BIFURCATIONS OF PERIODIC ORBITS IN
THE GENERAL 3-BODY PLANETARY PROBLEM

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Abstract. In this paper we present cases of bifurcation of families of periodic orbits within
the framework of the general three body problem. We restrict ourselves to the 2/1 and 3/1
resonant cases of two-planet systems and we demonstrate bifurcations which cause critical
changes in the structure of families of periodic orbits, and, furthermore, in the topology of
phase space, as the planetary mass ratio $\rho$ varies. We consider the whole range $0 < \rho < \infty$
and, therefore, we include the passage from external to internal resonances.

1. INTRODUCTION

It is well known that resonant families of periodic orbits exist for the general three
body problem. These families can be obtained either by continuation from a family of
circular orbits (Voyatzis and Hadjidemetriou, 2005,2006) or by computing stationary
solutions of the averaged resonant Hamiltonian (Beauge et al, 2003; Michtchenko et
al, 2006). In a resent work (Voyatzis et al, 2008) we show that all these families can
be derived by continuation of the families of the circular and the elliptic restricted
problem.

The basic model, which is used here, is the general planar three body problem
(TBP) consisting of a star $S$ of mass $m_0$ and two planets $P_1$ (inner) and $P_2$ (outer).
The system is given in a rotating frame which reduces the system to three degrees of
freedom (Hadjidemetriou, 1975, 2006).

Since $m_1 \ll m_0$ and $m_2 \ll m_0$, the position of periodic orbits depends (in first
order) to the planetary mass ratio $\rho = m_2/m_1$. For $\rho = 0$ we get the restricted
problem and the case of external resonances (i.e. the massless body moves outside
the orbit of the massive planet). For $\rho = \infty$ (but keeping $m_2$ finite) we get the
case of internal resonances of the restricted problem (i.e. the massless body moves
inside the orbit of the massive planet). By varying $\rho$, the existing families of periodic
orbits are continued and obey bifurcations at critical values of $\rho$. Such bifurcations
are demonstrated in the present paper for the 2/1 and 3/1 resonances. For each case,
details are given in Voyatzis et al. (2008) and Voyatzis (2008), respectively.
2. BIFURCATIONS IN 2/1 RESONANCE

Considering the 2/1 external resonance of the restricted problem \((\rho = 0)\) we mention the existence of the following families: (i) in the circular problem we have a symmetric family \(I\) from which an asymmetric family \(A_1\) bifurcates. (ii) in the elliptic problem we have a symmetric family \(S_1\) and an asymmetric family \(A_2\), which bifurcate from family \(I\) (iii) an asymmetric family \(A_3\) which bifurcates from \(S_1\) within the framework of the elliptic problem. In the general problem, for \(\rho \approx 0.275\), \(A_3\) and \(A_2\) go through a bifurcation and generate a new family \(A_{32}\).

The evolution of the family \(A_1\), as \(\rho\) increases, is regular without structural changes up to \(\rho \approx 0.37\). In Fig. 1a, which corresponds to \(\rho = 0.3\), it is shown that the family \(A_1\) has come close to the family \(A_{32}\). At \(\rho \approx 0.37\) the two families collide and two new families are generated, namely the family \(A_4\) and the family \(A_{123}\) (Fig. 1c). We call such a bifurcation as a collision-bifurcation. In this case only the family \(A_{32}\) has an orbit of critical stability, which separates the family in a stable (a) and in an unstable part (b). The family \(A_1\) is whole stable and there is no clear border between its parts c and d. After the bifurcation, an orbit of critical stability is shown only in the new family \(A_4\), while the family \(A_{123}\) is whole stable and starts and ends at bifurcation points of the symmetric family \(S_1\).

Now, we restrict ourselves to the evolution of the family \(A_{123}\) for \(\rho > \rho_2\). As it is shown in Fig. 2a, as \(\rho\) increases, the ending points of the family move on along the family \(S_1\) in opposite direction and the family shrinks and, finally disappears at \(\rho \approx 1.034\). In Fig. 2b we present the above transition by considering the stability index \(b_2\) (Hadjidemetriou, 2006) for the orbits along the family \(S_1\). In this case only the index \(b_2\) indicates the stability, since it is \(-2 < b_1 < 2\). The horizontal axis of the associated plot indicates the eccentricity of the periodic orbits along the family \(S_1\). As \(\rho\) increases, the curve of \(b_2\) values is raised continually and for \(\rho > 1.034\) is located above the value \(b_2 = -2\). Thus, the unstable part of \(S_1\) disappears and, consequently, the family \(A_{123}\) disappears too. We call such a bifurcation break-bifurcation. After this bifurcation, as \(\rho \to \infty\), the family \(S_1\) does not show any structural changes and can be assumed to coincide the family \(S_1'\) for the internal 2/1 resonance.
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Figure 2: a) The evolution of the families $A_{123}$ and $S_1$ as $\rho$ passes the critical value $\rho \approx 1.034$ and takes large values. b) The stability index $b_2$ along the family $S_1$, which determines the interval of instability ($b_2 < -2$) and the bifurcation points (at $b_2 = -2$) of the family $A_{123}$.

3. BIFURCATIONS IN 3/1 RESONANCE

In Voyatzis and Hadjidemetriou (2006), it is shown that the 3:1 resonance shows four families $S_i$, $i = 1, \ldots, 4$ of symmetric periodic orbits which bifurcate from the circular family. Voyatzis (2008) computed these families for a wide range of the mass ratio $\rho$ and verified the results obtained by Michtchenko et al (2006). In the following we consider the evolution of families of asymmetric periodic orbits as $\rho$ increases.

In Fig. 3(left) we present the 3/1 resonant asymmetric family $A_4$ and $A_{43}$, which bifurcate from the symmetric family $S_4$, for various values of $\rho$. The bifurcation point $B_4$, as $\rho$ varies, forms the characteristic curve $B_4$. We can obtain that at the critical value $\rho \approx 0.52$ the characteristic curves show a structural change at the point $C$. For $\rho < 0.52$ we have the family $A_4$ which extends up to high values of eccentricities. For $\rho > 0.52$ we have the family $A_{43}$, which terminates at the bifurcation point $B_3$ that belongs to the symmetric family $S_3$.

In Fig. 3(left) there is a region which is not occupied by the families $A_4$ and $A_{43}$. This region is indicated by the text “families $A_3$ and $A_0$” and contains new families of asymmetric periodic orbits, which, as $\rho$ increases, evolve as it is shown in the panels of Fig. 3(right). We showed above that the bifurcation points $B_3$ are ends of the families $A_{43}$ for $\rho > 0.51$. For lower values of $\rho$, the points $B_3$ are bifurcation points of new families, called families $A_3$, which exist as $\rho \to 0$. Starting from small values of $\rho$ (e.g. panel (a)), apart from $A_3$ we obtain the asymmetric family $A_{00}$ whose characteristic curve forms a loop. For $\rho \approx 0.175$ the families $A_{00}$ and $A_3$ involve in a collision-bifurcation at point $B_c$ (panel (b)) but, in this case, the two families join after the bifurcation and form one family, called again $A_3$ (since it still bifurcates from $B_3$). For $\rho > 0.52$, family $A_3$ shows a structural change and now the family is called $A_0$ and its bifurcation point is computationally undetermined.

Combining the family structures shown in Figs. 3(left) and 3(right), we can obtain that for $\rho < 0.52$ we have the families $A_3$ and $A_4$. At $\rho \approx 0.52$ the two families go
4. CONCLUSIONS

In this paper we demonstrated some important bifurcations which takes place in resonances of the three body problem of planetary type. The so called collision-bifurcations, take place when, by varying the ratio of the planetary masses, two families collide in the space of initial conditions. After such a collision, new families are generated and the characteristic curves of periodic orbits show substantial changes, which definitely are followed by topological changes of phase space.

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