CONTINUOUS DYNAMICAL SYSTEMS

$$\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n, t)$$
$$\dot{x}_2 = f_2(x_1, x_2, \ldots, x_n, t)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, x_2, \ldots, x_n, t)$$

$$[\quad \dot{x} = f(x,t) \quad]$$

ODEs

$$x_1 = x_1(t; x_{10}, \ldots, x_{n0})$$
$$x_2 = x_2(t; x_{10}, \ldots, x_{n0})$$
$$\vdots$$
$$x_n = x_n(t; x_{10}, \ldots, x_{n0})$$

$$[\quad x = x(t, x_0) \quad]$$

Solution

$$\vec{f} = (f_1, f_2, \ldots, f_n)$$

Vector field of the system

AUTONOMOUS SYSTEMS

The vector field of an autonomous system is constant in time

$$\frac{\partial f_i}{\partial t} = 0 \qquad \forall i = 1, 2, \ldots, n$$

ODEs

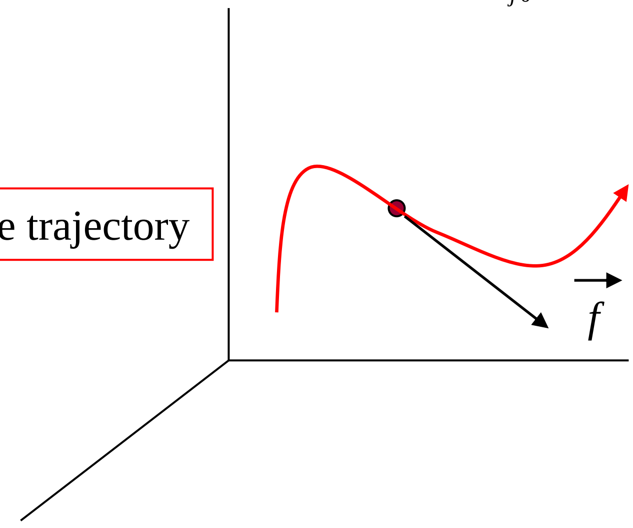
$$\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n)$$
$$\dot{x}_2 = f_2(x_1, x_2, \ldots, x_n)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, x_2, \ldots, x_n)$$

Initial conditions

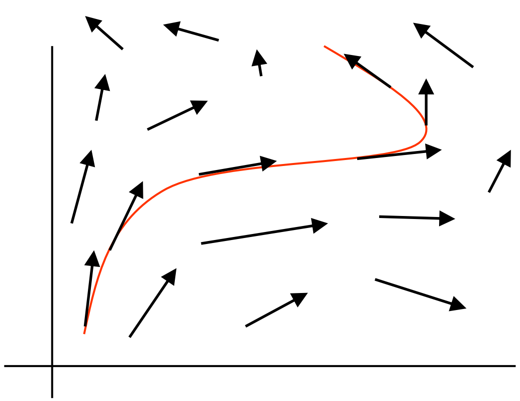
$$x_1(0) = x_{10}$$
$$x_2(0) = x_{20}$$
$$\ldots\ldots\ldots\ldots$$
$$x_n(0) = x_{n0}$$

Vector field

$$\vec{f} = (f_1, f_2, \ldots, f_n)_{x_{j0}}$$



Vector ***f*** is tangent to the trajectory

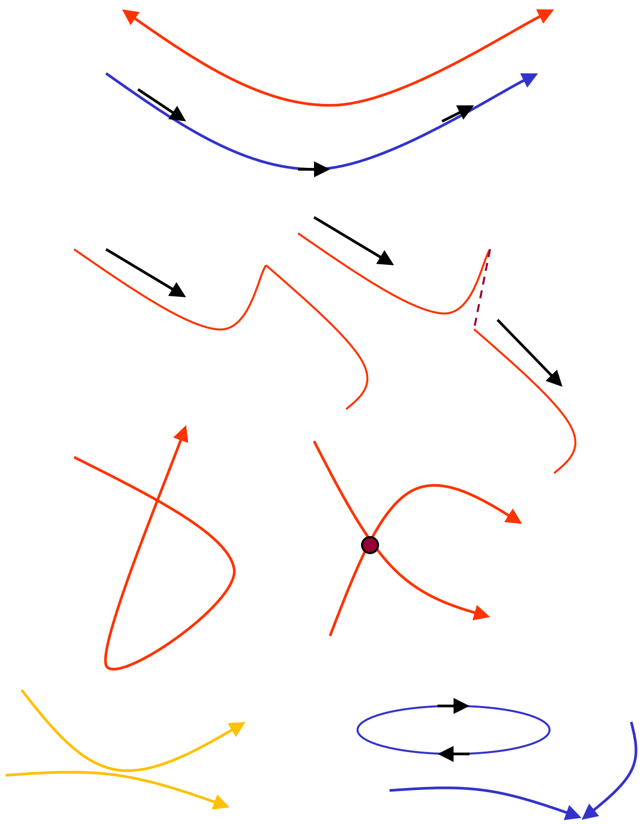


Trajectories follows tangentially the vector field

AUTONOMOUS SYSTEMS

>> Consequences due to the continuity of the vector field and Cauchy Theorem.

conventional and non-conventional trajectories



- The vector field flow has the same direction along the orbit.
- Trajectories are continuous and differentiable curves in phase space for any time (smooth curves)
- Any point of the phase space ***E*** belongs to a unique trajectory. Trajectories do not intersect transversally their self or other trajectories.
- Independently of the initial time value t_o, from a particular point in phase space ***E*** is defined only one solution (one trajectory passes). The dynamical system is **invariant in time translation**. So we can set always $\boldsymbol{t_0=0}$.

Autonomous systems – Special solutions

A. Equilibrium points: The singular (critical) points of the vector field

$$f_i(x_{10}, x_{20}, ..., x_{n0}) = 0, \qquad (x_{10}, x_{20}, ..., x_{n0}) : \quad \textit{equilibrium point}$$

$$\Leftrightarrow \quad \dot{x}_i = 0 \quad \Leftrightarrow \quad x_i(t) = x_{i0} \text{ (const)}$$

- Equilibrium solutions are found by solving algebraic and not differential equations.

Stability (draft definition) : If there exist initial conditions in the neighborhood of an equilibrium point providing orbits which diverge from the equilibrium point as $t \to \infty$, then the equilibrium point is called **unstable** otherwise is called **stable**.

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B. Periodic Solutions: Trajectories which are repeated per equal time intervals

$$x_i(t+T) = x_i(t), \quad \forall i = 1,..,n, \quad \forall t \in R \qquad (T : \text{period})$$

- If an orbit with initial state $\{x_{io}\}$ after time T is found again at the same state $\{x_{io}\}$ in phase space, afterwards it will evolve exactly the same as in the beginning. This is because the initial conditions and the vector field will be the same.
- **A periodic trajectory is represented by a “closed” curve in phase space**.

➢ A periodic solution is an **invariant set** in phase space

NON-AUTONOMOUS SYSTEMS

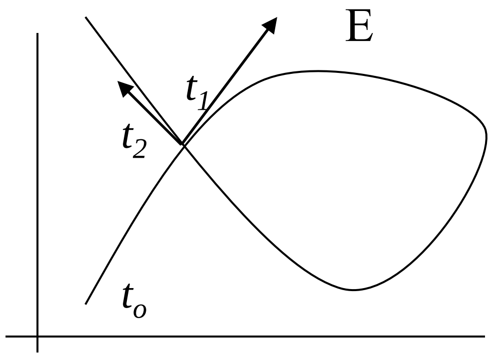
ODEs

$$\dot{x}_1 = f_1(x_1, x_2, ..., x_n, t)$$
$$\dot{x}_2 = f_2(x_1, x_2, ..., x_n, t)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, x_2, ..., x_n, t)$$

$$\exists i, \quad \frac{\partial f_i}{\partial t} \neq 0$$

Initial Conditions

$$x_1(t_0) = x_{10}$$
$$x_2(t_0) = x_{20}$$
$$\ldots\ldots\ldots$$
$$x_n(t_0) = x_{n0}$$



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- The vector field varies in time.

- The trajectories maybe intersect in phase space. They do not intersect in the extended phase space $\boldsymbol{R}$x$\boldsymbol{R}^n$

- If $\boldsymbol{f}$ =0 at the point **x*** at time $t=t^*$ then, generically, $\boldsymbol{f} \neq 0$ for $t \neq t^*$ and thus **x*** is not an equilibrium point.

- If a trajectory returns to its initial point $\mathbf{x}_0=(x_{10}, x_{20}, \ldots, x_{n0})$ at a time $t=t_0+\tau$, the evolution will be different because, genericly, $\boldsymbol{f}(\mathbf{x}_0, t_0) \neq \boldsymbol{f}(\mathbf{x}_0, t_0+\tau)$. Thus, the orbit is not periodic in general.

- If the variation of the vector field is periodic, with period T_f then the periodic orbits of the system, if they exist, should have period equal to $k\,T_f$, $k=1,2,\ldots$ This is proved by the fact that the trajectory is found at the same point $\mathbf{x}_0$ at time $t=t_0+k\,T_f$ when the vector field is the same with its initial state at t_0.

Qualitative classification of trajectories

$$\dot{x}_i = f_i(x_j, t), \quad x_i(0) = x_{i0} \quad \Rightarrow \quad x_i = x_i(t) = x_i(t; x_{j0}, t_0) \quad i = 1,..,n$$

trajectory : Oriented parametric curve in phase space, with parameter the time *t*, which starts from the point of initial conditions and evolves either **for increasing time** (future) or decreasing time (past).

- **Periodic trajectory** (Cycle): $x_i(t + kT; x_{j0}, t_0) = x_i(t; x_{j0}, t_0) \quad \forall i = 1,..,n, \quad \forall k \in N$

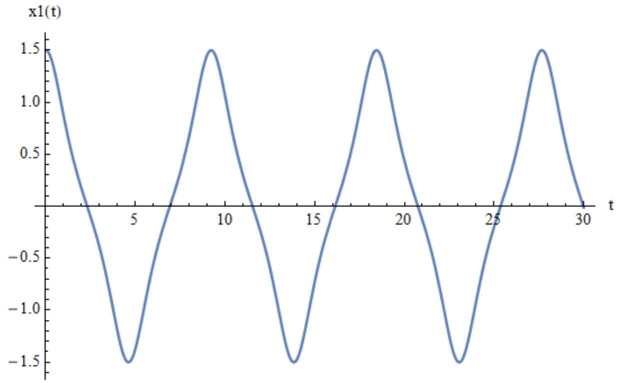
- **Bounded trajectory**: $| x_i(t; x_{j0}, t_0) - x_{i0} | < M \quad \forall i = 1,..,n, \quad \forall t > t_0$

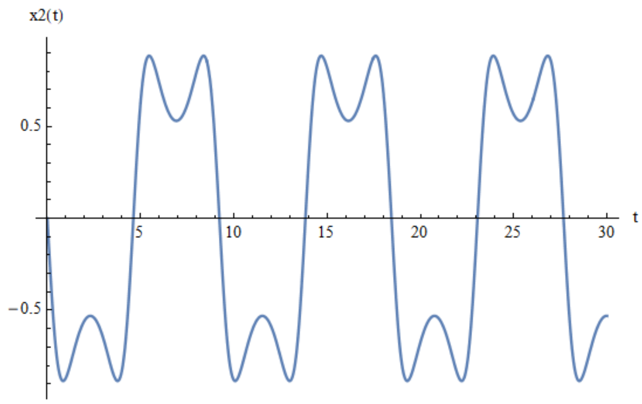
- **Unbounded trajectory** : $\exists i \in [1,..,n], \quad x_i(t; x_{j0}, t_0) \to \pm\infty \quad \gamma\iota\alpha \; t \to \infty$

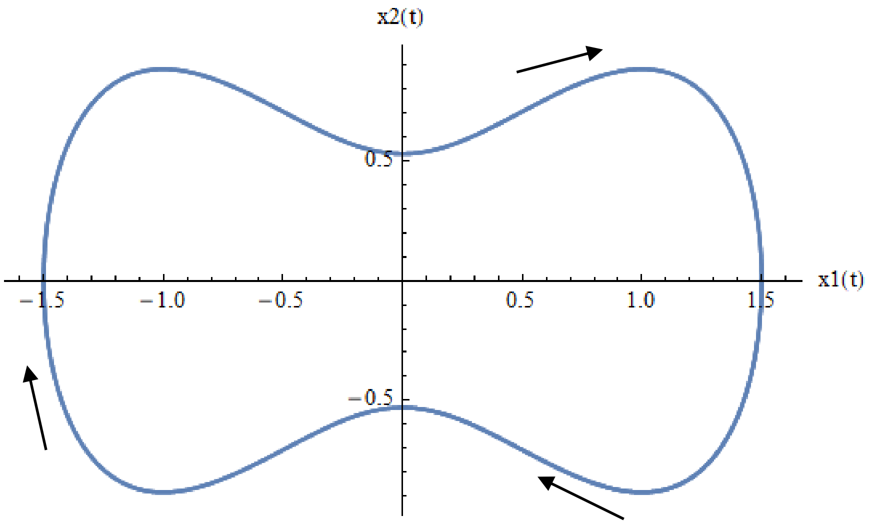
- **Asymptotic trajectory towards a point** : $\forall i = 1,..n, \quad \lim_{t \to +\infty} x_i(t; x_{j0}, t_0) = x_i^*$

- **Asymptotic trajectories towards to a phase space subset** $U^* \in R^{n-l}$, $0<l<n$: For any neighborhood *V* of the subset U^* in phase space, there exists t^* such that $(x_1(t), x_2(t), ..., x_n(t)) \in V, \quad \forall t > t^*$

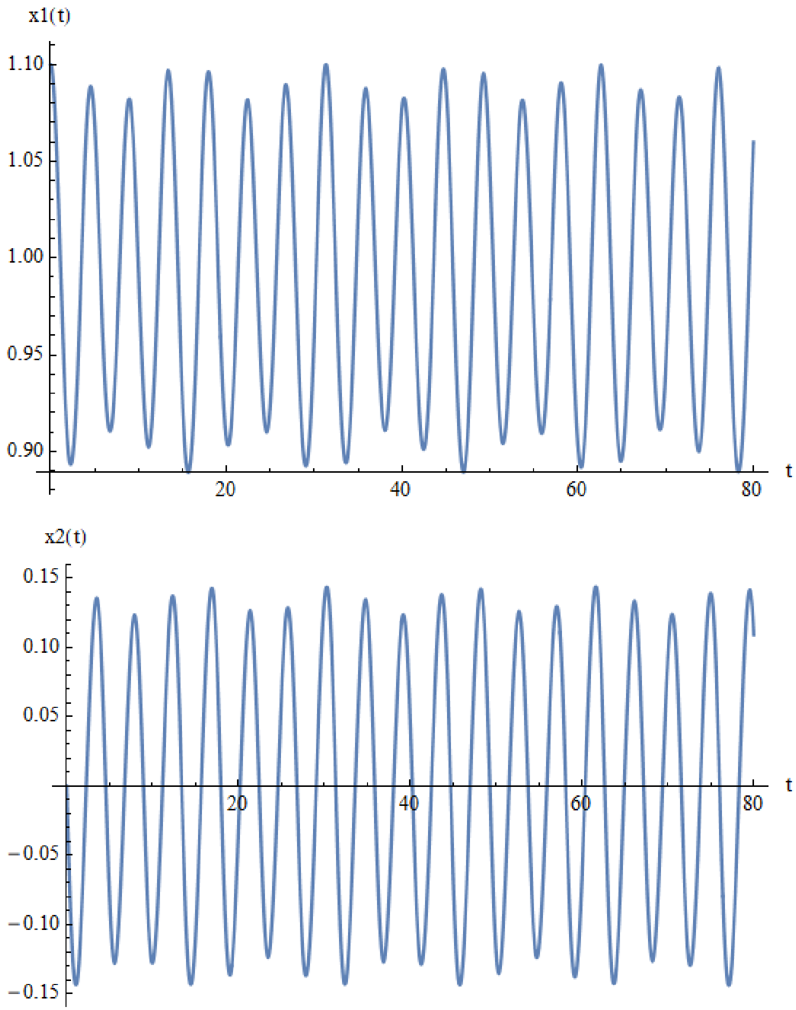
Periodic trajectory

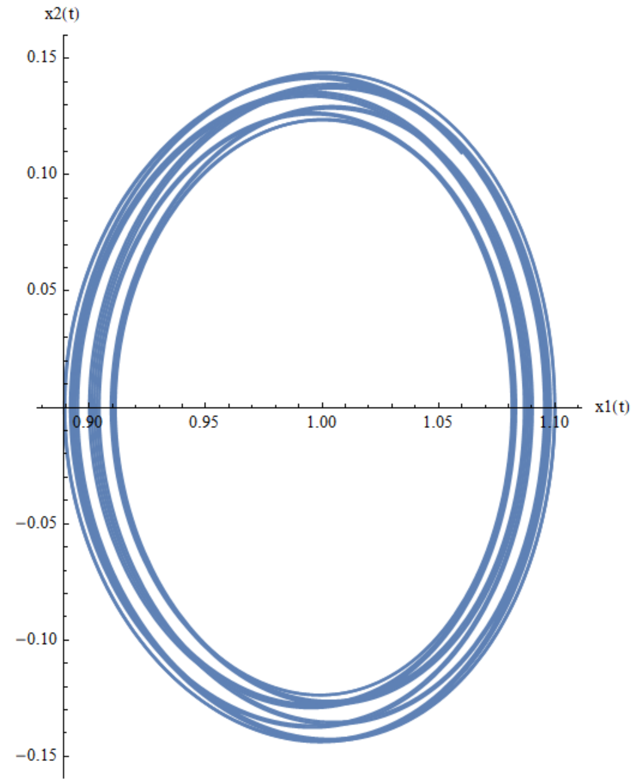




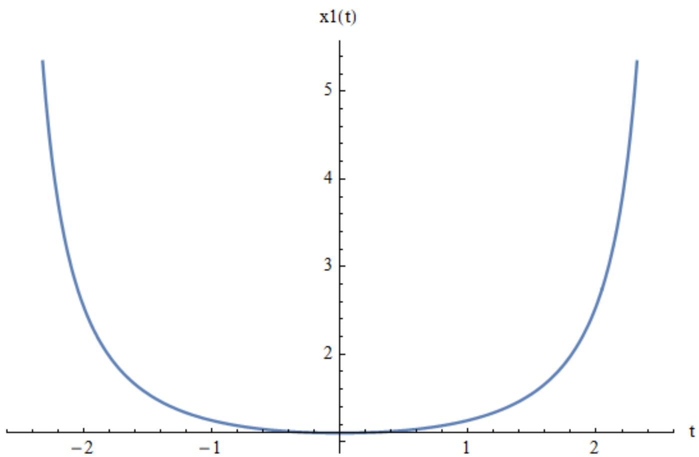


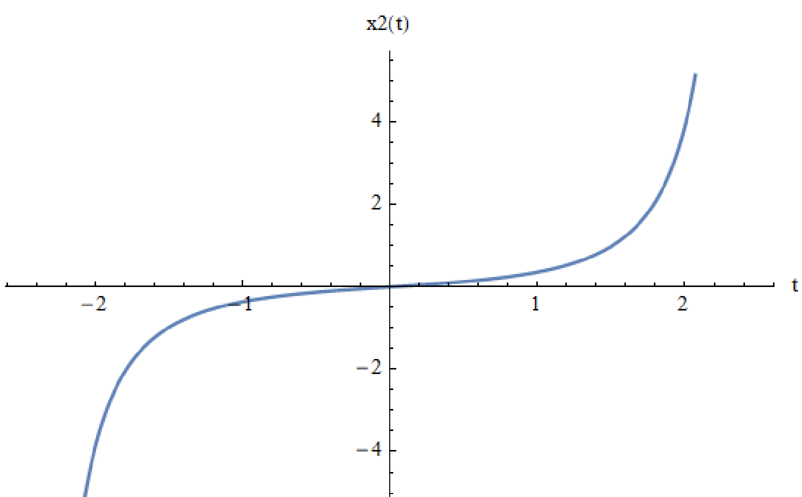
bounded trajectory but not periodic

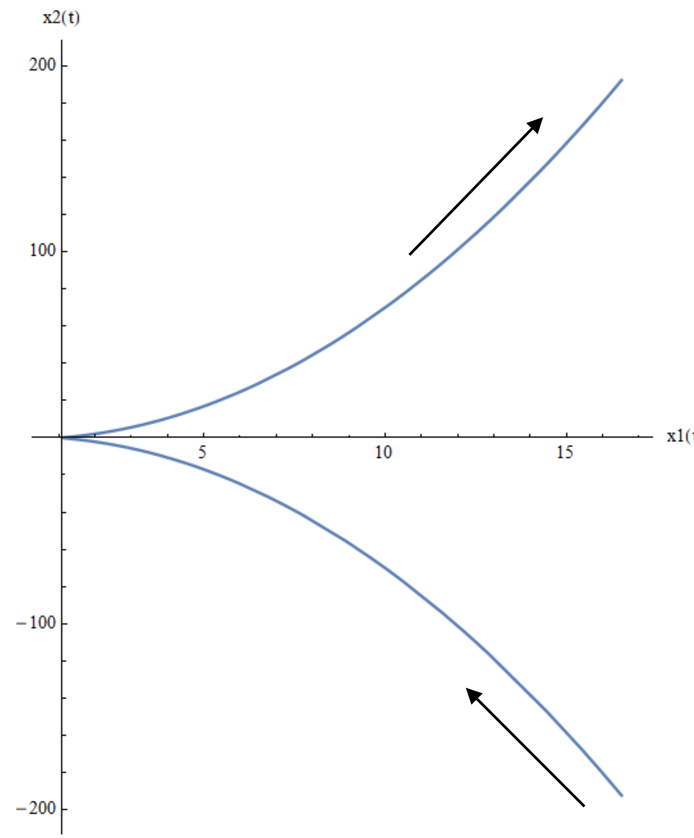




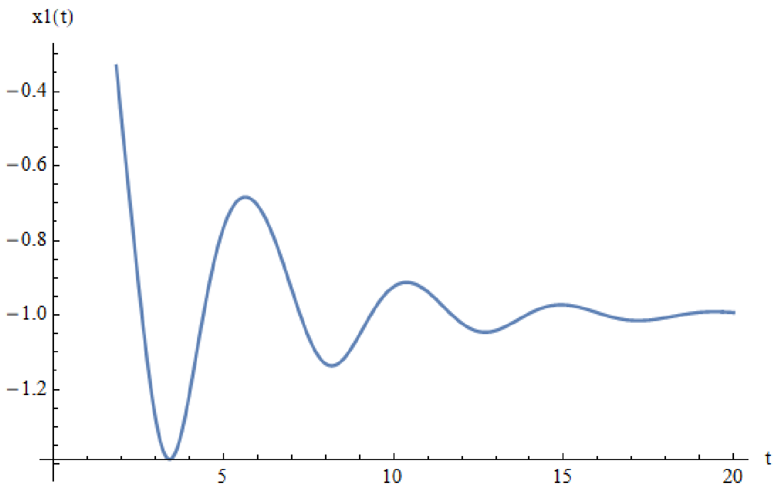
Unbounded trajectory

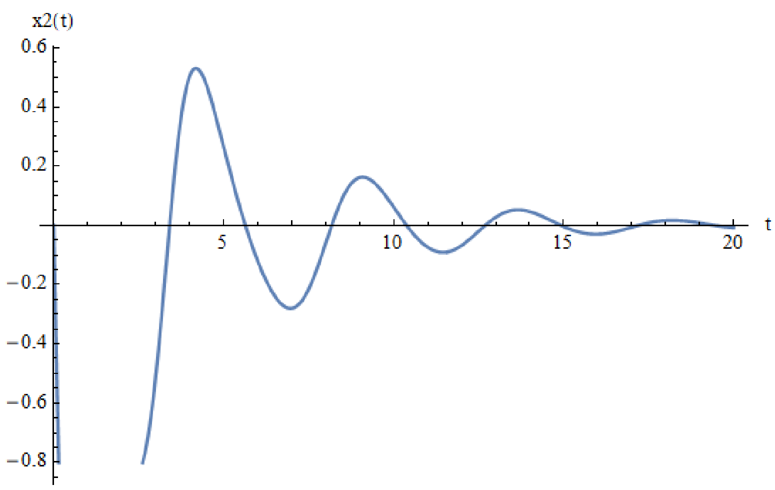


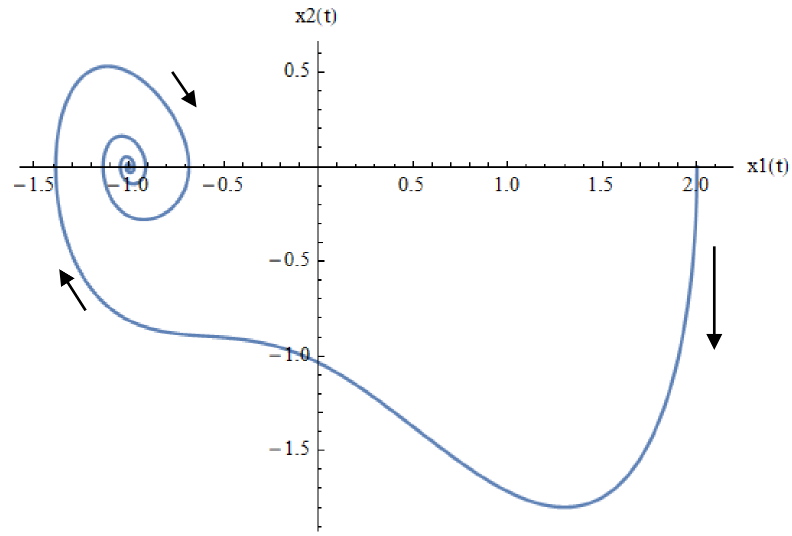




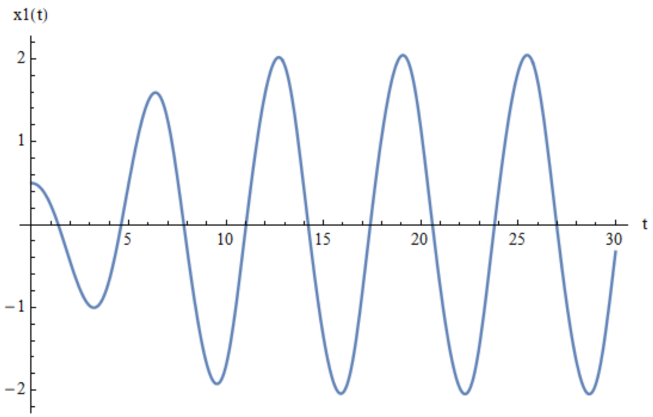
Asymptotic orbit to a point

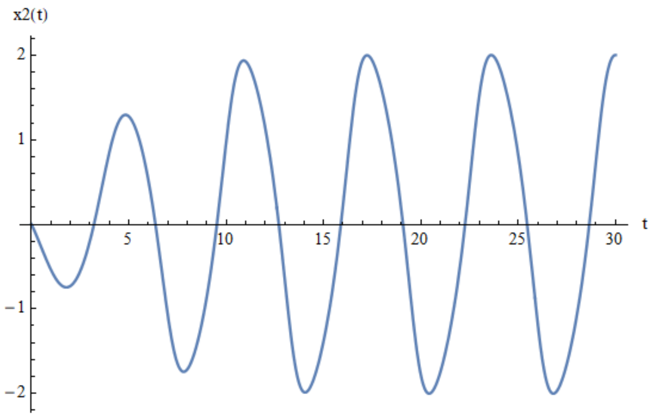


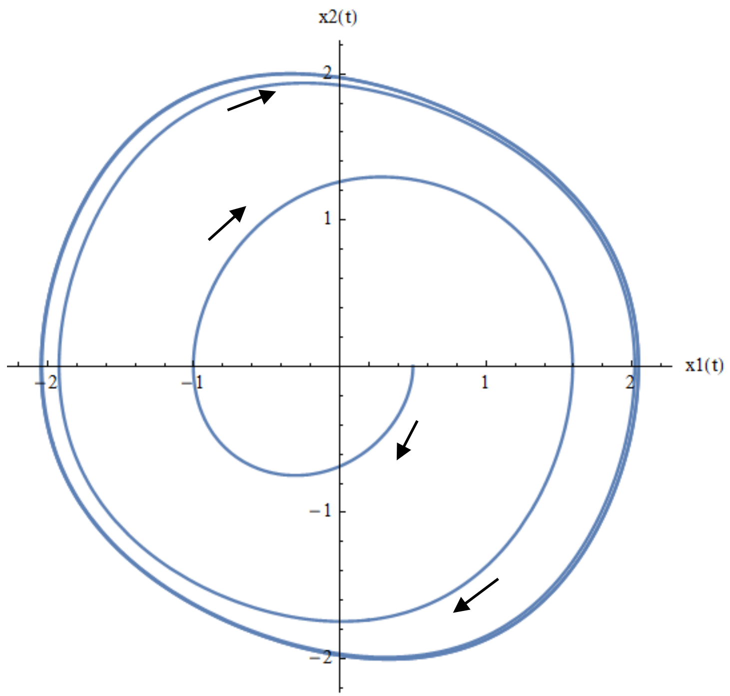




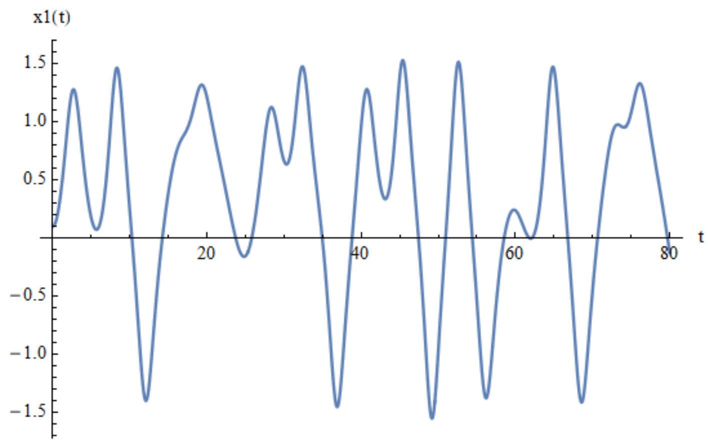
Asymptotic orbit to a cycle

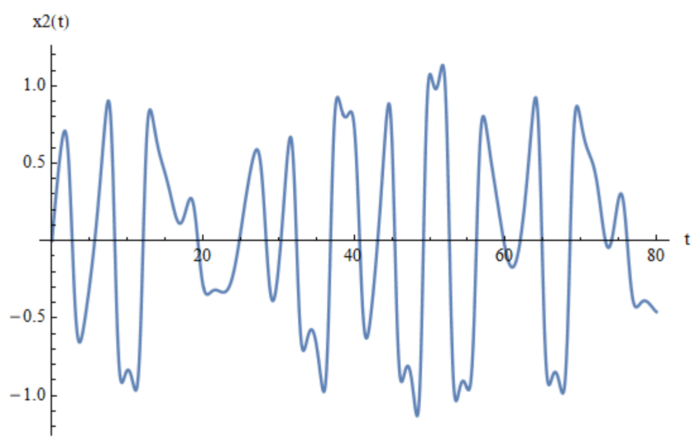


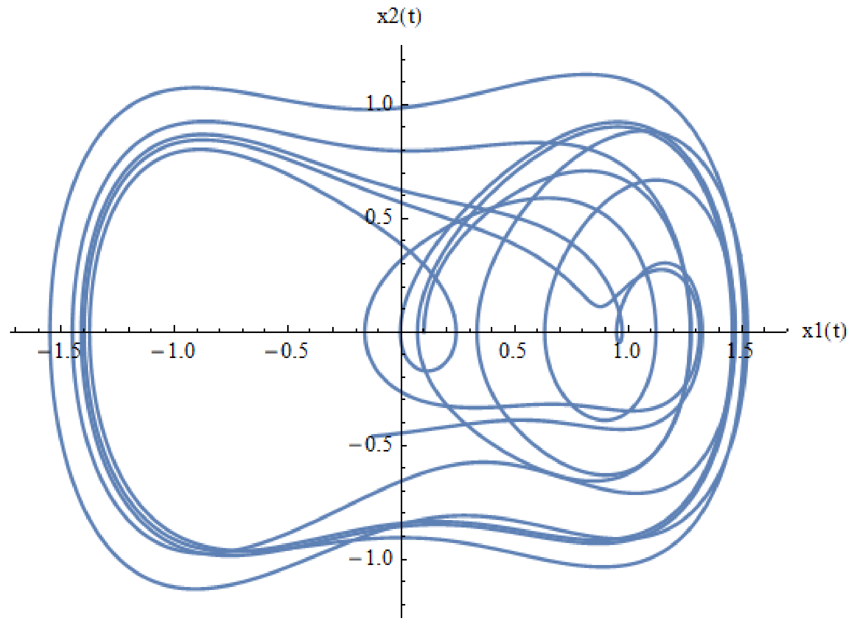




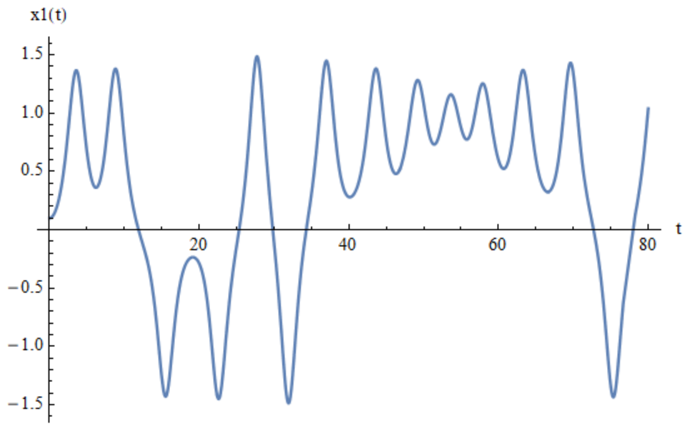
Chaotic trajectory towards an attractor

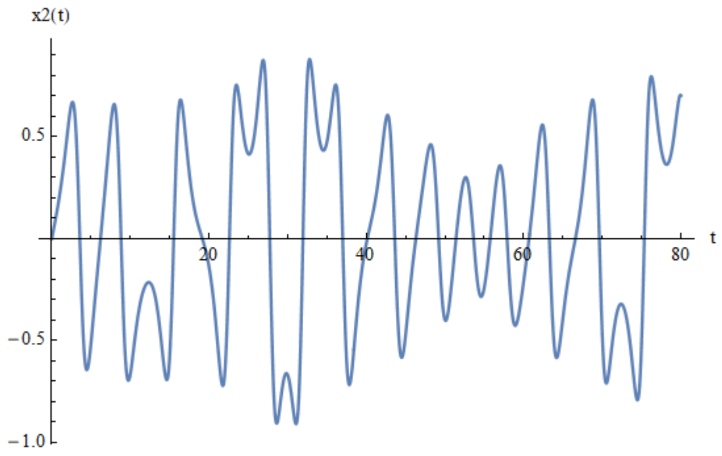


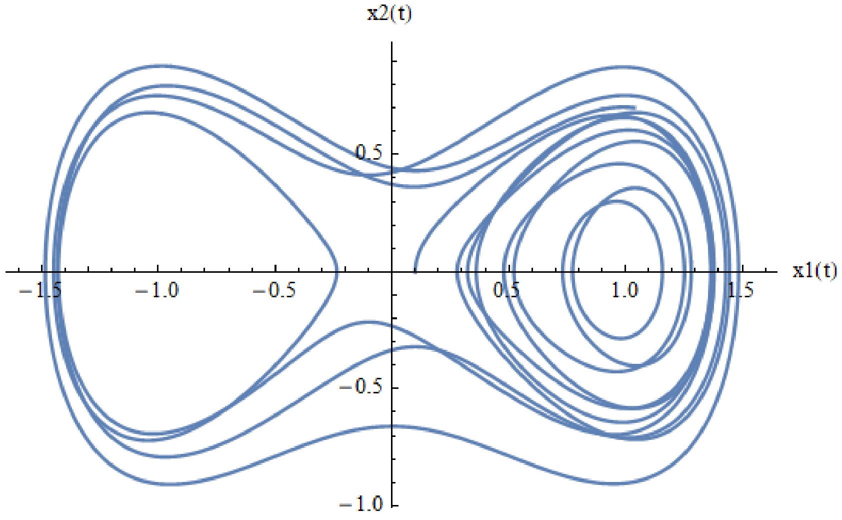




Chaotic trajectory







AREA PRESERVING – NON AREA-PRESERVING SYSTEMS

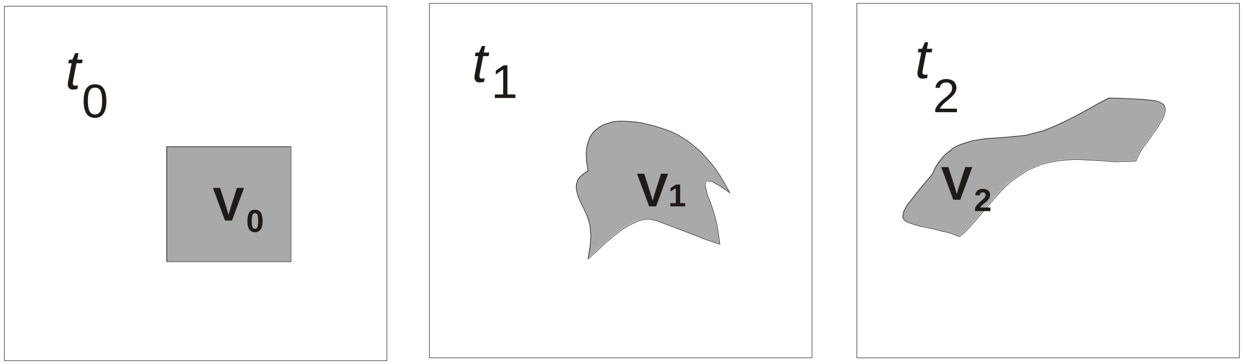
Phase Space = flow

$$\frac{d\rho}{dt} + \rho \, div\vec{\upsilon} = 0, \quad \vec{\upsilon} \equiv \vec{f}$$

(Continuity equation)

- **area preserving** (conservative)
- **dissipative**
- Explosive
- "area dependent preservation" (*dissipative*)

$$div\,\vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \ldots + \frac{\partial f_n}{\partial x_n} = \begin{cases} = 0 \\ < 0 \\ > 0 \\ a(x_i) \end{cases}$$



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