Nonlinear Autonomous Dynamical systems of two dimensions

Part C

Limit Cycles

From equilibrium points

to **isolated** periodic trajectories

The limit cycle solution

Example 1.

polar coord $(x = r\cos\theta, y = r\sin\theta)$

$$x^2 + y^2 = r^2$$
$$y/x = \tan\theta$$
$$x\dot{x} + y\dot{y} = r\dot{r}$$
$$x\dot{y} - \dot{x}y = r^2\dot{\theta}$$

$$\dot{x} = -y + \frac{x}{\sqrt{x^2+y^2}}(1 - x^2 - y^2)$$
$$\dot{y} = x + \frac{y}{\sqrt{x^2+y^2}}(1 - x^2 - y^2)$$

$$\Rightarrow \quad x\dot{x} + y\dot{y} = \sqrt{x^2+y^2}(1 - x^2 - y^2), \quad x\dot{y} - y\dot{x} = x^2 + y^2 \quad \Rightarrow \quad \dot{r} = 1 - r^2, \quad \dot{\theta} = 1$$

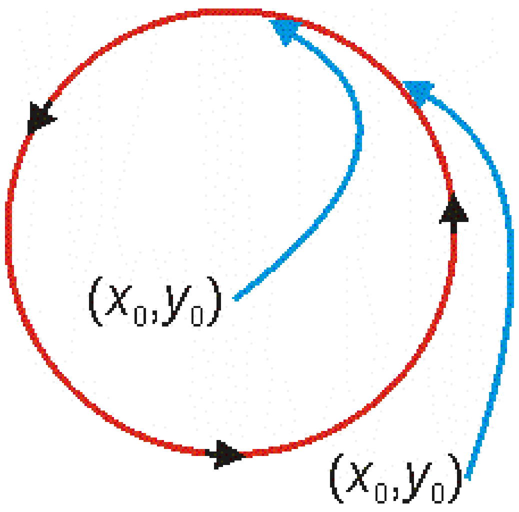
(0,0) : equilibrium

for r=1 is dr/dt=0,
so $r(t)$=1 is a solution

Solution

$$r = \frac{c - e^{-2t}}{c + e^{-2t}}, \quad c = (1 + r_0)/(1 - r_0)$$
$$\theta = t + \theta_0$$

$$\lim_{t\to\infty} r = 1, \quad \forall r_0, \theta_0$$



The limit cycle solution

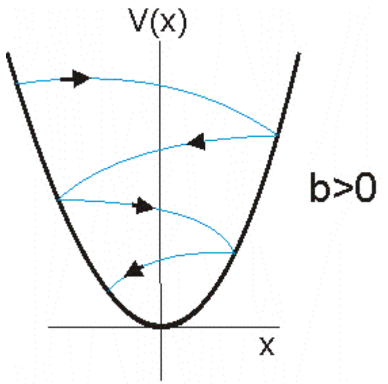
Example 2. Harmonic oscillator with damping

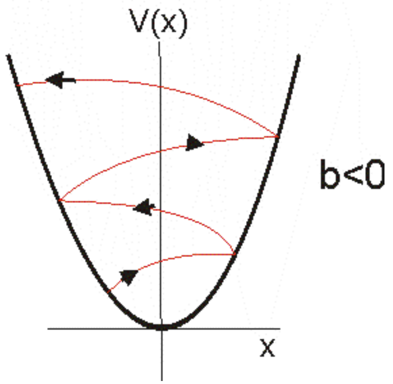
$$\dot{x} = y$$
$$\dot{y} = -kx - by, \quad k > 0$$

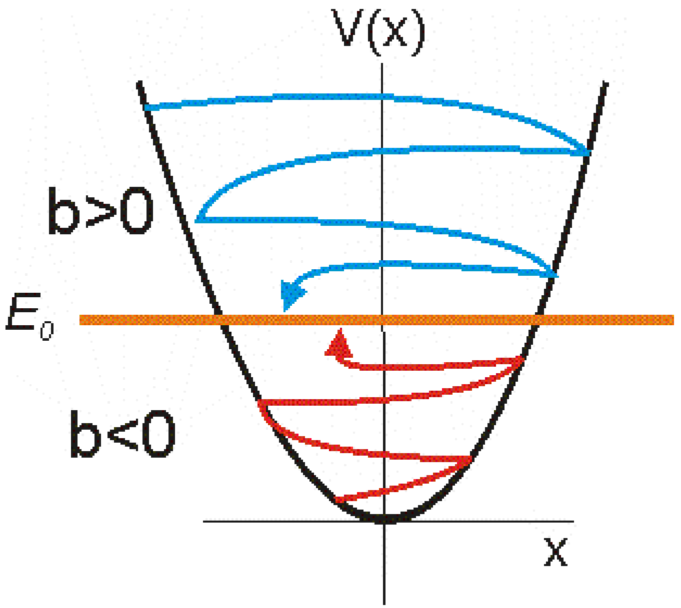
$$E = \frac{1}{2}y^2 + \frac{1}{2}kx^2$$

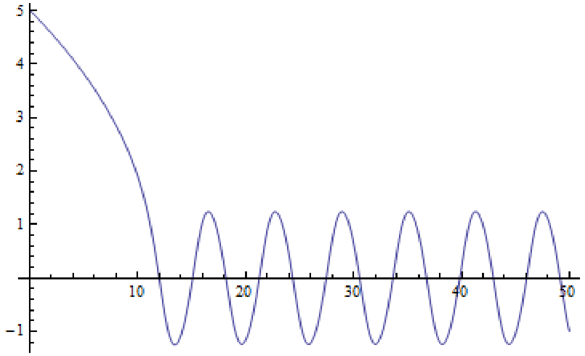
$$b = b(x, y) = E - E_0 = \frac{1}{2}y^2 + \frac{1}{2}kx^2 - E_0$$

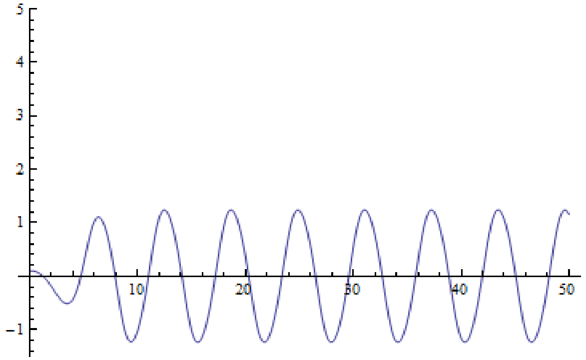
$$\dot{x} = y$$
$$\dot{y} = -kx - \left(\frac{1}{2}y^2 + \frac{1}{2}kx^2 - E_0\right)y, \quad k > 0$$





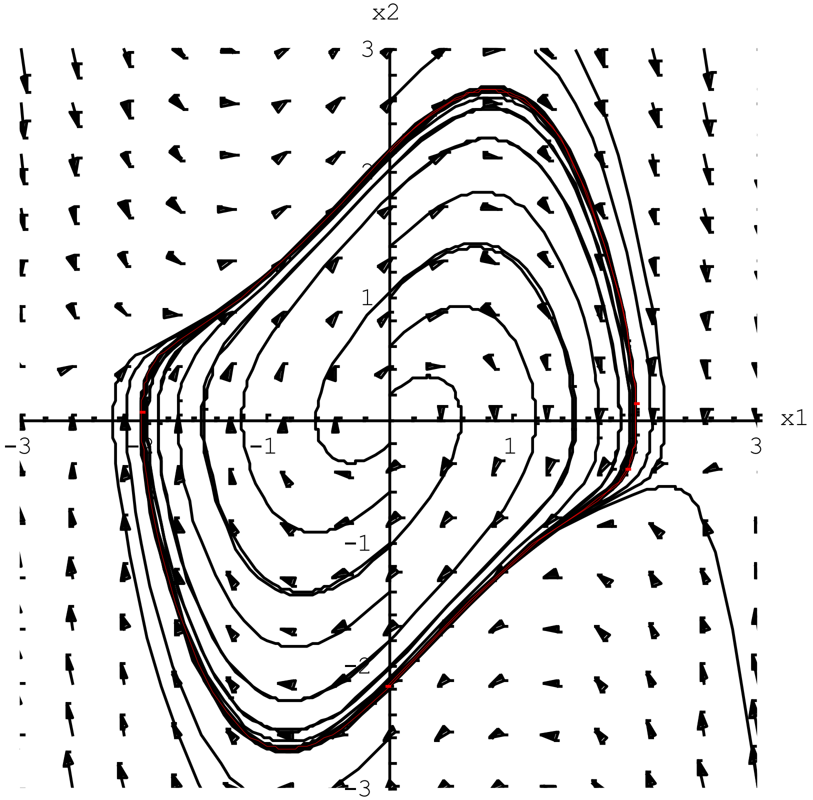




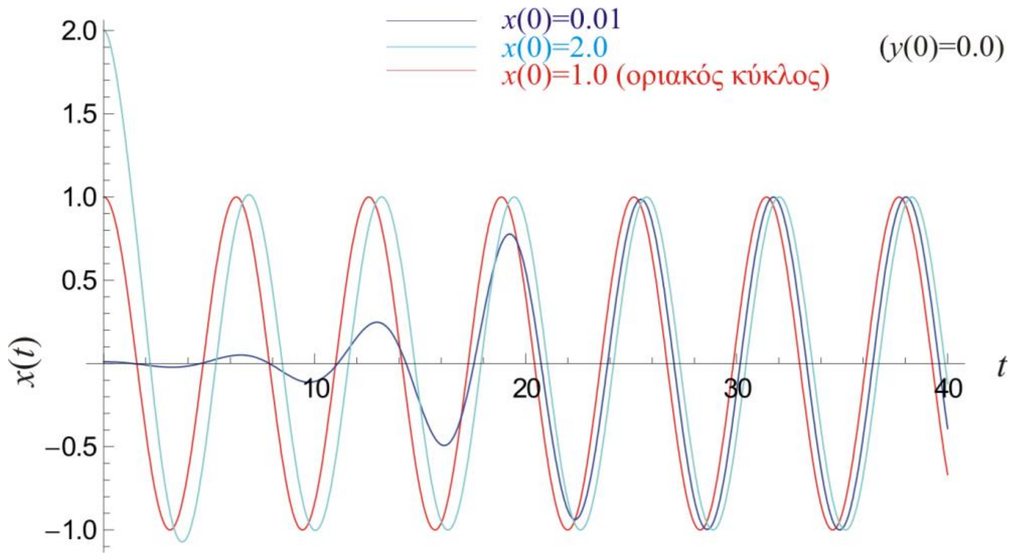


The limit cycle solution

A **stable limit cycle** (LC) is an isolated closed phase space trajectory (*or an isolated periodic solution*) which is included in a domain D such that $\forall$ $(x_0,y_0) \in$ D the corresponding solution ($x(t)$,y(t)) tends asymptotically, as $t\rightarrow\infty$, to the solution of the LC.



- The vector field is smooth on the solution of LC
- **The domain included inside the LC should contains at least one equilibrium point (focus or node)**

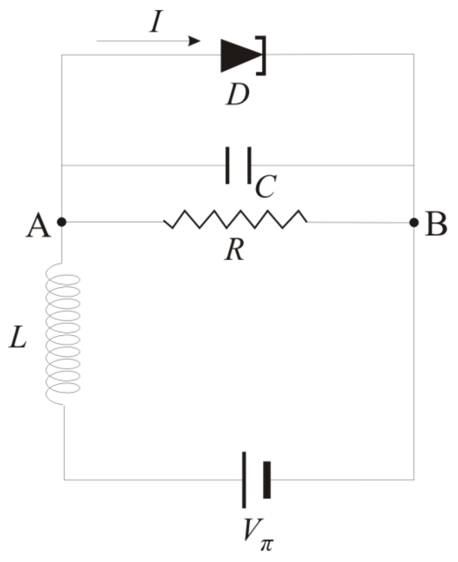


* if the solution ($x(t)$,y(t)) tends asymptotically to the LC as $t\rightarrow -\infty$, the LC is called **unstable**

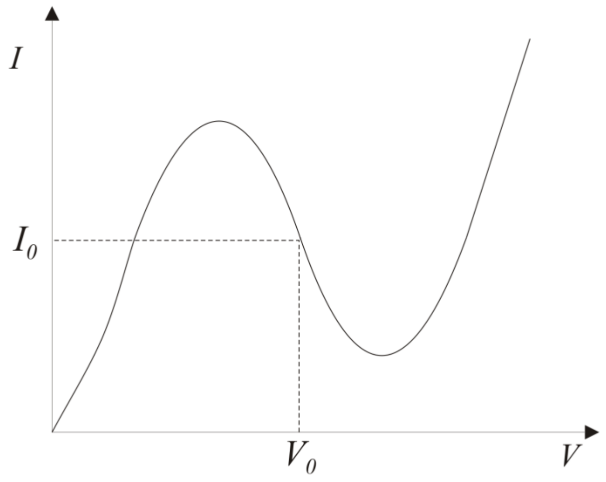
The limit cycle solution

example : the van der Pol equation

$$\ddot{x} = \mu(1 - x^2)\dot{x} - x \qquad x \sim V_{AB}, \qquad \mu = \mu(R, L, C, ...)$$



$$I = -aV + bV^3$$



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Production of electrical oscillation from a DC input.

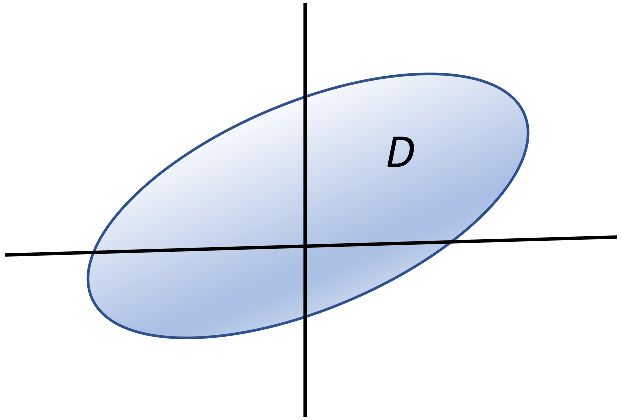
Non-Existence of a limit cycle solution

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

- **No equilibrium points $\Rightarrow$ No limit cycle solution (for 2D systems!)**

$$\dot{x} = x^2 + y^2 + 1$$
$$\dot{y} = x^2 - y^2$$

- **The negative criterion of Bendixson**



$$div\mathbf{f} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \neq 0 \quad \forall (x, y) \in D \quad \Rightarrow$$

there does not exist a limit cycle solution which is located completely in the domain *D*

$$div\mathbf{f} < 0 \quad or \quad div\mathbf{f} > 0$$

Example

$$\dot{x} = x^3 - y^3$$
$$\dot{y} = -3x + 2y + y^3$$

$\rightarrow$ $div\mathbf{f} = 3x^2 + 3y^2 + 2 > 0 \quad \forall (x, y) \in R^2$ $\rightarrow$ **no Limit Cycle**

(0,0): equilibrium

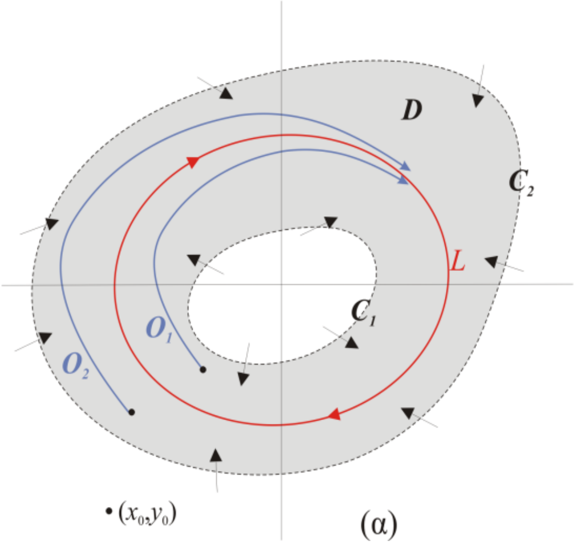
Existence of a limit cycle solution

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$
$$\mathbf{F} = (f, g)$$

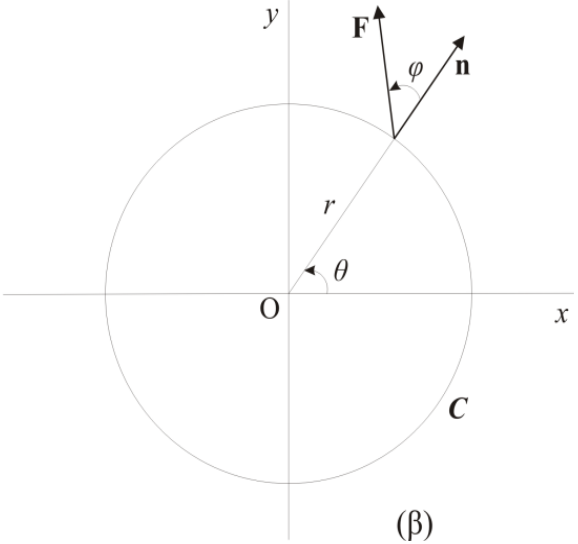
The Poincare – Bendixson theorem

We consider a bounded single connected domain $D \subset R^2$. If a phase space trajectory with initial conditions $(x_0, y_0) \in D$ is bounded as $t \to +\infty$ (or $t \to -\infty$) in the domain D then

- *The trajectory is closed (periodic solution)* ***or***
- *The trajectory tends to a closed trajectory (limit cycle) as* $t \to +\infty$ (or $t \to -\infty$) ***or***
- *The trajectory tends to an equilibrium point* $P \in D$ *as* $t \to +\infty$ (or $t \to -\infty$)



(α)



(β)

If we can define a **ring** with borders C_1 and C_2 such that

i) the ring does not contain an equilibrium

ii) the flow is directed outward at the inner border C_1

iii) the flow is directed inward at the outer border C_1

then there exist a limit cycle in the ring

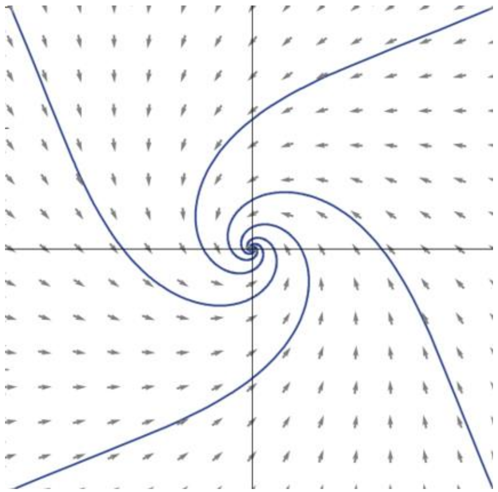
$$\cos\varphi = \frac{\mathbf{n}\cdot\mathbf{F}}{|.|} \quad \rightarrow \quad \begin{array}{ll} \mathbf{n}\cdot\mathbf{F} < 0 & \textit{inward direction} \\ \mathbf{n}\cdot\mathbf{F} > 0 & \textit{outward direction} \end{array}$$

example

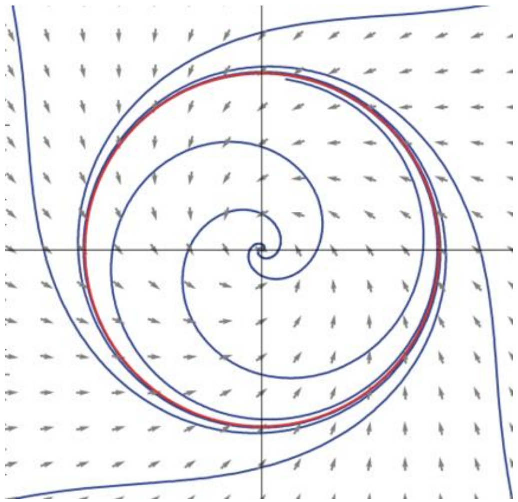
$$\dot{x} = x + y - x(x^2 + 2y^2)$$
$$\dot{y} = -x + y - y(x^2 + 2y^2)$$

The Hopf bifurcation

$$\dot{x} = f(x, y; \mu)$$
$$\dot{y} = g(x, y; \mu), \qquad \mu \in R$$



$\mu<\mu_0$: stable focus



$\mu>\mu_0$: unstable focus + Limit cycle

Example

$$\dot{x} = -y + x(\mu - x^2 - y^2)$$
$$\dot{y} = x + y(\mu - x^2 - y^2)$$

$$\begin{pmatrix} \dot{r} = r(\mu - r^2) \\ \dot{\theta} = 1 \end{pmatrix}$$

(0,0) : equilibrium, $\lambda=\mu\pm i$