Autonomous Mechanical systems of 1 degree of freedom

$$\ddot{x} = f(x, \dot{x}), \quad x \in \mathbb{R}$$

x : position
y : velocity
f : force function

$$\dot{x} = y \qquad \dot{y} = f(x, y)$$

$$\left(\eta' \quad \begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) \end{matrix} \right)$$

Solution : $x = x(t; c_1, c_2), \quad y = y(t; c_1, c_2), \qquad c_i = c_i(x_0, y_0)$

Integral of motion : $F(x, y) = c, \qquad c = c(x_0, y_0)$

Analytic solutions / integrals ?

Autonomous Mechanical systems of 1 degree of freedom

Example 1.

$$\ddot{x} = x \cdot \dot{x} \quad \acute{\eta} \qquad \dot{x} = y, \quad \dot{y} = xy$$

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \frac{dy}{dx}y$$

$$y\frac{dy}{dx} = xy \quad \overset{y\neq 0}{\Rightarrow} \quad dy = xdx \quad \overset{\int}{\Rightarrow} \quad y = \frac{1}{2}(x^2 + c_1) \qquad \Rightarrow \qquad F(x,y) = 2y - x^2 = c_1$$

Integral of motion

We consider $c_1 > 0$

$$y = \frac{dx}{dt} = \frac{1}{2}(x^2 + c_1) \quad \Rightarrow \quad \frac{dx}{x^2 + c_1} = 2dt \quad \overset{\int}{\Rightarrow} \quad \tan^{-1}(x/\sqrt{c_1}) = 2\sqrt{c_1}t + c_2$$

or

$$x(t) = \sqrt{c_1}\tan(2\sqrt{c_1}t + c_2) \quad \text{and} \qquad y(t) = \frac{dx}{dt} = \frac{2c_1}{\cos^2(2\sqrt{c_1}t + c_2)} \qquad (c_1 > 0)$$

.

$$c_1 = \frac{x_0^2}{\tan^2 c_2}, \quad c_2 = \sin^{-1}\sqrt{\frac{2x_0^2}{y_0}}$$

Autonomous Mechanical systems of 1 degree of freedom

Example 2.

$$\ddot{x} = x(1+\dot{x})$$

$$\ddot{x} = y dy / dx$$

$$\frac{y dy}{1+y} = x dx \quad \overset{\int}{\Rightarrow} \quad y - \ln|1+y| = \frac{1}{2}x^2 + c_1$$

$$\Rightarrow \quad y - \ln|1+y| - \frac{1}{2}x^2 = c_1$$

Integral of motion

$$\Rightarrow \quad \frac{dx}{dt} = y = \ldots.?$$

Autonomous Mechanical systems of 1 degree of freedom

$$\ddot{x} = f(x, \dot{x}), \quad \text{or} \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x, y) \end{aligned}$$

General features

- *Dimensions* : 2 (dynamical variable *x* and *y*)
- *Phase Space* : The plane position-velocity (*x,y*)
- *Autonomous system* : phase space trajectories do not intersect on the plane (*x,y*)
- *Area Preservation* $div\,\vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \frac{\partial y}{\partial x} + \frac{\partial f(x,y)}{\partial y} = \frac{\partial f(x,y)}{\partial y}$

$$div\,\vec{f} = 0 \quad \Leftrightarrow \quad \frac{\partial f}{\partial y} = 0$$

conservative systems

Part A. Conservative Systems

$$\ddot{x} = f(x) \qquad or \qquad \dot{x} = y, \quad \dot{y} = f(x)$$

- The **energy integral** $E = \frac{1}{2} y^2 + U(x) \;= E_0 = \sigma\tau\alpha\theta., \qquad E_0 = E(x_0, y_0)$

  where *U=U(x)* the potential function $U(x) = -\int f(x)\,dx$

- **Time Evolution – analytic solution**

  $x(0) = x_0, \quad E = E_0$ (initial conditions)

$$t = \int_{\chi = x_0}^{x} \frac{d\chi}{\sqrt{2(E_0 - U(\chi))}} \;\Rightarrow\; t = t(x; x_0) \;\overset{inv}{\Rightarrow}\; x = x(t; x_0)$$

- **Equilibrium solutions (points)** $x(t) = x_0, \quad y(t) = y_0 \quad \forall t$

  critical points of vector field $y_0 = 0, \quad f(x_0) = 0$

see cmath31.nb

$$f(x_0) = 0 \;\Leftrightarrow\; f(x) = -\left.\frac{dU}{dx}\right|_{x = x0} = 0$$

extrema of potential function

Part A. Conservative Systems

- **Phase space trajectories** *f(x,y)=c* are given analytically by the energy integral

$$E = \frac{1}{2}y^2 + U(x) \quad = E_0$$

  - Each trajectory corresponds to a constant value of E_0.
  - An energy value E_0 may corresponds to more than one trajectories (different orbits with the same energy)
  - Phase space trajectories are symmetric with respect to axis y=0

$$y = \pm\sqrt{2(E_0 - U(x))}$$

  - Flow is directed from left to right for *y*>0 and the opposite for *y*<0
  - If a trajectory intersects the axis *y*=0, this intersection should be perpendicular i.e. the vector field on the y=0 axis is vertical to it, **f**=(0 , *f(x)*)

Phase space diagram or **Phase Portrait** : a sufficient collection of trajectories for identifying all possible trajectories

see cmath32.nb

Part A. Conservative Systems

- **Restriction of range of motion** (valid *x*-interval)

$$E - U(x) \geq 0$$

$-\infty < x < \infty$ unbounded motion to both directions

$x_{\min} \leq x < \infty$ unbounded motion on the right

$-\infty < x \leq x_{\max}$ unbounded motion on the left

$x_{\min} \leq x \leq x_{\max}$ **bounded** motion

* At $x=x_{min}$ and $x=x_{max}$ is $y=0$

bounded motion $\rightarrow$ **closed phase space trajectory** $\rightarrow$ periodic motion (**oscillation**)

- **period of oscillation**

$$T = 2\int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{2(E-U(x))}}$$

see *cmath33.nb*

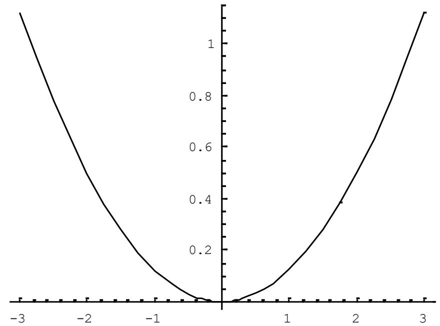
Conservative **Linear** Systems

1. Harmonic oscillator (elliptic system)

$$\ddot{x} = -kx, \quad k > 0 \quad \text{or} \quad \ddot{x} + \omega^2 x = 0, \quad \omega = \sqrt{k}$$

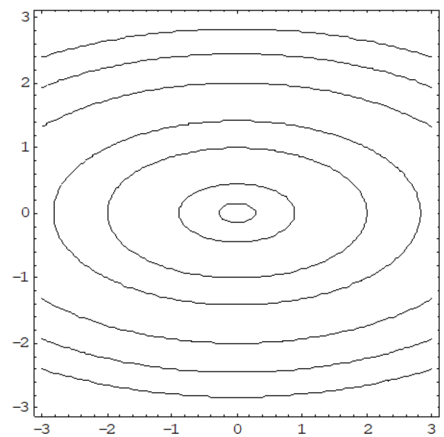
solution : $x(t) = x_0 \cos(\omega t) + \dfrac{y_0}{\omega}\sin(\omega t), \qquad y(t) = y_0 \cos(\omega t) - x_0 \omega \sin(\omega t)$

Energy : $U = \dfrac{1}{2}\omega^2 x^2, \qquad E = \dfrac{1}{2}y^2 + U,$



trajectories : $\dfrac{y^2}{2E} + \dfrac{x^2}{2E/\omega^2} = 1$

Range : $-\dfrac{\sqrt{2E}}{\omega} \le x \le \dfrac{\sqrt{2E}}{\omega}$



period : $T = \dfrac{2\pi}{\omega}$ independent of the initial conditions !

see math35harmonic.nb

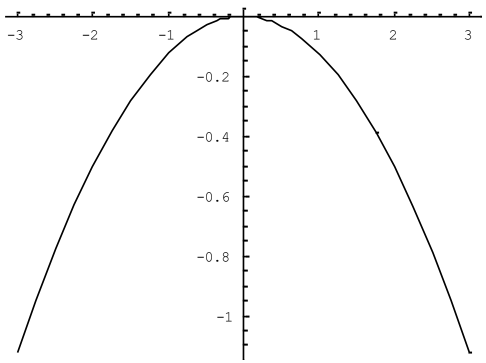
Conservative **Linear** Systems

1. Repulsive force proportional to the distance (hyperbolic system)

$$\ddot{x} = kx, \quad k > 0 \qquad or \qquad \ddot{x} - a^2 x = 0$$

solution : $x(t) = x_0 \cosh(at) + \frac{y_0}{a}\sinh(at), \qquad y(t) = y_0 \cosh(at) + x_0 a \sinh(at)$

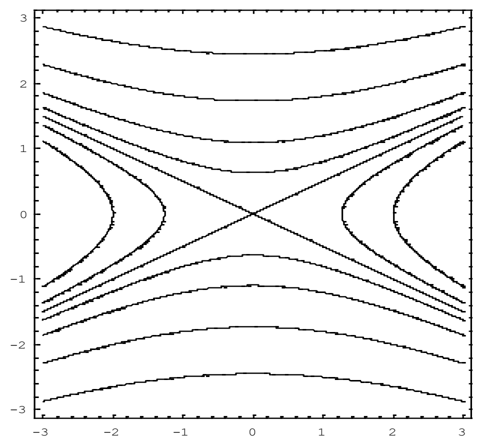
Energy : $U = -\frac{1}{2}a^2x^2, \qquad E = \frac{1}{2}y^2 + U,$



trajectories : $\frac{y^2}{2E} - \frac{x^2}{2E/a^2} = 1$

Range : $if \quad E > 0 \;\Rightarrow\; -\infty < x < \infty$

$if \quad E < 0 \;\Rightarrow\; x \le -\frac{\sqrt{-2E}}{a} \quad or \quad x \ge \frac{\sqrt{-2E}}{a}$



Asymptotic orbits : $E = 0 \;\Rightarrow\; y = \pm ax$

see cmath35hyperbolic.nb

Part A. Conservative Systems
Linear stability of equilibrium points

$$ODEs \quad : \quad \dot{x} = y, \quad \dot{y} = f(x)$$

equilibrium solution (EQP): $x=x_0$, $y=0$

solution near the equilibrium solution : $x=x_0 + \Delta x$, $y=0 + \Delta y$

$$ODEs \quad \Rightarrow \quad \Delta\dot{x} = \Delta y \quad , \quad \Delta\dot{y} = f(x_0 + \Delta x)$$

equations of deviations from EQP

(after 1st order Taylor expansion of f)

$$\Delta\dot{x} = \Delta y, \quad \Delta\dot{y} = -k\Delta x \quad , \quad k = -\left.\frac{df}{dx}\right|_{x_0} = \frac{d^2V}{dx^2},$$

$$or \qquad \Delta\ddot{x} = -k\Delta x$$

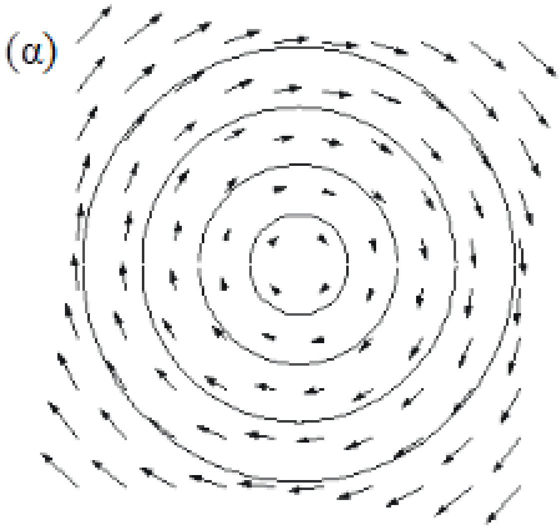
Linear ODE of deviations

k : linear stability index

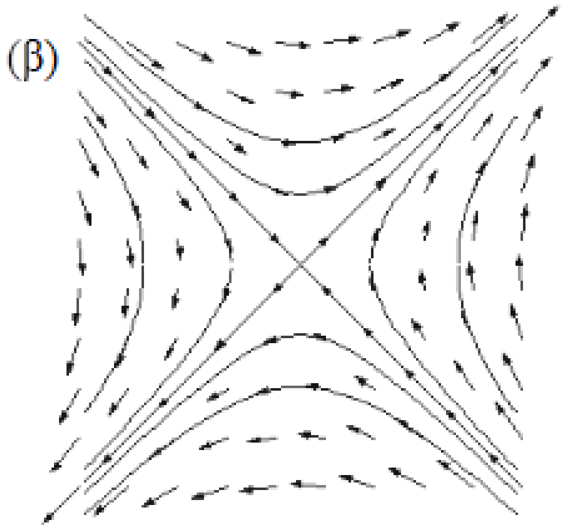
$k>0$ $\Rightarrow$ harmonic oscillations $\Rightarrow$ linearly **stable** EQP
$k<0$ $\Rightarrow$ hyperbolic motion $\Rightarrow$ linearly **unstable** EQP
$k=0$ $\Rightarrow$ critical case

Part A. Conservative Systems
Linear stability of equilibrium points

- Phase space topological structure near Equilibria



Linear stability



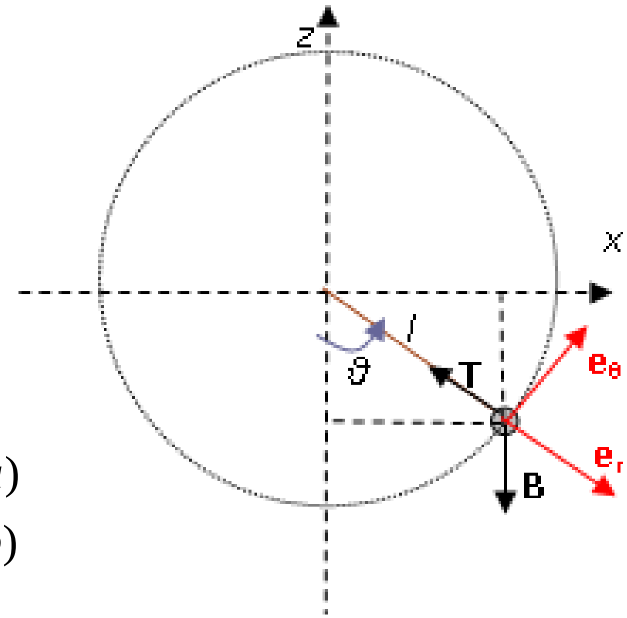
Linear instability

Part A. Conservative Systems
General study of Dynamics

- Find EQPs and their linear stability
- Plot The phase space portrait including sufficient orbits *(Energy levels : below, between and above EQPs)*.
- Determine the trajectories of asymptotic solutions, *which have the same energy level with the unstable EQPs*. The set of the above asymptotic trajectories (for each EQP) is called the **separatrix curve**
- Separatrix = stable + unstable manifold
- The separatrix may form **close loops** containing closed-periodic trajectories
- Determine the periods of the periodic trajectories. Close to the stable EQP the period is approximated given by the linear system. As the separatrix is approximated then $T \to \infty$.

Example 1 : Strongly nonlinear oscillator	*math37example1.nb*
Example 2 : 3-mode oscillations in polynomial potential	*math37example2.nb*
Example 3 : Oscillations in the Yukawa potential	*math37example3.nb*

The dynamics of the simple pendulum



$$m\ddot{\mathbf{r}} = \mathbf{F}, \qquad \mathbf{F} = \mathbf{B} + \mathbf{T}, \qquad \mathbf{B} = mg\mathbf{k}$$

in polar coordinates

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta$$

$$\mathbf{B} = mg\cos\theta\mathbf{e}_r - mg\sin\theta\mathbf{e}_\theta$$

$$\mathbf{T} = -T\mathbf{e}_r$$

$$\Rightarrow \quad \begin{aligned} -ml\dot{\theta}^2 &= mg\cos\theta - T \qquad (1a) \\ ml\ddot{\theta} &= -mg\sin\theta \qquad (1b) \end{aligned}$$

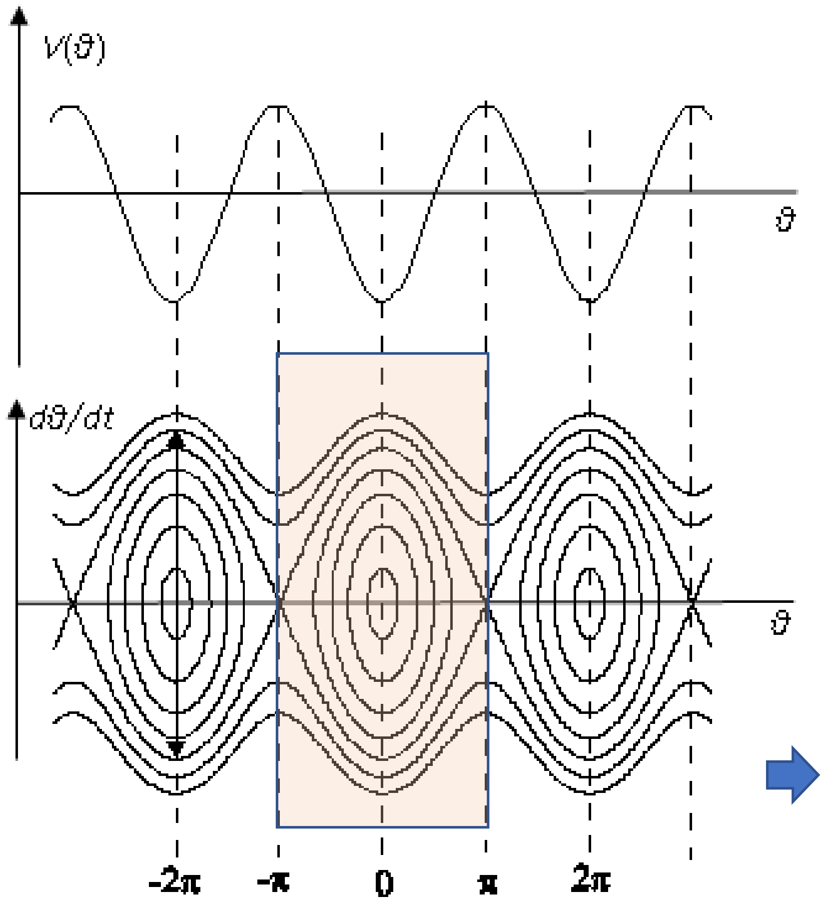
$$T = \frac{1}{2}ml^2\dot{\theta}^2, \quad V = mgz = -mgl\cos\theta \qquad \Rightarrow \qquad \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = E^* \quad (const)$$

$$(1b) \quad \Rightarrow \quad \ddot{\theta} = -\omega_0^2 \sin\theta, \qquad \omega_0 = \sqrt{g/l}$$

$$f = -\omega_0^2 \sin\theta \quad \Rightarrow \quad V = -\int f\, d\theta = -\omega_0^2 \cos\theta \qquad \Rightarrow$$

$$\frac{1}{2}\dot{\theta}^2 - \omega_0^2 \cos\theta = E \quad \text{(Energy integral)} \qquad (E = E^*/ml^2)$$

The dynamics of the simple pendulum



Equilibrium points : $\sin\theta=0$

$\theta_0 = 0, 2\pi, 4\pi, \ldots \qquad V'' = \omega_0^2 > 0 \qquad$ ευσταθεια

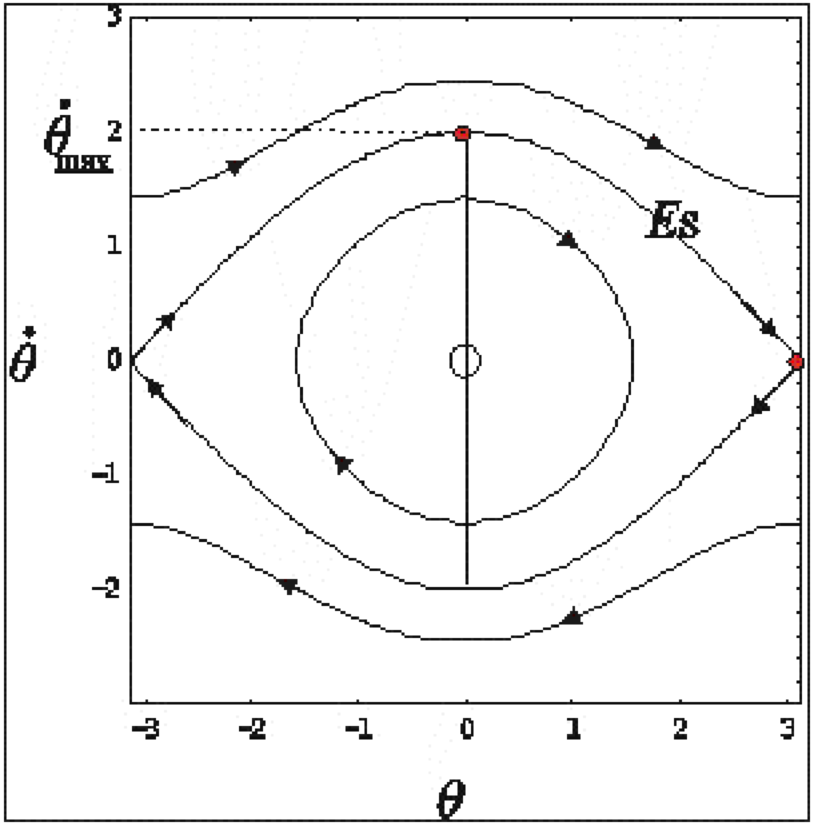
$\theta_0 = \pi, 3\pi, 5\pi, \ldots \qquad V'' = -\omega_0^2 < 0 \qquad$ ασταθεια

Remark: We observe that the ODE of motion is invariant under the translation

$$\theta \to \theta + 2k\pi, \quad k \in \mathbf{Z}$$

i.e. the system can be studied in the interval $-\pi \le \theta \le \pi$

The dynamics of the simple pendulum



Energy of Separatrix = Energy of unstable equilibrium point

$$E_S = -\omega_0^2 \cos\pi = \omega_0^2$$

Width of Separatrix $\Delta\dot{\theta} = 2\dot{\theta}_{max} = 4\omega_0$

Types of motion

$E<E_S$: Librations

$E>E_S$: Rotations

$E=E_S$: Asymptotic motion

Amplitude of libration $\theta_{max} = -\theta_{min} = \arccos\left(-\frac{E}{\omega_0^2}\right), \quad E < E_S$ $\quad A_\vartheta = \vartheta_{max} - \vartheta_{min} = 2\vartheta_{max}$

Period of librations

$$T = 4\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{2(E+\omega_0^2\cos\theta)}}$$

Period of Rotation

$$T = 2\int_0^{\pi} \frac{d\theta}{\sqrt{2(E+\omega_0^2\cos\theta)}}$$

PND01.NB

Εξάσκηση : Ενδεικτικές Ασκήσεις του Βιβλίου που δεν απαιτούν Mathematica

3.3.1, 3.3.2, 3.3.4 (χωρίς το *T*), 3.4.1