ANGULAR DERIVATIVES AND COMPACTNESS OF COMPOSITION OPERATORS ON HARDY SPACES

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Abstract. Let $D_o$ be a simply connected subdomain of the unit disk and $A$ be a compact subset of $D_o$. Let $\phi$ be a universal covering map for $D_o \setminus A$. We prove that the composition operator $C_\phi$ is compact on the Hardy space $H^p$ if and only if $\phi$ does not have an angular derivative at any point of the unit circle. This result extends a theorem of M.M. Jones.

1. Introduction

A holomorphic self-map $\phi$ of the unit disk $\mathbb{D}$ induces a composition operator defined by the equation

$$C_\phi f = f \circ \phi$$

which, by the Littlewood subordination principle, is bounded on the classical Hardy space $H^p$, $0 < p < \infty$. The main theme in the study of composition operators is to find relations between geometric-analytic properties of $\phi$ and operator theoretic properties of $C_\phi$. The basic results of this theory are presented in the books [3] and [14]. Here we are interested in the compactness of composition operators and how it is related to the angular derivatives of $\phi$.

The connection between compactness and angular derivatives was discovered by J.H. Shapiro and P.D. Taylor [15]. They proved that if $C_\phi$ is compact on $H^p$ for some $p$, then $C_\phi$ is compact on $H^p$ for every $p$, and moreover, compactness of $C_\phi$ implies that $\phi$ does not have angular derivative at any point of the unit circle. The converse is not true; see [14, Chapter 10] and references therein. If, however, $\phi$ is univalent (or boundedly valent), then it was proved by B.D. MacCluer and J.H. Shapiro [8] that nonexistence of the angular derivative of $\phi$ characterizes the compact composition operators $C_\phi$. The general characterization of compact composition operators was
discoverd by Shapiro [13]: \( C_{\phi} \) is compact on \( H^p \) if and only if

\[
\lim_{|w| \to 1^-} \frac{N_{\phi}(w)}{\log(1/|w|)} = 0,
\]

where \( N_{\phi} \) denotes the Nevanlinna counting function for \( \phi \); see subsection 2.1 for the definition.

M.M. Jones [6, 7] considered the case when \( \phi \) is a universal covering map of \( D \) onto a finitely connected domain \( D \subset \mathbb{D} \). He proved that \( C_{\phi} \) is compact on \( H^p \) if and only if \( \phi \) does not have angular derivative at any point of \( \partial \mathbb{D} \). Moreover, if \( A \) is the union of the bounded complementary components (the “holes”) of \( D \), Jones considered a Riemann map \( \psi \) of \( D \) onto the simply connected domain \( D_0 = D \cup A \) and proved that if \( C_{\phi} \) is compact on \( H^p \), then so is \( C_{\psi} \).

Jones used tools such as Fuchsian groups, Dirichlet fundamental polygons, and Poincaré series. We will use a different set of tools (Green functions, subordination, prime ends) to prove a stronger result:

**Theorem 1.** Let \( D_0 \subset \mathbb{D} \) be a simply connected domain. Let \( D \subset D_0 \) be a domain and assume that \( D_0 \setminus D \) is a compact subset of \( D_0 \). Let \( \phi \) be a universal covering map of \( \mathbb{D} \) onto \( D \) and let \( \psi \) be a Riemann map of \( \mathbb{D} \) onto \( D_0 \). The following are equivalent:

(a) \( C_{\phi} \) is compact on \( H^p \).
(b) \( C_{\psi} \) is compact on \( H^p \).
(c) \( \phi \) does not have an angular derivative at any point of the unit circle.
(d) \( \psi \) does not have an angular derivative at any point of the unit circle.

The domain \( D \) in the theorem may be infinitely connected. The main assumption is that \( D_0 \setminus D \) is a compact subset of \( D_0 \); that is, the holes of \( D \) do not accumulate on the boundary of \( D_0 \). This assumption cannot be omitted. Indeed, consider the set of dyadic points

\[
A = \{ w_{nk} = \left(1 - \frac{1}{2^n}\right) \exp\left(\frac{i\pi(2k - 1)}{2^n}\right) : k = 1, 2, \ldots, 2^n, \ n = 1, 2, \ldots \}.
\]

Let \( \phi \) be a universal covering map of \( \mathbb{D} \) onto \( \mathbb{D} \setminus A \). Then \( \phi \) does not have an angular derivative at any point \( \zeta \in \partial \mathbb{D} \) because if it had, then by a theorem of Ch. Pommerenke [11, p.291], [13, p.383] \( \phi(\mathbb{D}) \) would contain a small angular region with vertex at \( \phi(\zeta) \); this cannot happen because every such region contains dyadic points. On the other hand, \( \phi \) is an inner function (see [2, p.37]) and therefore \( C_{\phi} \) is not compact on \( H^p \) (see [13, p.382]).

The proof of Theorem 1 is in Section 3 after some background material presented in Section 2.
2. Background material

2.1. Green and Nevanlinna functions. A bounded planar domain $D$ possesses a Green function $g_D(z,w)$, $z, w \in D$, $z \neq w$; see [9], [12]. An important property of the Green function is the Lindelöf principle:

Suppose that $D$ and $\Omega$ are bounded domains in the complex plane. Let $\phi$ be a holomorphic function mapping $D$ into $\Omega$. If $w \in \Omega$, we denote by $z_j$ the pre-images of $w$ under $\phi$ with the usual convention that each pre-image is repeated as many times as its multiplicity. If $a \in D$, then for all $w \in \Omega \setminus \{\phi(a)\}$,

$$\sum_j g_D(z_j, a) \leq g_\Omega(w, \phi(a)).$$

Moreover, if $D = \mathbb{D}$ and $\phi : \mathbb{D} \to \Omega$ is a universal covering map of $\mathbb{D}$ onto $\Omega$, then equality holds in (2.1). Additional information about this result and its applications can be found in [1] and the references therein.

If $\phi : \overline{\mathbb{D}} \to \mathbb{D}$ is holomorphic, the Nevanlinna counting function for $\phi$ is defined for all $w \in \mathbb{D} \setminus \{\phi(0)\}$ by:

$$N_\phi(w) = \sum_j \log \frac{1}{|z_j|}.$$ 

Since $g_\mathbb{D}(z,0) = -\log|z|$, the Lindelöf principle implies that

$$N_\phi(w) \leq g_{\phi(\mathbb{D})}(w, \phi(0))$$

with equality if $\phi$ is a universal covering map.

2.2. Angular derivatives. Let $\phi : \overline{\mathbb{D}} \to \mathbb{D}$ be a holomorphic function. We say that $\phi$ has an angular derivative at a point $\zeta \in \partial\mathbb{D}$ if there is a point $\omega \in \partial\mathbb{D}$ such that the angular (non-tangential) limit

$$\angle \lim_{z \to \zeta} \frac{\phi(z) - \omega}{z - \zeta}$$

exists (finitely). This limit, if it exists, is called the angular derivative of $\phi$ at $\zeta$. The main theorem concerning the angular derivative is the Julia-Carathéodory theorem which asserts that the following three conditions are equivalent:

(a) $\phi$ has an angular derivative at $\zeta$.
(b) $\phi$ has angular limit of modulus 1 at $\zeta$ and $\phi'$ has angular limit at $\zeta$.
(c) The limit

$$\liminf_{z \to \zeta} \frac{1 - |\phi(z)|}{1 - |z|}$$

exists.

We refer to the books [3], [9], [10], [14] for presentations of the theory of the angular derivative.
2.3. **Subordination.** Let $\sigma, \phi$ be holomorphic functions on $D$. We say that $\sigma$ is subordinate to $\phi$ if there exists a holomorphic function $\omega : D \to D$ with the properties

$$|\omega(z)| \leq |z|, \quad \sigma(z) = \phi(\omega(z)), \quad z \in D.$$ 

We will need two basic results (see [5, Section 2.8]):

(a) If $\sigma$ is holomorphic and maps $D$ onto a domain $D$ and $\phi$ is a universal covering map of $D$ onto $D$ with $\sigma(0) = \phi(0)$, then $\sigma$ is subordinate to $\phi$.

(b) If $\sigma$ is subordinate to $\phi$, then

$$\max_{|z|=r} |\sigma(z)| \leq \max_{|z|=r} |\phi(z)|, \quad 0 < r < 1.$$ 

3. **Proof of Theorem 1**

The implications (a)$\Rightarrow$(c) and (b)$\Rightarrow$(d) come from the result of Shapiro and Taylor. The implication (d)$\Rightarrow$(b) follows from the result of MacCluer and Shapiro. Also, (b)$\Rightarrow$(a) holds by subordination; see Section 2.

**Proof of (c)$\Rightarrow$(a)**

Since nonexistence of angular derivative and compactness of composition operators are not affected by a composition with a conformal automorphism of $D$, we may assume that $0 \in D$ and $\phi(0) = \psi(0) = 0$. By Shapiro’s theorem, we need to show that

$$\lim_{|w| \to 1} \frac{N_\phi(w)}{\log(1/|w|)} = 0. \quad (3.1)$$

Let $(w_n)$ be a sequence in $D$ with $|w_n| \to 1$. Since the prime end compactification $\hat{D}_o$ of $D_o$ is a compact set, the sequence $(w_n)$ has a subsequence converging to a point in $\hat{D}_o$. The limit does not belong to $D_o$ because $|w_n| \to 1$. Therefore, the limit is a prime end. We use now an elementary fact about sequences of real numbers: If every subsequence of a sequence $(a_n)$ has a subsequence converging to $a$, then $a_n \to a$. Thus, to prove (3.1), we may assume that $w_n \to P$, where $P$ is a prime end of $D_o$ whose impression contains at least one point of the unit circle. We may further assume that this point is 1. By Carathéodory’s theorem [2, Chapter 9], [10, Section 9.2], we may extend the conformal map $\psi$ to a function mapping $\hat{D}$ onto the prime end compactification $\hat{D}_o$ of $D_o$. We may assume that $\psi(1) = P$.

Set $A = D_o \setminus D$, the union of the holes of $D$. Since $A$ is a compact subset of $D_o$, $\psi^{-1}(A)$ is a compact subset of $D$. So there exists a positive number $r_0 < 1$ such that the set $\Delta := \{z \in D : |z - 1| < r_0\}$ does not intersect $\psi^{-1}(A)$. Set $\Omega = \psi(\Delta)$. Then $\Omega \subset D_o \setminus A = D$.

We may assume that $\Omega$ contains all the points $w_n$. Fix a point $w_o \in \Omega$ and let $z_o = \psi^{-1}(w_o) \in \Delta$. Let also $z_n := \psi^{-1}(w_n) \in D$. For every $w_n$, let $z_{n,j}$, $j = 1, 2, \ldots$ be the preimages of $w_n$ under $\phi$. 

By the Lindelöf principle, the domain monotonicity and the conformal invariance of the Green function, for every $w_n$,

\begin{equation}
N_\phi(w_n) = \sum_j \log \frac{1}{|z_{n,j}|} = \sum_j g_D(z_{n,j}, 0) \\
\leq g_D(w_n, 0) \leq g_D(w_n, 0) = g_D(z_n, 0).
\end{equation}

By a boundary Harnack principle for the Green function (see [4, Lemma 7]),

\begin{equation}
g_D(z_n, 0) \leq C g_\Delta(z_n, z_\Omega) = C g_\Omega(w_n, w_\Omega),
\end{equation}

where $C > 0$ is a constant independent of $n$. It follows from (3.2) and (3.3) that

\begin{equation}
N_\phi(w_n) \leq C g_\Omega(w_n, w_\Omega).
\end{equation}

The prime end $P$ is determined by a null-chain of cross cuts of $D_\alpha$. These cross-cuts belong to $\Omega$ and therefore they determine a prime end of $\Omega$ which we continue to denote by $P$. Consider the conformal mapping $\sigma$ of $\mathbb{D}$ onto $\Omega$ with $\sigma(0) = w_\Omega$ and $\sigma(1) = P$. Choose a point $\zeta_\Omega \in \mathbb{D}$ with $\phi(\zeta_\Omega) = w_\Omega$. Let $\tau$ be the conformal automorphism of $\mathbb{D}$ with $\tau(0) = \zeta_\Omega$ and $\tau(1) = 1$. Then the function $\phi_1 = \phi \circ \tau$ is a universal covering map of $\mathbb{D}$ onto $D$ with $\phi_1(0) = w_\Omega$. Set $z_n' := \sigma^{-1}(w_n)$. By conformal invariance,

\begin{equation}
g_\Omega(w_n, w_\Omega) = g_\mathbb{D}(z_n', 0) = \log \frac{1}{|z_n'|} \leq 2(1 - |z_n'|).\end{equation}
Since $\sigma(\mathbb{D}) = \Omega \subset D = \phi_1(\mathbb{D})$ and $\sigma(0) = \phi_1(0) = w_0$, the function $\sigma$ is subordinate to $\phi_1$; see subsection 2.3. Therefore,

$$\text{(3.6)} \quad \max_{|z|=r} |\sigma(z)| \leq \max_{|z|=r} |\phi_1(z)|, \quad 0 < r < 1.$$ 

Let $z''_n \in \mathbb{D}$ be such that

$$|z''_n| = |z'_n| \quad \text{and} \quad |\phi_1(z''_n)| = \max_{|z|=r} |\phi_1(z)|.$$ 

Then, by (3.6), $|\sigma(z'_n)| \leq |\phi_1(z''_n)|$. Hence, (3.4) and (3.5) yield

$$\text{(3.8)} \quad \frac{N_\phi(w_n)}{\log(1/|w_n|)} \leq 2C \frac{1 - |z'_n|}{1 - |\sigma(z''_n)|} \leq 2C \frac{1 - |z''_n|}{1 - |\phi_1(z''_n)|}.$$ 

Consider the Carathéodory extension $\sigma : \hat{\mathbb{D}} \to \hat{\Omega}$ of the conformal mapping $\sigma$. Since $\sigma(1) = P$ and $w_n \to P$, we infer that

$$|z''_n| = |z'_n| = |\sigma^{-1}(w_n)| \to 1, \quad \text{as} \quad n \to +\infty.$$ 

Now we use the assumption (c) that $\phi$ does not have an angular derivative at any point of the unit circle. It follows that the same is true for the function $\phi_1 = \phi \circ \tau$. So, by the Julia-Carathéodory theorem (see subsection 2.2),

$$\text{(3.10)} \quad \lim_{n \to +\infty} \frac{1 - |\phi_1(z'_n)|}{1 - |z'_n|} = +\infty.$$ 

It follows from (3.8) and (3.10) that

$$\text{(3.11)} \quad \lim_{n \to +\infty} \frac{N_\phi(w_n)}{\log(1/|w_n|)} = 0.$$ 

So (3.1) has been proved.

**Proof of (a) $\Rightarrow$ (b)**

Suppose that $C_\phi$ is compact. As above, to prove that $C_\psi$ is compact it suffices to show that

$$\text{(3.12)} \quad \lim_{n \to +\infty} \frac{N_\psi(w_n)}{\log(1/|w_n|)} = 0,$$

where $(w_n)$ is a sequence in $D_o$ with $|w_n| \to 1$ and $w_n \to P$, where $P$ is a prime end of $D_o$. Using the notation we set above, the conformal invariance of the Green function, and the boundary Harnack principle, we obtain

$$\text{(3.13)} \quad g_{D_o}(w_n, 0) = g_{\hat{\mathbb{D}}}(z_n, 0) \leq C g_{\Delta}(z_n, z_o) = C g_{\hat{\Omega}}(w_n, w_o).$$ 

It follows from (3.13) and the Lindelöf principle (equality case) that

$$\text{(3.14)} \quad \frac{N_\psi(w_n)}{\log(1/|w_n|)} = \frac{g_{D_o}(w_n, 0)}{\log(1/|w_n|)} \leq 2 \frac{g_{D_o}(w_n, 0)}{1 - |w_n|} \leq 2C \frac{g_{\hat{\Omega}}(w_n, w_o)}{1 - |w_n|}.$$
Consider the conformal automorphism $\tau$ of the unit disk defined above. Then $\phi_1 := \phi \circ \tau$ is a universal covering map of $D$ with $\phi_1(0) = w_0$. For each $n$, let $\zeta_{n,j}$ be the sequence of pre-images of $w_n$ under $\phi_1$. By the Lindelöf principle (see subsection 2.1),

$$g_D(w_n, w_o) = \sum_j g_D(\zeta_{n,j}, 0) = N_{\phi_1}(w_n).$$

Combining (3.14) and (3.15), we obtain

$$N_{\psi}(w_n) \leq 2C N_{\phi_1}(w_n) \leq 4C \frac{N_{\phi_1}(w_n)}{\log(1/|w_n|)}.$$

Since $C_{\phi}$ is compact, so is $C_{\phi_1}$. Now (3.12) follows from Shapiro’s theorem and (3.16).

4. Remarks

Remark 1. If $\phi$ is a universal covering map of $\mathbb{D}$ onto a domain $\Omega \subset \mathbb{D}$ and $b : \mathbb{D} \to \mathbb{D}$ is an inner function with $\phi(0) = 0$, the equality holds in Lindelöf’s principle for the function $\phi \circ b$; see [1, Theorem 1.3]. Therefore

$$N_{\phi \circ b}(w) = g_{\Omega}(w, \phi(0)), \quad w \in \Omega \setminus \{\phi(0)\}.$$

This equality has the following consequences:

1a) Let $\Omega_1, \Omega_2$ be two domains in $\mathbb{D}$. Suppose that the logarithmic capacity of the set $(\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1)$ is zero so that the Green functions of $\Omega_1, \Omega_2$ coincide. Let $\phi_1 : \mathbb{D} \to \Omega_1, \phi_2 : \mathbb{D} \to \Omega_2$ be the corresponding universal covering maps and let $b$ be an inner function with $b(0) = 0$. Then $C_{\phi_1 \circ b}$ is compact on $H^p$ if and only if $C_{\phi_2 \circ b}$ is compact on $H^p$. This fact provides a short proof of Theorem 3 in [6].

1b) If $f$ is a holomorphic self-map of $\mathbb{D}$, let $\|C_f\|_e$ denote the essential norm of $C_f$ on $H^p$. By Shapiro’s well-known formula [13],

$$\|C_f\|_e = \limsup_{|w| \to 1-} \frac{N_f(w)}{\log(1/|w|)}.$$

By using (4.1), we see that for universal covering maps the essential norm is conformally invariant. More precisely: If $g$ maps a domain $D_1 \subset \mathbb{D}$ conformally onto a domain $D_2 \subset \mathbb{D}$ and $\psi_1 : \mathbb{D} \to D_1$ is a universal covering maps, then

$$\|C_{\psi_1}\|_e = \|C_{g \circ \psi_1}\|_e.$$

Remark 2. Suppose that $D$ be a domain and $\phi : \mathbb{D} \to D$ be a universal covering map satisfying the assumptions of Theorem 1. The proof of Theorem 1 shows that there exists a constant $C > 0$, that depends only on the
geometry of $D$, such that

\[(4.4) \quad \|C_{\phi}\|_{\ell}^2 \leq C \sup \left\{ \frac{1}{|\phi'(\zeta)|} : \zeta \in \partial D \right\}; \]

cf. [13, p.386]. An inequality in the opposite direction holds for general $\phi$; see [13, p.385].

References


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