

Linear autonomous systems of 2-dimensions

$$\begin{aligned}\dot{x} &= a x + b y \\ \dot{y} &= c x + d y\end{aligned}\quad a, b, c, d \in \mathbb{R} \quad (\text{constants})$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \det \mathbf{A} \neq 0, \quad \mathbf{x} = (x, y) \in \mathbb{R}^2$$

Linear autonomous systems of 2-dimensions

➔ All linear systems with constant coefficients have solutions in closed form, expressed with typical functions, and related with the **eigenvalues** and **eigenvectors** of the matrix **A**

[cmath60.nb](#)

➔ There exist only one equilibrium position at (0,0)

➔ Vector field and area preservation/dissipation

$$\mathbf{f} = (f_x, f_y) \rightarrow \begin{cases} f_x = ax + by \\ f_y = cx + dy \end{cases}, \quad \mathit{div}\mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = a + d = \mathit{trace}\mathbf{A}$$

➔ The phase space trajectories or an integral of motion $F(x,y)=const.$ can be derived from the solution of the 1st order ODE

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by} \quad (1)$$

[cmath61.nb](#)

(1) is an homogeneous equation and can be solved by setting $z=y/x$

Linear autonomous systems of 2-dimensions

Solutions for particular cases

A. Real and distinct eigenvalues, $\lambda_1 \neq \lambda_2, \lambda_i \in \mathbb{R}$

Existence of two eigenvectors $\mathbf{u}_1 = (u_{11}, u_{12}) \in \mathbb{R}^2$, $\mathbf{u}_2 = (u_{21}, u_{22}) \in \mathbb{R}^2$

Solution

$$\begin{aligned}x(t) &= c_1 u_{11} e^{\lambda_1 t} + c_2 u_{21} e^{\lambda_2 t} \\y(t) &= c_1 u_{12} e^{\lambda_1 t} + c_2 u_{22} e^{\lambda_2 t}\end{aligned} \quad \mathbf{x}(t) = \sum_{i=1}^2 c_i \mathbf{u}_i e^{\lambda_i t}$$

Special cases

- $c_1 = 0$ $x(t) = c_2 u_{21} e^{\lambda_2 t}, \quad y(t) = c_2 u_{22} e^{\lambda_2 t} \Rightarrow y = \left(\frac{u_{22}}{u_{21}} \right) x$
- $c_2 = 0$ $x(t) = c_1 u_{11} e^{\lambda_1 t}, \quad y(t) = c_1 u_{12} e^{\lambda_1 t} \Rightarrow y = \left(\frac{u_{12}}{u_{11}} \right) x$

The trajectory solutions $y=kx$ are called **invariant linear subspaces**

Linear autonomous systems of 2-dimensions

Solutions for particular cases

A. Real and distinct eigenvalues, $\lambda_1 \neq \lambda_2, \lambda_i \in \mathbb{R}$

Invariant linear subspace solution for $\lambda \in \mathbb{R}$ and $u = (u_1, u_2)$

$$x(t) = c(x_0, y_0)u_x e^{\lambda t}, \quad y(t) = c(x_0, y_0)u_y e^{\lambda t} \quad \Rightarrow \quad y = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} x$$

if $\lambda < 0 \Rightarrow \lim_{t \rightarrow \infty} (x, y) = (0, 0)$ Stable subspace (E^s)

if $\lambda > 0 \Rightarrow \lim_{t \rightarrow \infty} (x, y) = (\pm\infty, \pm\infty)$ Unstable subspace (E^u)

Linear autonomous systems of 2-dimensions

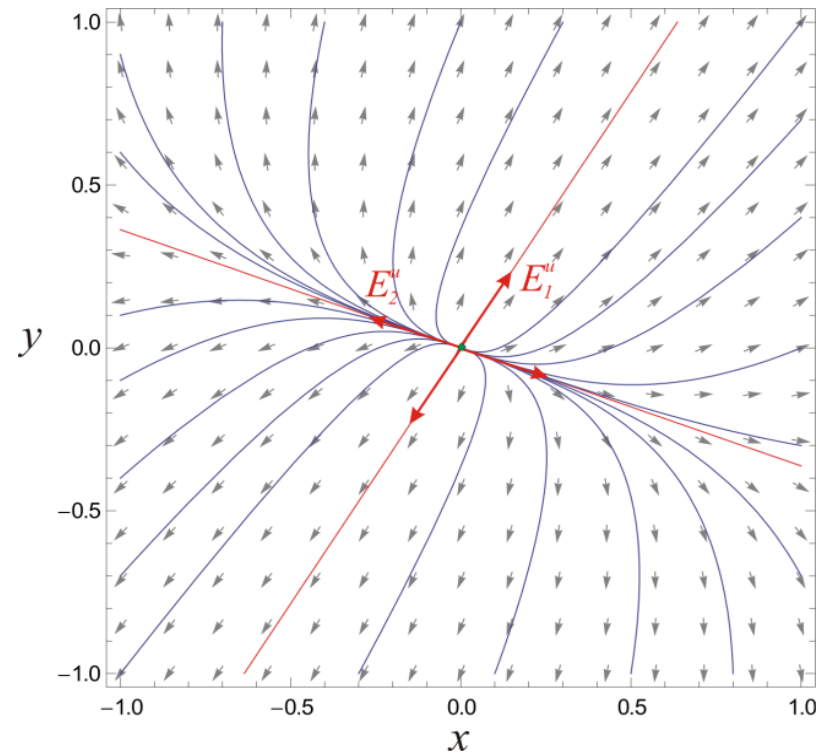
Solutions for particular cases

A. Real and distinct eigenvalues, $\lambda_1 \neq \lambda_2, \lambda_i \in \mathbb{R}$

A1. $\lambda_1 > 0$ and $\lambda_2 > 0$

two unstable linear subspaces =

Unstable node



A2. $\lambda_1 < 0$ and $\lambda_2 < 0$

two stable linear subspaces = **Stable node**

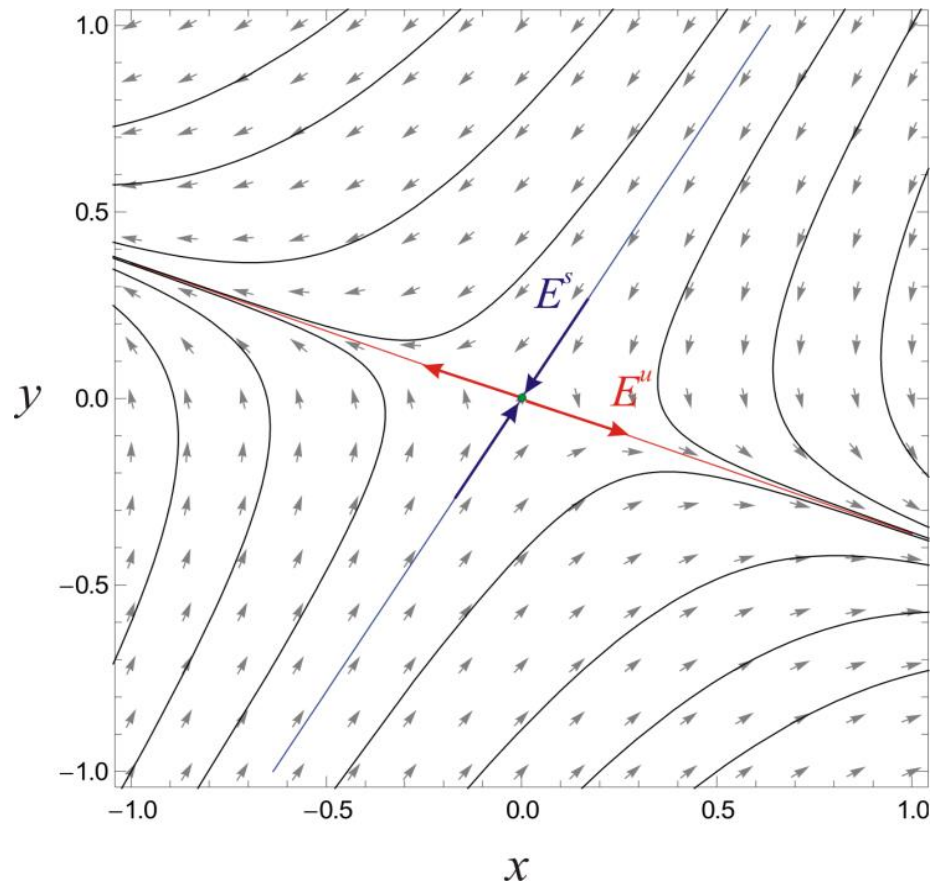
Linear autonomous systems of 2-dimensions

Solutions for particular cases

A. Real and distinct eigenvalues, $\lambda_1 \neq \lambda_2, \lambda_i \in \mathbb{R}$

A3. $\lambda_1 > 0$ and $\lambda_2 < 0$

one unstable and
one stable subspace = **saddle**



Linear autonomous systems of 2-dimensions

Solutions for particular cases

B. Equal eigenvalues, $\lambda_1 = \lambda_2 = \lambda$, $\lambda \in \mathbb{R}$

b1) If there exist two eigenvectors we obtain a stable or an unstable node

b2) If there exist one eigenvector $\mathbf{u} = (u_1, u_2)$ then

Solution

$$x(t) = (c_1 + c_2 t)u_1 e^{\lambda t} + c_2 w_1 e^{\lambda t}$$

$$y(t) = (c_1 + c_2 t)u_2 e^{\lambda t} + c_2 w_2 e^{\lambda t}$$

$$\mathbf{x}(t) = (c_1 + c_2 t)\mathbf{u}e^{\lambda t} + c_2 \mathbf{w}e^{\lambda t}$$

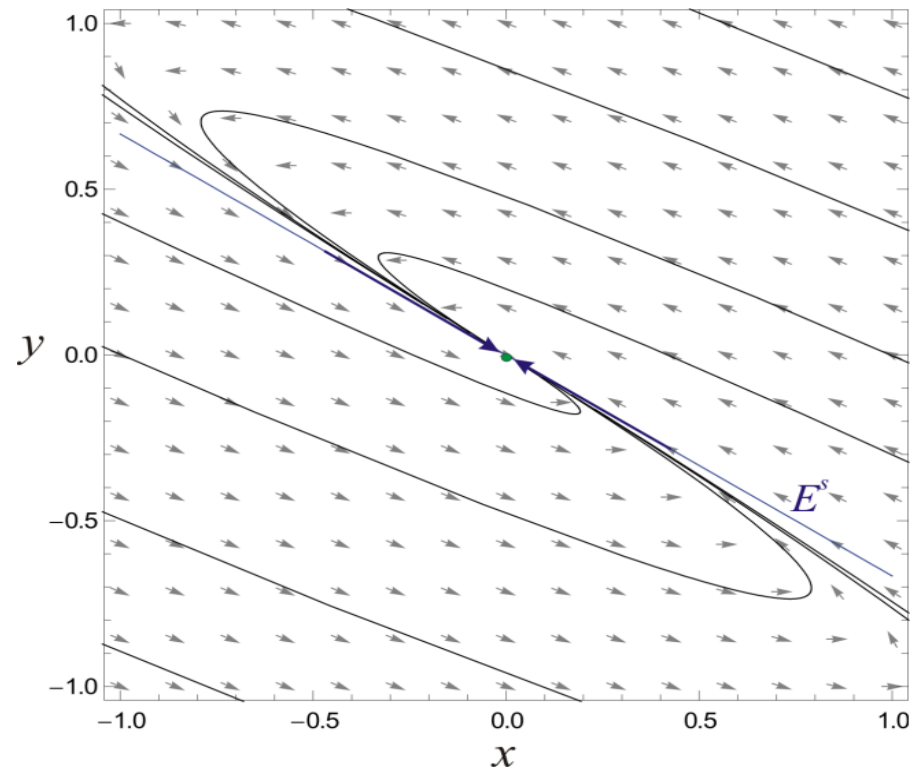
Linear Subspace for $c_2 = 0$

$$y = \left(\frac{u_2}{u_1} \right) x$$

if $\lambda < 0$: stable subspace

if $\lambda > 0$: unstable subspace

improper node



Linear autonomous systems of 2-dimensions

Solutions for particular cases

C. Complex eigenvalues : $\lambda_1 = a + ib$, $\lambda_2 = a - ib$ $b \neq 0$

Complex eigenvectors $u_1 + iv_1$ και $u_2 + iv_2$

Solution $x(t) = c_1 e^{at} (u_1 \cos bt + v_1 \sin bt) + c_2 e^{at} (-u_1 \cos bt + v_1 \sin bt)$
 $y(t) = c_1 e^{at} (u_2 \cos bt + v_2 \sin bt) + c_2 e^{at} (-u_2 \cos bt + v_2 \sin bt)$

Periodic terms ($T = 2\pi/b$) with amplitude $A \sim e^{at}$

- $a > 0$: oscillations of $x(t)$ and $y(t)$ with exponentially increasing amplitude
- $a < 0$: damped oscillations of $x(t)$ and $y(t)$
- $a = 0$: periodic oscillations of $x(t)$ and $y(t)$

Linear autonomous systems of 2-dimensions

Solutions for particular cases

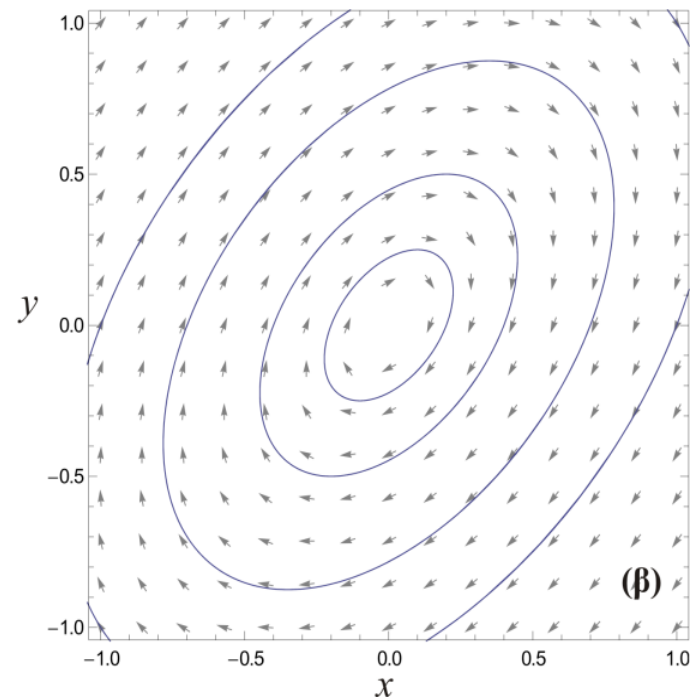
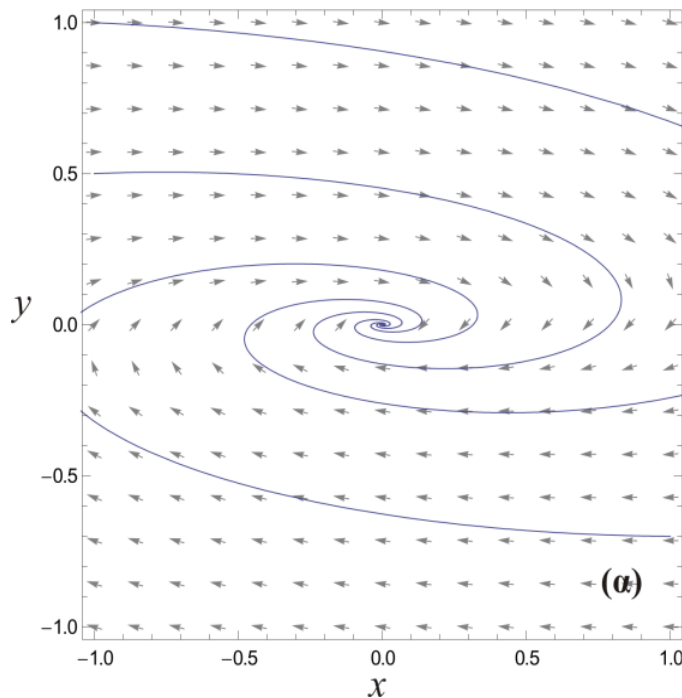
C. Complex eigenvalues : $\lambda_1 = a+ib$, $\lambda_2 = a-ib$, $b \neq 0$

$a > 0$: **unstable focus**

$a < 0$: **stable focus (sink)**

$a = 0$: **center**

if $A_{12} > 0 \Rightarrow$
clockwise evolution



Το γραμμικό σύστημα ονομάζεται **υπερβολικό** αν το πραγματικό μέρος όλων των ιδιοτιμών του πίνακα A είναι διάφορο του μηδενός

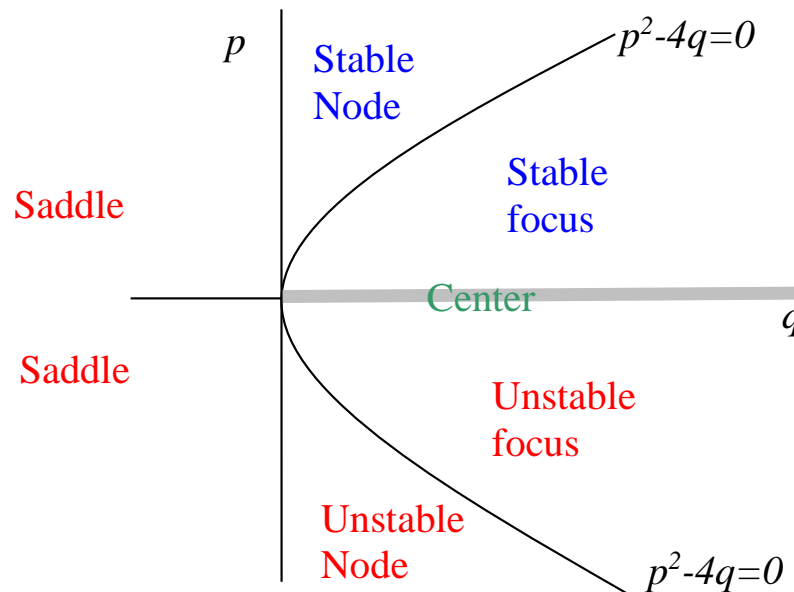
A linear system is called hyperbolic if the real part of all eigenvalues is non-zero.

Linear autonomous systems of 2-dimensions

Parametric space of Stability

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$p = -\text{trace}\mathbf{A} = -(a+d) \quad , \quad q = \det\mathbf{A} = ad-bc$$



[math64.nb](#)

[math65.nb](#)

* for $p=0$ the system is area preserving (includes centers ($q>0$) and saddles ($q<0$))

Άσκηση 1 (φασικό διάγραμμα γραμμικού συστήματος)

$$\dot{x} = -\frac{1}{2}x + y$$

$$\dot{y} = -x - 3y$$

Άσκηση 2 (φασικό διάγραμμα γραμμικού συστήματος)

$$\dot{x} = -x + y$$

$$\dot{y} = -2x - 2y$$

Άσκηση 3 (φασικό διάγραμμα γραμμικού συστήματος)

$$\dot{x} = -x + y$$

$$\dot{y} = -x - 3y$$