THE PROBLEM-SOLVING ELEMENT IN YOUNG STUDENTS' WORK RELATED TO THE CONCEPT OF AREA

Joanna Mamona-Downs*          Ioannis Papadopoulos**
University of Patras, Greece*   University of Macedonia, Greece**

There are extant many studies that examine primary and early secondary students' existing and developing understanding of the concept of area for plane figures. In this paper, the focus is shifted to consider the problem-solving skills that may accrue from exposition to tasks related to the calculation of area. In particular, the working of two 7th grade students on one specific task is examined vis-à-vis certain executive control issues about the selecting, handling and adaptation from a body of previously known methods concerning area determination.

INTRODUCTION

Empirical studies made by educators investigating students' understanding of mathematical concepts often utilise non-standard tasks in order to test and challenge fully the students' cognition about the notions under examination. However, the introducing of non-standard tasks might bring in aspects of problem solving based on strategy making, which does not necessarily contribute any new insight about the concept image. (See, e.g., Tall & Vinner, 1981, for the idea of the concept image.) On the other hand, the stratagems made will tend to refer to the relevant conceptual backdrop implied in the task environment. When the strategy facet dominates the students' attention, we shall say that the students are engaged on problem-solving activity ancillary to a concept; this is to be contrasted with concept enhancement evinced from exposition to non-standard tasks. As the strategy facet is still relevant to the concept, though in a more operational or functional role, it can be regarded intermediate to teaching about problem solving and teaching via problem solving, a distinction often made in the literature (for example, Schroeder & Lester, 1989).

In this paper, we shall present part of a study that, by careful design of the tasks given to the participating students, encouraged problem-solving activity ancillary to the concept of area. This study was motivated by an earlier research project involving primary students geared to examine and to extend understanding of area and its preservation under certain actions. The results from this earlier undertaking, though revealing many new interesting angles through its novel, computer aided, teaching material, had similarities to those found in other papers treating the same topic at the same level (for example, Baturo & Nasons, 1996; Clements & Stephan, 1998). However, it came to our attention that the students had problems in setting up their methods in order to apply their conceptual knowledge (beyond appropriately evoking it). This problem-solving aspect deserves explicit examination, but this perspective
Mamona-Downs & Papadopoulos

has not been taken up explicitly in research as yet. Hence the new study presented here.

The students were 7th graders who had participated in the project in earlier years; the topic area still deals with area measurement. Given the limitation of the length of the paper, we restrict ourselves to one case study involving one task and two students. This case is chosen because it displays much interesting executive control exercised in the handling, and even adaptation, of a body of methods in calculating or comparing areas known by the students. (For a full exposition of executive control, see Schoenfeld, 1985.) Clearly, focusing on these methods means that the control must relate to the conceptual background. Given this, the following issues are of particular significance:

- Did the students exploit the source of previously known methods and if so, how?
- What was the quality of the rationale behind the changes of direction in approach that the students made?
- What modes of verification did the students employ?

The paper will describe the problem-solving activity ancillary to the area concept evident in the two students' working, especially with regard to the three issues listed above concerning executive control.

STUDENTS’ BACKGROUND, DESCRIPTION OF THE TASK AND METHODOLOGY

The two participants (Nikos and Katerina) were 7th graders attending secondary school in Greece. Through their regular classes in mathematics, they were taught some basic concepts of geometry, including certain shapes and their properties (mostly limited to triangles, quadrilaterals, and circles). They also knew the formulas for the calculation of the area of each kind of these shapes.

However, in addition both students took part in earlier stages of our research project that took place in parallel with the normal teaching. Their attendance of this gave them experience in the usage of various tools enabling them to calculate area of more irregular plane figures. The tools available to them included: the usage of grids in a geo-board, the subdivision of an area unit (usually a square) into sub-units, length measuring tools that allowed calculating area especially for the separate parts of a decomposed region, and the cut and paste method. For the limited problem-solving component at this stage of the project, both these students showed themselves particularly adept. Our expectation, then, was that they would fare better than other students when the design of the tasks emphasized control skills; such design aspects were aimed for in the final stage of the project. In this paper we select one of the tasks from the final stage and present the students' problem-solving behavior for it.
Figure 1: Presentation of the task

Our rationale in forming the above task is the following:

The students were familiar with the cut and paste technique, and indeed had experience with tasks where several cut and paste actions were required. However they were never confronted with the circumstance where multiple actions were required in such a way that the different actions were inter-dependent, as it is in the case of the task given here. In terms of problem solving the special interest of the task is to see whether the students could coordinate the two actions; the making of the first action must anticipate the second. Another interest is whether the students would attempt to answer using other methods despite the directions given in the task statement.

This task is drawn from a session of about two hours where four other tasks were given. The students worked individually and were asked to vocalise their thoughts while they were performing the task. This was conducted on the lines of protocol analysis as set out by Simon & Ericsson (1984). Protocol analysis gathered in non-interventive problem solving sessions is especially appropriate for documenting the presence or absence of executive decisions in problem solving, and demonstrating the consequences of those executive decisions (Schoenfeld, 1992). Protocol analysis in character minimises the interference of the interviewers (the authors), but it was desirable to use more direct questioning concerning the motivation of the students’ working. In order to do this, we interviewed the students a few days after the session. Both sets of data were tape-recorded, transcribed and translated from Greek into English for the purpose of this paper.

RESULTS

Katerina’s problem solving processes.

After reading the task, Katerina started immediately on a putative solution procedure. She initially constructed the grid squares lying completely in either the given triangle or rectangle. An influence for this is that over the last two years, there was an accumulated experience where the task environment included such an array of dots, hence the construction of the grid squares was a familiar strategy for her. The next step (also according to past experience) was to divide the partial square units into...
Katerina very quickly made clear that the c-part is the common area between the two shapes, a finding that would help her to proceed to the solution of the problem. But the way she approached the solution after this decision was purely arithmetical. She tried to estimate the area of each sub-shape based again on grounds of square units formed from the array of dots provided. The existence of partial square units within the shapes was an obstacle; as a response, she made a second change of direction. She noticed the right angle in the d-part of the rectangle outside the triangle. Accordingly, she decided to 'move' the a-part (a right-angled triangle) inside the 'excess' part of the rectangle such that the right angles coincide.

A question remained how she managed to draw the hypotenuse of the triangle in its new position. In the interview her response was:

“I counted the dots. I knew that the right angle fits perfectly in the upper left corner so I counted the distance between the edges”.

Then, she transferred the b-part as it is shown in Figure 2. Finally, Katerina appreciates the conservation of area in this context. In the question concerning which area was bigger, her response was: “Just the same. Since no piece is left over”.

Niko’s Problem Solving Processes.

Nikos initially spent an amount of time to be familiar with the task before deciding how to proceed. His initial thought was that the ABX triangle was an isosceles one (Figure 3). He tried to prove it by measuring the length of the two supposedly ‘equal’ sides. This was based on the dots of the array provided but his measurements were inaccurate because the array allows exact measurement only
horizontally or vertically. He obtained approximations of the real length. This finally prompted him to declare that his effort was useless. At this point he made his first shift:

N.4.17 Perhaps I have to cut this triangle... that is outside and put it in the interior of the other shape....

He drew then the BDW and EWX triangles. He had already made an appropriate dissection but as yet did not see how to transfer the resulting parts into the rectangle:

N.4.46 I have to find a triangle that is exactly the same with the AEC one.

Because of his inability to work in a geometrical context, he turned to an arithmetical one, for a second time during this session, by comparing lengths. At this point he stated that the point E is the middle of the line segment AX.

N.4.71 The E point separates the AX segment line into 2 equal parts. So, the AE segment line fits exactly to the EX one.

N.4.72 A region is left over... It’s a right angle

N.4.76 It means that I have to put the triangle so as the EX side to be adjacent the EA side.....

N.4.81 It seems logical that the two triangles EWX and BDW will have together the same area with the AEC triangle.

N.4.82 I have to verify that this is true. I will find the area of the two former triangles and I will compare it with the area of the AEC one.

Despite that his intuition informed him that the two triangles together had the same area with the third one, he felt that he had to be sure about that. He again resorted to an arithmetical approach. He measured approximately bases and heights, he applied the known formula for the calculation of the area of a triangle but the two outcomes were different. He accredited this to inaccurate measurements.

N.4.89 It means that I probably made some errors during the process of the calculation of the area of one of the shapes.

N.4.90 I have to look it again, to try again.

His instinct, then, made him to insist to show that his initial intuition was correct, suggesting that this intuition was so strong to make him to assume that his failure for verification was due to erroneous calculations. Indeed when we asked him later why he insisted, he said.

“I was pretty sure that the area of EWX and DWB together was the same as the area of AEC so, when I could not confirm it arithmetically I was convinced that it was due to my erroneous calculations and consequently I had to try again with numbers”.

Figure 3. Nikos’ partition of the triangle
Finally he came back to the geometrical approach.

N.4.114 I will cut the DBW triangle and I will adjust it to the ACE angle.
N.4.115 Then, I will cut the EXW triangle and I will put it so as the side EX to be adjacent to the AE side and the W vertex to look towards the C point.

Nikos’ explanation why he delayed to reach the solution is interesting:

N.4.122 I think I delayed to reach the solution because I dealt from the very beginning with the formulas and I did not consider it as a single shape.
N.4.123 I did not try to find the relationship between the shape I was asked to make through the transformation and the already existed one.

DISCUSSION

Below we interpret the results from the fieldwork:

1) What techniques did the students employ?

Katerina read the problem and immediately chose to apply one method from the stock of previously met methods dealing with area measurement. There is evidence that this decision was influenced by the fact that an array of dots were provided in the presentation of the task, and this acted as a cue to argue in terms of completing and counting unit squares lying completely in the figure. (In Mamona-Downs, 2002, it is claimed that some configurations (‘cues’) act as a mental trigger to access particular domains of knowledge.) This was done despite of the direction to use the cut and paste method. Nikos’s initial behaviour on encountering the task contrasts with that of Katerina. First, he took some time in familiarising himself with the task environment and in making some preliminary exploration. This could be related to Polya’s suggested first step 'getting Acquainted' in obtaining a solution in problem solving, (see Polya, 1973, p. 33). Second, Nikos’s starting point involves a structural conjecture (a particular triangle is isosceles) and brings in his past experience in measuring lengths within the context of the array of dots as a validation device. In fact both students employed this method to check on their geometric ideas in other places. However, perhaps more significant was that in the end both students assigned the first transfer of region not exactly fitting in with the cut and paste protocol. This is because the shape is put into the frame of the rectangle, but not such that one side is shared with the figure that it had been cut from. The heuristic in Polya about ‘can you use the result?’ seemed to influence them to widen old methodology into one that is more flexible and approaches the more general image of dissection, as described in Hartshorne, 2000, p. 213.

2) The decision making

Both students made various changes in approach in their solving activity. Katerina rejects her original idea employing the grid because this method did not meet with the task specification of ‘two movements’. However, this reason would seem a side consideration towards explaining her previous remark: ‘What I have done is useless’. Clearly, she found difficulties in her method, but instead of trying to articulate these she picks out a task specification that she had previously neglected. However, this
proved to be in practice a useful act of control; if a method that one is using does not seem to be working, look back at the task formulation to make sure that there is not a clue there how to proceed in another direction. This allowed Katerina to completely change her focus, and she creates a partition of the triangle into three parts. She states a motive for doing this; one part is common to both the triangle and the rectangle and so would be invariant, leaving just the other two parts to be transferred. This could be regarded as an act of control, based on exploiting perceived structural similarities (Mamona-Downs & Downs, 2005). Nikos started his work by trying to show a triangle was isosceles, where there did not seem much purpose in doing this vis-à-vis the task requirement. (Taking such blind directions and their effect have been reported in Schoenfeld, 1985.) However, Nikos soon rejects this approach, but like Katerina, does this out of practical considerations: the soundness of the basic idea remains unchallenged. Later, Nikos encounters a clash between some data obtained from measurement and his geometric intuition. He makes a decision: to regard the measurement as unsound, and to direct his attention to strengthen his argument based on visualisation (such that the role of measurement would become redundant). This he does quite convincingly in the end. As a final note, one of Nikos' s closing remarks (N.4.123) suggests that he felt in retrospect his working should have been more directed towards what was required, again pointing to the heuristic 'Can you use the result'.

3) Verification

In previous exercises for which these students encountered the method of cut and paste, the ‘transferral of area’ was perceived by eye; the more sophisticated context of this task, though, made both students feel the need to verify that the two transferrals indeed achieved what was desired. This in itself is an important act of control; the matching of the pieces was not so transparent that it could be left unargued. The two students finally did the verification differently; Katerina’s was based on measurement, Nikos on visualisation. Notice that Katerina’s line of verification was far more utilitarian compared to Nikos’, so likely Nikos’ final apprehension of the solution was the more insightful.

CONCLUSIONS

Executive control is concerned with the solver’s evaluation of the status of his/her current working vis-à-vis the solver's aims. In general, this requires mature deliberation in projecting the potential of the present line of thought, married with an anticipation how this might fit in with the system suggested from the task. Schoenfeld (1985) has indicted that many undergraduate mathematics students have very poor executive control skills. On the other hand, some quite young students do seem to have some ability to make deliberate decisions that lead to effective changes in approach. (Schoenfeld, 1992). The paper contributes to the research question: What executive control skills can we expect from younger students? Our study, involving 12 year olds, reveals in particular:
Mamona-Downs & Papadopoulos

(i) Students can have the ability to adapt and extend known methods in response to a novel problem-solving situation, via understanding that the situation affords a broader approach.

(ii) Students can affect changes in approach, but the evidence from this study suggests that these changes are mostly motivated by not being able to advance rather than pin-pointing why the approach is not functioning as wished. Students are able to take advantage of overt structural features appearing within the task environment to frame their strategies.

(iii) When students understand in outline a likely way of solving the task, they can insist on forming verifications rather than just assuming that their ‘mental’ plan will work out.

References


