ARE USELESS BRACKETS USEFUL TOOLS FOR TEACHING?
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In this paper two populations of different cultural settings (Italy and Greece) are examined. The aim is to explore whether the presence of brackets affects the pupils’ performance on simple equality number sentences as well as whether brackets influence their understanding of the symmetric property of equality. The findings in both countries showed that when pupils work with brackets they get better scores in the success rates and give evidence of relational thinking.

INTRODUCTION
Hoch and Dreyfus (2004) working with students in the context of solving equations found that students react differently in the presence or absence of brackets. They found that the presence of brackets by giving a clue where to look and by focusing the students’ attention affected positively the use of structure sense, acting more or less as a supporting prosthesis. Thus, the students dealing with a compound term as a single entity, and trough an appropriate substitution, recognise a familiar structure in a more complex form and this is a fundamental ingredient for obtaining structure sense (Hoch & Dreyfus, 2010). This is in accordance with Radford (2000) who following previous studies found that instead of seeing signs as the reflection mirrors of internal cognitive processes they rather are considered as tools or prostheses of the mind to accomplish actions as required by contextual activities in which the individuals engage. So, the practice of brackets as “external components” could be an appropriate way for pointing out this aspect as far as the “internal components” are concerned. In other words, brackets as “external components” facilitate the recognition of internal components. Moreover since brackets are a typical feature of the first steps in algebraic thinking, it is a question whether their presence affects also the relational thinking (Molina, Castro & Mason, 2008) (pupils’ thinking makes use of relations between the elements in the sentence and sentences are considered as wholes instead of as processes to carry out step by step).

However, even though brackets are a kind of signs used very frequently in mathematics, researchers are not yet able to find reports or results on how the brackets affect pupils’ success in solving equations (Hoch & Dreyfus, 2004). Given that there is a lack of such a research in primary school we wanted to explore whether and in which direction the presence of brackets affects the performance of children when working on simple examples of equality relations (equations), in grades 2 and

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3, i.e., before an explicit use of brackets in their first steps towards algebraic learning. Additionally we would like to record instances showing the presence of relational thinking in the pupils’ work and whether this is connected to the presence of brackets. The study took place in two different cultural settings (Italy and Greece). The innovative nature of this study relies on considering lower primary school grades (instead of ones in secondary or high school) so as to see whether the aforesaid cognitive aspect is still present in these early school years and in what extent this aspect could be developed properly even from the very beginning of mathematical learning.

DESCRIPTION OF THE STUDY

This study is only a facet of a wider and multipurpose investigation about equality. The questionnaire we used is a modified version of the one proposed by Parslow-William and Cockburn (2008) and it is comprised of two parts that include number sentences as ‘equations’ with one-digit numbers. The first part (Level 2a) deals with addition whereas the second (Level 2s) deals with subtraction. Perhaps this choice seems limitative, but a careful reading of the questions shows that the way the questions are formulated requires simultaneously an awareness of both addition and subtraction. From the original questionnaire we obtained a new version adding useless brackets (Table 1), in order to detect whether in lower school grades the presence of brackets affect the results (possibly in a positive manner).

<table>
<thead>
<tr>
<th></th>
<th>Level 2a with brackets</th>
<th>Level 2a</th>
<th>Level 2s</th>
<th>Level 2s with brackets</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>7 + 2 = ☐</td>
<td>(7 + 2) = ☐</td>
<td>9 - 3 = ☐</td>
<td>(9 - 3) = ☐</td>
</tr>
<tr>
<td>b)</td>
<td>5 + ☐ = 8</td>
<td>(5 + ☐) = 8</td>
<td>7 - ☐ = 2</td>
<td>(7 - ☐) = 2</td>
</tr>
<tr>
<td>c)</td>
<td>☐ + 4 = 9</td>
<td>(☐ + 4) = 9</td>
<td>☐ - 4 = 5</td>
<td>(☐ - 4) = 5</td>
</tr>
<tr>
<td>d)</td>
<td>☐ - 3 = 4</td>
<td>☐ - (3 + 4) = 4</td>
<td>☐ - 4 = 1</td>
<td>☐ - (4 - 1)</td>
</tr>
<tr>
<td>e)</td>
<td>5 - ☐ + 1</td>
<td>☐ = (☐ + 1)</td>
<td>5 - ☐ = 1</td>
<td>5 - (☐ - 1)</td>
</tr>
<tr>
<td>f)</td>
<td>8 - 5 + ☐</td>
<td>8 - (5 + ☐)</td>
<td>5 - 7 - ☐</td>
<td>5 - (7 - ☐)</td>
</tr>
<tr>
<td>g)</td>
<td>5 + 4 - ☐ = 8</td>
<td>(5 + 4) - (☐ = 8)</td>
<td>5 - 4 = ☐</td>
<td>(5 - 4) - (☐ - 8)</td>
</tr>
<tr>
<td>h)</td>
<td>6 + 2 + 3 = ☐</td>
<td>(6 + 2) + (3 + ☐)</td>
<td>6 - 2 = ☐</td>
<td>(6 - 2) = (9 - ☐)</td>
</tr>
<tr>
<td>i)</td>
<td>1 + ☐ - 6 = 2</td>
<td>(1 + ☐) - (6 + 2)</td>
<td>7 - ☐ = 8 - 2</td>
<td>(6 - ☐) = (8 - 2)</td>
</tr>
<tr>
<td>j)</td>
<td>☐ - 3 = 7 - 2</td>
<td>(☐ - 3) - (6 + 2)</td>
<td>☐ - 3 = 7 - 5</td>
<td>(☐ - 3) = (7 - 5)</td>
</tr>
<tr>
<td>k)</td>
<td>5 + ☐ = 6 + ☐ - 7</td>
<td>(5 + ☐) = (☐ - 7)</td>
<td>6 - 8 - ☐</td>
<td>(6 - ☐) = (8 - ☐)</td>
</tr>
<tr>
<td>l)</td>
<td>9 = ☐</td>
<td>9 = ☐</td>
<td>5 - 4 - 7 - ☐ = ☐</td>
<td>(5 - 4) - (7 - ☐) = ☐</td>
</tr>
<tr>
<td>m)</td>
<td>5 + 4 + 6 - ☐ = ☐</td>
<td>(5 + 4) + (6 - ☐) = ☐</td>
<td>8 - 5 = ☐</td>
<td>(8 - 5) = (6 - ☐)</td>
</tr>
<tr>
<td>n)</td>
<td>4 + 3 = 2 + ☐ - ☐ - 1</td>
<td>(4 + 3) - (2 + ☐) - (☐ - 1) = ☐</td>
<td>8 - 5 = ☐</td>
<td>(8 - 5) = (6 - ☐)</td>
</tr>
</tbody>
</table>

Table 1: A unified version of the two levels of the questionnaire
During school year 2009/2010 we administered the questionnaire to 292 pupils including 140 of grade 2 and 152 of grade 3. The classes involved were 7 for grade 2 and 7 for grade 3. All the Greek participants were pupils of two primary schools in Thessaloniki whereas the Italian participants attended primary school in various villages near Parma. The worksheets were given to the pupils in two sessions in different days. During the first week the 2nd graders completed firstly the test of addition and then that of subtraction; the duration of each session was about 15 minutes. In the next week 3rd graders followed in the same order and the duration of each session was about 10 minutes. Both the length of the questionnaire and the time given for each sheet performance were chosen deliberately for our research aims, since we assume that shortage of time might reveal in a better way the child’s deep beliefs when engaging in a problem solving activity.

Each of the participating class was divided into two parts roughly equivalent. During the session one of the two parts worked on the without brackets version whereas the other one worked on the version of the same test, with brackets. This ensured the comparison of the results of the different tests. In this way, we get for Level 2a, 283 protocols (141 without brackets and 142 with brackets; for Level 2s, 276 protocols (142 without brackets and 134 with brackets). It is not strange that there are different numbers for the different levels since the sessions took place in different days and so it was not possible for various reasons to keep the participants’ number constant.

**Analysis of the questionnaire.**

A main element of the current questionnaire is the usage of an empty square for denoting the unknown. Considering the available ways of denoting the unknown and according to the existed literature we resulted to the widespread usage of an empty square, known also as the ‘scaffolding style’ (Hejný & Slezáková, 2007). We expected that this choice might engender possible misunderstandings: the expectation that the pupils could interpret every time the same sign in the same way in case they had an empty square more than one times in the same expression. For example, in items 2a.k, 2a.m and 2a.n and 2s.k, 2s.l and 2s.m there exist, at least, two empty squares in the same expression. This is not something new in mathematics. There is an implicit rule for the pupils that whenever a sign is present a lot of times in a series of tasks then the certain way of interpreting the sign in the first task is the rule for interpreting the same sign in all the tasks. But, despite that the issue of using the multiple presence of the unknown in the same number sentence has been the focus for various research studies (Sáenz-Ludlow & Walgamuth, 1998; Radford, 2000; Hejný & Slezáková, 2007) the case of the identical interpretation of the symbols for the unknown has not been considered yet.

From the morphological point of view we can easily classify the questions based on different criteria: i) the number of addition/subtraction signs, ii) the number of equality signs and, iii) the number of empty boxes. Another feature that plays a crucial role is the relative place of the box(es) in the sentences in relation to the signs
of +, -, and =. All the aforementioned morphological features can give hints about: the combinatorial ideas that inspired the questionnaire; the way the items were organized so as to examine whether the position of the empty square in the number sentence and more generally the types of sentence structure could influence pupils; the gradual increase in the items level of difficulty - the first six questions in each part are at the same time the simpler ones from the cognitive point of view and the shortest ones (except 2a.1) and they deserve a particular ‘attention’- these six questions play an introductory role to familiarize pupils with the forthcoming increase in difficulty of the next questions.

Due to the limitation in the number of pages we restrict ourselves to present just one possible classification of the tasks that is relative to the operation(s) needed to solve them (Table 2).

<table>
<thead>
<tr>
<th>Level</th>
<th>one addition</th>
<th>one subtraction</th>
<th>first addition then subtraction</th>
<th>first subtraction then addition</th>
<th>two subtractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2a</td>
<td>a), d), b), c), e), f), g), h), i), j), m).</td>
<td>n)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2s</td>
<td>c), e)</td>
<td>a), b), d), f), g), j), h), i)*, l), m)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Operation algorithm(s) involved in the standard solution of the tasks.

Items in bold, when seen as equations, can be solved if the unknown (empty square) is interchanged with the number that is in the other side of the equal sign. This algorithm can be followed since the empty square is at the right of the minus sign. The items k) are not included in this Table (for both Levels) since this equation is actually a Diophantine equation. Item 2a.1 is not included in Table 2 since it is about an equation presenting at the same time its solution. For items 2s.i and 2s.i-bra it could be mentioned that actually the algorithm is similar to the rest of the items in bold font.

We do not restrict our expectations to these known algorithms since we know that young students are able very often to surprise us by choosing other ways for solving the tasks that are completely irrelevant to the standard algorithms, obtaining thus even quickly the correct answer. Some of them could be the Subtraction by complement (for example, items 2a.b, 2a.c, 2a.e, 2a.f, 2a.g, 2a.h, 2a.i, 2a.j, 2a.m, 2s.b, 2s.f) and the Invariantive property of addition or subtraction (balance) (for example 2a.g, 2a.h, 2a.i, 2a.j, 2a.k, 2a.m, 2s.g, 2s.h, 2s.i, 2s.j, 2s.k, 2s.l, 2s.m).

Moreover, we expect our findings to give some evidence concerning pupils’ early relational thinking. We will focus on the symmetry of equality relation. It is necessary to make a distinction between the pair 2.a.b and 2.a.f (5 + \( \bar{y} \) = 8 and 8 = 5+\( \bar{y} \) respectively) and the remaining five pairs (2a.a-2a.d, 2s.a-2s.d, 2s.b-2s.f, 2a.c-2a.e, 2s.e-2s.e). In the first pair both items have as their solution the number 3. Therefore,
the role of symmetry is obvious since the involved numbers are identical (‘strict’
symmetry). In the remaining five pairs we can speak for symmetry ‘at large’ but only
for the structure of the number sentences and not for the numbers involved. For
example, a solver of the 7-\( \tilde{y} \) = 2 who has trouble with 5 = 7-\( \tilde{y} \), can think of this
second task in terms of 7-\( \tilde{y} \) = 5 to find the right answer. These tasks give us the
chance to detect which pupils are ‘blinded to the symmetric property of the equality’,
i.e., incomplete understanding of the symmetric property of the equality (Attorps &
Tossavainen, 2007) either in its ‘strict’ or ‘at large’ meaning: a (correct) answer at
only one question of these pairs may suggest a lack of an ‘at large’ application of the
symmetry in the specific pair at the trial (or success) level.

RESULTS-DISCUSSION

The collected protocols showed two significant types of data: we call them trial and
success. When a child tries to cope with a problematic situation in mathematics then
there are two options: either to get round the task or to attempt finding a solution.
Both reactions belong to the realm of the behaviour and this is why it is not an easy
task to investigate the reasons that make a pupil to give up from being engaged in a
problem solving process. There are a lot of potential explanations: tiredness,
inadequate mastery of skills relevant to the certain mathematical topic, lack of
available time for completing a task, lack of motivation, state of health and so on.
The presence of answers in the test’s items (no matter whether it is about a correct or
wrong answer) is indicative of the pupil’s positive disposition towards the task, of a
familiarity, self-reliance, self-confidence. We adopted the term ‘trial’ as a measure of
the positive disposition towards the task and we denote with T its percentage.
‘Success’ refers to the right answers to the test’s items and we denote with S its
corresponding percentage.

The answers collected can give some evidence about the learners’ awareness as far as
the specific mathematical topic is concerned. Obviously, it is not possible to know in
a dogmatic way whether a wrong answer is an evidence of incomplete mastery of
certain skills or knowledge. When a statistical elaboration takes place, then, it may
happen some times to consider lack of answer and wrong answer as identical events,
otherwise a distinguish between them has to be made.

For the evaluation purpose we adopted a twofold kind of characterization. The first
one is related to trial rate (T) whereas the second to success rate (S). Thus, in relation
to the first kind, a task is characterized as adequate if T \( \geq \) 60\%, and inadequate in the
remaining case. For the second, a task is characterized as very hard if S \( \leq \) 30\%, hard
if 30\% < S \( \leq \) 60\%, easy if 60 \% < S \( \leq \) 80\% and very easy if 80\% < S.

According to these kinds of evaluation our data showed that in a global level the
questionnaire was adequate for both Levels: Level 2a (T = 91.44\%) and Level 2s (T =
88.88\%), but it was hard: Level 2a (S = 56.99\%) and Level 2s (S = 52.26\%). Given
that our focus was on the potential contribution of brackets we split the data in two
groups: brackets and no-brackets versions. The corresponding percentages obtained
were as in Table 3. It can be seen that the test was again considered as adequate and hard for each of its versions no matter the cultural setting. With a $\chi^2$-test we obtained a statistically significant difference only between the two versions of Level 2a (=0.04%).

<table>
<thead>
<tr>
<th>T(rial) whole sample</th>
<th>S(sucess) whole sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>2a no bracket</td>
<td>91.19%</td>
</tr>
<tr>
<td>2a bracket</td>
<td>91.70%</td>
</tr>
<tr>
<td>2s no bracket</td>
<td>89.06%</td>
</tr>
<tr>
<td>2s bracket</td>
<td>88.69%</td>
</tr>
</tbody>
</table>

Table 3. Items’ global evaluation concerning trial and success

The presence of brackets seemed to be useful for 22 of the items in both levels. There were items such as 2a.a., 2a.l., 2s.c., 2s.h. and 2s.i. that got better success rate in the no-bracket version. In more details the success for the entire questionnaire is presented below in Table 4:

<table>
<thead>
<tr>
<th></th>
<th>very hard</th>
<th>hard</th>
<th>easy</th>
<th>very easy</th>
</tr>
</thead>
<tbody>
<tr>
<td>2a no bracket</td>
<td>g), k), m), n)</td>
<td>h), i), j), l)</td>
<td>d), e), a), b), c), e), f)</td>
<td></td>
</tr>
<tr>
<td>2a bracket</td>
<td>m), n)</td>
<td>g), h), i), j), k), l)</td>
<td>a), b), c), d), e), f)</td>
<td></td>
</tr>
<tr>
<td>2s no bracket</td>
<td>g), j), l), m)</td>
<td>e), h), i), j), k)</td>
<td>c), d), f)</td>
<td>a), b)</td>
</tr>
<tr>
<td>2s bracket</td>
<td>g), j), m)</td>
<td>h), i), k), l)</td>
<td>c), e)</td>
<td>a), b), d), f)</td>
</tr>
</tbody>
</table>

Table 4. Items’ global evaluation in relevance to success for Levels 2a and 2s

Item in italics indicate that for this item the obtained success rate was the highest among the two versions (bracket and no-bracket).

From the overall results we have evidence that the use of (useless) brackets supported better performance in terms of success rate. It seems that brackets’ function was to give ‘unity’ to different terms connected by a sign of operation. It can be said that they are useful for understanding correctly the equality sign since this can be easily misunderstood as another operation sign. The case of item 2.a.c ($\Box +4 = 9$) is a suitable example. In the version without brackets we found pupils from both grades (2nd and 3rd) who proposed number 13 as the solution for the empty square. Obviously this was the result of the operation $4+9=13$. However, no one in the no-brackets version (($\Box +4 = 9$) suggested 13 as the solution of the task.

There were also examples with lower scores in the brackets version rather than in the no-brackets one. As an example we present item 2.s.i: $7-\Box = 8-2$ (no brackets
version) and \((6 - □) = (8 - 2)\) (brackets version). For both versions the commonest wrong answer was 6, with relative frequency of \(\frac{25}{65}\) (38.46%) for the bracket version and \(\frac{33}{65}\) (51.54%) for the other one. The fact that the different numbers were used in each version possibly was the reason for getting lower scores in the bracket version. This is connected to the known fact that handling of zero (as a solution) is a great obstacle for younger pupils. Therefore, instead of using zero as the wanted number in the expression \((6 - □)\) they were lead to the 6 by a mistaken interpretation of item 2.s.i working only on a part of it: \(□ = 8 - 2\).

Another surprising finding was that exactly the same task, in the same place in both versions got different success rate. The item 2.a.1 was presented in the form \(9 = □\) in both versions. A potential explanation for the wrong answers in the brackets version (success rate 45.77% vs 56.74% for the no-bracket version) could be that this task was the only one without brackets in a worksheet where all the other items included brackets. The commonest wrong answer was 10 and this could be the result of reading the equality sign as a ‘separator’ along the line number.

For the relational thinking we mentioned earlier that for both items 2a.b and 2a.f the correct solution is 3. In case there was a correct answer only for one of the two items this could be attributed to an incomplete mastery of the formal property of equality. For the remaining pairs we presume that a right answer to one question of the pair and the firm awareness of the symmetry of equality can suggest a good strategy for solving the other question of the couple, even if the numbers are not identical. The simplicity of both tasks (2a.b and 2a.f) had as a consequence great success rate: 95% (99%) for 2a.b and 84% (88%) for 2a.f; percentages in brackets show the rate that corresponds to items in the bracket version. The percentages for those who did not give identical solutions to these two items were 18% and 13% for the no-brackets and brackets versions respectively. These numbers mean that there exists blindness concerning ‘strict’ symmetry. The accumulated rates for the other five pairs (that show blindness concerning ‘symmetry at large’) for the whole levels 2a and 2b were respectively 24% (16%) and 34% (30%). These percentages strengthen the thesis that the usage of useless brackets contribute to the reduction of the phenomenon of ‘blindness’ the symmetric property of equality.

**CONCLUSIONS**

It is widely accepted that an understanding of equality is crucial in developing pupils’ algebraic thinking. A typical feature for the first steps in this development is the usage of brackets in numerical expressions. In this study we tried to use ‘useless’ brackets aiming to highlight their potential contribution to pupils’ performance as well as to detect presence of relational thinking. Our findings showed that in general the presence of brackets increased the success rates of the involved pupils showing at the same time a reduction in phenomena of ‘blindness’ concerning the symmetric property of equality. The findings were more or less similar for both countries something that gives an extra support to our hypothesis that the usage of brackets
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could affect positively pupils no matter the cultural setting. However, it must be said that after a comparison that was made between the official curriculums in both countries and from the cognitive point of view, the situation found to be similar so much for the Greek as well as for the Italian pupils.

Our origin was the view that brackets help pupils see structure but the major research findings mainly concern older students. However, in this study we tried to add and demonstrate another aspect for using brackets and at the same time to legalize this ‘useless’ way of applying brackets when a teacher is engaged in fostering early algebraic thinking to very young students. Such a perspective constitutes useless brackets valuable teaching tools in the teaching practice and make teachers able to respond the their students’ needs for an early understanding of equality either in terms of performance or acquiring relational thinking.

References


