Linear autonomous systems of 2-dimensions

$$\dot{x} = a\,x + b\,y$$
$$\dot{y} = c\,x + d\,y$$

$a, b, c, d \in R$ (constants)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}\,, \qquad \det \mathbf{A} \neq 0, \qquad \mathbf{x} = (x, y) \in R^2$$

Linear autonomous systems of 2-dimensions

- All linear systems with constant coefficients have solutions in closed form, expressed with typical functions, and related with the **eigenvalues** and **eigenvectors** of the matrix **A**

cmath60.nb

- There exist only one equilibrium position at (0,0)

- Vector field and area preservation/dissipation

$$\mathbf{f} = (f_x, f_y) \rightarrow \begin{array}{l} f_x = ax + by \\ f_y = cx + dy \end{array}, \qquad \mathit{div}\mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = a + d = \mathrm{trace}\mathbf{A}$$

- The phase space trajectories or an integral of motion $F(x,y)=const$. can be derived from the solution of the 1st order ODE

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by} \qquad (1)$$

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(1) is an homogeneous equation and can is solved by setting $z=y/x$

Linear autonomous systems of 2-dimensions
Solutions for particular cases

A. Real and distinct eigenvalues, $\lambda_1 \neq \lambda_2$, $\lambda_i \in \mathrm{R}$

Existence of two eigenvectors $\mathbf{u}_1=(u_{11},u_{12}) \in \mathrm{R}^2$, $\mathbf{u}_2=(u_{21},u_{22}) \in \mathrm{R}^2$

Solution

$$x(t) = c_1 u_{11} e^{\lambda_1 t} + c_2 u_{21} e^{\lambda_2 t}$$
$$y(t) = c_1 u_{12} e^{\lambda_1 t} + c_2 u_{22} e^{\lambda_2 t}$$

$$\mathbf{x}(t) = \sum_{i=1}^{2} c_i \mathbf{u}_i e^{\lambda_i t}$$

Special cases

- $c_1 = 0$ $\quad x(t) = c_2 u_{21} e^{\lambda_2 t}, \quad y(t) = c_2 u_{22} e^{\lambda_2 t} \quad \Rightarrow \quad y = \left(\dfrac{u_{22}}{u_{21}}\right) x$
- $c_2 = 0$ $\quad x(t) = c_1 u_{11} e^{\lambda_1 t}, \quad y(t) = c_1 u_{12} e^{\lambda_1 t} \quad \Rightarrow \quad y = \left(\dfrac{u_{12}}{u_{11}}\right) x$

The trajectory solutions $y=kx$ are called invariant linear subspaces

Linear autonomous systems of 2-dimensions
Solutions for particular cases

A. Real and distinct eigenvalues, $\lambda_1 \neq \lambda_2$, $\lambda_\iota \in$ R

Invariant linear subspace solution for $\lambda \in$ R and $u=(u_1,u_2)$

$$x(t) = c(x_0, y_0)u_x e^{\lambda t}, \qquad y(t) = c(x_0, y_0)u_y e^{\lambda t} \quad \Rightarrow \quad y = \left(\frac{u_2}{u_1}\right)x$$

$$\text{if} \quad \lambda < 0 \quad \Rightarrow \quad \lim_{t\to\infty}(x, y) = (0, 0)$$

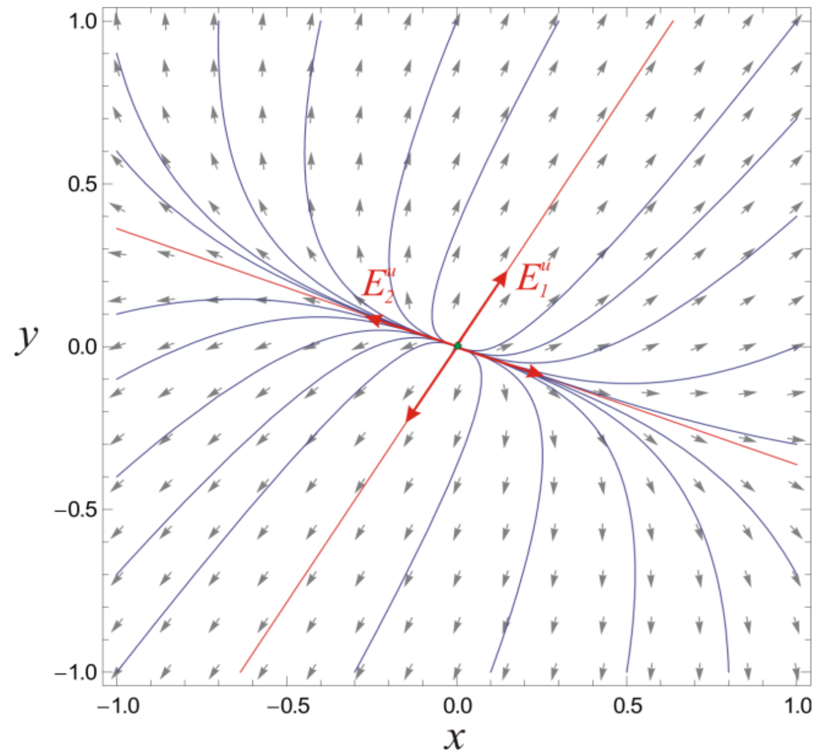
Stable subspace (E^s)

$$\text{if} \quad \lambda > 0 \quad \Rightarrow \quad \lim_{t\to\infty}(x, y) = (\pm\infty, \pm\infty)$$

Unstable subspace (E^u)

Linear autonomous systems of 2-dimensions
Solutions for particular cases

A. Real and distinct eigenvalues, $\lambda_1 \neq \lambda_2$, $\lambda_i \in$ R



A1. $\lambda_1>0$ and $\lambda_2>0$

two unstable linear subspaces =

Unstable node

A2. $\lambda_1<0$ and $\lambda_2<0$

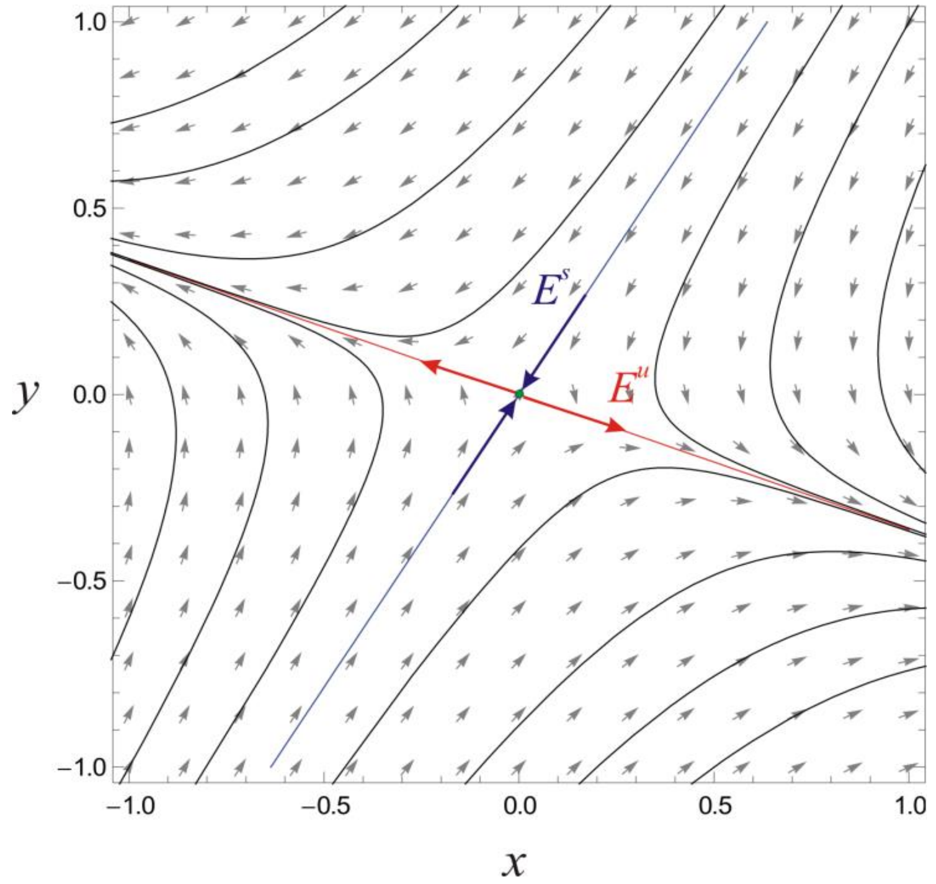
two stable linear subspaces = **Stable node**

Linear autonomous systems of 2-dimensions
Solutions for particular cases

A. Real and distinct eigenvalues, $\lambda_1 \neq \lambda_2$, $\lambda_\iota \in$ R

A3. $\lambda_1 > 0$ and $\lambda_2 < 0$

one unstable and
one stable subspace = **saddle**



Linear autonomous systems of 2-dimensions
Solutions for particular cases

B. Equal eigenvalues, $\lambda_1=\lambda_2=\lambda$, $\lambda \in$ R

b1) If there exist two eigenvectors we obtain a stable or an unstable node

b2) If there exist one eigenvector **u**=(u_1,u_2) then

Solution

$$x(t)=(c_1+c_2 t)u_1 e^{\lambda t}+c_2 w_1 e^{\lambda t}$$
$$y(t)=(c_1+c_2 t)u_2 e^{\lambda t}+c_2 w_2 e^{\lambda t}$$

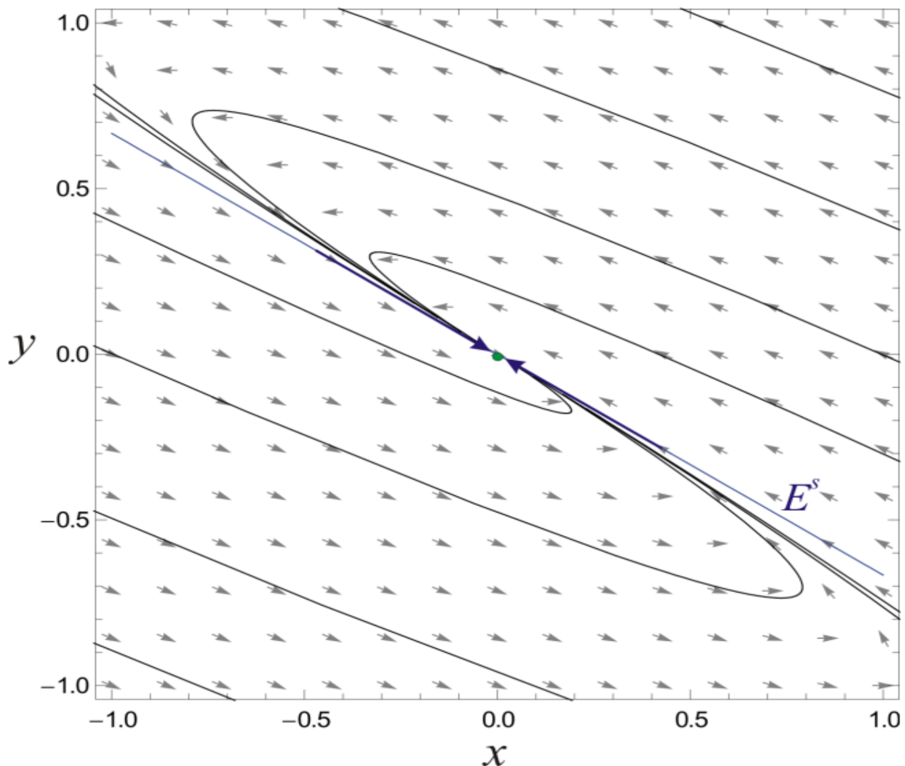
$$\mathbf{x}(t)=(c_1+c_2 t)\mathbf{u}e^{\lambda t}+c_2\mathbf{w}e^{\lambda t}$$

Linear Subspace for $c_2=0$

$$y=\left(\frac{u_2}{u_1}\right)x$$

if λ<0 : stable subspace

if λ>0 : unstable subspace



improper node

Linear autonomous systems of 2-dimensions
Solutions for particular cases

C. Complex eigenvalues : $\lambda_1 = a + ib$, $\lambda_2 = a - ib$ $\quad b \neq 0$

Complex eigenvectors $u_1 + iv_1$ και $u_2 + iv_2$

Solution
$$x(t) = c_1 e^{at}(u_1 \cos bt + v_1 \sin bt) + c_2 e^{at}(-u_1 \cos bt + v_1 \sin bt)$$
$$y(t) = c_1 e^{at}(u_2 \cos bt + v_2 \sin bt) + c_2 e^{at}(-u_2 \cos bt + v_2 \sin bt)$$

Periodic terms ($T=2\pi/b$) with amplitude $A \sim e^{at}$

- $a>0$: oscillations of $x(t)$ and $y(t)$ with exponentially increasing amplitude
- $a<0$: damped oscillations of $x(t)$ and $y(t)$
- $a=0$: periodic oscillations of $x(t)$ and $y(t)$

Linear autonomous systems of 2-dimensions

Solutions for particular cases

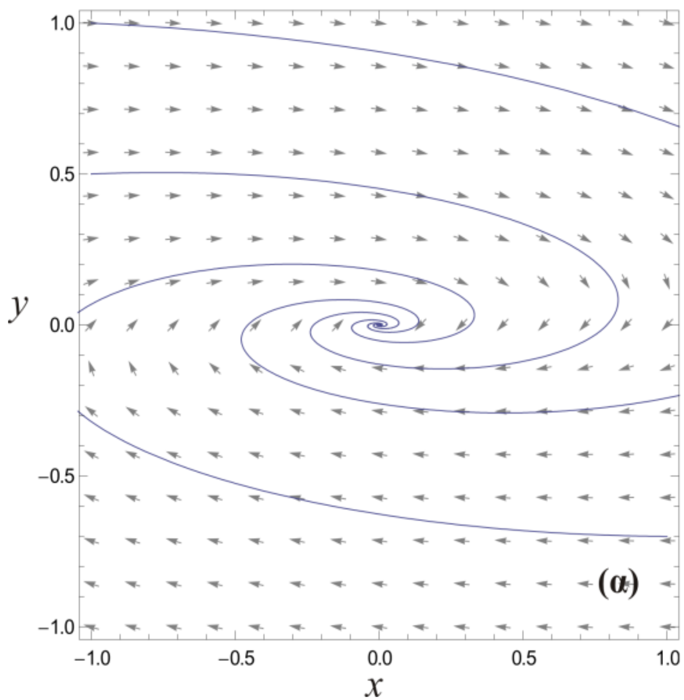
C. Complex eigenvalues : $\lambda_1 = a + ib$, $\lambda_2 = a - ib$, $b \neq 0$

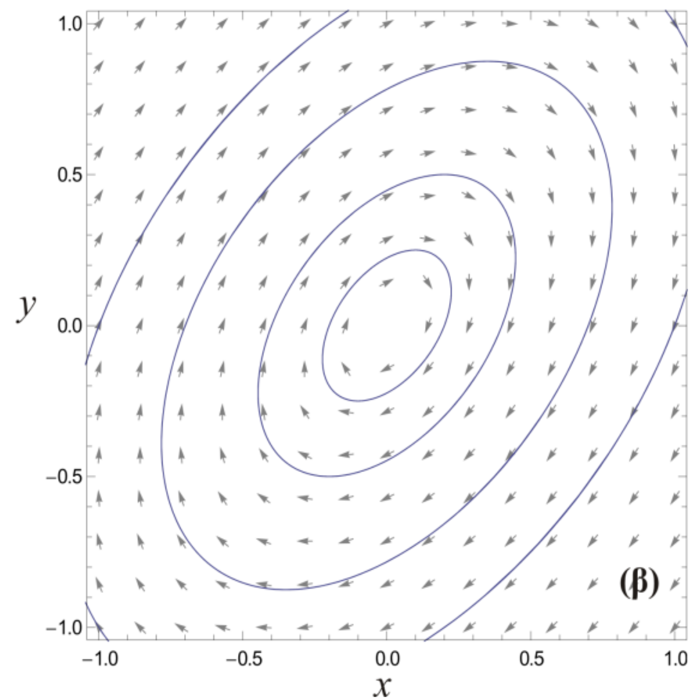
if $A_{12} > 0 \Rightarrow$
clockwise evolution

$a > 0$: **unstable focus**

$a < 0$: **stable focus** (sink)

$a = 0$: **center**





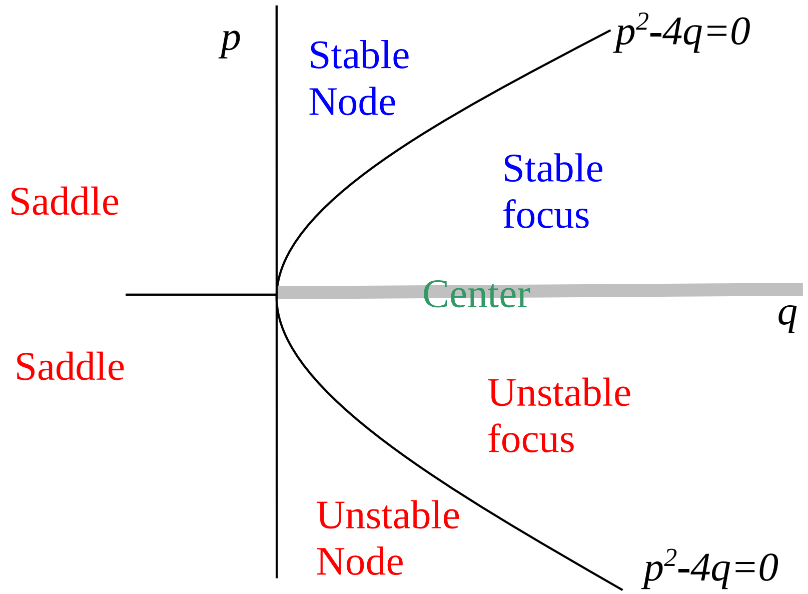
Το γραμμικό σύστημα ονομάζεται **υπερβολικό** αν το πραγματικό μέρος όλων των ιδιοτιμών του πίνακα Α είναι διάφορο του μηδενός

A linear system is called hyperbolic if the real part of all eigenvalues is non-zero.

Linear autonomous systems of 2-dimensions

Parametric space of Stability

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad p = -\mathrm{trace}\mathbf{A} = -(a+d) \;, \quad q = \det\mathbf{A} = ad - bc$$



math64.nb

math65.nb

* for p=0 the system is area preserving (includes centers (q>0) and saddles (q<0))

Canonical form of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad or \quad \dot{\mathbf{X}} = \mathbf{AX}$$

$$\dot{\mathbf{X}} = \mathbf{AX} \quad \rightarrow \quad \mathbf{X} = \mathbf{PY} \quad \rightarrow \quad \dot{\mathbf{Y}} = \mathbf{BY} \quad , \quad \mathbf{Y} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$$ (Similarity transformation)

$\lambda_1 \neq \lambda_2 \, , \lambda_\iota \in R$ | $\lambda_1 = \lambda_2 = \lambda, \; \lambda_\iota \in R$ | $\lambda_{1,2} = a + ib \in C \backslash R$

$$\mathbf{P} = \begin{pmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{pmatrix} \quad or \quad \begin{pmatrix} u_1 & w_1 \\ u_2 & w_2 \end{pmatrix}^* \quad or \quad \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad or \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad or \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

* case for one eigenvector, and $\mathbf{w}=(w_1,w_2)$ is one solution of the equation $(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{w}=0$

$$\dot{x}' = \lambda_1 x' \qquad \dot{x}' = \lambda x' + y' \qquad \dot{x}' = ax' + by'$$

$$\dot{y}' = \lambda_2 y' \qquad \dot{y}' = \lambda y' \qquad \dot{y}' = -bx' + ay'$$

Άσκηση L1 Σχεδιάστε το φασικό διάγραμμα παρακάτω γραμμικού συστήματος και γράψτε το σε κανονική μορφή

(a)

$$\dot{x} = -\frac{1}{2}x + y$$
$$\dot{y} = -x - 3y$$

(b)

$$\dot{x} = -x + y$$
$$\dot{y} = -2x - 2y$$

(c)

$$\dot{x} = -x + y$$
$$\dot{y} = -x - 3y$$

* επιλέξτε **ένα σύστημα** και αποστείλετε ένα **pdf** με όνομα αρχείου που ξεκινάει με το κωδικό της άσκησης στη συνέχεια το επίθετό σας (με λατινικούς χαρακτήρες) στο email **voyatzis@physics.auth.gr**