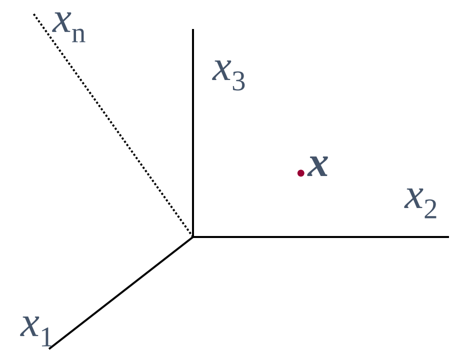
DYNAMICAL SYSTEMS

Dynamical variables (quantities) : $x_1, x_2, x_3, \ldots, x_n$ n: Dimension of the system

$x_i = x_i(t) \in R, \quad \mathbf{x} \in E \subseteq R^n, \quad t \in R$

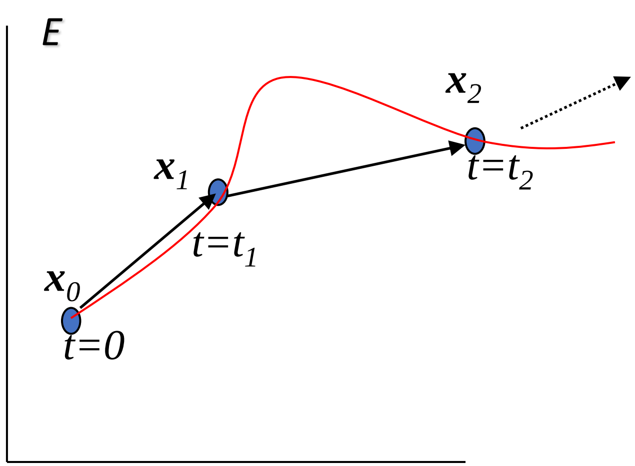
$x_{i0} = x_i(t_0) \rightarrow (x_{10}, x_{20}, \ldots, x_{n0})$: Initial conditions/state at $t = t_0$

$x_i = x_i(t) \rightarrow (x_1, x_2, \ldots, x_n)$: State of system at time t



$E \subset R^n$: **Phase Space**

Dynamical system : Mapping $\varphi : R \times E \rightarrow E$



flow $\varphi_t(x)$
ή $\varphi(t,x)$ ⟶ Phase space tracectory

$$\varphi_{t0}(x) = x_0$$

,

$$\varphi_{t_2} \circ \varphi_{t_1}(x) = \varphi_{t_1 + t_2}(x)$$

- Continuous systems $\quad t \in R$
- Discrete systems $\quad t \leftrightarrow \tau \in N$

- Deterministic systems
- Stochastic systems

CONTINUOUS DYNAMICAL SYSTEMS

$$\dot{x}_1 = f_1(x_1, x_2, ..., x_n, t)$$
$$\dot{x}_2 = f_2(x_1, x_2, ..., x_n, t)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, x_2, ..., x_n, t)$$

$$[\quad \dot{x} = f(x,t) \quad] \quad \Rightarrow$$

$$x_1 = x_1(t; x_{10}, ..., x_{n0})$$
$$x_2 = x_2(t; x_{10}, ..., x_{n0})$$
$$\vdots$$
$$x_n = x_n(t; x_{10}, ..., x_{n0})$$

$$[\quad x = x(t, x_0) \quad]$$

Trajectory is given in parametric form

$t \in \mathbf{R}$,

- $t < t_0$: past
- $t = t_0$: present
- $t > t_0$: future

$$\vec{f} = (f_1, f_2, ..., f_n)$$

Vector field of the system or velocity field

Cauchy Theorem

Existence and continuity of $f_i, \dfrac{\partial f_i}{\partial x_j}$, $\forall i, j, \quad t \in R$ ⟶ Existence and uniqueness of solutions in the extended phase space $R^{n+1} = R \times E$

in the following

We always assume domains where the Cauchy conditions for uniqueness and existence of solutions are satisfied by the vector field of the ODEs

Example: Chemical reactions (oregonators)

$$\dot{X} = k_1 aY - k_2 XY + k_3 aX - k_4 X^2$$

$$\dot{Y} = -k_1 aY - k_2 XY + \frac{3}{4} k_5 Z$$

$$\dot{Z} = 2k_3 aX - k_5 Z$$

Normalization : reduction of parameters)

$$\dot{x} = \frac{1}{\varepsilon}(x + y - qx^2 - xy)$$

$$\dot{y} = -y + z - xy \qquad (t \to \tau)$$

$$\dot{z} = \frac{1}{p}(x - z)$$

Example: The planar three body problem

$$\dot{x}_1 = v_{x1}$$
$$\dot{y}_1 = v_{y1}$$
$$\dot{x}_2 = v_{x2}$$
$$\dot{y}_2 = v_{y2}$$
$$\dot{x}_3 = v_{x3}$$
$$\dot{y}_3 = v_{y3}$$

$$\dot{v}_{x1} = -\frac{Gm_2}{r_{12}^2}(x_1 - x_2) - \frac{Gm_3}{r_{13}^2}(x_1 - x_3)$$

$$\dot{v}_{y1} = -\frac{Gm_2}{r_{12}^2}(y_1 - y_2) - \frac{Gm_3}{r_{13}^2}(y_1 - y_3)$$

$$\dot{v}_{x2} = -\frac{Gm_2}{r_{21}^2}(x_2 - x_1) - \frac{Gm_3}{r_{23}^2}(x_2 - x_3)$$

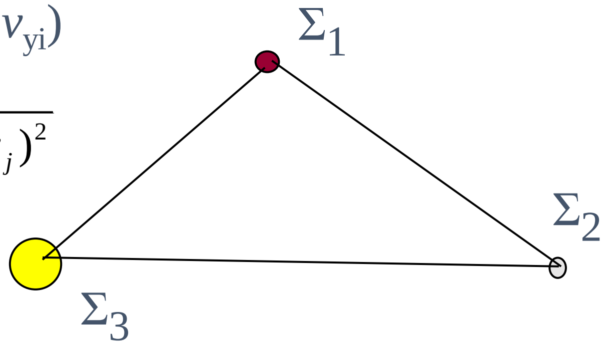
$$\dot{v}_{y2} = -\frac{Gm_2}{r_{21}^2}(y_2 - y_1) - \frac{Gm_3}{r_{23}^2}(y_2 - y_3)$$

$$\dot{v}_{x3} = -\frac{Gm_3}{r_{31}^2}(x_3 - x_1) - \frac{Gm_1}{r_{23}^2}(x_3 - x_2)$$

$$\dot{v}_{y3} = -\frac{Gm_3}{r_{31}^2}(y_3 - y_1) - \frac{Gm_1}{r_{23}^2}(y_3 - y_2)$$

Σ_i: $r_i=(x_i,y_i)$, $v_i=(v_{xi},v_{yi})$

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$



Cauchy conditions are not satisfied at **collisions**

Reduction of number of equations

$$\Sigma_3 = f_{KM}(\Sigma_1, \Sigma_2)$$

$$x_3 = -\frac{1}{m_3}(m_1 x_1 + m_2 x_2), \quad v_{x3} = -\frac{1}{m_3}(m_1 v_{x1} + m_2 v_{x2})$$

$$y_3 = -\frac{1}{m_3}(m_1 y_1 + m_2 y_2), \quad v_{y3} = -\frac{1}{m_3}(m_1 v_{y1} + m_2 v_{y2})$$

Normalization of units

$$e.g. \quad r_{13}(0) = 1, \quad m_1 + m_2 + m_3 = 1, \quad G = 1$$

The two main categories of Dynamical systems

A. Linear systems

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + b_i \longrightarrow$$

Solutions given by
•Standard functions
•Special function
• Convergent series

B. Non-linear systems $\longrightarrow$?

If we can compute the general analytic solution of a dynamical system we know its evolution **for all** initial conditions and **all** time intervals

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In general **we cannot find analytic solutions** for non-linear systems of 2 or more dimensions with standard functions and **numerical solutions should be computed**

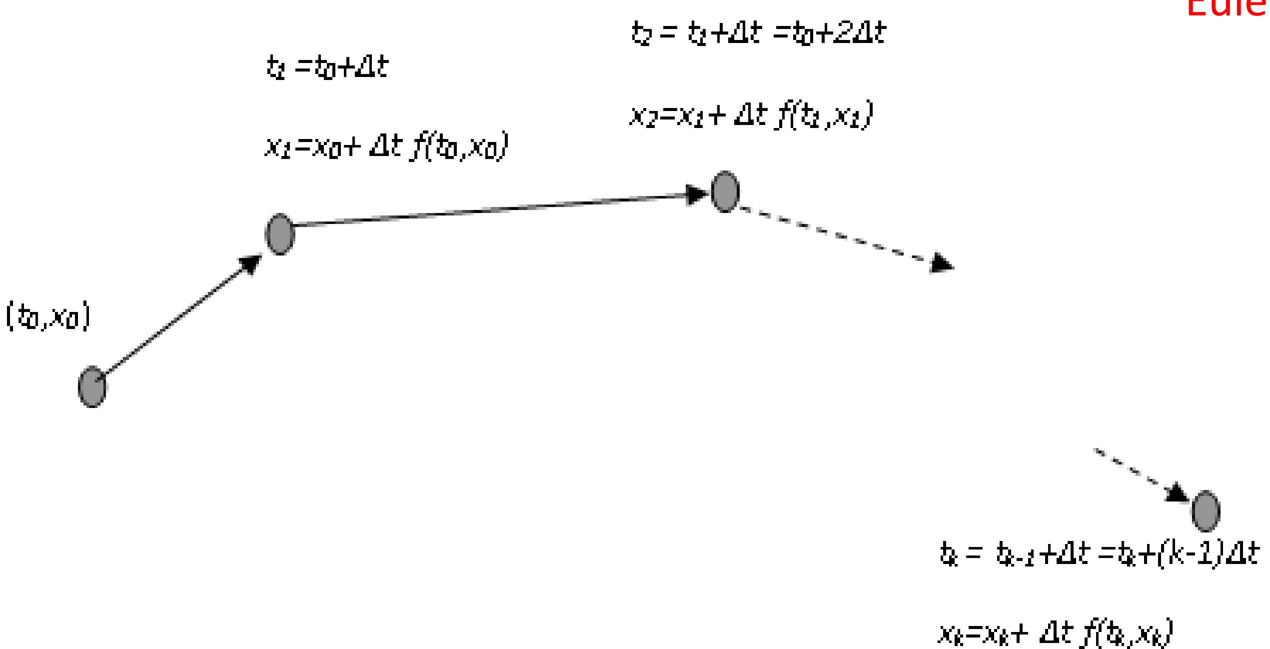
The basic notion of numerical solution

ODE $\dfrac{dx}{dt} = f(t,x)$ initial condition $x(t_0)=x_0$

It holds $\left.\dfrac{dx}{dt}\right|_{t=t_0} = f(t_0,x_0)$ and $\left.\dfrac{dx}{dt}\right|_{t=t_0} = \lim_{\Delta t\to 0} \dfrac{x(t_0+\Delta t)-x(t_0)}{\Delta t} \simeq \dfrac{x(t_0+\Delta t)-x(t_0)}{\Delta t} \quad (\Delta t \ll 1)$

So $x(t_0+\Delta t) \simeq x(t_0) + \Delta t\, f(t_0,x_0)$

Euler's approximation



The basic notion of numerical solution

A numerical solution

- is a discrete map of a partial solution of an initial value problem
- consists of a set of points $A(t,x) = \{ (t_k, x_k) \in R^{n+1}, k = 0,1,\ldots,N\}$
- Is valid in a restricted time interval $t_{min} \leq t \leq t_{max}$
- suffers from numerical errors

<u>Numerical solutions in Mathematica</u>

NDSolve[*Initial Value problem*, $\{x_1, x_2, \ldots\}$,{t, tmin, tmax}]

Output : **Interpolating function,** one for each variable

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