ABSTRACT
The present paper presents a new fast and efficient algorithm for the solution of the dynamic optimization problem resulted from the implementation of a model predictive control (MPC) framework in highly nonlinear dynamic systems. The solution procedure involves the discretization of the system differential equations using orthogonal collocation on finite elements with the optimal control problem formulated as a nonlinear program with an objective function consisted of a final state term and a time varying term. The sequence of the optimal control actions is obtained through the solution of the parameterized set, with respect to the initial point, of the Karush-Kuhn-Tucker (KKT) conditions for the nonlinear program. The parameterized set of KKT equations is formed as a homotopy problem that utilizes the optimal control solution at the previous control interval solved using a continuation method. Efficient sparse solvers enable a quick calculation of the optimal solution thus allowing the implementation of a nonlinear model predictive control (NMPC) framework in challenging mechanical engineering problems such as the control of a cart with a double pendulum.

KEYWORDS: Model predictive control, Nonlinear model, Continuation method

1. INTRODUCTION
Model predictive control is a powerful control algorithm that enables the satisfaction of complex control objectives, and the systematic consideration of all process dynamic interactions for multivariable control systems /1/. The implementation of nonlinear model predictive control in dynamic systems that possess fast dynamics is prohibitive as the solution to the associated dynamic optimization problem is comparable to the duration of the control interval suitable for adequate control performance. The computational delay appears to be a considerable factor for the deterioration of the NMPC performance /2/. Therefore, either a delayed control action is applied to the system or a simplified model, usually a linear realization of the nonlinear dynamic model is used in the model predictive control algorithm. In both cases, however, the performance of the controller diminishes as crucial process dynamics may be ignored through the utilization of a simplified (linear) model or the adoption of a delayed control action.

Over the years a large number of efficient real-time strategies have been proposed such as explicit NMPC, Newton-type controllers and NLP sensitivity-based controllers. Explicit NMPC approaches compute off-line the optimal control actions and through sensitivity considerations the entire state space is divided into regions with similar control action characteristics /3/. However, as the problem size increases the combinatorial problem associated with the identification of all possible regions within the state space grows significantly in size. Newton-based controllers have achieved considerable improvement in the time for the solution of the NMPC optimization problem /4-5/. Recently, the advanced-step NMPC /6/ achieved significant improvement in the speed for the
solution of the optimization problem as the control interval is used in order to guide the solution of the problem towards the real optimum using the prediction model before the measurements become available. A Newton-based algorithm would then correct the solution using the obtained measurements.

In the current work, a different formulation for the NMPC problem is sought based on a constructed homotopy for the optimization problem. A homotopy continuation method follows the optimal control actions from the current control interval to the next one by parameterizing the KKT optimality conditions of the NMPC problem with respect to the initial conditions. The formulation requires the complete discretization of the dynamic model using a suitable technique such as orthogonal collocation on finite elements. The equivalent to the dynamic program conventional NLP problem is then formulated as a set of equations representing the KKT optimality conditions. Starting from a known optimal point for the current set of initial conditions for the state variables the optimal solution is pathfollowed using a sparse continuation method to a new set of initial conditions that reflect the new plant state. Obviously, as in the advanced step algorithm the pathfolowing to the predicted new set of initial condition can be performed during the control interval and only a small correction in the initial vector values is required after the acquisition of the new measurements. The method appears to provide significant improvement in the speed of solution with an enhanced robustness in terms of convergence. Homotopy continuation methods are known for their increased robustness especially with nonlinear problems and difficult to converge problems /7/.

The paper is organized as follows: Section two provides a presentation of the NMPC algorithm, section three introduces the discretization scheme of the dynamic model, and section four outlines the details of the algorithm. Section five shows the merits of the proposed algorithm through an illustrative example.

2. NMPC FORMULATION

The NMPC problem can be formulated as follows:

\[
\begin{align*}
\text{Min} \quad J &= \phi(x_N, y_N) + \sum_{i=0}^{N-1} \phi(x_i, y_i, u_i) \\
\text{s.t.} \quad h(x_i, y_i, u_i) &= 0 \\
\quad g(x_i, y_i, u_i) &\leq 0 \quad i = 0, K, N \\
\quad x_0 &= p
\end{align*}
\]

(1)

Vector \( x \) is the real-valued state vector of dimension \( n \), \( y \) is the real-valued output vector of dimension \( l \), \( u \) is the real-valued input (manipulated variables) vector of dimension \( m \), and \( x_0 \) is the initial values vector for the state variables of dimension \( n \). Functional \( J \) represents the objective of the control problem consisted of a terminal term, \( \Phi(\cdot) \) and a time varying term expressed using nonlinear functions \( \phi(\cdot) \). Vector \( h \) represents the set of nonlinear modeling equations describing the dynamic behavior of the system, whereas \( g \) represents the set of inequality constraints. The model predictions extend into the future for \( N \) time intervals, whereas the optimal sequence of manipulated variables extends to \( M \) time intervals with \( M < N \).

According to the NMPC algorithm, given an initial point for the system the sequence of optimal control actions is calculated from the solution of problem (1). Problem (1) considers the effects of previous control actions through the realization of the initial conditions and calculates a series of future control actions that minimize the performance index, \( J \). The control action that corresponds to the first control interval is then implemented in the plant. At the end of the duration of the control interval a new set of measurements is acquired for the output variables from the plant.
instrumentation. Provided that the system is observable, the set of measurements is then used to estimate the error between the model predictions and the plant output. The prediction horizon moves one time step further and the procedure is repeated. A schematic of the model predictive control algorithm is shown in Figure 1.

**Figure 1:** Schematic of the model predictive control framework.

The plant output includes all sources of disturbances such as the unmeasured disturbances affecting the plant and measurement noise. It is assumed that the plant output has the form:

\[ h(\xi, x, y, u_i) + w_i = 0 \quad i = 1, K, N \] (2)

Vector \(w_i\) represents the unmeasured plant disturbances. Based on the plant-model error a new set of values for the state variables can be calculated at the end of the current time interval. This new set of values for the state variables thus becomes the initial conditions for problem (1) for the next time interval. Therefore, the solution of problem (1) can be viewed then as a parameterized optimization problem with respect to the initial conditions vector, \(p\). Such property has been exploited by Zavala and Biegler /7/ in order to improve the efficiency of NMPC solution strategies.

3. DISCRETIZATION OF DYNAMIC MODEL

The solution of NMPC problem (1) can be achieved by either the sequential or the direct solution methods. In the sequential solution method /8/ a control vector parameterization is sought for the system. An iterative procedure is then implemented that evaluates the system dynamic behavior for a given control sequence as well as the sensitivity of the state and output variables with respect to the control actions. The latter information is then passed to an optimization step that calculates the control actions that minimize the objective function. The procedure continues till convergence in the control actions between successive iterations is achieved. This procedure suffers in cases that unstable open-loop behavior is present, handles implicitly trajectory inequality constraints but maintains feasibility during the procedure. The direct solution methods discretize the differential equations by transforming them into a large set of nonlinear algebraic equations /9/. The optimization problem thus takes the form of a conventional nonlinear program. The main
advantage of the method is the explicit handling of trajectory constraints but the method suffers from the large dimensionality of the associated problem.

In the present work, a direct solution method is applied with the differential equations of the plant model discretized using an orthogonal collocation on finite elements (OCFE) approximation technique. The prediction horizon is partitioned in $NE$ finite elements and within each finite element the state profiles are approximated by Lagrange polynomials. The state profiles are then expressed as functions of the state variables defined at specified time instances; namely the collocation points as follows:

$$\tilde{x}_i(t) = \sum_{j=0}^{n} W_{i,j}(t)x_j(t_{i,j}), \quad t_{i,0} \leq t \leq t_{i,\text{end}}, \quad i = 1, \ldots, NE$$

$$W_{i,j}(t) = \prod_{k=0, k \neq j}^{n} \frac{t - t_{i,k}}{t_{i,j} - t_{i,k}}, \quad j = 0, \ldots, n$$

Lagrange polynomials $W_{i,j}(t)$ are equal to zero at collocation points $t_{i,k}$ when $k \neq j$ and to unity when $k=j$. Within each finite element, a Lagrange interpolation polynomial of different order can be used. The time derivatives of the state profiles are then calculated as follows:

$$\frac{d\tilde{x}_i(t)}{dt} = \sum_{j=0}^{n} W_{i,j}(t)\frac{dx_j(t_{i,j})}{dt}, \quad t_{i,0} \leq t \leq t_{i,\text{end}}, \quad i = 1, \ldots, NE$$

The collocation points are selected as the roots of orthogonal polynomials, usually Legendre polynomials, of order equal to the number of collocation points within each finite element. The shape and characteristics of the approximated variable profiles (e.g., linear or irregularly shaped profiles, steep fronts) basically determine the required order of polynomial approximation. Moreover, the adaptive placement of the element breakpoints may alter the density of collocation points within specific time regions by altering the element sizes. A schematic of the time domain partition in finite elements with collocation points is shown in Figure 2. Naturally, the element boundaries coincide with the time intervals in the digital implementation of the NMPC algorithm but also integer multiples of finite elements can form one time (control) interval. Constraints at various time instances can be best accommodated within the approximation scheme by either placing a collocation point or a finite element breakpoint at the specific time instance.

Figure 2: Time domain partition using orthogonal collocation on finite elements.

OCFE modeling equations preserve the same form and residuals associated with the modeling balances are required to vanish only at the collocation points. Zero-order continuity for the state
profiles is imposed at the element boundaries. The approximation of the state profiles and the discretization of the dynamic equations using OCFE transform the NMPC problem to an equivalent nonlinear program.

\[
\min_{\mathbf{u}} J = \Phi(\mathbf{x}_i, y_i) + \sum_{i=0}^{N_t-1} \Phi(\mathbf{x}_i, \mathbf{y}_i, \mathbf{u}_i)
\]

s.t. \( h(\mathbf{x}_i, \mathbf{y}_i, \mathbf{u}_i) = 0 \)
\( g(\mathbf{x}_i, \mathbf{y}_i, \mathbf{u}_i) \leq 0 \) \( i = 1, \ldots, NE \)
\( x_0 = \mathbf{p} \) \( (6) \)

4. PARAMETERIZED NONLINEAR MODEL PREDICTIVE CONTROL ALGORITHM

Problem (6) of the time discretized NMPC algorithm can be solved using conventional nonlinear program algorithms. During the implementation of the algorithm the solution of problem (6) is sought at different initial points utilizing basis information from the previous control interval. Even though efficient algorithms have been developed over the years, the nonlinear nature of the problem may cause significant computational delay and sometimes failures to converge in the optimal solution. An alternative method for the solution of (6) relies on the formulation of the problem as a set of nonlinear Karush-Kuhn-Tucker optimality conditions parameterized with respect to the initial conditions /10/.

\[
F = \begin{bmatrix}
\nabla J + \lambda^T \nabla h + \mu^T \nabla g_A \\
h \\
g_A \\
\frac{p_i + p_{i+1}}{p_{i+1}} - \theta \zeta
\end{bmatrix} = 0
\]

(7)

In equation (7) the first entry in the vector corresponds to the gradient of the Lagrangian with respect to \( x, y \) and \( u \) of problem (6). Vectors \( \lambda \) and \( \mu_A \) denote the equality and active inequality constraints Lagrange multipliers, respectively. The second entry corresponds to the feasibility condition under the assumption that the linear independence constraint qualification holds (i.e. the gradients of the equality and active inequality constraints are linearly independent). The third entry describes the feasibility constraint for the active inequality constraints. The strict complementarity condition is ensured with the positiveness constraint for the Lagrange multipliers corresponding to the active inequalities. Finally, the last entry in equation (7) determines the transition from the initial point \( p_i \) (at current time interval, \( i \)) to the target initial point \( p_{i+1} \) (time interval \( i+1 \)) along the direction \( \theta \) in the multi-dimensional space defined by the system states. Symbol \( \zeta \) denotes the independent continuation parameter for the problem. Hence, starting from a known optimal solution for the initial vector \( p \), the trajectory of optimal solutions for problem (6) as \( \zeta \) moves from zero (corresponding to initial vector \( p \)) to unity that corresponds to initial vector \( p_{i+1} \) estimated after the measurements have been obtained.

Equation (7) is an under-determined set of nonlinear equations as the overall number of variables exceeds by one the overall number of equations. The set is solved using a continuation method as implemented in PITCON /11/. PITCON uses a predictor-corrector solution method that automatically selects the continuation parameter (usually \( \zeta \)). However, the large dimensionality of the resulted problem necessitates the introduction of efficient sparse solution solvers for the Newton step during the corrector step of the solution algorithm. The numerical solver UMFPACK...
has been adapted within the PITCON software for improved speed of solution. Furthermore, the Jacobian of equation set $F=0$ involves the Hessian of the modeling equations $h=0$ and $g<0$ which has been analytically evaluated for enhanced accuracy and computationally performance. According to Seferlis and Hrymak /10/ at every continuation point the algorithm performs checks for possible violation of inactive inequality constraints including variable bounds, and for the satisfaction of the strict complementarity condition, i.e. $\mu_\lambda > 0$. Any violation of inequality constraint, variable bounds and sign changes for Lagrange multiplier corresponding to active inequalities results in an optimal solution active set change. In such a case equation set (7) must be suitably modified to reflect such change by either including a new active inequality or removing an inequality that ceased to be active. In addition, turning points in the optimal solution path or bifurcation points may result to optimality loss or violation of the LICQ. The procedure enables the identification of such critical points so that the solution path reaches the final point that reveals the optimal control actions for the next time interval.

5. EXAMPLE: CART WITH DOUBLE PENDULUM CONTROL

The nonlinear MPC position control with minimum sway of a cart with a double pendulum as shown in Figure 3 is considered.

Using Lagrange’s method the following modeling differential equations are derived for the system:

\[
\begin{aligned}
(M + m_z + m_2) \ddot{\theta}_1 &= (m_1 + m_2) l_1 \left[ \ddot{\theta}_1 \cos \theta_1 - \ddot{\theta}_2 \sin \theta_1 \right] + m_2 l_2 \left[ \ddot{\theta}_2 \cos \theta_2 - \ddot{\theta}_2 \sin \theta_2 \right] + b \dot{\theta}_1 = F \\
(m_1 + m_2) l_1^2 \ddot{\theta}_1 &= (m_1 + m_2) l_1 \left[ \ddot{\theta}_1 \cos \theta_1 \right] + m_2 l_1 l_2 \left[ \ddot{\theta}_2 \cos (\theta_1 - \theta_2) + \ddot{\theta}_2 \sin (\theta_1 - \theta_2) \right] + (m_1 + m_2) g l_1 \sin \theta_1 = 0 \\
m_2 l_2 \ddot{\theta}_2 &= m_2 l_2^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[ \ddot{\theta}_1 \cos (\theta_1 - \theta_2) + \ddot{\theta}_1 \sin (\theta_1 - \theta_2) \right] + m_2 g l_2 \sin \theta_2 = 0
\end{aligned}
\]

The state variables for the system are selected as follows:
\[ z_1 = x, \quad z_2 = \dot{x}, \quad z_3 = \theta_1, \quad z_4 = \dot{\theta}_1 = \ddot{x}, \quad z_5 = \theta_2, \quad z_6 = \dot{\theta}_2 = \ddot{x} \]  

(8)

The performance criterion for the system involves a penalty term for the deviation of the final position and velocity of the cart from the desired values in addition to time varying terms penalizing deviations for the sway angles \( \theta_1 \) and \( \theta_2 \) and the control effort.

\[
J = w_1 (x_N - x_f)^2 + w_2 (\theta_N - \theta_f)^2 + \sum_{i=0}^{N-1} \left[ w_3 (\theta_{i+1} - \theta_i)^2 + w_4 (\theta_{i+2} - \theta_{i+1})^2 + w_5 u_i^2 \right]
\]  

(9)

The prediction horizon is selected equal to one second. The discretization of the modeling equations is achieved through twenty finite elements utilizing 5th order Lagrange interpolating polynomials. Each finite element represents a control interval with duration of 50ms. Initially, the accuracy of the discretization method is verified through the comparison of the calculated state profiles for a given control sequence with the profiles obtained from numerical integration of the full order system using DASSL /13/. Figure 4 shows adequate accuracy of the time domain approximation technique exhibiting a maximum absolute error of \( 5.7 \times 10^{-4} \).

![Figure 4: Comparison of OCFEM and DASSL state profiles.](image)

The solution of the control problem as defined in problem (1) has been performed using the augmented Lagrangian solver MINOS 5.5 /14/ for the discretized system (6) and a sparse version of PITCON /11/ for the parameterized KKT conditions of equation (7). The two problem formulations involve 721 (equation 6) and 1415 (equation 7) variables, respectively. Two hundred simulation time steps have been calculated with actual control interval equal to 50ms, thus leading to an overall simulation time span of 10s. No computational delay has been considered in the implementation of the NMPC algorithm. However, the computational time has been recorded in order to compare the performance of the two procedures in terms of solution effort. DASSL has been used for the integration of the differential equations of the plant, whereas a model with mismatch has been utilized in the NMPC algorithm. The update of the state vector is based on the estimation of a bias term equivalent to an integration disturbance model. The control profiles and the state profiles for the NMPC are shown in Figure 5.

The NMPC algorithm successfully placed the cart close to the new target position while maintain the cart velocity and the two suspended masses swaying within acceptable limits. The computational time required for the solution of the NMPC problem (6) with MINOS 5.5 is equal to 130ms per control interval whereas the respective solution time for the solution of problem (7) with a sparse implementation of PITCON was 4.4ms (4GHz Intel processor). The proposed solution approach exhibited significantly better robustness as it fully converged at all simulation steps. On the contrary, the NLP solver MINOS 5.5 failed at a small number of simulation steps to converge to an optimal solution.
6. CONCLUSIONS

This paper presents a fast and efficient solution approach for nonlinear model predictive control applications. The proposed algorithm involves the formulation of the NMPC optimization problem as a set of nonlinear equations that represents the KKT optimality conditions after a suitable discretization of the system differential equation is achieved. The KKT optimality conditions are therefore parameterized with respect to the system's initial conditions. A sparse continuation method aims to trace the NMPC optimal solution path from a known point at the initial values set of the previous control time interval to the target point at the current initial values set. The identification of active set changes and the satisfaction of constraint qualifications and second-order optimality conditions are performed explicitly along the solution trajectory. The procedure exhibits superior performance in both solution effort and convergence robustness than solution methods based on conventional nonlinear programming when implemented in the benchmark control problem of a cart with a suspended double pendulum.
7. REFERENCES