panel a).
Figure 5. Mapping of the mapping at the 2/1 resonance in in the axes
$e \cos(\sigma + \nu)$, $e \sin(\sigma + \nu)$. Overlap of the secondary resonances: (a) planar model, (b)
three dimensional model. (c) The evolution of $e$ and $i$ corresponding to (b).

is shown in the lower row of figure 6). This means that an asteroid can be
trapped in this region of phase space, which corresponds to values of the
eccentricity in the region $e = 0.15$ to $e = 0.25$, which is not very far from
the observed Hilda family.

6. Discussion

The aim of this work is to make a comparative study of the 2/1 and the 3/2
resonances in the asteroid belt, by making use of a mapping model. The sim-
plest version is the planar circular model, and from first sight one is tempted
to say, looking at the Poincaré maps of figure 2, that the 3/2 resonance is
more chaotic than the 2/1 resonance. Indeed, for the same eccentricity of
the resonant orbit, we have ordered motion up to eccentricities as high as
$e = 0.30$ for the 2/1 resonance, while large chaotic regions appear at the 3/2
resonance, even for small eccentricities. The observations however of real
asteroids lead to the opposite conclusion and we tried to explore further
the dynamics at these two resonances. We studied, next, the elliptic model,
whose phase space is four dimensional, by taking suitable projections on two
dimensional spaces. It turned out that the secondary resonances provided
such a tool, and although, in general, the 3/2 resonance is more chaotic, it is
more ordered than the 2/1 resonance in the region of the secondary
resonances, as we explained above. This means that trapping can take place
at the 3/2 resonance, at the region of phase space corresponding to the
secondary resonances.

The above results have been obtained by a mapping model which is based
on perturbations from Jupiter only and no perturbations from other planets.
However, even this simple model proved to have the basic dynamics of the
real system and its study helped us to find those special regions in phase
space where there exists ordered motion in the 3/2 resonance, as compared to the 2/1 resonance. In particular, the role of the secondary resonances was made clear. The existence of this more stable region at the 3/2 resonance has been verified by more elaborate models by other investigators, mentioned in the introduction and the qualitative results that we obtained by the mapping model have been refined by them, so that they correspond closely to the observations.

Acknowledgements

We thank the referee for critically reading the manuscript and many useful suggestions.

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