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Systematic approaches to experimentation: The case of Pick's theorem

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ABSTRACT

In this paper two 10th graders having an accumulated experience on problem-solving ancillary to the concept of area confronted the task to find Pick's formula for a lattice polygon's area. The formula was omitted from the theorem in order for the students to read the theorem as a problem to be solved. Their working is examined and emphasis is given to highlighting the students' range of systematic approaches to experimentation in the context of problem solving and aspects of control that are reflected in these approaches.

1. Introduction

Mathematics to a certain extent is an experimental science as the well-known mathematician Halmos (2007, p. 94) states. The modern way of presenting mathematics in the educational context is mainly influenced by the 'Bourbaki' group. This group of mathematicians developed the system of "definition-theorem-proof" via textbooks which had tremendous influence over a multitude of topics in mathematics, mathematical teaching and mathematical pedagogy in general (Beaulieu, 1990). This effort, to present the mathematical knowledge in uniform way, left experimentation hidden. Fortunately, there are psychological theories of learning that are talking about experimentation as a factor that plays an important role in learning (Freudenthal, 1979). Historically, experimentation has been connected with the proof of theorems. Mathematicians usually are first convinced of the truth of a statement on the basis of experimentation before proceeding to its proof. It is reasonable to ask for conviction before spending considerable time generating a proof. In this spirit Polya (1962) in his *Mathematical Discovery* claims that if one wants a description of scientific method in three syllables the answer is "Guess And Test" (p. 156). Then he gives the rule: First guess then prove. This prompt means (always according to Polya) that one has to test conjectures, to examine particular cases, to consider and not neglect analogies, to combine observations, to try and try again (Polya, 1954). In one word: to experiment.

However, in the classroom setting a theorem that is introduced to the students demanding its proof creates a paradox. The students are asked to attempt to prove the truth of a statement without previously being convinced. This became the starting point of this study. It is true that in the context of problem solving a theorem could be considered as a problem that has to be solved. So, in this study, instead of giving a theorem and asking the students to prove it, Pick's theorem was modified and his formula for the calculation of area of lattice polygons was omitted. So, the theorem now became a problem for the students that had to be solved. This latter way of presenting a theorem allows students to experiment and formulate conjectures that later may be proved. This kind of working with theorems might be an alternative way to convince students about the necessity and importance of proof. Profoundly, this kind of situation does not contribute to the enhancement of the concept that is behind the theorem; however, it retains ties to that concept. The adequate content knowledge relevant to

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the specific concept is presupposed. Mamona-Downs and Papadopoulos (2011) call this situation problem-solving ancillary to a concept or ancillary problem solving.

Thus, the research questions for this study are: (i) How do students experienced on problem solving organize their experimentation while solving a task? and (ii) What aspects of control do they develop during their experimentation? In the next section the theoretical background for experimentation and problem solving will be presented together with some relevant research findings. What follows is the description of the study and a section devoted to the description of the students' working. After that in Section 5 the findings will be examined and commented on in detail. The paper ends with conclusions and some implications for future research.

2. Selected literature as background material

2.1. The theme of experimentation

Experimentation is strongly related to the progress in mathematics. Lakatos (1976) established a new tradition in the domain of the philosophy of mathematics by incorporating the notion of experiment in this science. According to him all the non-typical proofs are actually mental experiments, which mean syllogisms that are intuitively convincing but cannot be validated by using a strict logical proof. He used the term *thought experiment* for describing the situation mentioned above. Schoenfeld (1994) describes mathematics as: "an inherently social activity in which a community of trained practitioners (mathematical scientists) engages in the science of patterns-systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems defined axiomatically or theoretically ('pure math') or models of systems abstracted from real-world objects ('applied math')." (p. 60). According to De Villiers (2010) experimentation is employed when (i) mathematical conjectures and/or statements are numerically or visually evaluated by means of special cases, geometric construction and measurement, and (ii) conjectures, generalizations or conclusions are made on the basis of intuition, analogy or experience obtained through any of the preceding methods. He also distinguishes a series of functions of experimentation which are quite often closely linked:

- *conjecturing* (looking for an inductive pattern, generalisation, etc.);
- *verification* (obtaining certainty about the truth or validity of a statement or conjecture);
- *global refutation* (disproving a false statement by generating a counter-example);
- *heuristic refutation* (reformulating, refining or polishing a true statement by means of local counter-examples);
- *understanding* (the meaning of a proposition, concept or definition or assisting with the discovery of a proof.

This last one allows a distinction between the use of experimentation as a pedagogical approach for discovering certain properties or relationships or even solving a problem and its use in leading towards a proof.

In Buchberger's (1989) work which demonstrates the path of discovery of mathematical knowledge a *Creativity Spiral* scheme is proposed. This includes three phases: experimentation, exactification and application. The *phase of experimentation* includes a series of certain steps: Firstly, producing examples through known algorithms; then through examples we observe properties; finally, these properties are expressed as a conjecture. In the remaining two phases conjectures are turned to theorems (exactification) and algorithms are applied to real data (application).

Actually by experimentation a very high level of conviction may sometimes be reached even in the case of a lack of a rigorous proof (see for instance in Davis & Hersh (1981) the case of the unproved Riemann Hypothesis in terms of numerical evidence). Most mathematicians spend considerable time thinking about and analyzing particular examples which motivates future development of theory and gives a deeper understanding of existing theory. Two historical examples that are in favor of the importance of experimentation could be the cases of Gauss and Euler. Gauss claimed that his way to mathematical truths was through systematic experimentation. Euler made important discoveries on infinite series, in the Theory of Numbers, by observation, daring guess and shrewd verification. Experimentation can contribute to the discovery of a hidden clue or underlying structure of a problem (De Villiers, 2010), can facilitate the acquisition of a substantial part of knowledge (Kutzler, 2000), and is an integral part of doing mathematics (Wu, 1994). However, it is interesting that this facet of mathematical discovery is excluded from the main body of published work (Epstein & Levy, 1995). Very recently an interest towards experimentation emerged because of computers. The computer, as an extremely powerful tool for experimental exploration, has in the past few decades revolutionised mathematical research in several areas, resulting in many new and exciting results. One of its main advantages is that it provides powerful visual images and intuitions that can contribute to a person's growing mathematical understanding of that particular research area. Furthermore, the computer provides a unique opportunity for the researcher to formulate a great number of conjectures and to immediately test them by only varying a few parameters of a particular situation (De Villiers, 2010). It is more than profound that in such conditions we can refer to experimental explorations rather than to formal manipulation of objects and we can talk about experimental mathematics in the sense of utilizing advanced computing technology to explore mathematical structures, test conjectures and suggest generalizations (Bailey & Borwein, 2001). Kutzler (2000) comparing experimentation in the frameworks of today's curricula (paper and pencil) and of electronic assistant found that in the latter one there is no limit to the number of examples the students can do and this assistant guarantees the properness of the results. Especially for young students and according to Sfard and Leron (1996) working with computer corresponds to kid's notion of fair play since they are free to experiment and the rules for action (i.e., relationships between input and output) are clearly set and cannot be changed. Another consequence of computer experimentation that has been observed is psychological—the theory achieves concreteness and immediacy. It is different when the theory interacts with examples. Even if the computer is only presenting lists of numbers and symbols to one person, the fact that he/she has connected the theory with a physical object, a computing machine, can make him/her feel different about it. Contact with data and examples helps to keep one on solid ground and helps to avoid the crass blunders to which we are nearly all prone when we adopt purely formal methods of reasoning (Epstein & Levy, 1995).

The research finding that are related to experimentation in a non-technological environment are rather limited. Hiebert, Morris, and Glass (2003) found that treating lessons as experiments provides a more systematic way to engage students in activities by focusing attention on, and making more explicit, the process of forming and testing hypotheses, a process that is contained in most definitions of "experiment." They highlighted the fact that experimentation can be used as the basis for making more informed decisions later. Koleza and Iatridou (2004) working with pre-service teachers examined and analyzed the mechanisms of experimentation followed by them, working in four-person group, during mathematical problem-solving process. Their main finding was that the teachers used different types of representational systems to shape and present their thoughts starting with visual representations. Miyazaki (2008) tried to capture the perplexity and puzzlement some students feel when trying to establish the universality of propositions through experimentation/measurement. Some researchers use the term *investigation* which may seem to one as relevant to experimentation. Indeed, as mathematical activities, investigations and problem solving are very close and many people use these terms interchangeably. Although both notions refer to complex mathematical processes and point towards problematic activity, Ponte (2001, 2007) suggested that they also involve some differences. When one starts working on an investigation, the question and the conditions are usually not completely clear and making them more precise is the first part of the work. That is, investigations involve an essential phase of problem posing by the pupil—something that in problem solving is usually done by the teacher. Finally, Hanna (2000), criticizing the tendency among the educators to consider that deductive proof in geometry should be downplayed or abandoned in favour of an entirely experimental approach to mathematical justification makes a call for the need for a clear view of the relationship between experimentation and deduction. She claims that with the increased use of an experimental approach in the classroom there is increased scope for misunderstanding with epistemology at its root. So, it is a challenge for the educator to convey very clearly to students the interplay of deduction and experimentation and to convey that the experimentation in classroom aims to provide a chance for them to engage in deductive reasoning and to arrive at an understanding of existing relationships.

2.2. The theme of George Pick's Theorem

George Pick introduces two systems of parallel straight (in our case perpendicular) lines in the plane, having the same distance, named lattice (or grid) lines. A point at the intersection of two lattice lines is called lattice point. A polygon whose vertexes are lattice points is called lattice polygon. This means that all the sides of a lattice polygon are lattice lines. In 1899 Pick proved his known theorem. According to this theorem there exists a formula for calculating the area $A(P)$ of any lattice polygon: $A(P) = I + B/2 - 1$ ('I' denotes the points in the interior of the polygon whereas 'B' the points on its boundary).

This formula has been from time to time the object of different studies in different contexts. Brodie (1995) in the setting of concept acquisition, worked with 9th graders using a geoboard on tasks aiming to enhance students' acquisition of the concept of area and perimeter. Students were asked to find a general rule for the relationship between the number of internal and boundary points and the area of the shapes (i.e., Pick's Formula). The students were not able to resolve the conflict that emerged between the rule they developed and the actual area of the shapes, so an intervention from the teacher took place. Brodie suggests that it is necessary to view teaching as integral to learning rather than disruption of an otherwise spontaneous process. Bruckheimer and Arcavi (1995) dealt with Pick in the context of Number Theory. They followed Pick's conviction that educational advantage could be gained from bringing together apparently distinct mathematical topics (interdisciplinary approach). Pick in his original work used lattices and his area formula to prove the Fundamental Theorem of Number Theory. In addition the authors prove all the elementary properties of Farey series using Pick's geometrical approach. Bagni (1997) examined the introduction of the main concepts by Pick in high school (17–18 year-old) students. He emphasized the importance of lattice geometry in the didactics of mathematics. Hanna and Jahnke (2004) examined the role of thought experiments in general and a specific thought experiment for deriving Pick's formula. Finally Koleza and Iatridou (2004) examining preservice teachers' work on Pick's theorem from the experimentation point of view found the predominance of visual representations as "tools to think with". These representations were each group's most familiar geometrical figures gradually increasing in difficulty, at the same time avoiding figures with no obvious way for calculating their area. The fact is that most of their effort on experimentation was influenced by the lack of adequate knowledge of area of certain shapes.

2.3. The theme of problem solving

The framework within which this study took place is the one of problem-solving ancillary to a concept (Mamona-Downs & Papadopoulos, 2011). There are two traditions in problem solving: One regards problem solving as result oriented. The focus is on how the solution enhances the understanding of a certain concept. The other is strategy based and the focus is on how the solution itself is obtained. Schroeder and Lester (1989) refer to these two situations as "teaching via problem solving" and "teaching about problem solving," respectively. The second tradition is mainly represented by the work of Polya (1973) and Schoenfeld (1985) for which notions such as heuristics and executive control take central roles. But it does not usually place emphasis on a constant mathematical theme. The former one on the contrary tends to keep a succession of tasks relevant to a certain concept hoping that this could contribute to the acquisition and enhancing of the concept. However, sometimes this hope is not valid. If the solver possesses the required conceptual backdrop then his/her main focus is on how he/she could handle the already assimilated methodology associated with the concept. So on the one hand the problem-solving aspect has a context imposed by the concept and thus the solver acts in a particular sphere. On the other hand this connection to the concept does not mean a direct dependence on it. This is where problem-solving ancillary to a concept occurs. Usually, when a new concept is taught, a collection of techniques relevant to the concept is introduced didactically aiming to promote the understanding of the concept. This presupposes that the students are exposed for a long period to tasks related to the aimed concept (in our case that of the area). These tasks cannot be accomplished by merely being based on known formulas. Therefore, gradually, the students develop certain techniques for solving the problems and finally they create a collection of available techniques related to the concept. Papadopoulos (2010) working with 11–12 year-old students on the area of irregular plane figures found that this collection included techniques such as: division of a shape to subshapes, usage of the square grid, subdivision of an area unit, transforming the irregular shape to a known one, and so on. However, after a while, the usage of these techniques no longer enhances the acquisition of the concept and the solver starts to employ his/her mastery of the concept to organize the collection of the techniques he/she already possesses in order to tackle given tasks. That is, the techniques themselves, and not the concept, become the focus of examination and may be adapted. Mamona-Downs and Papadopoulos (2011) found that at this stage the students: act on the task's environment in order to allow the application of a technique; adapt known techniques; work in tandem with more than one technique for the same task; lay down the foundations of new techniques and apply known techniques not only to calculate/estimate areas but also to verify their solution. This is indicative of a process where problem solving evolves from an orientation towards the mere result to the appreciation of the strategy taken. The existent collection of techniques serves as a unified base that is rarely found in the tradition of "teaching about problem solving." Consequently the demand for some more exacting factors such as mature metacognition and executive control might be reduced. It is known that knowledge and awareness of one's own thinking strategies develop with age in children. Young children have a poor sense of their own thinking strategies and this improves as the children mature. Performance on many tasks is positively correlated with the degree of one's metaknowledge (Schoenfeld, 1985). However, in the context of ancillary problem solving the existent collection of techniques could serve as counterbalance reducing thus the demand for the above mentioned control factors. Ancillary problem solving possesses a ready-to-hand well-connected knowledge basis and this gives unusual support to the solvers' reasoning. This is in accordance with the current "state of art" in many nations as far as the problem solving in early years is concerned. Problem solving is viewed as important even at the very early stages of primary school so that the students start to develop relevant skills and heuristics (Arcavi & Friendlander, 2007; Doorman et al., 2007).

3. Students' background and setting of the study

There are two 10th graders, a boy (Nikos) and a girl (Katerina), participating in this case study. In this grade, students in Greece are taught for their first time the notion of theorems and their formal proof. The way of teaching is as follows: the wording of the theorem is initially presented and immediately follows the proof. Usually this is accompanied by examples and exercises. The students have to accept a priori the truth of the theorem's statement. This is not fair for the students since this way of teaching gives the students the impression that theorem is some sort of magic performed by extraordinary people when it is the result of hard work and intuition after examining many special cases. Obviously this way of usually working in the Greek classroom does not permit students to experiment with the notion behind the theorem. Thus, they miss an important part of the process that allows one to be firstly convinced about the potential truth of a statement before proceeding to its rigorous proof. We believe that the problem-solving framework could allow the development of these preliminary attempts towards the formulation of a theorem via experimentation. So, it was important that the two students had an accumulated experience on problem solving. During their last two grades of primary and their first of secondary school and through their regular classes in mathematics, these students were taught some basic concepts of geometry, concerning the recognition of certain 'basic' shapes (mostly limited to triangles, quadrilaterals, and circles) and some of their properties. They were also familiar with the role of formulae as devices for calculating the area of these shapes. In parallel to the normal teaching they participated in a project conducted by Mamona-Downs and Papadopoulos (2006, 2011). This project had two stages. The first aimed to explore and enhance students' comprehension of the concept of area with an emphasis on problem-solving techniques for the estimation of the area of irregular plane figures. The students' participation in this phase of the project gave them on the one hand sufficient content knowledge that was prerequisite for problem-solving ancillary to the concept of area and on the other hand experience in the usage of various techniques enabling them to calculate the area of some irregular shapes. Some techniques that could be mentioned are the usage of grid, the subdivision of a shape to sub-shapes, the usage of units and sub-units, the cut-and-paste technique and the usage of length measuring tools based on the dots of the array provided in the tasks. In the second stage, given that the students already possessed the requisite conceptual backdrop, the focus shifted to the handling of the techniques themselves as already described in the previous section. During the above mentioned project these two students were generally successful in obtaining correct answers and when they had to overcome difficulties on the route they exposed modes of thinking that were essential to achieve the tasks.

They also used to show explicit differences in their problem-solving behaviour and it was expected that the same would also happen in this study. These two characteristics might explain why they were chosen. Taking account of this experience of our students we challenged them to discover Pick's theorem. More specifically students were asked to discover Pick's formula that allows the calculation of the area of any lattice polygon. Instead of presenting the theorem with the formula included in its statement, our idea was to modify and present it as a problem which had the formula as its solution. Here is the wording of the task:

> A point at the intersection of two or more grid lines is called lattice point. A lattice polygon is a polygon whose vertices are lattice points. In 1899 the known mathematician George Pick (1859–1942) proved that there is a formula for the calculation of the area of any lattice polygon. This formula includes the number of lattice points on the polygon's boundary as also the number of lattice points in its interior. Try to find this formula.

There were three reasons we chose this task. First, the negotiated concept is area and so there is a direct link with the students' past experience concerning problem-solving ancillary to this concept. Second, this task is suitable for experimentation since one can generate and test several examples so as to find the formula. Finally, we used Pick's Theorem as a situation where the two variables I (internal points) and B (boundary points) do not determine a single 'shape', so working merely on the variables is not sufficient to fully analyze the situation. So, in that sense, this perspective could be an instructive intervention from the teacher's point of view. This ensures that the students by initially discovering the formula and checking numerous cases will be convinced about the validity of their findings and then the typical proof will be the answer to their reasonable question: "Why does this happen?" Perhaps, it could be possible for the students by adopting an algebraic approach to set up a possible equation as $A = xI + yB + z$ and find the desired relationship algebraically based on the usage of simple cases. However, since our students (and according to the official curriculum) lacked the required algebraic knowledge, this approach was not an expected one.

The students worked individually for an hour without intervention from the researcher. They worked on a dot paper and they were asked to vocalize their thoughts while performing the task (thinking aloud protocol). This was conducted on the lines of protocol analysis as set out by Schoenfeld (1985) and Ericsson and Simon (1984). Given that students can only access a certain aspect of their thinking, the Thinking Aloud Protocol does necessarily mean a congruency between the students' thinking and their utterances. Protocol analysis gathered in non-intervention problem-solving sessions is especially appropriate for documenting the presence or absence of executive decisions in problem solving, and demonstrating the consequences of those decisions (Schoenfeld, 1992). Moreover protocol analysis in character minimises the interference of the researchers. Students were tape-recorded and then the data was transcribed for the purpose of this paper. Besides these transcribed protocols our data included the worksheets of the students which were collected after the problem-solving session. The students were asked to keep everything written on their worksheets without erasing anything and this is why more than one worksheet was provided to them. After collecting the data the students' protocols were parsed into episodes (always according to Schoenfeld's (1985) proposed *Framework for Analysis of Problem-Solving Protocols*). Each episode was characterized as one of the following: reading, analysis, planning, implementation, exploration, verification or transition (juncture between episodes). Emphasis was placed on the points at which:

- instances indicating that students organize their experimentation to facilitate the correct solution could be identified,
- actions showing aspects of control could be monitored.

4. Results

4.1. Katerina's path to Pick's formula

Katerina initially spent some time familiarizing herself with the task and making clear what the lattice polygons really are. Therefore, she started drawing a series of polygons. Her first shape was a square 3 × 3. But she was not yet able to experiment on the basis of the number of the points. Her second thought was to create the right triangle that was the half of the square. Observing analogies (i.e., the arithmetical relationship of 1:2 between the two shapes) was her starting point towards experimentation. The direction she followed (based on analogy) was that the desired formula is: *number of boundary points divided by two*. Obviously this did not work. Her next step was to turn to two extreme cases trying now to determine the role of the number of points in the interior and on the boundary. Thus, she drew the smallest square ((1 × 1), no internal points and four boundary ones) and a quite large rectangle (7 × 5). Very soon she rejected both. The small one did not provide much information. The large one was not manageable due to the complexity of the required calculations:

K.2.35 In these shapes there are such big numbers that it is not easy to make the necessary operations.

In this extract (and in each extract from now on) the first letter (K or N) refers to the name of the student (Katerina and Nikos). The number next to the capital letter refers to the number of the task since in our research project there was more than one task. Finally, the last number indicates the line of the transcribed protocol. Therefore, K.2.35 means that this is the 35th line of Katerina's protocol concerning the 2nd task.

Her next step was to try to find any relation between the numbers of interior (I) and boundary (B) points by simply executing various operations such as B/I, or $B/2 + I$. It is true that in the case of $B/2 + I$ she almost reached the correct formula

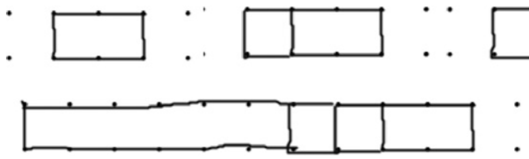

Fig. 1. Katerina's way of working.

Table 1
Katerina's collected data.

Area	2	3	4	10	12
Boundary	6	8	10	22	26
Interior	0	0	0	0	0

but this was an accidental result and this is why she did not pay the required attention. Thus, she decided to use a table to organize her data and she tried to check her first conjecture: the ratio (*area of the shape*):(*number of boundary points*) is constant. However, working with two polygons of area 9 and 4.5 she found that the number of boundary points was 12 and 9 respectively and this made her abandon this conjecture since $9/12 \neq 4.5/9$. Realizing that it was difficult to come to a conclusion she decided to eliminate one of the components of the asked formula. This is why she decided to work with polygons that have the same number of internal points. Firstly she considered polygons without any internal point. She started with a rectangle of area 2. Then she continued increasing the length of the rectangle so as to have always a rectangle of a bigger area but without internal points (Fig. 1). Every time a new polygon was considered she made the necessary calculation of the area and counted the number of boundary points (Table 1).

Looking at the numbers, what was firstly observed was that the bigger the number of boundary points the larger the area. But very soon she found that:

K.2.114 When the number of internal points is zero then the area of the polygon can be found applying the formula Area = $(B/2) - 1$.

Her next step was to increase the number of the internal points from zero to one. She started with a square of area 4, with 8 boundary points and one internal. She applied her previous finding:

K.2.116 Area 4, boundary 8, internal 1.
K.2.117 Then, boundary divided by 2 equals 4, minus 1 equals 3.
K.2.118 So, in order to find 4 for the area I have to add 1.

This resulted in her second statement:

K.2.119 When the number of internal points is one, then the formula is similar to the previous one (boundary points, divided by two, minus one) plus one (i.e., the number of internal points).

And she verified this result for a series of polygons with different number of boundary points and one internal.

Then she went one step further

K.2.119 I have to apply the same for other polygons with a different number of internal points.

Indeed she verified the validity of the formula for polygons with more than one internal point. She concluded as follows:

K.2.131 So, in any case the formula is: number of boundary points divided by two, minus one, plus the internal points, equals the area.

Finally, she thought that she had (and for the sake of the generality of her formula) to apply this formula on a completely new shape described by her as "random" one.

K.2.134 I have to check this now for a random, weird shape.

This is why she drew a triangle whose sides were not in the vertical–horizontal (prototype) position, expressing thus the randomness of the shape.

She ended her effort by wondering:

K.2.146 I am sure now that my formula is correct. But I do not know why this is true.

4.2. *Nikos's path to Pick's formula*

Nikos's first choice was to start with a series of squares. The first one was of side length 1 and he continued gradually increasing the side length, one unit per time (Fig. 2). He thoroughly examined each new square and transferred the requisite measurements to a table. Considering these numbers he concluded that changing the number of boundary points does not necessarily influence the number of internal points.

His second choice was to find the polygon with the minimum number of points. He found that this condition is satisfied with a polygon having three boundary and zero internal points (see the first triangle in Fig. 3). This shows how he purposely checked the 'number of boundary points' variable. Then he started to successively draw triangles with gradually growing

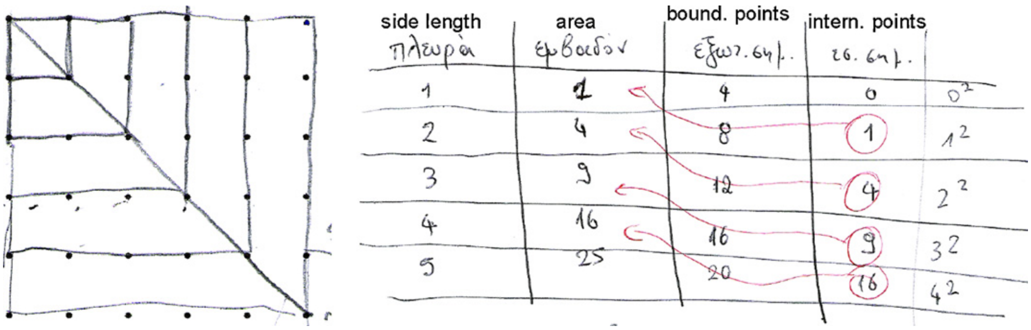

Fig. 2. Nikos's collected data.

number of internal points (0–5) but always with the same number of boundary points (i.e., 3) and made predictions about shapes with more internal points. His past experience on problem-solving ancillary to area helped him to calculate the area of these triangles by using a specific technique: He outlined each triangle by a rectangle and then he subtracted the triangles that were exterior to the initial one considering them as its complements in reference to the rectangle.

He organized the relevant data to a table and his main aim was to find a pattern in this data:

N.2.91 There is a constant rhythm in the produced numbers
N.2.95 Whenever the internal points are increased by one, the area of the triangle is also increased by one.

His next choice was to verify the validity of his previous finding working with quadrilaterals. Thus, he completed a table without drawing any quadrilateral. He predicted what would be their area if there were four (or five) boundary points and the internal ones follow the succession 0, 1, 2, 3, 4. This helped him to be aware of the fact that in the formula the number of internal points is always added. He explained for example in the case (3,0) (see Fig. 3) that the area is 0.5 whereas in the case (3,3) the area is 3.5. Subtracting the two areas, the difference (3.5 – 0.5 = 3) is the number of the internal points in the second shape (i.e., 3). This was verified several times and for a variety of polygons. Thus, what remained was to find the contribution of the number of boundary points to the formula. For this he subtracted the number of internal points from the area number. His aim was to guess how this difference was connected with the number of boundary points. He made an observation that seems arbitrary. Starting with his first triangle (boundary points: 3, internal: 0, area: 0.5) he explained:

N.2.170 The triangle is defined by two collinear points plus **one** more point
N.2.171 The square (*boundary points:4, internal:0, area:1*) is defined by two collinear points plus **two** more points.
N.2.173 For the triangle, **one** divided by two equals 0.5 (*i.e., the area*)
N.2.174 For the square, **two** divided by two equals one (*i.e., the area*)
N.2.181 So, I have to use letters. Let's denote with B the number of boundary points and with I the number of internal ones.
N.2.182 So, the formula is, Area = $(B-2)/2+I$ (*which actually is Pick's formula*).

After that Nikos applied his formula to irregular lattice polygons (Fig. 4) ("random" ones) verifying that his formula is a valid one.

N.2.184 In order to verify the formula I will check an irregular shape.

What followed was a post-reflection regarding his problem-solving process. He evaluated this process and emphasized on the significance of his decisions: (i) to work independently with the boundary and internal points, (ii) to summarize the data on tables and (iii) to ask for a pattern in the produced numbers.

In the end he also expressed his questioning:

N.2.240 It satisfied me that I found the formula. And I am sure that I could apply this formula for every polygon. However, I am not able to justify why my formula is the correct one.

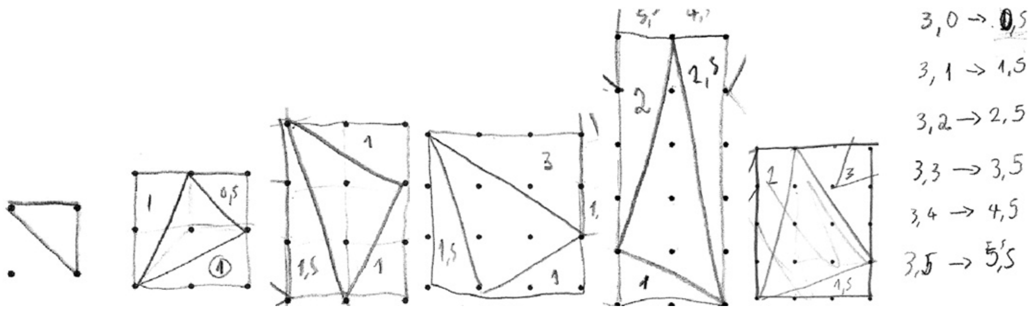

Fig. 3. Increasing the number of internal points.

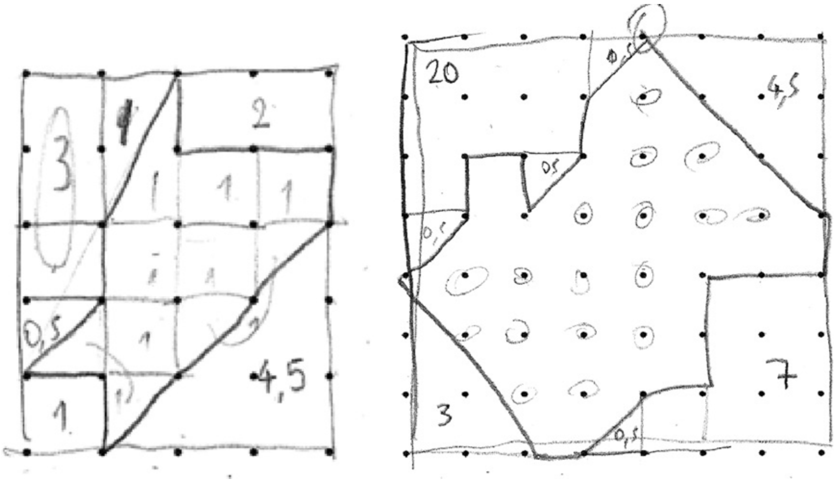

Fig. 4. Applying the formula to irregular polygons.

5. Discussion

5.1. Control aspects during experimentation

In Pick's Theorem there are two variables I (number of points in the interior of the shape) and B (number of points on the shape's boundary) that do not determine a single kind of shape. The students realized that working merely with these two variables was not sufficient for analyzing the situation. Therefore, they had to experiment by selecting various kinds of shapes. However, there is an aspect of control in choosing a variety of examples for experimentation.

Katerina started choosing a square. Square is a known shape and she was familiar with it and its relevant concepts such as the area of a square or units of area measurement. The fact that her only criterion was whether the shape was a familiar one shows that for the moment she was not able to experiment. Her second shape continued to make her feel comfortable since it had an arithmetical relation with the square (2:1) and this made her work in the context of her past problem-solving experience: *follow analogies*. Her analogical thought signalled her first step of experimentation by checking whether the required formula was: number of boundary points divided by two. Her next choices were two extreme cases of shapes something which was also in accordance with the spirit of the problem-solving tradition. These extreme cases were the smallest square (1×1) and a large rectangle (7×5). The first one was considered by her as inappropriate since it contained very little information. She also rejected the large shape due to the complexity of the required calculations (K.2.35). Her intuition seemed to be correct. Relevant results of Koleza and Iatridou (2004) showed that the groups of preservice teachers that dealt with large shapes delayed reaching the correct formula (in contrast to the groups with small shapes who did manage to find the formula sooner). Moreover, this rejection of the large shapes indicates the control ability as far as the selection of the shape is concerned. Gradually this experimentation resulted in the formulation of certain criteria concerning the shapes that serve the task's solution better. Finally she gathered her findings into a table elaborating them in two different ways. This was a crucial part that facilitated experimentation and even more the correct solution of the problem. One can start from the number of the points asking for relation that result in the wanted area or vice versa starting from a known area and asking for the relations that connect the number of boundary and internal points. The gathered results also facilitated the formulation of conjectures and at the same time these findings were used for checking or rejecting these conjectures. So, firstly she found the formula in the case of shapes without internal points (K.2.114), made some tests to check its validity and then progressively reached the final formula (K.2.116–K.2.119).

Nikos from the very first moment showed that he was able to experiment on the basis of his past experience. He decided to start with something simple and he gradually proceeded to more complex shapes, something that is important in experimentation. He organized his findings into a table trying to understand in depth any relationship occurring between the variables when changes are made to one of them. In a similar way he observed analogies (making predictions for shapes having internal points more than 5 on the basis of the previous findings) and asking for the simplest shape or the "extreme" one. Asking for the simplest shape is evidence of his awareness concerning the control of the "internal points" variable. His intention was to experiment altering the number of internal points so as to find a "rhythm" expressing the way area is changing (N.2.91, N.2.95). This way of experimenting supported him to discover the role of internal points to the wanted formula. As an experienced problem solver he always tried to explain and interpret the behaviour of his findings thus establishing the connection between what he found by calculation and what he predicted (N.2.170–N.2.174). Finally, it is interesting to note the emphasis he placed on the usage of variables regarding that using variable would make his result more valid and acceptable (N.2.181–N.2.182).

Nikos's work could be considered as more interesting than Katerina's. It is not just the fact that he uses more 'irregular' shapes which would seem more convincing grounds to believe the conjecture in general. But, there is an interesting dual control process. First to produce triangles with a given number of interior points, second to calculate their areas (with the tools that ancillary problem solving provides) and to check if it meets with the working conjecture, or at least, helps in recognizing a pattern.

5.2. Systematizing the approaches of experimentation

In the ancillary problem-solving framework students not only develop certain techniques associated with a certain concept but moreover they act on these techniques adapting, extending or combining them. Apparently, this special perspective of problem solving relates to more general aspects of problem solving such as forming conjectures, verification processes and assessing, exploring and executive control. The task of Pick's formula has its origin in the concept of area thus allowing the students to resort to their past experience gained by their participation in an earlier project within the framework of problem-solving ancillary to the concept of area. Both students showed a broad range of relevant actions exploiting the tools provided to them by ancillary problem solving. They used certain techniques for calculating the area of irregular polygons such as the decomposition of the shape to sub-shapes or the outline of the irregular polygon with a rectangle. They verified their findings systematically. They checked the validity of their partial findings applying them to polygons with larger number of points either on the boundary or the interior. They felt the need to post-reflect on their working path evaluating the significance of their decisions. The new element in this work is the organizing of their experimentation to cope with the task.

The two students followed different paths of experimentation but the key-concept of their approach was the same:

> Identify the structural components of the problem. Then keep all but one unchanged and experiment with this one, changing it in various ways so as to identify its role in the problem's solution. Then keep this component constant and change another one, and so on, until making clear how all these components contribute to the solution.

Katerina decided to temporarily ignore the number of internal points so as to identify how the number of boundary points is involved in the formula. For this, she selected shapes with no internal points and started to gradually change the number of boundary points. Then she continued increasing the number of the internal points by one. Examining the data gathered she managed to correctly find the formula. Nikos firstly defined what he called "the simplest shape," that is a triangle with only three boundary points (its vertices) and no internal points. This is in itself an important mathematical act indicating a deep knowledge of the structural elements of the task. He based his strategy on identifying the so-called 'simplest shape' that is in essence the basic shape constituted by the minimum number of boundary and internal points. Then he kept constant the number of boundary points and progressively increased the number of internal points discovering their role in the formula. Finally he dealt with the role the boundary points play in Pick's formula. So, it can be seen that in the first case the student completely eliminated the component of the internal points whereas in the second case the student neutralized the component of the boundary points. But, both of them obeyed the general approach of experimentation.

Following the preceding it seems useful to emphasise a difference between Nikos's and Katerina's problem-solving behaviour. This difference will explain why these two students were chosen. Even though they both get the right result/conjecture, the sample of examples they checked were qualitatively different in character. This actually showed that for Nikos the conjecture was more 'secure' than for Katerina.

However, there is a certain point the students failed to achieve: to go some way towards obtaining a general justification. Both of them examined a certain number of cases resulting in a conjecture verified by these specific cases. They considered that this was sufficient for accepting the universality of the formula, given that the formula was valid for a 'random'/'irregular' shape (K.2.134, N.2.184). They failed to realize that the success of a specific 'random'/'irregular' shape does not guarantee the generality of the formula but it constitutes the starting point for moving to the formal proof. However, we could attribute this to the fact that they were in their early stages of familiarizing themselves with formal proof. Besides, we did not expect them to provide a formal proof since this was beyond their abilities. However, we did expect them to wonder 'why' their formula was valid for any shape.

Although the students did not turn towards obtaining a rigorous justification, aspects of proof can be recognized in the students' experimental approach. The evident behaviour of both students to vary one variable whilst the other is held constant gave the chance for the students to make a local conjecture first, and to obtain the full 'version' when the parametric variable was also varied. Reaching the right conjecture is important for moving to proof. Experimental evidence frequently plays a role not only in the initial formulation of a conjecture but also in continuing efforts to prove particular results (De Villiers, 2010). There are aspects also of the process of solving a proof problem (Nunokawa, 2010) that can be mirrored in our students work. According to Nunokawa, by experimentation students obtain new information about the situation. In our case the role of internal and boundary points could be considered as a piece of new information. New information deepens students' understanding which in turn makes them able to build explanations based on that understanding (see for example Nikos's explanation about the simplest shape). Then, it often becomes clear which aspects are critical and the students can loosen or exclude non-essential conditions (for example, Nikos did not take account of the numbers in the last column in Fig. 2, presenting the 2nd power of the squares' side lengths). Finally it is worth mentioning that both students after obtaining the formula started to wonder "why" this formula is valid for every shape (K.2.146, N.2.240). This is important since it constitutes the starting point for moving from an experimental convincement to the typical proof. It is indicative of an understanding of the necessity of the proving process as something that comes to give an answer and explain why a statement is true.

6. Conclusion

The way theorems are usually taught in the Greek classroom is as follows: The wording of the theorem is presented to students and immediately the typical proof of the theorem follows accompanied by examples and exercises. This way does not give any insight into how a person could have dreamed up the theorem. Profoundly in such an approach there is no space for experimentation and in the case of the Pick's theorem it leaves the students with the impression of an extraordinary mind who invented the formula instantly. Instead of this we proposed students read the wording of the theorem as a problem that has to be solved. This approach allows students to experiment and through experimentation the solvers make some steps towards typical proof by exploring, deepening their understanding, explaining, conjecturing, verifying the validity of their conjectures.

Both students systematized their experimental approach. But for applying experimentation in an effective way the sufficient content knowledge relevant to the concept is prerequisite. Ancillary problem solving ensures the acquisition and enhancement of the concept and so, experimentation could not be hampered due to the lack of relevant knowledge. This perspective may lessen the problem described by Koleza and Iatridou (2004) where the participants' experimentation was mainly influenced by their lack of relevant knowledge leading them to other directions rather than the solution of the task. Both of our students expressed their systematic experimentation by controlling the components of the task that play a crucial role in the solution process. Two different kinds of control can be distinguished in their effort:

i. Experimentation by controlling the "shape" component. The students tried to control the variation either of the kinds of the selected shapes (triangles, squares, rectangles, irregular polygons) or of the shapes' size.
ii. Experimentation by controlling the relationship among the variables of the wanted formula (number of internal points – number of boundary points – area of the shape).

Obviously we cannot make generalizations since we worked with two students and this study could be better considered as a case study. However the richness of our findings gives strong support on the one hand to offer to mathematics teachers another approach of introducing theorems to their students and on the other hand to design a future research focusing on the importance of experimentation as an innate factor of successful problem solving. In this latter case it would make more sense to have more students with varying backgrounds trying to face this task. Then, it would be interesting to make connections between the students' backgrounds and their approaches since some of them might have needed considerable support, others much less, and others may not even have reached the correct relationship between *I* and *B* and the area.

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