Nonlinear Autonomous Dynamical systems of two dimensions

Part A

Nonlinear Autonomous Dynamical systems of two dimensions

$$\dot{x} = f(x,y), \qquad x(0) = x_0 \qquad \text{vector field}$$
$$\dot{y} = g(x,y), \qquad y(0) = y_0 \qquad \mathbf{F} = (f, g)$$

- *f*, *g* are continuous and differentiable in a subset of R^2
- Dimension=2 (x_1=*x*, x_2 =*y*) dynamical variables)
- Phase space= the plane (*x*,*y*)
- Trajectories = time-parameter curves (*x*(t),*y*(t)) (invariant sets)
- *div* ***F*** = 0 (area preserving) or *div* ***F*** ≠ 0 (dissipative)
- **Analytic solutions** : Generally, no solutions in closed form can be found, numerical solutions are used.
- In special cases some particular solutions can be obtained after a careful inspection of the ODEs. These solutions may be important for constructing the phase space.

math70.nb

Nonlinear Autonomous Dynamical systems of two dimensions

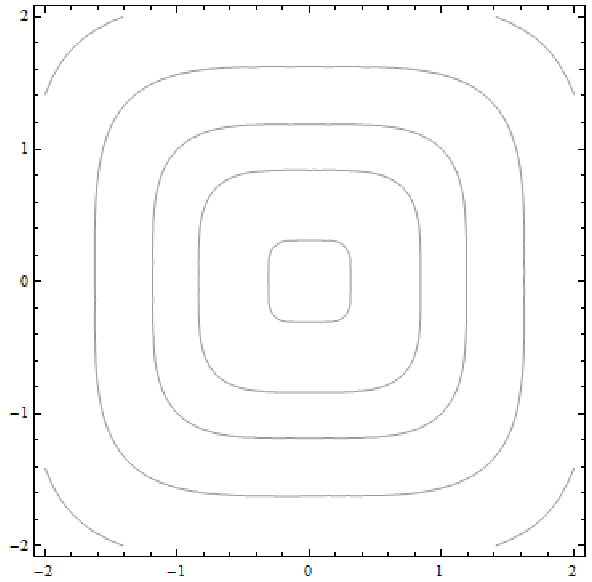
$$\dot{x} = f(x,y), \qquad x(0) = x_0$$
$$\dot{y} = g(x,y), \qquad y(0) = y_0$$

A function Φ(*x,y*) is an **integral of the system** if $\forall t \quad \Phi(x,y) = c, \qquad c = c(x_0, y_0)$

A first integral can be computed as solution of the 1st order ODE,

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

e.g. $\dot{x} = -y^3$, $\dot{y} = x^3 \quad \Rightarrow \quad x^4 + y^4 = c$

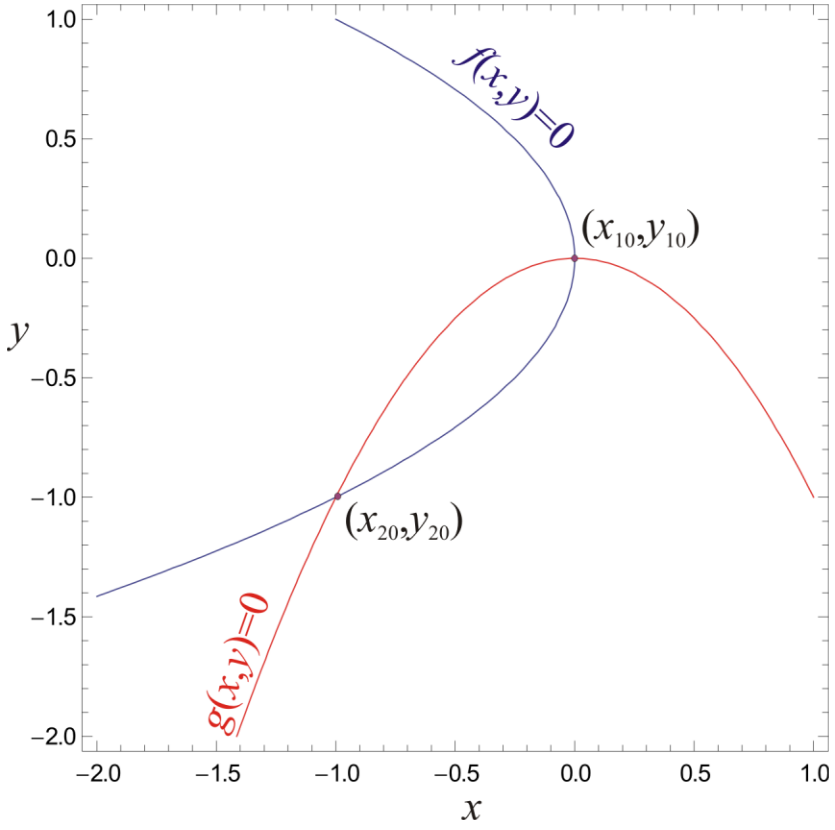


- A first integral presents the analytic closed form of the phase space trajectories
- If Φ(*x,y*) is known then the phase portrait is computed as contour plot of function Φ

Nonlinear Autonomous Dynamical systems of two dimensions

Equilibrium points : real solutions (x*,y*) of the system

$$\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned} \quad \Rightarrow \quad x = x^*, \quad y = y^*$$



(x_{10}, y_{10}), (x_{20}, y_{20}) two equilibrium solutions in the domain ***T***={(-2,1)×(-2,1)}

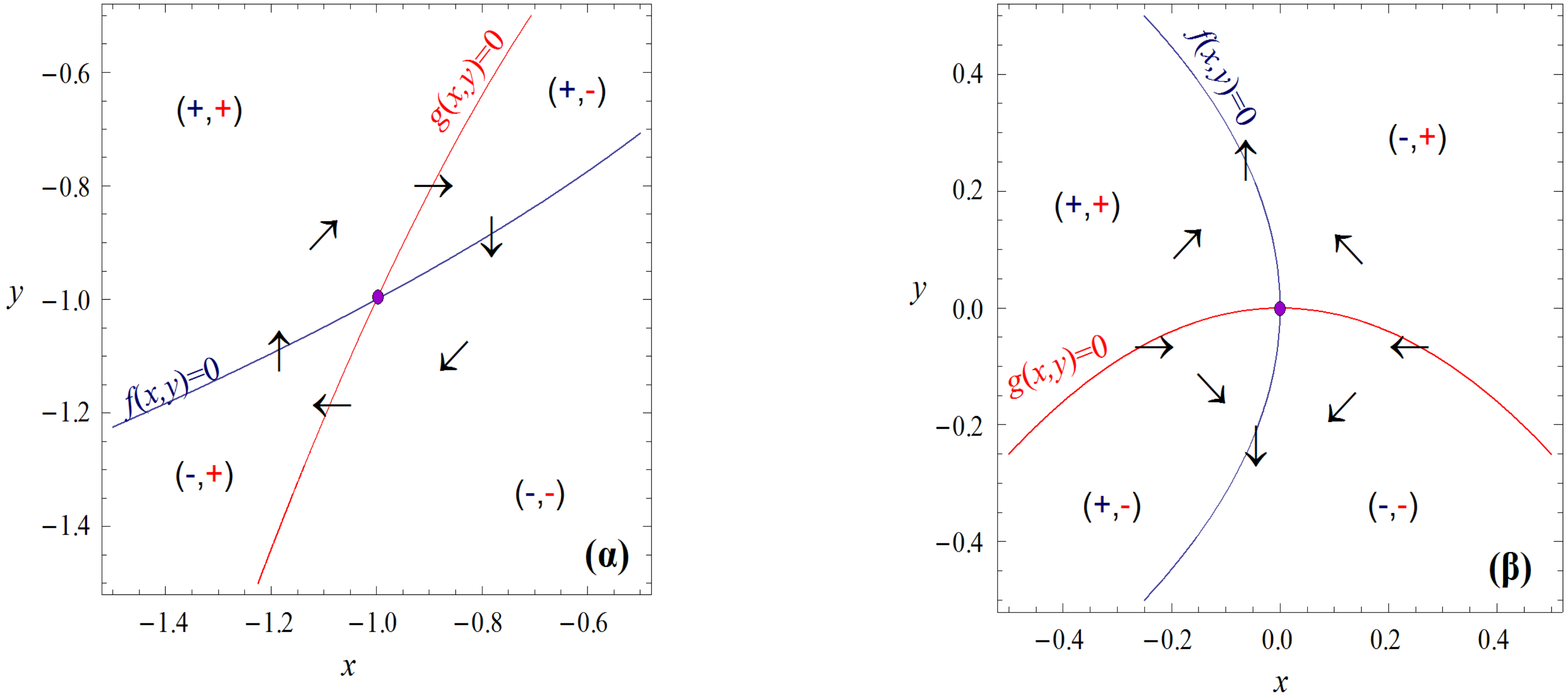
Mathematica note

If *f,g* polynomial functions NSolve finds all roots, else FindRoot should be applied in sufficient and proper subdomains of the plane domain ***T***.

Nonlinear Autonomous Dynamical systems of two dimensions

$$\dot{x} = f(x,y),\quad \dot{y} = g(x,y) \qquad \mathbf{f} = (f,g)$$

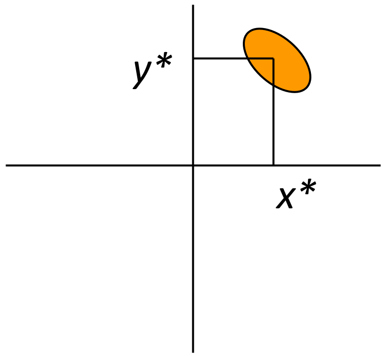
Flow around equilibria



Linearized system around equilibrium points

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

$$f(x,y) = 0$$
$$g(x,y) = 0 \quad \Rightarrow \quad x = x^*, \quad y = y^*$$



Initial conditions and solution near (*x**,*y**)

$$x = x^* + \delta x, \quad y = y^* + \delta y$$

$$ODEs \;\Rightarrow\; \begin{aligned} \dot{x}^* + \delta\dot{x} &= f(x^* + \delta x,\, y^* + \delta y) \\ \dot{y}^* + \delta\dot{y} &= g(x^* + \delta x,\, y^* + \delta y) \end{aligned} \;\xrightarrow[\dot{x}^*=\dot{y}^*=0,\quad f(x^*,y^*)=g(x^*,y^*)=0]{Taylor\ expansion\ up\ to\ 1st\ order}$$

$$\delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_0 \delta x + \left.\frac{\partial f}{\partial y}\right|_0 \delta y$$
$$\delta\dot{y} = \left.\frac{\partial g}{\partial x}\right|_0 \delta x + \left.\frac{\partial g}{\partial y}\right|_0 \delta y \quad \rightarrow$$

$$\dot{x} = \left.\frac{\partial f}{\partial x}\right|_* x + \left.\frac{\partial f}{\partial x}\right|_* y$$
$$\dot{y} = \left.\frac{\partial g}{\partial x}\right|_* x + \left.\frac{\partial g}{\partial x}\right|_* y \quad , \qquad \mathbf{A}(x^*,y^*) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*,y^*)}$$

Linearized system around equilibrium (x*,y*)

Linearized system around equilibrium points

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbf{A}(x^*, y^*)\begin{pmatrix}x\\ y\end{pmatrix} \quad , \qquad \mathbf{A}(x^*, y^*) = \begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}_{(x^*, y^*)}$$

λ_1, λ_2 eigenvalues of **A**

Linear stability or **type** of equilibrium points

$\lambda_1 \neq \lambda_2$, $\lambda_i \in$ R, $\lambda_1 > 0$, $\lambda_2 > 0$: unstable node

$\lambda_1 \neq \lambda_2$, $\lambda_i \in$ R, $\lambda_1 < 0$, $\lambda_2 < 0$: stable node

$\lambda_1 \neq \lambda_2$, $\lambda_i \in$ R, $\lambda_1 < 0$, $\lambda_2 > 0$: saddle

$\lambda_1 = \lambda_2$, $\lambda_i \in$ R, $\lambda_1 > 0$, $\lambda_2 > 0$: improper unstable node

(one eigenvector)

$\lambda_1 = \lambda_2$, $\lambda_i \in$ R, $\lambda_1 < 0$, $\lambda_2 < 0$: improper stable node

$\lambda_1 = a + bi$, $\lambda_2 = a - bi$, $a > 0$: unstable focus

$\lambda_1 = a + bi$, $\lambda_2 = a - bi$, $a < 0$: stable focus

$\lambda_1 = +bi$, $\lambda_2 = -bi$, $a = 0$: center

math71.nb

For mechanical systems of one degree of freedom

$$\dot{x} = y, \quad \dot{y} = f(x)$$

the matrix of the linearized system is

$$A = \begin{pmatrix}0 & 1\\ -k & 0\end{pmatrix}, \quad k = -\left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x^*}$$

with eigenvalues $\lambda_{1,2} = \pm\sqrt{-k}$

So $k>0$: center, $k<0$: saddle

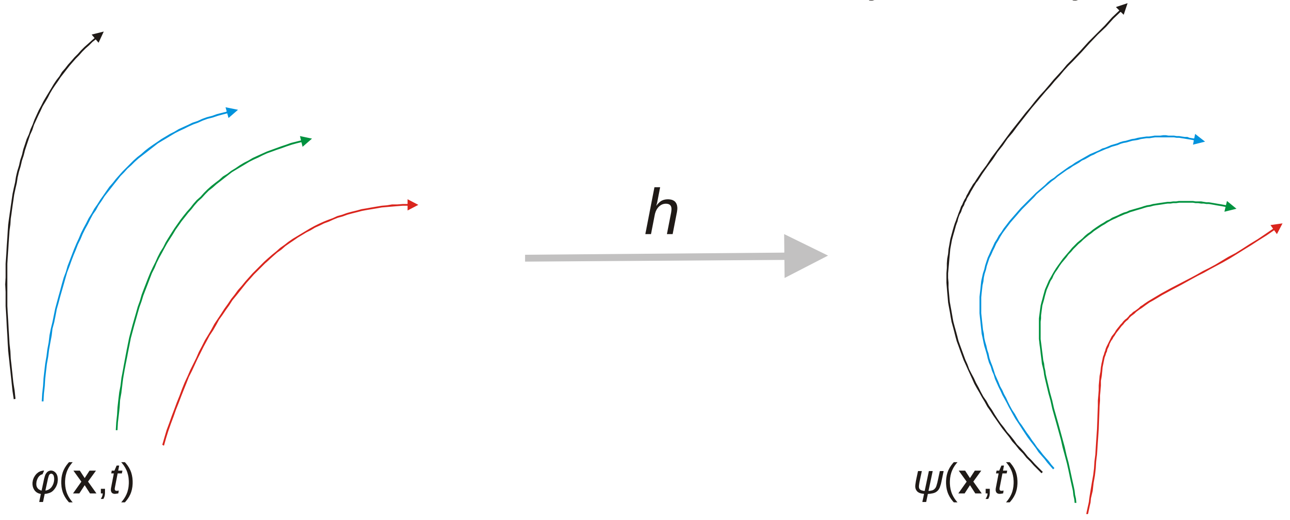
Topological conjugacy/ equivalence

A function $h: A \rightarrow B$ between two topological spaces is called a **homeomorphism** or **topological isomorphism** if :

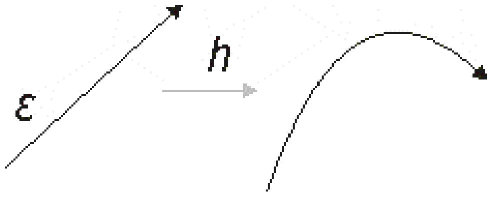
i) h is 1-1, ii) h is continuous, iii) h^{-1} is continuous.

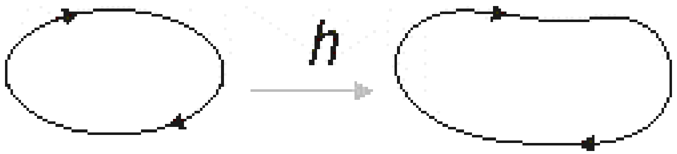
Dynamical system A

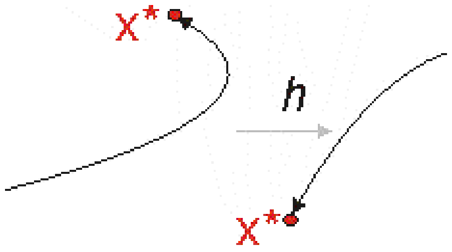
Dynamical system B



If there exist an h between the phase spaces of two dynamical systems A and B, the dynamical systems are called **topologically conjugate** and the phase spaces of A and B, when the direction of time is preserved, are **topologically equivalent**







From linear to nonlinear system

The Hartman-Grobman Theorem (a)

- The topology of trajectories near an equilibrium point **x*** of a nonlinear system is equivalent with the topology of the linearized system **if all eigenvalues** λ of the linearized matrix at **x* have nonzero real part**.

i.e. there exist a homeomorphism *h* between the trajectories of the nonlinear system and the linear system and $h(\mathbf{x}^*)=\mathbf{0}$.

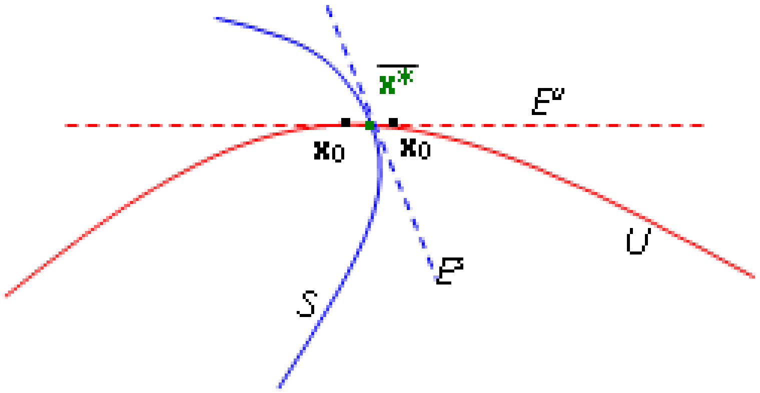
- Any equilibrium point with $Re(\lambda_i)\neq 0$ is called **hyperbolic point**
- In a planar dynamical system (dim=2), all equilibria (presented in previous slides) are hyperbolic <u>except centers.</u> All “critical” cases with $\lambda=0$, which has not been presented are <u>non-hyperbolic.</u>

From linear to nonlinear system

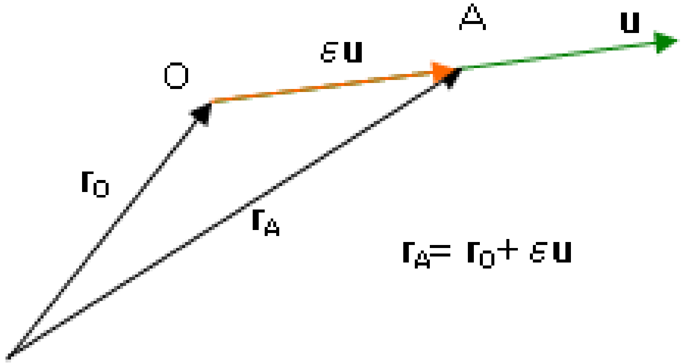
The Hartman-Grobman Theorem (b)

- If λ_i are real and nonzero, there are asymptotic solutions (**invariant manifolds**) that are tangent to the linear subspaces of the linearized system at the equilibrium point. If the subspace is stable or unstable the invariant manifold is also stable (S) or unstable (U), respectively.

$$\mathbf{S} = \{\mathbf{x}_0 \mid \quad \mathbf{x}(t,\mathbf{x}_0) \to \mathbf{x}^* \quad \text{for} \quad t \to \infty \quad \text{and} \quad \mathbf{x}(t,\mathbf{x}_0) \in \mathbf{S}, \forall t \geq 0\}$$
$$\mathbf{U} = \{\mathbf{x}_0 \mid \quad \mathbf{x}(t,\mathbf{x}_0) \to \mathbf{x}^* \quad \text{for} \quad t \to -\infty \quad \text{and} \quad \mathbf{x}(t,\mathbf{x}_0) \in \mathbf{S}, \forall t \leq 0\}$$



Initial conditions that approximately belong to the invariant manifold solutions



$$x(0) = x^* \pm \varepsilon u_x, \quad y(0) = y^* \pm \varepsilon u_y,$$
$$\text{where} \quad |\varepsilon u_x| << 1, \quad |\varepsilon u_y| << 1$$

- If the manifold is unstable we integrate the system for $t \to \infty$
- If the manifold is stable we integrate the system for $t \to -\infty$

math72.nb

Studying 2D Nonlinear systems: phase portraits and dynamics

A. If a first integral $\Phi(x,y)$ **is known**

- We compute equilibrium points (x^*,y^*)
- Invariant manifolds correspond to contours with value equal to $\Phi(x^*,y^*)$ (centers are excluded)
- Phase space is computed by contours of Φ.

math75.nb

- The dynamical system dx/dt=f(x,y), dy/dt=g(x,y) is called **reversible** if it is invariant under the transformation $t \to -t$, $y \to -y$, *i.e.*

$$f(x,-y) = -f(x,y), \quad g(x,-y) = g(x,y)$$

All systems
$$\dot{x} = y, \quad \dot{y} = f(x)$$
are reversible

Theorem : If an equilibrium point of a reversible system is center (according to the linearized system) then in the nonlinear system the equilibrium point is surrounded by closed trajectories in phase space.

Example:
(see math75)

$$\dot{x} = -x - y^2, \quad \dot{y} = y + x^2 \quad \xrightarrow{x=\frac{\sqrt{2}}{2}(X-Y),\quad y=\frac{\sqrt{2}}{2}(X+Y)} \quad \dot{X} = Y - \sqrt{2}XY, \quad \dot{Y} = X + \frac{\sqrt{2}}{2}(X^2+Y^2)$$

(reversible)

Studying 2D Nonlinear systems: phase portraits and dynamics

B. If a first integral $\Phi(x,y)$ **is not known**, then

- We compute equilibrium points (x^*,y^*) and their linear stability.
- We compute by numerical integration and plot the corresponding invariant manifolds – Initial conditions are approximated from the linear subspaces close to the (x^*,y^*).
- We construct phase space by computing trajectories for various initial conditions numerically.

math76.nb

Application. Lotka-Volterra systems : Interaction of species (prey and predator)

LotkaVolteraFishes.pdf , Fishes_model1.nb, Fishes_model2.nb