Exercise 1 //Nonlinear dynamical systems

D. Logistic Model with Predation

One of the more predominant pests in Canadian forests is the spruce budworm. These insects, which are moths in the adult stage, lay eggs in the needles of the trees. After hatching, the larvae eventually tunnel into the old foliage of the tree. Later, they spin a webbing in the needles and devour new growth until they are interrupted. Although they do not destroy the tree, budworms weaken the trees and make them susceptible to disease and forest fires. We can model the rate of change in the size of the budworm population by building on the logistic model. Let $W(t)$ represent the size of the budworm population at time t. One factor that helps control the budworm population is predation by birds (that is, birds eat budworms). Consider the model,

$$\frac{dW}{dt} = rW\left(1 - \frac{1}{k}W\right) - P(W)$$

where $P(W)$ is a function of W describing the rate of predation. Notice that for low population levels, the model demonstrates exponential growth rate r. Based on the logistic model, k represents the carrying capacity of the forest. In the absence of predation, this gives a maximum size of the budworm population.

Properties of the Predation Function To develop a formula for $P(W)$, we make two assumptions. First, if $W = 0$, then $P(W) = 0$ (that is, if there are no budworms, the birds have none to eat). Second, $P(W)$ has a limiting value (that is, the birds have a limited appetite. Even if the budworm population is large, the birds eat only what they need). A function that satisfies both of these properties is $P(W) = aW^2/(b^2 + W^2)$. Therefore, our model becomes:

$$\frac{dW}{dt} = rW\left(1 - \frac{1}{k}W\right) - \frac{aW^2}{b^2 + W^2}, \quad W(0) = W_0.$$

1. What is the limiting value of $P(W) = aW^2/(b^2 + W^2)$ as $W \to \infty$?
2. Using the indicated parameter values, approximate the equilibrium solutions. Sketch a phase line in each case.
   (a) $r = 1, k = 15, a = 5, b = 2$
   (b) $r = 1, k = 20, a = 5, b = 2$.
3. Using the values $r = 1$, $a = 5$, $b = 2$, approximate the value of k that leads to three equilibrium solutions. (Do this experimentally by selecting values for k and then plotting. This value is a bifurcation point.) Sketch the phase line.