Pairing numbers: An unconventional way of evaluating arithmetic expressions

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We have analyzed the written solutions to four arithmetic expressions comprising addition/subtraction and multiplication of 4 to 7 numbers from 235 students (112 in Greece and 123 in Sweden, 11-12 years old). Beside the order-of-operations and sequential (left-to-right) calculations, several students seem to calculate in an unconventional way by pairing numbers. That is, there are students who combine the numbers two-and-two, seemingly regardless of the operations, and then operate on these pairs when evaluating an expression. In our data, pairing seems to be as frequent as sequential calculations. We identify the characteristics of three qualitatively different ways of pairing. The different ways seem to be applied independent of the length of the numerical expression.

Keywords: Arithmetic, Order of operations, Precedence rules.

Introduction and theoretical background

In mathematics, we use conventions for arithmetic expressions to be unambiguously perceived. However, to design successful teaching on how to perceive and calculate arithmetic expressions, we need to first learn what ideas are present among the students. In this report, we present a study of those ideas and can show that evaluating numerical expressions by number pairing seems to be more significant in students’ ideas than previously reported, and that it can emerge in different types.

Without the mathematical conventions like, e.g., the rules for the order of operations, there would be different ways in which numerical expressions can be evaluated. Accordingly, it has been shown that students who have not yet internalized these conventions use different ways in their calculations (Blando, Kelly, Schneider, & Sleeman, 1989; Glidden, 2008; Gunnarsson, 2016; Headlam, 2013; Libenberg, Linchevski, Sasman, & Olivier, 1999). This might result in errors such as exchanging a number with another number, exchanging an operation with another operation, or, simply, making computational errors. Blando et al. (1989) call these errors “careless” and “substitution errors”. But, there are also errors that are more related to the structure of the expression. Linchevski and Livneh (1999) showed that students, while being inconsistent in the computational aspects, can be very consistent in these structural errors.

There has been substantial research devoted to students’ handling of arithmetic expressions from perception and cognitive points of view. It has been shown in several studies that the mere proximity of two numbers can result in that these are calculated first, regardless of the operator in between (Gómez et al., 2014; Jiang, Cooper, & Alibali, 2014; Landy & Goldstone, 2010; Rivera & Garrigan, 2016). More generally, it has been shown that the visual appearance of the mathematical notation can
cause students to disregard mathematical rules (Kirshner & Awtry, 2004). A particular effect of the visual appearance was pointed out by Linchevski and Livneh (1999). They showed that some students can dissociate two parts of an expression, often induced by an operation sign (typically minus), like, e.g., when $50 - 10 + 10 + 10$ is evaluated as $50 - (10 + 10 + 10)$.

In addition, there are several educational research studies showing that students can perceive the structure of numerical expressions in different ways. Kieran (1979) has shown that the rule ‘to be calculated first’ can be perceived as ‘to be put to the left’. Conversely, it appears as if students perceive what is read first (i.e., to the left, if that is the reading order) should be calculated first. At least, several studies describe that students calculate expressions sequentially from left to right (Blando et al., 1989; Headlam, 2013; Liebenberg et al., 1999).

There have also been brief reports that some students pair up numbers when processing arithmetic expressions (Gunnarsson, 2016; Liebenberg et al., 1999; de Villiers, 2015). While de Villiers (2015) mentions it as one type of erroneous calculation (without further discussion) where order of operations should have been used, Liebenberg et al. (1999) describe students who break up expressions in parts. As example, they find students calculating $4 + 5 + 5 \times 2 \times 6 + 4$ by grouping numbers in pairs and then processing these pairs as $4 + 5 + 5 \times 2 \times 6 + 4 \rightarrow (4 + 5) + (5 \times 2) \times (6 + 4) = 9 + 10 \times 10$. However, we have noted that, apart from the example above, the expressions used in most previous studies are quite short and contain two (or maximum three) different operators. Typically, the numerical expressions are of the form $a + b \times c$, occasionally $a + b \times c + d$. The exception found in the study of Liebenberg et al. (1999) with longer expressions (up to seven numbers, thus six operations), may have enabled them to observe a breaking-up-into-parts effect. Hence, in this study we would like to explore this effect further by using long expressions.

Therefore, with the idea that teaching arithmetic conventions must take its point of departure in students’ actual ideas on how calculations should be completed, the aim of this study is to further explore students’ idea of pairing when calculating arithmetic expressions. In short, we would like to find what forms of pairing mechanism occur when students evaluate arithmetic expression. This study is part of a larger project to design teaching to improve students’ arithmetic computational competence.

**Method**

This study engaged 112 Greek students and 123 Swedish students, 11-12 years old (grade 5 in Sweden, grade 6 in Greece). In total students from nine different schools (five Greek schools and four Swedish schools) were included. The schools were chosen from mixed background to represent all achievement levels. According to the curriculum and the teaching traditions in the two countries, the students should have been taught the rules for the order of operations prior to the study. The students were given a written test and sufficient time (no time limit) to answer it. The test comprised several tasks, all typeset with the equation editor of a conventional word-processor, to avoid unintended influence from the visual appearances of the expressions. The test was aimed to test students’ understanding of arithmetic conventions. Four of the tasks were specifically designed to explore students understanding of the order of operations by expressions of different lengths. These tasks are shown in Figure 1.
The data, comprising the students’ written answers to the test – their written evaluations of the expressions in Figure 1 – was subject to a qualitative content analysis as described, e.g., by Leedy and Ormrod (2015). In the analysis, the correctness of the answers was not considered, instead the underlying mathematical idea were in focus for each solution of a numerical expression. In the first step we were looking for well-known ideas, like following the order of operations, sequential calculations (left-to-right), detachment (from Linchevski & Livneh, 1999), and signs of pairing. Hence, data, for each task separately, was first divided into different groups depending on what ideas seemed to have governed the students while solving the task. In some cases, the students used different ways throughout the process of evaluating a single expression. Then, the different groups were combined based on the main underlying mathematical idea the solutions seemed to be based on. This resulted in a few different categories of foundational mathematical ideas.

In the second round of analysis, we focused only on the solutions that have a character of number pairing. Here, no fixed categories were defined prior to the categorization. The categories of ways of pairing numbers emerged from the data in the analysis process. Data were examined separately by the two researchers and discussed to resolve discrepancies. We did not see any significant difference in the data of Swedish and Greek students. Hence, in this study we have not separated the data of the groups. The numbers below are the combined total numbers for the entire data set.

**Results and discussion**

First, we note that only a minor part of the errors in the students’ solutions is due to arithmetic errors. We do find the same type of errors as reported in previous studies (like, e.g., by Headlam, 2013), such as changing operators (e.g., multiplication is replaced by addition), changing numbers and simple arithmetic errors. But, the major part of the operations explicitly described in the data is correctly calculated. Even in the cases of changing numbers or simple arithmetic errors we find that it is easy to trace back the intention of the operation – the actual underlying mathematical idea. This is in agreement with the findings of Linchevski and Livneh (1999). Hence, we can conclude that to a large extent the data (the result of the evaluated numerical expressions) originate from students’ ideas on how numerical expressions should be evaluated.

Second, we find three main ‘ideas’ that seem to influence students’ solutions: *order of operations*, *sequential calculation* and *pairing*. In our data, we find students who follow the convention and use the (correct) order of operations, and we find students who use a sequential calculation (left-to-right), see Figure 2. As an example, a student that would have calculated the second expression (b in Figure
1) according to \(((((5 \times 4) \times 3) + 6) \times 2) + 1\) is considered to have followed a left-to-right, sequential, strategy. This does not mean that the specific student is always following that strategy, but at least did so in this specific expression.

Some students indeed are consistent in their calculations, in the sense that they calculate all four expressions in the same way, or with the same apparent idea behind the calculation. Most of the students who are consistent have used the order of operations. But there are also students who consistently have used a sequential calculation on all four expressions. For instance, two students consistently used one specific type of pairing (which we denote ‘entangled pairing’ further down in the report). In total, we find that the order of operations was used in 47% of all expressions, sequential calculations in 12% of the expressions, and that pairing seemed to be the main idea in 16% of the expressions in the students’ solutions.

Typically, students’ difficulties with numerical expressions from other aspects of precedence, like e.g. detachment (as from Linchevski & Livneh, 1999) or visual salience (Kirshner & Awtry, 2004), are included in the category named “other”. In task c, \(2 \times 3 \times 5 \times 4 \times 2 + 6\), we note that many students evaluate the expression as \((2 \times 3 \times 5) - (4 \times 2 + 6)\). We believe this is directly related to the detachment-effect reported by Linchevski and Livneh (1999). However, the number of solutions that seem to be based on detachment is also the reason why the “other” category is relatively much more frequent for the (c) expression, as shown in Figure 2.

Additionally, as will be shown further down (Fig 4 c), there are examples in the data where students have used a pairing strategy in the first term \((2 \times 3 \times 5)\), but then, seem influenced by detachment and used that in the continuation of the calculation. Hence, it could be a combined pairing and detachment effect. In Figure 2, these examples have been counted as “pairing”.

Particularly, in the data from the first task we found several students’ responses that could be described as due to pairing. Although, we cannot exclude the possibility that there are other ideas that have influenced the students’ behavior and made them evaluate the expression as \((9 - 2) \times (3 - 2)\). When we explored the mathematical ideas, we found that the pairing mechanism can come in qualitatively different forms.
**Pairing overall**

The first type, and perhaps the easiest to separate from the others, is a mechanism of pairing throughout the entire expression. It is a pairing that goes overall. This occurs when the pairs span across different operations. It seems as when one pair of numbers has been “used”, the attention is directed to the next pair. No consideration is taken to the precedence of operations. Instead, it seems that numbers are the focus of the attention. Figure 3 shows three different examples where students have used pairing overall to evaluate the expressions. The brackets were made spontaneously by the students.

![Figure 3: Three examples of students' answers where pairs have been formed across the entire expression without consideration of what operation should precede the other (“pairing overall”)](image)

This type was particularly evident when there was an even number of numbers in the expression, as in the expressions a – c (cf. Figure 1). We also find examples where students seem to have started with the pairing approach in the expression d with odd number of numbers. One such example is shown in Figure 3(c). Here, the student has paired all the numbers, except the last, which could not form a pair in the same way. However, we also find examples in the data of students’ solutions that indicate that single numbers are explicitly included in the later step of the calculation, as, e.g., the student who explicitly writes \((5 \times 3) + (2 \times 4) \times (5 + 3) \times 2 = (15 + 8) \times (8 \times 2) = 23 \times 16 = 368\), with brackets inserted by the student. Hence, the student has indicated pairing as a mechanism that is applied overall the entire solution.

**Pairing within terms**

Another type of pairing-mechanism observed in the data is pairs formed within the terms, that is, not across different operations. To some extent, this preserves the order of operations, not breaking the rule that multiplication should precede addition/subtraction. In that respect, this category appears to be some intertwining of the order of operations and the pairing mechanism. Figure 4 shows three examples of pairing within terms that we attribute to this category.
Pairing within terms is particularly evident when three factors are to be multiplied. Then many students seem to invent some pair where one number is disregarded, as in Figure 4(a), or added/multiplied in the end, similar to the pairing mechanism reported by Liebenberg et al. (1999). However, some students used the numbers twice, as in Figure 4(b, c). Hence, in the latter two examples, there is a double counting of one of the numbers (the centre factor of the term). In Figure 4(b), the number “4” is used both in $2 \times 4$ and in $4 \times 5$, separately. In Figure 4(c), the number “3” is double counted. To us, it appears as if this type of pairing can be such a strong idea in the students’ minds that they form pairs regardless if the numbers are used more than once, as in Figure 4(b, c).

Unfortunately, our test did not include any term with four factors. We did not foresee this alternative. If such example would be used, we hypothesize that one could find students who paired the factors within that term without the invention of double-counting.

**Entangled pairing**

In our data, we find students who are operating with each adjacent pairs of numbers. So far, we have only seen two students do like this, but, on the other hand, they are very consistent in this strategy. In this type of pairing mechanism, the operations are subordinate. No consideration is taken to what operations there are in the expression. In the end, the pairs are added. Examples of this type of idea of how numbers should be paired is shown in Figure 5.

Hence, compared to the “pairing overall”-strategy, this type of pairing disregards the operations even more, as pairing is conducted across all operations. In addition, similarly to the “pairing within terms”-type of strategy, double counting of numbers does not seem to be a problem for these students. In this “entangled”-strategy all numbers (except the first and last) are operated with twice.
Concluding remarks

Apart from Liebenberg et al. (1999) and a short example by de Villiers (2015), previous research on students’ solutions to numerical expressions have focused on arithmetic errors and on sequential (left-to-right) strategies. However, our data suggest that pairing is a much more abundant idea in students’ evaluation of numerical expressions than previously shown, at least for longer expressions. We find that 16%, as an average over all students’ solutions, are solved with pairing being the underlying mathematical idea. Hence, it seems that the idea of pairing could be as frequent as the idea of sequential calculations.

Moreover, we find three qualitatively different types of pairing ideas. We find ways of pairing bound within terms, and ways of pairing applied in the expressions overall. In addition, we have shown that expressions with an odd number of numbers in the terms can trigger an apparent adaption of the pairing to the order of operations – the double counting of numbers within terms. The origin of pairing seems a deeply rooted idea. We find that despite being taught the order of operations, a considerable amount of the students apply the idea of pairing nevertheless. In addition, as previous research has shown, the mere proximity of numbers can trigger these to be calculated first (Gómez, et al., 2014; Jiang, Cooper & Alibali, 2014; Landy & Goldstone, 2010). Hence, pairing could be closely related to deep cognitive processes. Nonetheless, pairing is a persistent and strong idea and needs to be considered and handled when designing teaching on how to perceive and calculate arithmetic expressions.

References


