Nonlinear Autonomous Dynamical systems of two dimensions

Part B

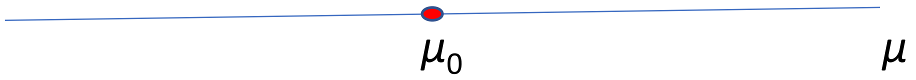
Bifurcations

Structural stability and Bifurcations

$$
\begin{aligned}
\dot{x} &= f(x, y; \mu) \\
\dot{y} &= g(x, y; \mu), \qquad \mu \in R
\end{aligned}
$$

If in the parameter domain (μ_1,μ_2) the system shows qualitatively (topologically) the same phase space portrait, e.g. the same equilibrium points with the "same stability", the system is called **structurally stable.**

If for $\mu_0 \in (\mu_1,\mu_2)$ the system shows qualitatively (topologically) different phase space portrait in the two intervals (μ_1,μ_0) and (μ_0,μ_1), namely different equilibrium points with respect to their number or stability, we say that <u>μ_0 is a **bifurcation point**</u> for the system**.**



If for all equilibrium points the **real parts of eigenvalues*** of the linearized system are **non-zero** for $\mu=\mu'$, then the system is **structurally stable** at least in a small parameter interval around μ'.

* For area preserving systems, where centers exit with $\lambda_{1,2} = \pm ib$, structural stability is guaranteed when $\lambda \neq 0$

Bifurcations. Typical cases on plane

A. Saddle-Node bifurcation

$$\dot{x} = \mu - x^2,$$
$$\dot{y} = y$$

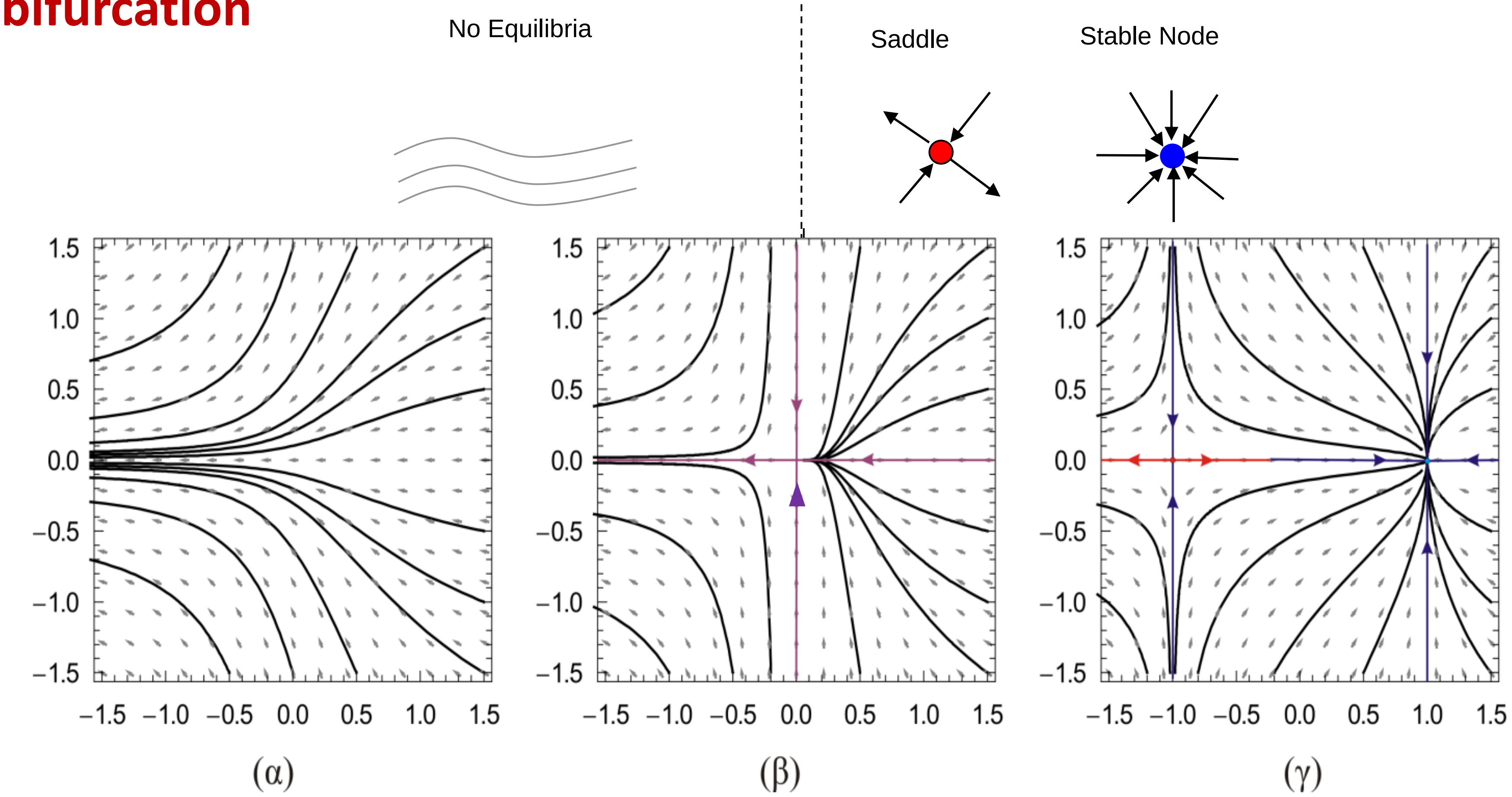
if $\mu < 0$ no EQPs

if $\mu = 0$ EQP=(0,0)

if $\mu > 0$

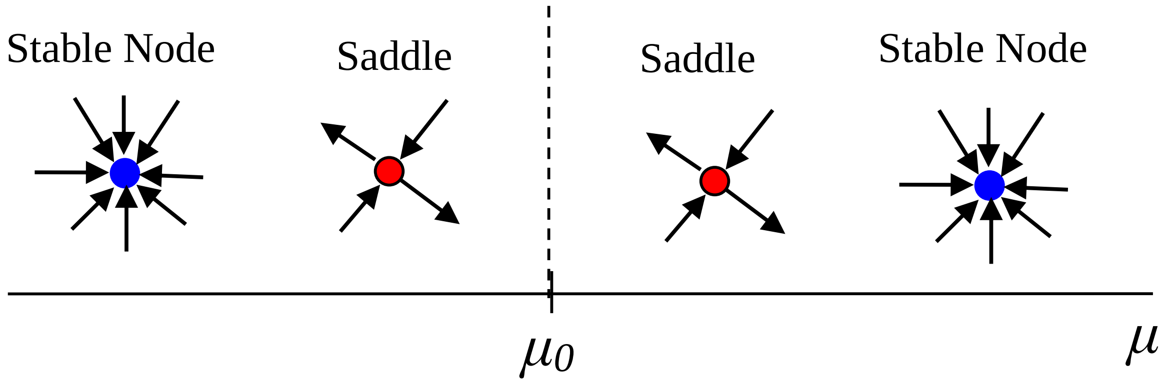
1stEQP $(-\sqrt{\mu}, 0)$

2ndEQP $(\sqrt{\mu}, 0)$



Bifurcations. Typical cases on plane

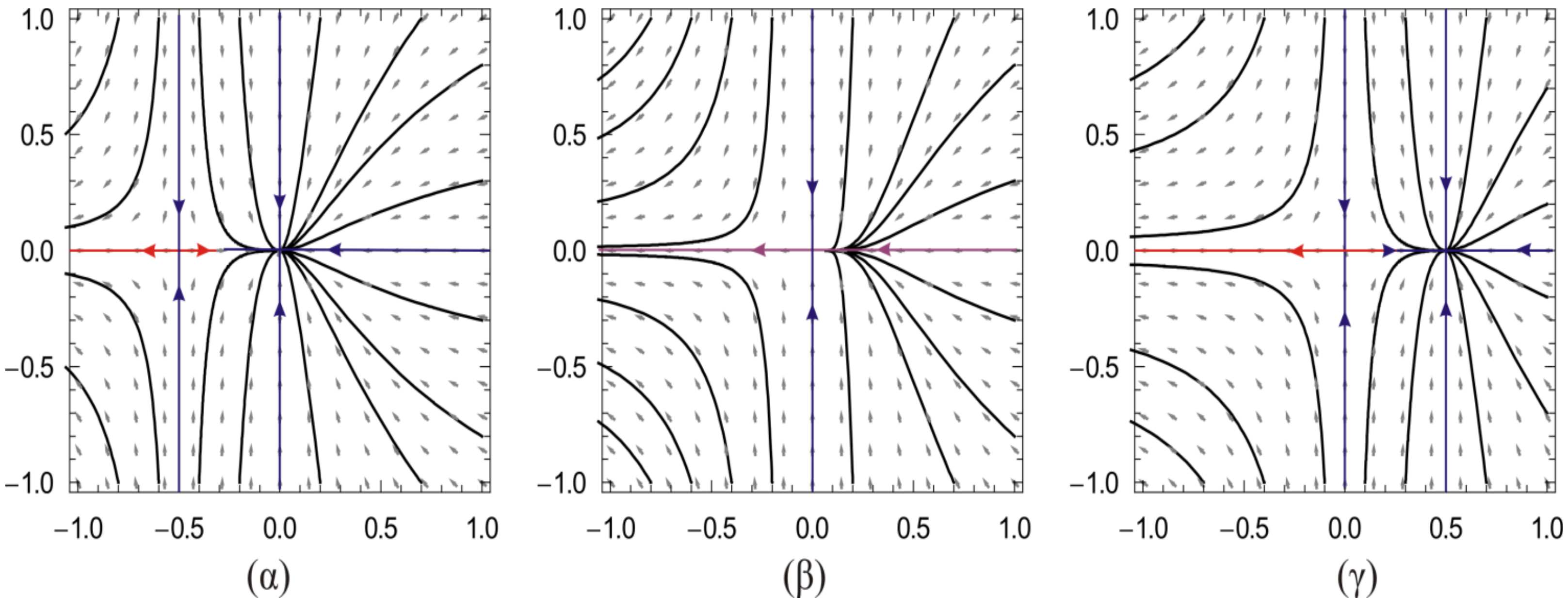
B. Transcritical bifurcation



$$\dot{x} = x(\mu - x),$$
$$\dot{y} = -y$$

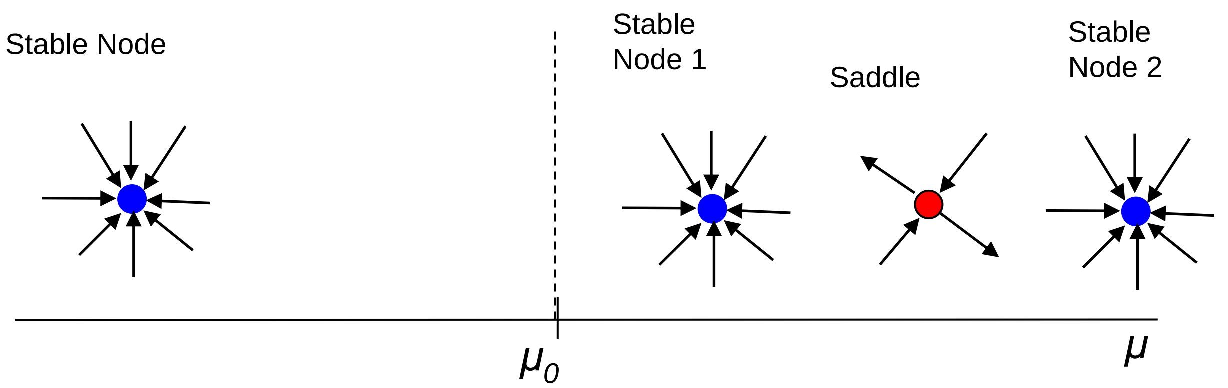
1stEQP $(0,0)$

2ndEQP $(\mu,0)$



Bifurcations. Typical cases on plane

C. Pitchfork bifurcation



$$\dot{x} = \mu x - x^3,$$

$$\dot{y} = -y$$

if $\mu < 0$ EQP=(0,0)

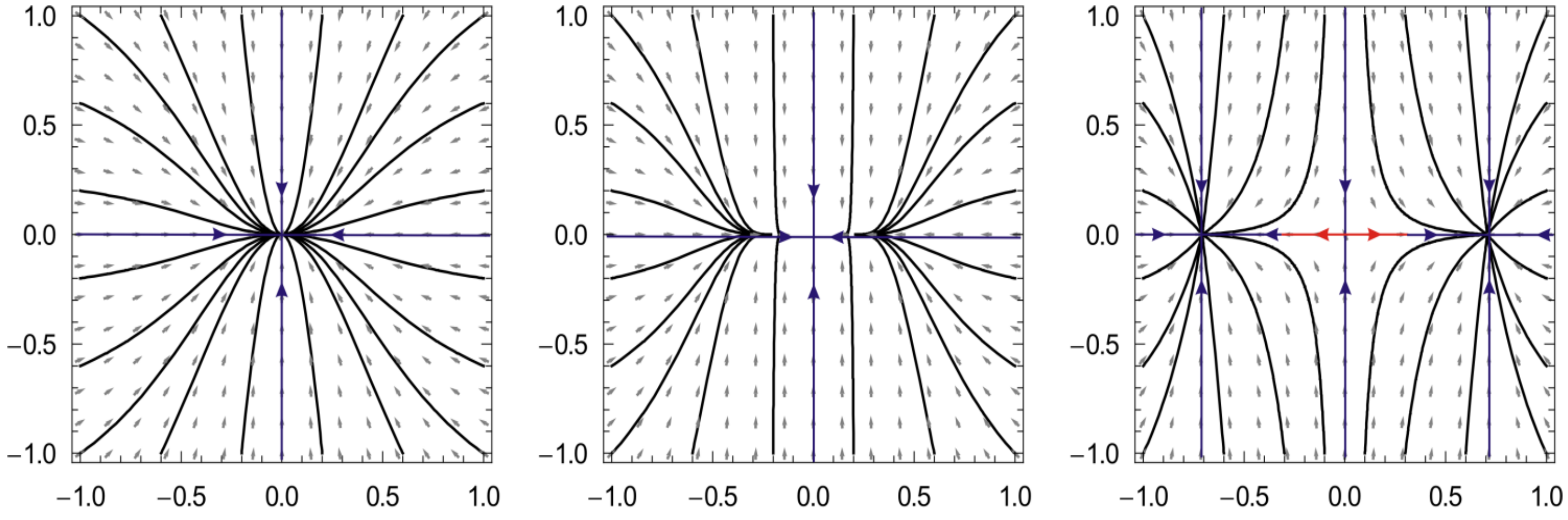
if $\mu = 0$ EQP=(0,0)

if $\mu > 0$

1stEQP $(-\sqrt{\mu}, 0)$

2ndEQP $(0, 0)$

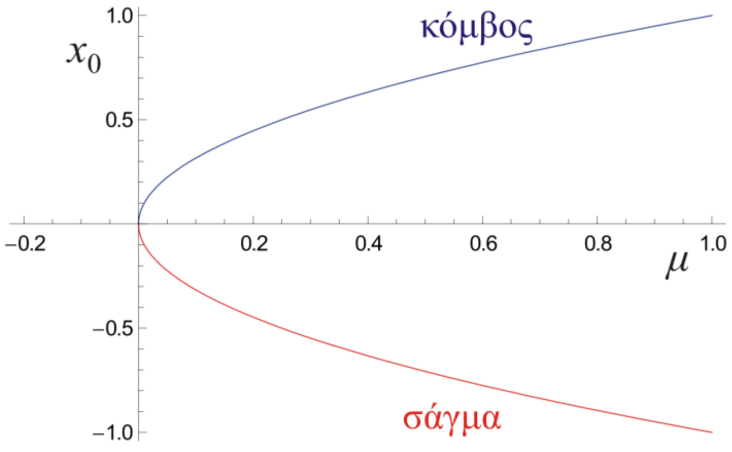
3rdEQP $(\sqrt{\mu}, 0)$



(α) (β) (γ)

Bifurcation Diagrams : *Parameter/position/stability*

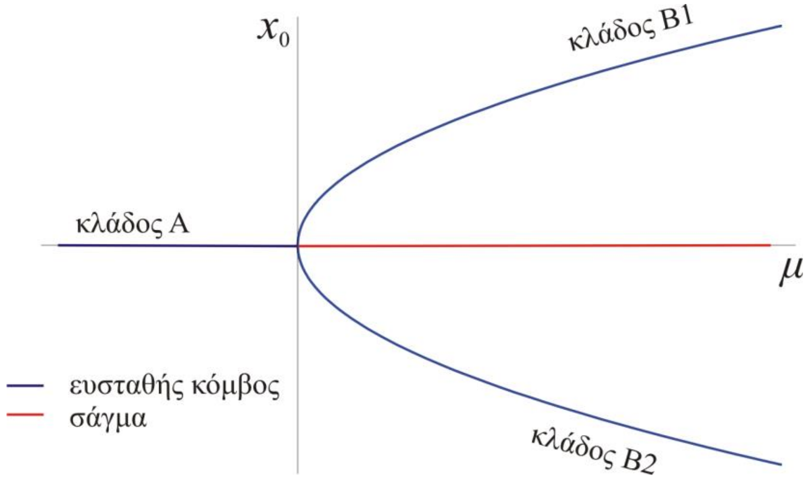
$$\dot{x} = \mu - x^2 \quad \dot{y} = y \quad \Rightarrow \quad \begin{aligned} (x_0, y_0) &= (\sqrt{\mu}, 0), \quad \mu > 0 \quad (stable\, node) \\ (x_0, y_0) &= (-\sqrt{\mu}, 0), \quad \mu > 0 \quad (saddle) \end{aligned}$$



$$(x_0, y_0) = (0,0) \quad \Rightarrow \quad \begin{aligned} \mu < 0 \quad (stable\, node) \\ \mu > 0 \quad (saddle) \end{aligned}$$

$$\dot{x} = \mu x - x^3 \quad \dot{y} = -y \quad \Rightarrow \quad \begin{aligned} (x_0, y_0) &= (\sqrt{\mu}, 0), \quad \mu > 0 \quad (stable\, node) \\ (x_0, y_0) &= (-\sqrt{\mu}, 0), \quad \mu > 0 \quad (stable\, node) \end{aligned}$$

(Pitchfork)

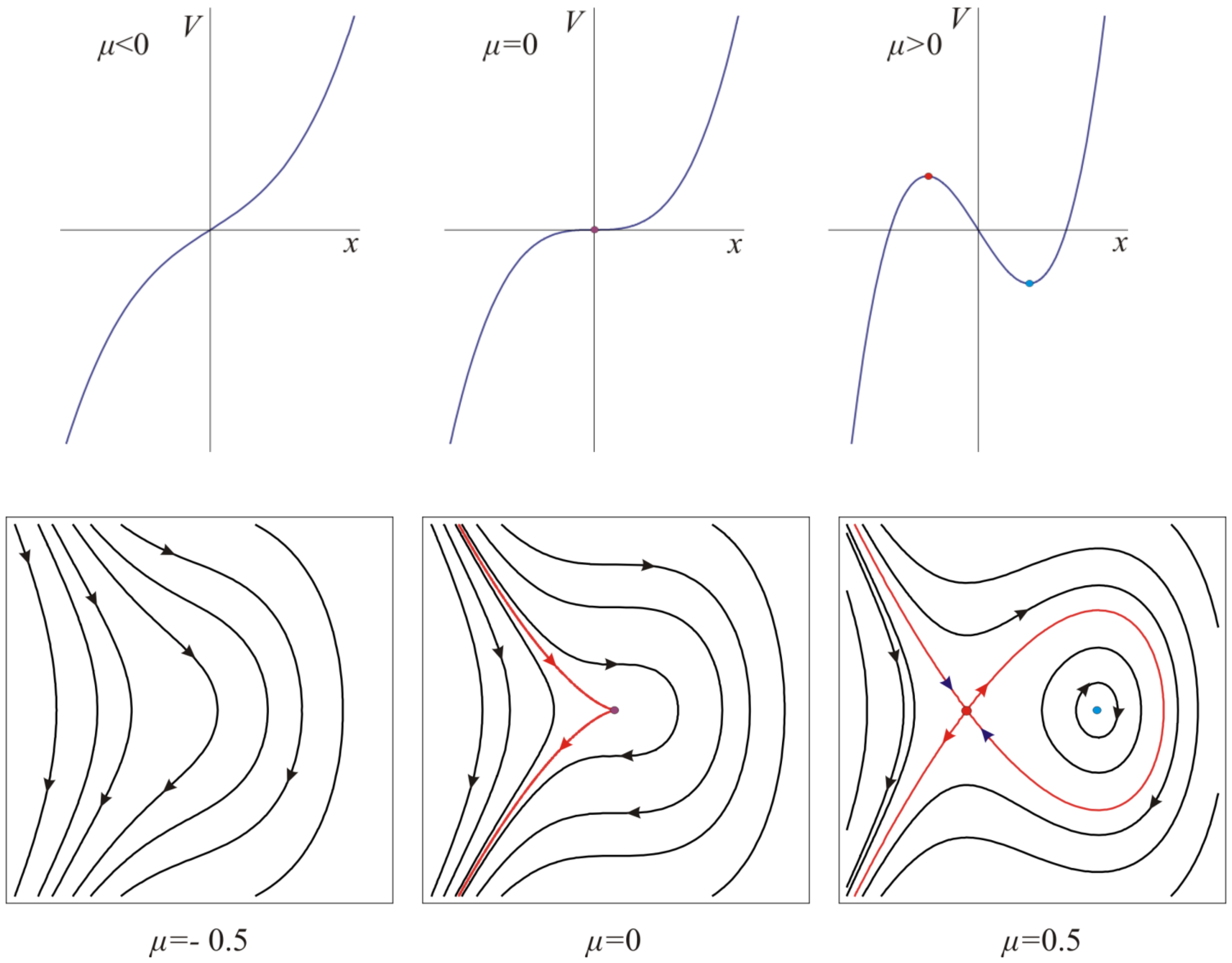


Bifurcations in mechanical systems

$$\ddot{x} = f(x), \qquad V(x) = -\int f(x)dx$$

- **fold bifurcation**

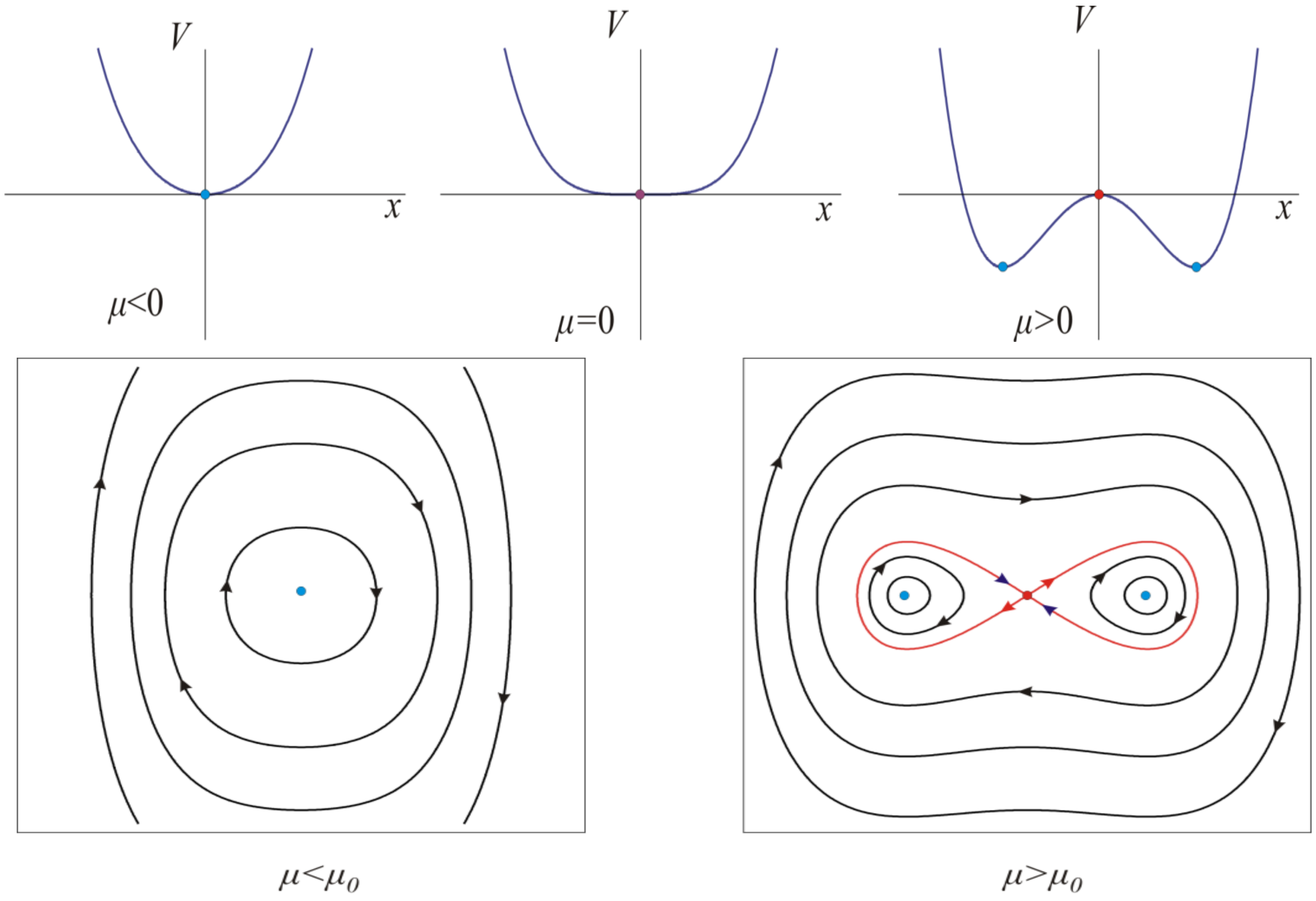
(saddle-center bifurcation)



Bifurcations in mechanical systems

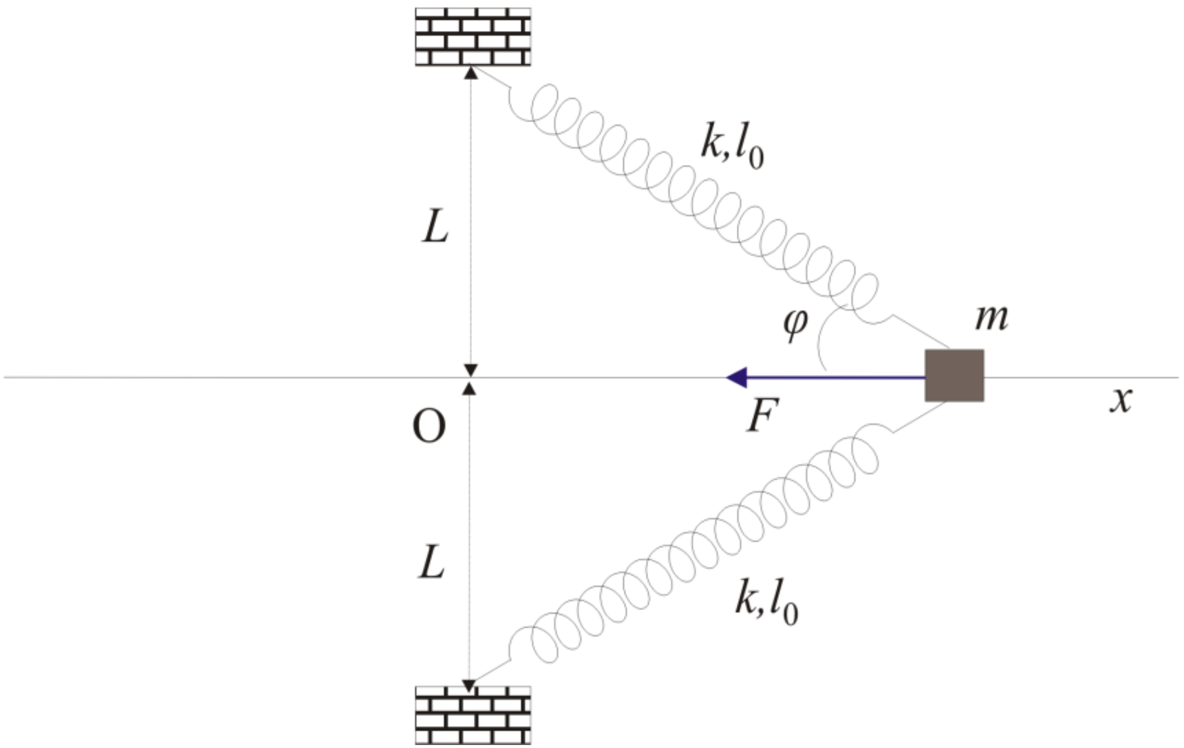
$$\ddot{x} = f(x), \qquad V(x) = -\int f(x)dx$$

- **Pitchfork bifurcation**



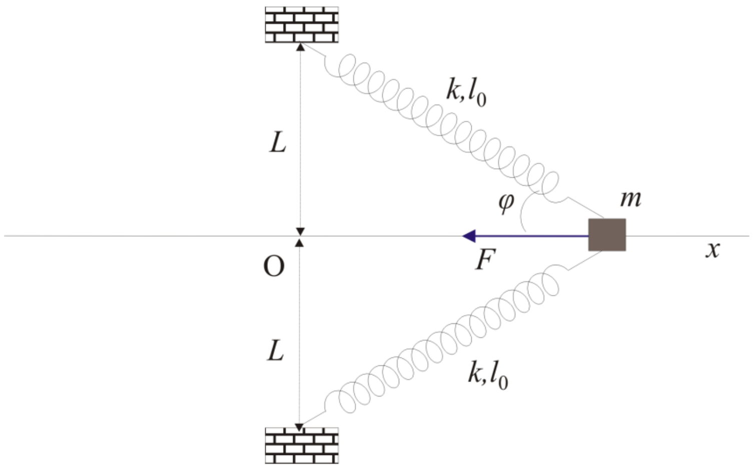
Structural stability and Bifurcations

example : a nonlinear oscillator

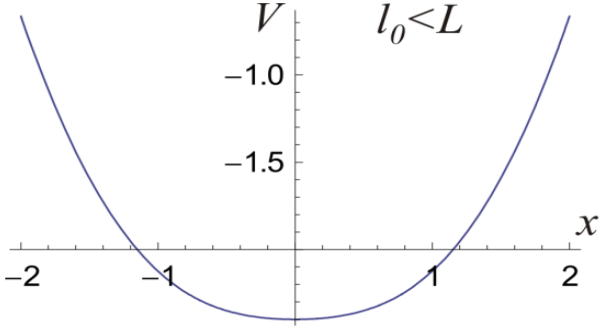


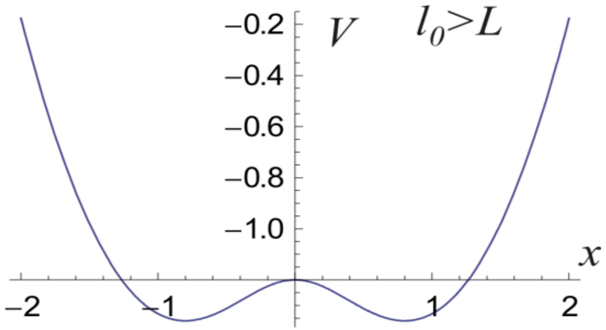
Structural stability and Bifurcations

example : a nonlinear oscillator



$$m\ddot{x} = -2k\left(1 - \frac{l_0}{\sqrt{x^2 + L^2}}\right)x, \qquad V = -\int F dx = k\left(x^2 + L^2 - 2l_0\sqrt{x^2 + L^2}\right)$$

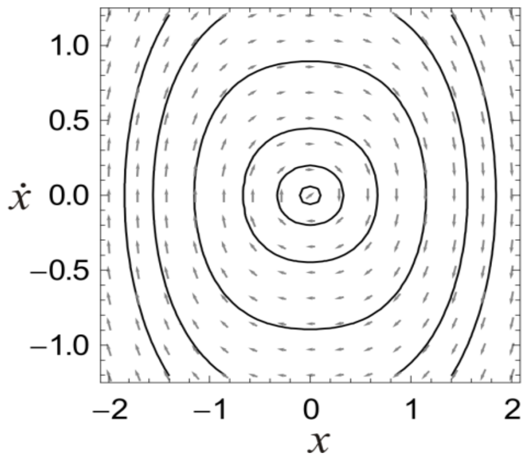


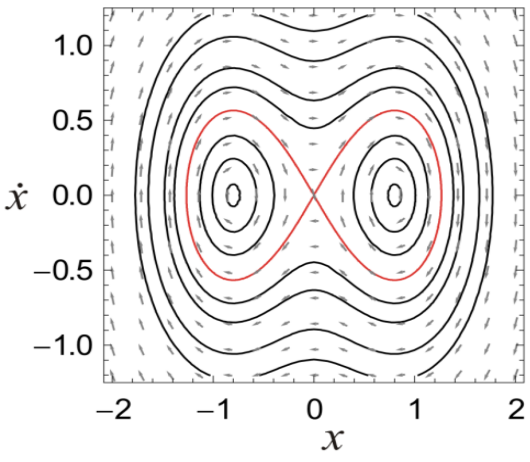


Equilibrium points and stability

$$x_{01} = 0, \qquad x_{02}, x_{03} = \pm\sqrt{l_0^2 - L^2}$$

$$\kappa(x_{01}) = 2k\left(1 - \frac{l_0}{L}\right), \quad \kappa(x_{02,03}) = 2k\left(1 - \frac{L}{l_0}\right)$$





if $l_0 < L$, $\kappa > 0$ x_{01} is stable (center)
if $l_0 = L$, $\kappa = 0$ critical stability (bifurcation)
if $l_0 > L$, $\kappa < 0$ x_{01} is unstable (saddle)
x_{02} , x_{03} are stable (centers)

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