Calculation of Eddy Current Losses in Nonlinear Ferromagnetic Materials

DIMITRIS LABRIDIS, STUDENT MEMBER, IEEE, AND PETROS DOKOPOULOS, MEMBER, IEEE

Abstract—**This paper presents a complex analysis of the nonlinear diffusion problem in ferromagnetic materials under steady-state excitation. The problem is solved by considering an equivalent fictitious material where the relative permeability is assumed to be constant in time but different from point to point and is related to the nonlinear *B–H* characteristic curve with the help of the stored magnetic co-energy density. Eddy current losses are calculated in a one-dimensional thick steel plate. A comparison made with results obtained from the classical step-by-step method shows a good agreement.**

I. Introduction

The FIELD problems involving ferromagnetic materials are complicated by the nonlinear relationship between flux density and the magnetic field intensity. Even in the steady-state ac operation, time has to appear as an explicit variable in the diffusion equation. The solution will become easier if an equivalent material with non-time-varying permeability could be found. As a result, time effective calculations can be made by using phasor quantities.

The problem of introducing a new concept for the material permeability has been approached earlier. In [1] the average magnetic energy density has been used to obtain a constant permeability. In [2] the flux density has been used to obtain an rms reluctivity. However, when the saturation increases the magnetic energy seems to be hardly influenced. On the other hand, the magnetic co-energy is a better measure of the degree of saturation.

The purpose of this work is the computation of the eddy current losses in a one-dimensional nonlinear diffusion problem. The finite difference method has been used to obtain a solution and Frohlich approximation has been used for the nonlinear *B–H* relation.

The results have been compared with the classical step-by-step solution [3] and the agreement in the losses was excellent.

II. The Problem

A classic one-dimensional eddy current problem is the solution for a thick steel plate with a sinusoidal magnetic field applied to its surfaces. The model shown in Fig. 1 consists of a semi-infinite plate of thickness $2d$. The only

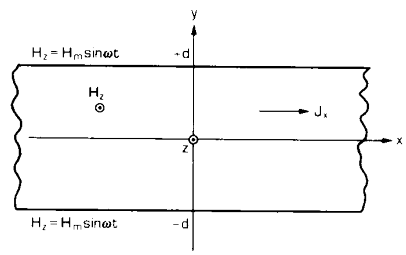

Fig. 1. Cross section of a semi-infinite thick steel plate.

component of the magnetic field is assumed to be in the z direction and is constrained to be a spatial function of y alone. The z component of the magnetic field intensity in the material is governed [4] by the diffusion equation

$$\frac{\partial^2 H_z}{\partial y^2} = \sigma \frac{\partial B}{\partial H} \frac{\partial H_z}{\partial t} \tag{1}$$

assuming a uniform conductivity σ.

In order to approximate the nonlinear *B–H* curve, the Frohlich representation

$$B = \frac{H}{\alpha + \beta |H|} \tag{2}$$

seems to be the best compromise between accuracy and simplicity.

The excitation for the plate is a sinusoidal magnetic field intensity $H_m \sin \omega t$ applied to both surfaces.

III. Equations in the Complex Domain

Sinusoidal time variation of the field quantities enables us to substitute $j\omega$ for $\partial/\partial t$. Thus (1) becomes

$$\frac{d^2 H}{dy^2} = j\omega\sigma\mu_0\mu_r H \tag{3}$$

and this is the well-known linear diffusion equation where H is a z-component phasor in the complex domain. However, in nonlinear problems this equation is complicated because of the time dependence of the relative permeability μ_r. There are two possibilities to overcome this difficulty:

i) To introduce a complex permeability μ, using the notation [5] $\mu = \mu_r - j\mu_i$.

ii) To introduce a constant permeability along the period T.

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The authors are with the Power Systems Laboratory, Department of Electrical Engineering, Aristotelian University of Thessaloniki, 54006 Thessaloniki, Greece.
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In both cases, the new permeability must lead to a solution of the diffusion equation (3) that results in the same value of losses that would be obtained from the solution of the explicit form of (1).

The constant permeability approximation has been chosen in this analysis because it allows the use of the classical solution methods applied in (3). Hence the problem that has to be solved is as follows:

Assume a fictitious material with a constant unknown relative permeability, which is a spatial function of y alone. This permeability is related to the nonlinear B-H characteristic curve through the unknown values of the magnetic field intensity at every point. The condition to be fulfilled is that the linear fictitious material has the same eddy current average loss density as the nonlinear material has at every point.

Using the superscript f for the fictitious material, (3) becomes

$$\frac{d^2H^f}{dy^2} = j\omega\sigma\mu_0\mu_r^f(H^f)H^f. \quad (4)$$

IV. Magnetic Co-Energy Density and Nonlinearity

The existence of the two unknowns H^f and μ_r^f in (4) leads to an iterative solution. Thus a relation must be established between these two unknowns and the nonlinear B-H curve of the material.

Consider a point i in the fictitious material at which the magnetic field intensity is a sinusoidal function of time with maximum value H_{mi}^f. Since this material has a constant relative permeability at this point which is equal to μ_{ri}^f, the maximum value of the associated sinusoidal flux density will be

$$B_{mi}^f = \mu_0\mu_{ri}^f H_{mi}^f. \quad (5)$$

In the nonlinear material neither H nor B are sinusoidal functions of time. They are periodic functions containing harmonics and they are assumed to be related through the nonlinear Frohlich equation (2). This relation, using the same maximum value H_{mi}^f of the fictitious material, gives a maximum value for the flux density

$$B_{mi} = \frac{H_{mi}^f}{\alpha + \beta H_{mi}^f}. \quad (6)$$

At this point, two definite integrals will be considered. The interval for the integration is a quarter of the period T and it is assumed that $H_i^f(t = 0) = 0$ and $H_i^f(t = T/4) = H_{mi}^f$.

From Fig. 2(a) it is easily seen that the magnetic co-energy density during this quarter of period at the point i is

$$w_{1i} = \int_0^{H_{mi}^f} b\, dh. \quad (7)$$

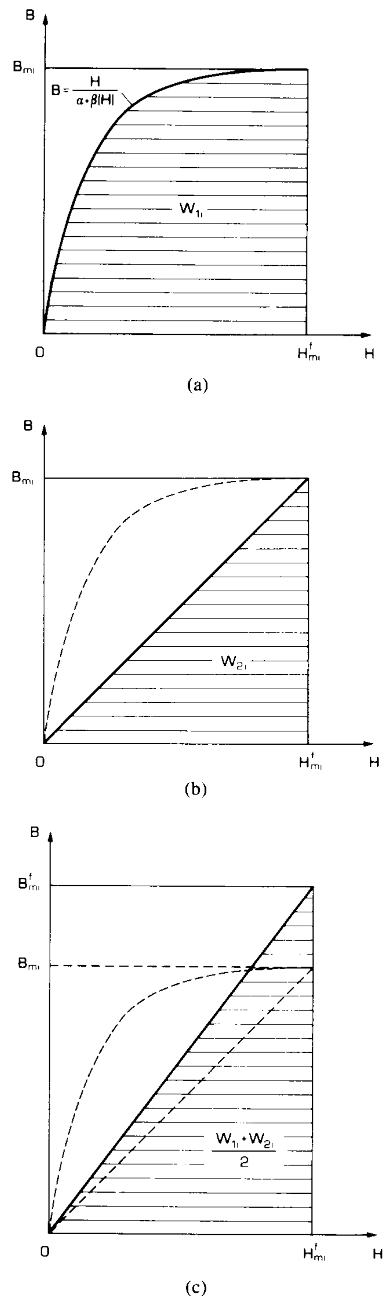

Fig. 2. (a) Magnetic co-energy density during a quarter of period T using the actual B-H curve. (b) Magnetic co-energy density during a quarter of period T using the average value of the slope dB/dH. (c) Magnetic co-energy density during a quarter of period T of the equivalent fictitious material.

Using (2) for the B-H relation and after integration, (7) becomes

$$w_{1i} = \frac{H_{mi}^f}{\beta} - \frac{\alpha}{\beta^2}\ln\frac{\alpha + \beta H_{mi}^f}{\alpha}. \quad (8)$$

The second integral is related to the average value of the slope dB/dH of (2) during the same time interval, that is

$$\left\langle \frac{dB}{dH} \right\rangle = \frac{1}{H^f_{mi}} \int_0^{H^f_{mi}} \frac{db}{dh}\, dh = \frac{B_{mi}}{H^f_{mi}} \tag{9}$$

where B_{mi} is given in (6). The magnetic co-energy density of a material having this average slope during the same quarter of period is, as shown in Fig. 2(b),

$$w_{2i} = \frac{1}{2} B_{mi} H^f_{mi} \tag{10}$$

and using (6) for the relation between B_{mi} and H^f_{mi}, (10) becomes

$$w_{2i} = \frac{1}{2} \frac{(H^f_{mi})^2}{\alpha + \beta H^f_{mi}}. \tag{11}$$

Our concern is to find linear fictitious materials which lead to upper and lower bounds for the estimation of the losses. A fictitious material, that has a constant relative permeability μ^f_{ri} and the same maximum field intensity H^f_{mi} at point i, is related to a magnetic co-energy density during the same quarter of the period equal to

$$w^f_i = \tfrac{1}{2}\mu_0 \mu^f_{ri} (H^f_{mi})^2. \tag{12}$$

If this magnetic co-energy density is set equal to w_{2i}, the fictitious material will have a relative permeability at node i equal to

$$\mu^f_{r2i} = \frac{1}{\mu_0(\alpha + \beta H^f_{mi})} \tag{13}$$

as it can be seen from (6) and Fig. 2(b). The eddy current losses of a plate of a linear magnetic material, taking both sides into account, is given in [4] by

$$P_1 = \frac{H_m^2}{\sigma\delta} \frac{\sinh\gamma - \sin\gamma}{\cosh\gamma + \cos\gamma} \tag{14}$$

where $\gamma = 2d/\delta$ and $\delta = (1/\pi f \sigma \mu_0 \mu_r)^{1/2}$. From (14) it can be seen that when μ_r increases the losses also increase. The nonlinear material during a quarter of a period changes its relative permeability from a maximum value (when H is zero) to a minimum (when H is maximum). Since μ^f_{r2i} given in (13) represents a minimum value of relative permeability, we expect that the equation of w^f_i and w_{2i} may lead to a fictitious material having lower losses than the nonlinear material has, i.e., we expect w_{2i} to be a lower bound for the loss estimation.

In order to consider a corresponding upper bound, the magnetic co-energy density w_{1i} given by (7) seems to be a reasonable choice. Indeed, w_{1i} takes into account the saturation of the material, since it can be seen from Fig. 2(a), w_{1i} increases with the increase of H^f_{mi}. On the contrary, the magnetic energy density w_e defined by

$$w_e = \int_0^{B_{mi}} h\, db \tag{15}$$

tends to a limit that is hardly influenced by an increase in H, because B has a limiting saturation value for each material.

To check the hypothesis of lower and upper bound, the losses of the thick steel plate have been computed using independently equations $w^f_i = w_{1i}$ and $w^f_i = w_{2i}$. They have led to a corresponding over- and underestimation, having almost equal and opposite difference compared to the loss value that was computed with the classical method of [3]. So one reasonable estimation of the magnetic co-energy density of the fictitious material is the average of w_{1i} and w_{2i}, given by

$$w^f_i = \frac{w_{1i} - w_{2i}}{2} \tag{16}$$

as shown in Fig. 2(c).

Using (12) and (16), the relative permeability of the fictitious material at point i is

$$\mu^f_{ri} = \frac{w_{1i} + w_{2i}}{\mu_0 (H^f_{mi})^2} \tag{17}$$

and it is a function of H^f_{mi} alone for a given Frohlich curve, as can be easily seen using (8) and (11).

V. The Iterative Procedure

The solution of (4) is based on an iterative procedure, since both μ^f_r and H^f are unknowns. This procedure contains six steps that will be explained in detail.

Step 1: The diffusion equation (4) is solved for the unknown values of H^f. In the first iteration the relative permeability μ^f_r of the fictitious material is set equal to the initial slope of the Frohlich curve, i.e.,

$$\mu^f_r = \frac{1}{\mu_0 \alpha}$$

and hence it has the same value at every point. For all the next iterations μ^f_r will be obtained from step 6 and it will be generally different from point to point.

Step 2: Using the values of H^f_{mi} from the solution at step 1 and (5), the maximum flux densities B^f_{mi} are calculated.

Step 3: Using (12), magnetic co-energy densities w^f_i are calculated at every point.

Step 4: Using (6) and the same values of H^f_{mi} from step 1, maximum flux densities B_{mi} are calculated at every point.

Step 5: Using (8) and (11), magnetic co-energy densities w_{1i} and w_{2i} are obtained, respectively, and the average

$$w_i = \frac{w_{1i} + w_{2i}}{2}$$

is computed.

Step 6: At every point of the material, the values of w_i and w^f_i are tested whether they differ more than a small quantity w_{err}. If $|w_i - w_i^f| \leq w_{err}$ at every point, the iteration procedure is terminated. If $|w_i - w_i^f| > w_{err}$ at point i, a new relative permeability is related to this point, equal to

$$\mu_{ri}^f = \frac{2w_i}{\mu_0 (H_{mi}^f)^2}.$$

This new value will be used at step 1 of the next iteration.

The value of w_{err} is related to the precision of the computation and it is a function of the applied field H_m on the surface of the plate.

VI. Results

The problem of the thick steel plate has been solved over a wide range of geometrical and physical data and comparison is made with the universal loss chart of [3]. In order to make a right comparison, the same numerical method of finite differences and the same B-H approximation of (2) were used. In all cases, the agreement of the two methods was excellent and the differences were between 0.5 and 2 percent.

In order to illustrate the validity of the new method, a case with the data shown in Table I is examined here in detail. In Figs. 3 and 4 the eddy current density and the relative permeability are shown as functions of time t at a distance $y = 0.5$ mm from the plate surface. In Figs. 5 and 6 the eddy current density and the relative permeability are shown as functions of distance y at a time $t = 2.5$ ms, equal to the eighth of the period T. In all cases, solid lines refer to the nonlinear material solved with the classical step-by-step method of [3] while dashed lines refer to the equivalent fictitious material proposed in this paper.

The step-by-step method of [3] gives a total value of losses

$$P = 1360.68 \text{ W/m}^2$$

taking into account both sides of the plate. The complex solution method proposed in this paper gives a total value of losses

$$P^f = 1385.17 \text{ W/m}^2.$$

The difference of the two values is therefore 1.8 percent, but 1/12 of the computing time was needed for the solution of the problem.

The value of w_{err} used was

$$w_{err} = 0.001 w_b$$

and the magnetic co-energy density w_b chosen as a reference was

$$w_b = \frac{1}{2}\frac{H_m^2}{\alpha}.$$

The convergence of the new method is very fast, as it can be seen from Table II. In all cases tested, after the fourth iteration the computed value of losses had an acceptable

TABLE I
Physical and Geometrical Data for the Thick Steel Plate

$\alpha = 156$	m/H
$\beta = 0.59$	1/T
$\sigma = 5 \times 10^6$	S/m
$T = 20$	ms
$d = 2.5$	mm
$H_m = 2644.1$	A/m

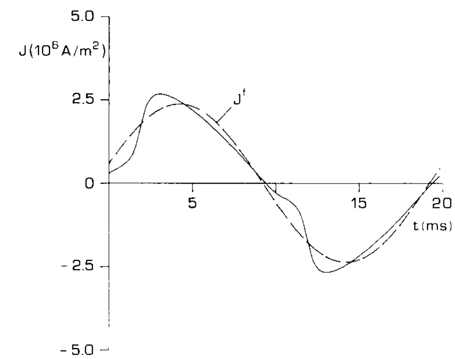

Fig. 3. Eddy current density J versus time at a distance $y = 0.5$ mm from plate surface.

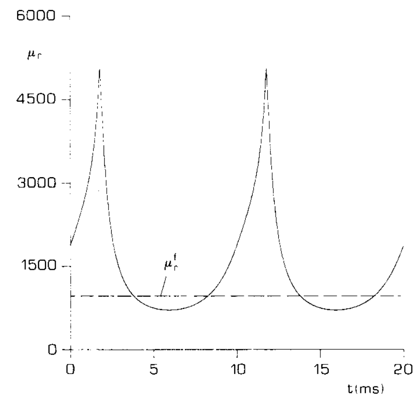

Fig. 4. Relative permeability μ_r versus time at a distance $y = 0.5$ mm from plate surface.

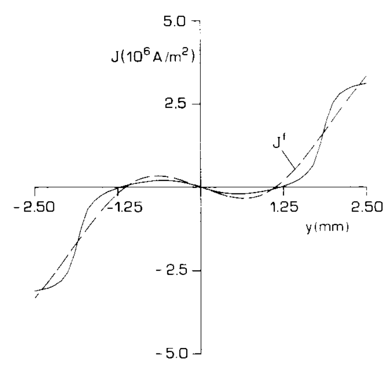

Fig. 5. Eddy current density J versus distance at a time $t = 2.5$ ms.

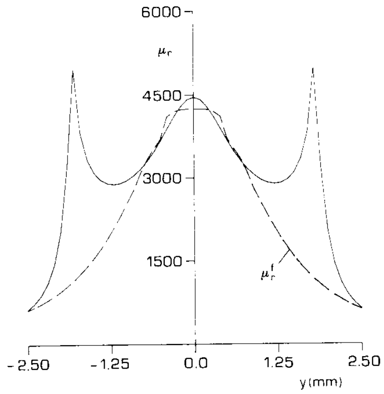

Fig. 6. Relative permeability μ_r versus distance at a time $t = 2.5$ ms.

TABLE II
CONVERGENCE OF THE METHOD ($P = 1360.68$ W/m^2)

Iteration	P^f(W/m^2)	$\frac{P^f - P}{P} \times 100$
1	3171.76	133.10
2	1626.36	19.53
3	1445.30	6.22
4	1402.00	3.04
5	1389.81	2.14
6	1386.06	1.87
7	1385.38	1.82
8	1385.17	1.80

TABLE III

Average Values of the Loss Density

$$\langle u(y) \rangle = \frac{1}{T} \int_0^T \frac{|J(y, t)|^2}{\sigma} dt$$ computed with the classical method of [3]

$$\langle u^f(y) \rangle = \frac{|J^f(y)|^2}{2\sigma}$$ computed with the new complex method

Distance from Surface y (mm)	$\langle u(y) \rangle$ (W/m^3)	$\langle u^f(y) \rangle$ (W/m^3)	Difference (%)
0.5	555853	563157	1.3
1.0	236380	230098	−2.7
1.5	70886	71679	1.1

difference of 3 percent with the corresponding value of losses given by the loss chart of [3].

Finally, the average loss densities for a period of time T were computed with both methods and at three different points of the steel plate. The results are shown in Table III and it can be seen that the new complex solution method introduces a material that has at every point the same average losses as the nonlinear material.

VII. Conclusions

The iterative procedure presented in this paper leads to accurate eddy current loss computations, using a complex analysis of the nonlinear diffusion equation in a one-dimensional thick steel plate problem.

An equivalent material with non-time-varying permeability is introduced and it is related to the nonlinear B-H curve of the steel with the help of the stored magnetic co-energy at every point. This equivalent material has the same time average losses as the nonlinear material at every point.

The use of phasor quantities for the solution of the problem makes this new method considerably faster than the classical step-by-step solution. Therefore, it may be applied to eddy current problems in nonlinear ferromagnetic materials, when the computation of the losses and not of the actual field is of importance.

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Dimitris Labridis (S'88) was born in Thessaloniki, Greece, on July 26, 1958. He received the Dipl.-Eng. degree from the Department of Electrical Engineering at the Aristotelian University of Thessaloniki, in November 1981.

Since 1982 he has been working as a research assistant at the Department of Electrical Engineering at the Aristotelian University of Thessaloniki. His special interests are power system analysis with special emphasis on the simulation of transmission systems and electromagnetic field analysis.

Petros Dokopoulos (M'74) was born in Athens, Greece, on September 16, 1939. He received the Dipl.-Eng. degree from the Technical University of Athens in July 1962 and the Ph.D. degree from the University of Brunswick, FRG, in 1967.

During 1962–1967 he was with the High Voltage Laboratory at the University of Brunswick, FRG, during 1967–1974 with the Nuclear Research Center at Jülich, FRG, and during 1974–1978 with the Joint European Torus. Since 1978 he has been full professor at the Department of Electrical Engineering at the Aristotelian University of Thessaloniki, Greece. He has worked as consultant to Brown Boveri and Cie, Mannheim, Siemens, Erlangen, FRG, Public Power Corporation, Greece, and to National Telecommunication Organization, Greece. His scientific fields of interest are dielectrics, power switches, generators, power cables, and alternative energy sources. He has 41 publications and seven patents on these subjects.