Combinatorics related to NF consistency

by

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Abstract. We elaborate on and refine certain aspects of the approach to NF consistency developed in [8], through coherent pairs and their extendibility properties. Some further results notions and problems are presented. First a quick proof of NF₃ consistency is provided. Next an alternative equivalent formalization of coherent pair, in terms of “coherent triples” of partitions is given. Extendibility is closely inspected and it is shown that instead of general partitions, only “simple partitions”, i.e., partitions consisting of infinite and one-element sets, can be used throughout. Also a property weaker than n-extendibility, called “n-augmentability”, is presented. Some particular n-augmentability questions are proved in the affirmative, while others, especially the appealing (0,0,n)-augmentability, remain open. A partial case of this question is settled, while the source of its hardness is discussed. Finally it is briefly sketched how all these questions can be phrased as combinatorial problems of ZFC alone, without any reference to models of TST.
1. Introduction

In this paper we elaborate on the approach to NF consistency developed in [8] and try to refine, sharpen and improve some of the notions and results presented there. The paper is organized as follows: In section 2 we survey the basic definitions and results of [8], giving in addition a slightly different formalization of pieces of type-shifting automorphisms, in terms of coherent triples of partitions. In section 3 we give a short proof of the consistency of the fragment NF₄ using coherent pairs adjusted to models of TST₃. In section 4 a closer examination of the key property of n-extendibility is attempted which leads to a reduction of partitions and coherent pairs to simple partitions and simple coherent pairs. In section 5 the weaker property of n-augmentability is considered, which follows naturally from the “unfolding” of the extendibility formulation. In subsection 5.1, the special cases of (n,0,0)- and (0,n,0)-augmentability are considered and proved for the trivial pair of a rich model of TST₄. In contrast (0,0,n)-augmentability of the trivial pair, for n > 2, is still open and in subsection 5.2 we discuss certain aspects of this question and prove a partial result. This is a particularly appealing and natural question whose affirmative answer would be a nice strengthening of Theorem 3.6 of [8], since the hard case of that result is equivalent to (0,0,2)-augmentability. All extendibility and augmentability questions are purely combinatorial in essence, asking how elements of finite Boolean algebras distribute over the atoms of corresponding similar Boolean algebras lying at next higher levels of a TST model. So in section 6 we describe briefly how all the preceding notions and questions can be phrased as combinatorial problems without any mention of TST models, just referring only to full models which are quite familiar objects of ZFC.

2. Survey of coherent pairs

Recall that as a consequence of the fundamental contributions [7] and [3] the following are equivalent:

(a) NF is consistent

(b) NF₄ is consistent

(c) There is a model \( \mathcal{A} = (A₀, A₁, A₂, A₃) \) of TST₄ with a type-shifting automorphism

\[
A₀ \xrightarrow{f₀} A₁ \xrightarrow{f₁} A₂ \xrightarrow{f₂} A₃,
\]
or just \( \in \) and \( \subseteq \)-preserving bijections

\[
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3.
\]

(Grishin observed that \( f_0 \) can be recovered from \( f_1 \) by setting \( f_0(a) = x \) if \( f_1(\{a\}) = \{x\} \)).

The idea then is to try to construct the automorphism \( A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \) by forcing for a suitable model \( \mathcal{A} = (A_0, A_1, A_2, A_3) \) of TST. Coherent pairs over a model \( \mathcal{A} \models \text{TST}_4 \) were introduced in [8] as finite approximations of a type-shifting automorphism \( A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \). They were intended to be used as forcing conditions, a generic subset of which would yield the required automorphism. However, in order for that to work, coherent pairs should be extendible in the ordinary sense. That is, given a pair \( p \) and an element \( t \in A_1 \cup A_2 \cup A_3 \), there should be a \( q \leq p \) that captures \( t \). But extendibility is proved to be an exceptionally hard combinatorial problem.

Throughout we use only standard transitive (henceforth s.t.) models of TST. As shown in [8, Lemma 1.2], one can confine oneself to such models without any serious loss of generality. E.g., for every \( A \neq \emptyset \), the sequence \( (A, P(A), P^2(A), \ldots, \in) \) is a s.t. model of TST. Such a model is called full and is denoted by \( \langle\langle A \rangle\rangle \). If \( A \) is infinite \( \langle\langle A \rangle\rangle \) is uncountable. To find a countable model we can take a countable elementary submodel of \( \langle\langle A \rangle\rangle \).

Such a model is standard but not transitive.

Though the intuitive meaning of coherent pairs is clear, the formal definition as given in [8] contains some inaccuracies which do not affect the proofs but might confuse the reader. Below we give a corrected and simplified version based on finite partitions and finite Boolean algebras.

**Definition 1.** Let \( A_1, A_2 \) be infinite sets such that \( |A_1| = |A_2| \), and \( w_1, w_2 \) be finite partitions of \( A_1, A_2 \) respectively. We say that \( w_1 \) and \( w_2 \) are similar and write \( w_1 \sim w_2 \), if there is a bijection \( p : w_1 \rightarrow w_2 \) such that \( |p(x)| = |x| \) for every \( x \in w_1 \). In that case we write \( p : w_1 \sim w_2 \).

Every finite partition \(^1\) \( w \) on a set \( A \) generates a finite Boolean algebra denoted \( B(w) \) whose set of atoms is \( w \). Conversely every (nontrivial) finite Boolean algebra \( B \) on \( A \) has a set of atoms, denoted by \( \text{Atom}(B) \), that constitutes a finite partition of \( A \).

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1. All finite partitions \( w \) on a set \( A \) considered throughout will be assumed to contain only nontrivial sets, i.e., for every \( x \in w, x \neq A, \emptyset \).
Definition 2. Two finite Boolean algebras $B_1, B_2$ on the sets $A_1, A_2$ respectively are said to be similar, notation $B_1 \sim B_2$, if the partitions produced by their atoms are similar.

It follows from the above definitions that for any partitions $w_1, w_2$ of $A_1, A_2$,

$$w_1 \sim w_2 \iff B(w_1) \sim B(w_2).$$

Also if $p : w_1 \sim w_2$ is a similarity, $p$ extends to $B(w_1)$ by setting for every $X \in B(w_1)$,

$$p^*(X) = \bigcup \{p(x) : x \in w_1 \land x \subseteq X\}.$$

$p^*$ is a Boolean-algebra isomorphism between $B(w_1)$ and $B(w_2)$, for which moreover $|p^*(X)| = |X|$. For simplicity we drop the star from $p$ and write $p : B(w_1) \sim B(w_2)$ instead of $p^* : B(w_1) \sim B(w_2)$. Note that $p : B(w_1) \sim B(w_2) \Rightarrow p : B(w_1) \cong B(w_2)$,

but not conversely.

Definition 3. Let $A = (A_0, A_1, A_2, A_3)$ be a model TST. A coherent pair (c.p. in brief) over $A$ is a pair $p = (p_1, p_2)$ of finite 1-1 mappings with the following properties:

(a) $\text{dom}(p_1)$ is a finite Boolean subalgebra of $A_1$, $\text{rng}(p_1) = \text{dom}(p_2)$ is a finite Boolean subalgebra of $A_2$, and $\text{rng}(p_2)$ is a finite Boolean subalgebra of $A_3$. We set $u_1 = \text{dom}(p_1)$, $u_2 = \text{rng}(p_1) = \text{dom}(p_2)$ and $u_3 = \text{rng}(p_2)$.

(b) $p_1 : u_1 \sim u_2$ and $p_2 : u_2 \sim u_3$.

(c) $p_1, p_2$ are $\in$-isomorphisms, i.e., for every $x \in u_1$ and $y \in u_2$,

$$x \in y \iff p_1(x) \in p_2(y).$$

Given c.p.’s $p = (p_1, p_2)$ and $q = (q_1, q_2)$ we say that $p$ extends $q$, and denote it by $p \preceq q$, if $p_1 \supseteq q_1$ and $p_2 \supseteq q_2$.

Instead of $p = (p_1, p_2)$ we often write more suggestively

$$p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3.$$

Before going on let us fix and make explicit some notational conventions which have already been used above and will facilitate greatly the reader.
Notational conventions. Given any model \( \mathcal{A} = (A_0, A_1, A_2, A_3) \) of TST\(_4\), the letters

- \( X, x, x_1, \text{ etc.} \) denote exclusively elements of \( A_1 \),
- \( Y, y, y_1, \text{ etc.} \) denote exclusively elements of \( A_2 \),
- \( Z, z, z_1, \text{ etc.} \) denote exclusively elements of \( A_3 \).

Also

- \( u_1, u_2, u_3 \) (as well as \( v_1, v_2, v_3 \)) are reserved for finite Boolean subalgebras of \( A_1, A_2, A_3 \), respectively, and
- \( w_1, w_2, w_3 \) are reserved for partitions included in \( A_1, A_2, A_3 \), that is, for partitions of the underlying sets \( A_0, A_1, A_2 \), respectively.

An alternative formalization: Coherent triples. Since to each c.p. \( p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3 \) there are associated three finite Boolean algebras \( u_1, u_2, u_3 \), the domains and ranges of \( p_1, p_2 \), we might alternatively consider “coherent triples” of Boolean algebras \( (u_1, u_2, u_3) \) instead of pairs of functions. Moreover, as we saw above, speaking about finite Boolean algebras is tantamount to speaking about finite partitions. So instead of triples of Boolean algebras, we may consider just triples of partitions \( (w_1, w_2, w_3) \) of \( A_0, A_1, A_2 \), respectively. Most often partitions come up as sequences rather than sets, e.g. \( w_1 = (x_1, \ldots, x_n) \), \( w_2 = (y_1, \ldots, y_n) \), \( w_1 = (z_1, \ldots, z_n) \). In such a case the elements of the algebras \( B(w_i) \) can be indexed by means of the sets \( I \subseteq [n] \) as follows (throughout \( [n] \) denotes the set \( \{1, \ldots, n\} \), for every \( n \geq 1 \): for every \( I \subseteq [n] \) let

\[
X_I = \bigcup \{x_i : i \in I\},
\]

and similarly for \( Y_I \in B(w_2) \) and \( Z_I \in B(w_3) \). Obviously for every \( X \in B(w_1) \) (resp. \( Y \in B(w_2) \), \( Z \in B(w_3) \)) there is a unique \( I \subseteq [n] \) such that \( X = X_I \) (resp. \( Y = Y_I \), \( Z = Z_I \)). 2 For a given \( I \), we often refer to \( X_I, Y_I, Z_I \) as “corresponding” sets, with respect to the correspondences \( p_1(x_i) = y_i \) and \( p_2(y_i) = z_i \), since clearly \( p_1(X_I) = Y_I \) and \( p_2(Y_I) = Z_I \) for each \( I \subseteq [n] \). The following definition can be used as an alternative equivalent to Definition 3:

2. The letters \( w_i \) denote, somewhat ambiguously, either a set or a sequence, depending on the context. For example in the notation “\( B(w_1) \)”, \( w_1 \) is just a set. But the indexing of the sets \( X \in B(w_1) \) by \( I \subseteq [n] \) clearly depends on a particular ordering of \( w_1 \). This is why for more clarity we should write \( X_I^{w_1} \) rather than \( X_I \), where now \( w_1 \) refers to a sequence. If \( w'_i \) is a permutation of \( w_1 \), then in general \( X_I^{w_1} \neq X_I^{w'_1} \). This notation is employed in the discussion of section 5.2.
Definition 4. Let \((w_1, w_2, w_3)\) be a triple of partitions of \(A_0, A_1, A_2\), respectively. \((w_1, w_2, w_3)\) is said to be a coherent triple, (c.t. for short), notation \(\text{Co}(w_1, w_2, w_3)\), if

(a) There are \(p_1 : w_1 \sim w_2\), and \(p_2 : w_2 \sim w_3\).

(b) Let \(w_1 = (x_1, \ldots, x_n)\), \(w_2 = (y_1, \ldots, y_n)\), \(w_3 = (z_1, \ldots, z_n)\), be enumerations of \(w_1, w_2, w_3\) such that \(p_1(x_i) = y_i\), and \(p_2(y_i) = z_i\). Then

\[
X_I \in y_i \Leftrightarrow Y_I \in z_i,
\]

for all \(i \in [n]\) and all \(I \subseteq [n]\).

Given triples of partitions \(w = (w_1, w_2, w_3)\), \(w' = (w'_1, w'_2, w'_3)\) we say that \(w' \) extends \(w\) and write \(w' \leq w\), if \(w'_1 \supseteq w_1\), \(w'_2 \supseteq w_2\) and \(w'_3 \supseteq w_3\), where \(w'_i \supseteq w_i\) means that \(w'_i\) refines \(w_i\), i.e., each element of \(w'_i\) is a subset of some element of \(w_i\).

Remarks

(a) Note that condition (c) of Definition 3, that \(\in\) is preserved by \(p_1, p_2\), is equivalent to

\[
X_I \in Y_J \Leftrightarrow Y_I \in Z_J,
\]

for all \(I, J \subseteq [n]\). However it is easy to check that condition (2) suffices for (3) to hold, that is, (2) and (3) are equivalent.

(b) The relation \(w' \leq w\) for c.t.’s is the analog of \(p \leq q\) for c.p.’s

(c) The relation between coherent pairs and coherent pairs is simply the following:

\[
\text{Co}(w_1, w_2, w_3) \Leftrightarrow \text{there is a c.p.
}
\]

\[
p = B(w_1) \xrightarrow{p_1} B(w_2) \xrightarrow{p_2} B(w_3).
\]

(d) Coherent pairs and triples over a model \(\mathcal{A}\) of TST are not elements of \(\mathcal{A}\), since they are “unstratified objects”. Their relationship to \(\mathcal{A}\) is that of proper classes to a model of ZFC. If one wants to treat them formally one has to extend TST to a “second-order” variant TST\(^c\) which is able to accommodate unstratified objects like coherent pairs. Models of TST\(^c\) have the form \((\mathcal{A}, C)\), where \(\mathcal{A}\) is a model of TST\(_4\) and \(C\) is a certain subset of \(\bigcup_{i=0}^{3} A_i\). For details see [8], p. 294.
Example 1. The simplest example of a c.p. is that in which \( u_1, u_2, u_3 \) are the trivial Boolean subalgebras of \( A_1, A_2, A_3 \), respectively, and \( p_1, p_2 \) are the trivial isomorphisms between them. Namely let: \( u_1 = \{ \emptyset, A_0 \} \), \( u_2 = \{ \emptyset, A_1 \} \), \( u_3 = \{ \emptyset, A_2 \} \), \( p_i(\emptyset) = \emptyset \), for \( i = 1, 2 \), \( p_1(A_0) = A_1 \), and \( p_2(A_1) = A_2 \). We denote this pair by \( \sigma^A \). I.e.,

\[
\sigma^A = \{ \emptyset, A_0 \} \xrightarrow{\sigma_1} \{ \emptyset, A_1 \} \xrightarrow{\sigma_2} \{ \emptyset, A_2 \}.
\]

We refer to \( \sigma^A \) as the trivial c.p. of \( A \).

Example 2. Let

\[
u_1 = (\emptyset, A_0, \{a\}, -\{a\}), \]

for some \( a \in A_0 \),

\[
u_2 = (\emptyset, A_1, \{x\}, -\{x\}), \]

for some \( x \in A_1 \) such that \( x \neq \{a\}, -\{a\} \),

\[
u_3 = (\emptyset, A_2, \{y\}, -\{y\}), \]

for some \( y \in A_2 \) such that \( y \neq \{x\}, -\{x\} \).

If \( p_1 : u_1 \rightarrow u_2, p_2 : u_2 \rightarrow u_3 \) are the mappings preserving the above orderings of \( u_i \), it is easy to check that \( p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3 \) is a c.p.

As already said above, coherent pairs (or coherent triples) are intended to be used as forcing conditions a generic subset of which would provide the required type-shifting automorphism of the model \( A \) of TST\textsubscript{4}. So the key property for (some of) them should be extendibility.

Definition 5. Let \( p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3 \) be a c.p. We say that \( p \) is extendible if for every \( t \in A_1 \cup A_2 \cup A_3 \), there is a pair \( v_1 \xrightarrow{q_1} v_2 \xrightarrow{q_2} v_3 \) such that \( q \leq p \) and \( t \in v_1 \cup v_2 \cup v_3 \). When such a \( q = (q_1, q_2) \) exists, we say for simplicity that \( q \) captures \( t \), and denote this by \( q \leq p \sim \{t\} \).

Are there extendible c.p.'s? We can prove that there are (Theorem 4 below), but it is far more easy to give examples of non-extendible c.p.'s rather than extendible ones.

Example 3. Consider the pair of Example 2 above:

\[
u_1 = (\emptyset, A_0, \{a\}, -\{a\}), \]

\[
u_2 = (\emptyset, A_1, \{x\}, -\{x\}),
\]
such that $x \neq \{a\}, \{-a\}$,

$$\begin{align*}
  u_3 &= (\varnothing, A_2, \{y\}, \{-y\}),
\end{align*}
$$

such that $y \neq \{x\}, \{-x\}$, with

$$\begin{align*}
  p_1 &: (\varnothing, A_0, \{a\}, \{-a\}) \rightarrow (\varnothing, A_1, \{x\}, \{-x\}) \\
  p_2 &: (\varnothing, A_1, \{x\}, \{-x\}) \rightarrow (\varnothing, A_2, \{y\}, \{-y\})
\end{align*}
$$

If $|x| \neq |y|$, $p = (p_1, p_2)$ is non-extendible. For if $q = (q_1, q_2)$ and $q \leq p \setminus \{x\}$, then necessarily $q_1(x) = y$, hence $|q_1(x)| = |x| = |y|$, a contradiction.

The preceding example gives an idea of the hardness of the extendibility problem. Extendibility is a “chain-reaction” generating property: If $p = (p_1, p_2)$ is a given pair and, say, $y \in u_2$, in order for $p$ to be extendible we must make sure that for any $x_1 \in y$ there exists a $y_1 \in p_2(y)$, as well as a $z_1$ so that $p$ extends to a $q$ that captures $x_1, y_1, z_1$; then for any $x_2 \in y_1$ we must find $y_2, z_2$ captured by an extension of $q$ and so on. It follows that extendibility alone, as defined above, is by no means adequate. Even if we are able to extend $p$ to $q$ to capture a new element $t$, $q$ need not be further extendible, and the procedure will stop. What we need is a property of iterated extendibility up to $\omega$ iterations.

**Definition 6.** Let $p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3$ be a pair.

- $p$ is said to be $1$-extendible if it is extendible.
- $p$ is said to be $(n + 1)$-extendible if for every $t \in A_1 \cup A_2 \cup A_3$ there is a pair $q = (q_1, q_2)$ such that, $q \leq p \setminus \{t\}$ and $q$ is $n$-extendible.
- $p$ is said to be $\omega$-extendible if it is $n$-extendible for all $n \geq 1$.

We shall see below that the trivial pair $\alpha^A$ is $1$-extendible for any sufficiently rich $A$. As to $n$-extendibility, for all $n \geq 1$, this is exactly the required property.

**Theorem 1 (Main Theorem [8]).** Let $M$ be a countable model of ZFC in which for every $n \in \mathbb{N}$, there is a s.t. model $A$ of TST that contains an $n$-extendible c.p. Then there is a generic extension $M[G]$ of $M$ that contains a model of NF. Conversely, if $M$ contains a model of NF, then in $M$ there is a s.t. model $A$ of TST that contains an $n$-extendible c.p., for every $n \geq 1$. 
The most natural candidate pair to be extendible would be the trivial pair \( o^A \) of Example 1. The main theorem above can be equivalently formulated as follows:

**Theorem 2 (Main Theorem [8]).** Let \( M \) be a countable model of ZFC in which for every \( n \in \mathbb{N} \), there is a s.t. model \( A \) of TST such that \( o^A \) is \( n \)-extendible. Then there is a generic extension \( M[G] \) of \( M \) that contains a model of NF. Conversely, if \( M \) contains a model of NF, then there is a s.t. model \( A \in M \) such that \( o^A \) is \( n \)-extendible, for every \( n \geq 1 \).

Roughly \( n \)-extendibility works as follows: If for every \( n \) there is model \( A_n \in M \) of TST such that \( o^{A_n} \) is \( n \)-extendible, then, by compactness, there is \( A \models TST_4 \) in \( M \) such that \( o^A \) is \( \omega \)-extendible. If further \( B \) is a saturated elementary extension of \( A \) in \( M \) and we set

\[
P_\omega = \{ p : p \text{ is } \omega \text{-extendible over } B \},
\]

then \( (P_\omega, \leq) \) is a forcing notion, and setting \( f = \bigcup G \), for any generic \( G \), \( f \) is the required type-shifting automorphism of \( B \). Thus \( (B, f) \) yields a model of NF in \( M[G] \).

Are there models \( A \) of TST having \( n \)-extendible pairs? More simply: Are there \( A \) such that \( o^A \) is \( n \)-extendible? All we know is that there exist \( A \) for which \( o^A \) is 1-extendible. In any case extendibility capabilities of \( o^A \) depend on properties of the underlying model \( A \). The properties of \( A \) mainly employed in [8] were “richness” and “regularity”. Here are the definitions:

**Definition 7.** A model \( A \) of TST is called **regular** if for every \( x \in A \),

\[
x \text{ is finite } \iff A \models \text{Fin}(x).
\]

**Definition 8.** The Boolean algebra \( A_{i+1} \) is said to be **rich** if for every infinite (with respect to the ground model) \( x \in A_{i+1} \), there is a \( x_1 \in A_{i+1} \) such that \( x_1 \subseteq x \) and both \( x_1 \) and \( x - x_1 \) are infinite.

The structure \( A \) is said to be **rich** if every level \( A_{i+1} \), for \( i \geq 0 \), is rich.

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3. One may ask whether \( (P_\omega, \leq) \) is a nontrivial forcing notion, that is, one producing a strict extension \( M[G] \supseteq M \) (e.g., whether \( (P_\omega, \leq) \) is separative, see [9]). The answer is that the question has no bearing on the issue of NF consistency. For if \( (P_\omega, \leq) \) is trivial and \( M[G] = M \), that simply means that the sought type-shifting automorphism \( f = \bigcup G \) is already in \( M \)!
If $\mathcal{A}$ is regular, then the property of richness is definable in $\mathcal{A}$. Moreover the following holds:

**Lemma 3.** Let $\langle \mathcal{D} \rangle$ be a full model of TST and let $\mathcal{A}$ be a standard transitive model isomorphic to an elementary submodel of $\langle \mathcal{D} \rangle$. Then $\mathcal{A}$ is regular and rich.

**Theorem 4 ([8]).** Let $\langle \mathcal{D} \rangle$ be a full model of TST (with infinite $D$) and let $\mathcal{A}$ be a standard transitive model isomorphic to an elementary submodel of $\langle \mathcal{D} \rangle$. Then the trivial pair $\sigma^\mathcal{A}$ is extendible.

### 3. A quick proof of NF₃ consistency

There are several proofs of the consistency of NF₃, due to V.N Grishin [3], M. Boffa and P. Casalegno [1] and R. Kay (see [2, p. 59]). In this section, as an application of coherent pairs adapted to the fragment NF₃, we give another short and simple proof of this result. The cost to be paid for the simplicity is that the model of NF₃ exists not in the ground model $M$ of ZFC but in a generic extension of it.

Recall that a model of NF₃ exists iff there is a model $(A₀, A₁, A₂)$ of TST₃ together with a type-shifting automorphism $A₀ \xrightarrow{f₀} A₁ \xrightarrow{f₁} A₂$. Grishin in [3] observed that in order for $(A₀, A₁, A₂) \models \text{TST}_3$ to be a model of NF₃, only a mapping $f₁ : A₁ \to A₂$ is needed, which will be a Boolean isomorphism. (This result was used also in the case [8] for the reduction of NF₄ consistency.)

**Lemma 5.** NF₃ is consistent iff there is a model $(A₀, A₁, A₂)$ of TST₃ such that there is Boolean-algebra isomorphism $f : A₁ \cong A₂$.

**Proof.** — The property is obviously necessary. Conversely, suppose there is a model $(A₀, A₁, A₂)$ of TST₃ and a Boolean-algebra isomorphism $f : A₁ \to A₂$. Put $f₁ = f$ and define $f₀ : A₀ \to A₁$ by setting

$$f₀(a) = x \iff f₁(\{a\}) = \{x\}.$$  

Then $f₀$ is a bijection because $f₁$ sends atoms to atoms. Moreover $(f₀, f₁)$ is a type-shifting automorphism from $(A₀, A₁)$ onto $(A₁, A₂)$. Indeed, by
the definition of \( f_0 \) we have that for every \( a \), \( f_1(\{a\}) = \{f_0(a)\} \). So for every \( a \in A_0 \) and \( x \in A_1 \)

\[
f_0(a) \in f_1(x) \iff \{f_0(a)\} \subseteq f_1(x) \iff f_1(\{a\}) \subseteq f_1(x).
\]

Since \( f_1 \) preserves \( \subseteq \), we have

\[
f_1(\{a\}) \subseteq f_1(x) \iff \{a\} \subseteq x \iff a \in x.
\]

Combining the above equivalences we get

\[
a \in x \iff f_0(a) \in f_1(x).
\]

This says that \((f_0, f_1)\) is a type-shifting automorphism. \( \square \)

Now we adjust the notion of coherent pair (defined initially over models of TST\(_4\)) to models of TST\(_3\). Necessarily it won’t be a pair of finite mappings \((p_1, p_2)\), but only a single mapping \(p\), namely an isomorphism between finite Boolean algebras such that, in addition, \( |p(x)| = |x| \) for every \( x \in \text{dom}(p) \). We shall keep, however, calling it “coherent pair”. The following is the adaptation of Definition 3 to TST\(_3\).

**Definition 9.** Let \( \mathcal{A} = (A_0, A_1, A_2) \) be a countable model of TST\(_3\). A coherent pair over \( \mathcal{A} \) is a 1–1 mapping \( u_1 \overset{p}{\rightarrow} u_2 \), such that

(a) \( u_1 = \text{dom}(p) \subset A_1 \), \( u_2 = \text{rng}(p) \subset A_2 \),

(b) \( u_1, u_2 \) are finite subalgebras of the Boolean algebras \( A_1, A_2 \) respectively, and

(c) \( p : u_1 \sim u_2 \), and in addition \( |p(x)| = |x| \) for all \( x \in u_1 \).

Obviously, given a model \( \mathcal{A} = (A_0, A_1, A_2) \) of TST\(_3\), coherent pairs \( u_1 \overset{p}{\rightarrow} u_2 \) over \( \mathcal{A} \) are finite approximations of an isomorphism \( A_1 \overset{f}{\rightarrow} A_2 \) which is required to turn \( \mathcal{A} \) into a model of NF\(_3\). Using these pairs as forcing conditions, we can force \( f \) to occur in a generic extension \( M[G] \), provided the forcing conditions are extendible. But this, in contrast to the hardness of extendibility in the NF\(_4\) case, can be easily shown to hold.

**Theorem 6.** Let \( M \) be a countable model of ZFC and let \( \mathcal{A} = (A_0, A_1, A_2) \in M \) be a countable rich and regular model of TST\(_3\). Then there is a generic extension \( M[G] \) of \( M \) containing an isomorphism \( f : A_1 \rightarrow A_2 \). Hence \( M[G] \) contains a model of NF\(_3\).
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Proof. — Let $M, A \in M$ be as in the hypothesis. Let $(P, \leq)$ be the forcing notion in $M$, where

$$P = \{ p : p \text{ is a coherent pair over } A \},$$

and $p \leq q := p \supseteq q$.

Claim 1. Each $p \in P$ is extendible, i.e.,

(a) for every $x \in A_1$, there is a $q \leq p$ such that $x \in \text{dom}(q)$, and

(b) for every $y \in A_2$, there is a $q \leq p$ such that $y \in \text{rng}(q)$.

Proof of Claim 1:

(a) Let $u_1 = \text{dom}(p)$, $u_2 = \text{rng}(p)$. By assumption $u_1, u_2$ are finite Boolean subalgebras of $A_1, A_2$ and $p : u_1 \sim u_2$. $p$ maps the atoms of $u_1$, onto the atoms $u_2$, and we can take enumerations of the sets of atoms of $w_1 = (x_1, \ldots, x_n)$, $w_2 = (y_1, \ldots, y_n)$ of the algebras $u_1, u_2$ so that, $w_1 = B(w_1)$, $w_2 = B(w_2)$ and $p(x_i) = y_i$, $w_1, w_2$ are partitions of the sets $A_0, A_1$ respectively and for each $i = 1, \ldots, n$, $|x_i| = |p(x_i)| = |y_i|$. Given $x \in A_1$, let $w_1 \oplus x$ be the smallest partition that refines $w_1$ and accommodates $x$. That is,

$$w_1 \oplus x = \{ x_i^0, x_i^1 : i = 1, \ldots, n \},$$

where $x_i^0 = x_i \cap y_i$, $x_i^1 = x_i \cap (A_0 - x_i)$, $i = 1, \ldots, n$. Let $v_1 = B(w_1 \oplus x)$. We have to find a set $y \in A_2$ and a mapping $q : v_1 \sim v_2 = B(w_2 \oplus y)$ such that $q \leq p$ and $q(x) = y$. What we need to do is just split each set $y_i$ of $w_2$ into two subsets $y_i^0$ and $y_i^1$, such that $|y_i^0| = |x_i^0|$ and $|y_i^1| = |x_i^1|$. This is always possible, because $|y_i| = |x_i|$ and our model $A$ is rich and regular. By regularity we do not bother about the internal and external meaning of finiteness. So if $x_i^0$ is finite with, say $m$ elements, we pick a $y_i^0 \subseteq y_i$ with $m$ elements. If both $x_i^0, x_i^1$ are infinite, then using richness we can split $y_i$ into two infinite subsets $y_i^0, y_i^1$. Then it suffices to define $q$ by setting $q(x_i^0) = y_i^0$ and $q(x_i^1) = y_i^1$. Clearly $q$ is the required extension of $p$.

(b) This case is quite similar to the previous one. Given $p$ and $y \in A_2$ we find as before $x \in A_1$ and $q : B(w_1 \oplus x) \sim B(w_2 \oplus y)$ such that $q \leq p$ and $q(x) = y$. This completes the proof of Claim 1.

In view of the Claim, $(P, \leq)$ is an ordinary forcing notion on $M$. If $G$ is a $P$-generic set and $f = \bigcup G$, then clearly $\text{dom}(f) = A_1$, $\text{rng}(f) = A_2$, so $f : A_1 \cong A_2$. \hfill \Box
4. Extendibility revisited

4.1. Unfolding extendibility

We are going here to examine more thoroughly the property of \( n \)-extendibility. Recall that given c.p.’s \( p = (p_1, p_2) \), \( q = (q_1, q_2) \) and elements \( t_1, \ldots, t_k \in A_1 \cup A_2 \cup A_3 \) the notation

\[
q \leq p \prec \{t_1, \ldots, t_k\}
\]

abbreviates the fact that \( q \) extends \( p \) and captures \( t_1, \ldots, t_k \), that is, \( q \leq p \) and \( \{t_1, \ldots, t_k\} \subseteq \text{dom}(q_1) \cup \text{dom}(q_2) \cup \text{rng}(q_2) \). Let us set

\[
\theta_0(p) := \text{“} p \text{ is a coherent pair”},
\]

\[
\theta_n(p) := \text{“} p \text{ is } n\text{-extendible”}.
\]

The definition of \( n \)-extendibility given in Definition 6 (page 116) is inductive. Namely:

\[
\theta_{n+1}(p) := (\forall t)(\exists q)(q \leq p \prec \{t\} \land \theta_n(q)).
\]

and

\[
\theta_\omega(p) := \bigwedge_n \theta_n(p).
\]

If we “unfold” \( \theta_n \) into a plain formula, we shall get inductively the formulas

\[
\theta_n(p) = (\forall t_1)(\exists q_1)(\forall t_2)(\exists q_2) \cdots (\forall t_n)(\exists q_n)
\]

\[
\left[ q_1 \leq p \prec \{t_1\} \land \bigwedge_{i=2}^n q_i \leq q_{i-1} \prec \{t_i\} \right],
\]

where \( q_i \) range over coherent pairs and \( t_i \) range over elements of \( A_1 \cup A_2 \cup A_3 \).\(^4\)

We see that \( \theta_n(p) \) is of high logical complexity, a sort of “\( \Pi_n \)-formula”, with \( n \) alternating quantifiers. On can easily verify by induction on \( n \) that properties \( \theta_n \), \( n \geq 1 \), become stronger and stronger as \( n \) grows:

\[
(\forall p)[\theta_{n+1}(p) \Rightarrow \theta_n(p)].
\]

It is open whether this hierarchy of \( \theta_n \) is proper or collapses at a certain level. The following is only a partial answer to the question.

\(^4\) As we have noticed in remark (d) on page 114, coherent pairs are not elements of models of TST, but of an extended theory TST\(^c\). For the same reason \( \theta_n \) (as well as the formulas \( \chi_n \) defined later) are not formulas of \( L_{\text{TST}} \) but of \( L_{\text{TST}\text{c}} \). So it doesn’t make sense to write \( A \models \theta_n(p) \). To relax the reader from the technicalities of using \( L_{\text{TST}\text{c}} \), we shall say instead that “\( \theta_n(p) \) holds with respect to \( A \)”.
Proposition 7. Let $M$ be a countable model of ZFC and let $A \in M$ be a model of TST$^4$ elementarily embeddable to a full model. Then

$$\left( \forall p \right) \left[ \theta_1(p) \Rightarrow \theta_2(p) \right]$$  \hspace{1cm} (8)

is false with respect to $A$.

\textbf{Proof.} — Suppose $A$ is as in the hypothesis and let (8) hold true.

\textbf{Claim 2.}

$$\left( \forall p \right) \left[ \theta_n(p) \Rightarrow \theta_{n+1}(p) \right].$$  \hspace{1cm} (9)

\textbf{Proof of the Claim.} — We prove (9) by induction on $n$. For $n = 1$, (9) is (8). Suppose (9) holds for $n$. Then, by (5),

$$\theta_{n+1}(p) = (\forall t)(\exists q)(q \leq p \vdash \{t\} \land \theta_n(q)),$$

and

$$\theta_{n+2}(p) = (\forall t)(\exists q)(q \leq p \vdash \{t\} \land \theta_{n+1}(q)).$$

Since by the induction hypothesis $\theta_n(q) \Rightarrow \theta_{n+1}(q)$ is true, it follows from the preceding equations that so is $\theta_{n+1}(p) \Rightarrow \theta_{n+2}(p)$.

Now by Theorem 4, $\theta_1(\sigma^A)$ holds true, so by (8) and (9), $\theta_n(\sigma^A)$ holds true for all $n \geq 1$, therefore $\theta_\omega(\sigma^A)$. This means (see the proof of [8, 2.8] for details) that if $B$ is a saturated elementary extension of $A$, and $P_\omega = \{ p : \theta_\omega(p) \text{ holds with respect to } B \}$, then forcing with $(P_\omega, \leq)$ yields a generic type-shifting automorphism $f$ for $B$, $f \in M[G]$. We have $B \equiv A \equiv \langle \mathbb{D} \rangle$ for some full model $\langle \mathbb{D} \rangle$ and $\langle \mathbb{D} \rangle \models \text{AC}$, where AC is the choice axiom adjusted to the language of TST. So $B \models \text{AC}$. Now $B$ is an ambiguous model, and if $B^*$ is the induced model of NF, the clearly $B^* \models \text{AC}$. But this contradicts Specker’s result [6] that NF $\vdash \neg \text{AC}$. \qed

Before elaborating further on properties $\theta_n$ we shall first deal with a simplification of all objects considered so far (c.p.’s, Boolean algebras, c.t. partitions, etc). All these objects are defined in terms of finite partitions. The simplification concerns the kind of partitions involved. The simplest kind of finite partitions are those whose sets are either infinite or singletons. We shall call such partitions “simple”. And we shall see that all extendibility questions about general c.p.’s can be reduced, without any loss of generality, to questions about “simple c.p.” only, that is, c.p.’s whose domains are (essentially) simple partitions.
4.2. Simple partitions and simple extendibility

Recall that all partitions $w_i$, $i = 1, 2, 3$, considered in this paper are non-trivial, in the sense that for every $x \in w_i$, $x \neq A_{i-1} \setminus \emptyset$. Each of the underlying sets $A_i$ is (countably) infinite, so every $w_i$ contains at least one infinite set and possibly several finite ones. To simplify things let us consider partitions whose sets are only either infinite or singletons.

**Definition 10.** A finite partition $w$ (resp. a finite Boolean algebra $u$) of an infinite set $A$ is said to be simple if each $x \in w$ (resp. each atom of the algebra) is either a singleton or an infinite set.

A c.p. $p = w_1 \rightarrow w_2 \rightarrow w_3$, is said to be simple if each $u_i$ is a simple Boolean algebra. Similarly a c.t. $w = (w_1, w_2, w_3)$ is simple if all partitions $w_i$ are simple.

Recall that for partitions $w, w'$ of a set $A$, $w \sqsubset w'$ denotes the fact that $w'$ is a refinement of $w'$. Also we have already defined in the proof of Theorem 6, for every finite partition $w = \{x_1, \ldots, x_n\}$ of a set $A$ and every $x \subseteq A$, the smallest refinement

$$w \oplus x = \{x \cap x_i : x_i \in w\} \cup \{(A - x) \cap x_i : x_i \in w\}$$

of $w$ that accommodates $w$.

**Definition 11.** Let $w$ be a partition of $A$. The *simple refinement* of $w$, denoted $sr(w)$, is the partition resulting from $w$ if we replace each finite $x \in A$ with the sets $\{a\}$, $a \in x$.

Clearly, if $w$ is finite, then $sr(w)$ is the $\sqsubseteq$-least simple partition that refines $w$.

Let $p = u_1 \rightarrow u_2 \rightarrow u_3$ be a simple c.p. and let $w_i = \text{Atom}(u_i)$, the set of atoms of the algebra $u_i$. Let $t$ be a new element of $A_1 \cup A_2 \cup A_3$, say $t = x \in A_1$. In order for $p$ to be extendible on $x$, it is necessary and sufficient that it be extendible on each of the elements of $w \oplus x$, and further on each of the elements of $sr(w \oplus x)$. So given a pair $p = w_1 \rightarrow w_2 \rightarrow w_3$, let

$$S(p) = \{t : (\exists t_1 \in w_i)(t \subset t_1) \wedge (|t| = 1 \lor |t| = \infty)\}.$$ 

In words, $S(p)$ is the set of all singletons or infinite sets which are proper subsets of elements of some $w_i$. 


Definition 12. A simple c.p. $p = (p_1,p_2)$ is said to be $n$-simply extendible if $\theta^n_0(p)$ holds, where

$$\theta^n_0(p) := "p \text{ is a simple c.p.}" ,$$

and

$$\theta^n_{n+1}(p) := (\forall t \in S(p))(\exists q)(q \preceq p \setminus \{t\} \land \theta^n_q(q)). \quad (10)$$

$p$ is $\omega$-simply extendible if $\theta^n_\omega(p)$ holds, where $\theta^n_\omega(p) := \bigwedge_n \theta^n_n(p)$.

Proposition 8. Let $(A,C)$ be a (recursively) saturated model of TST$^\omega$.

(a) If $\theta^n_\omega(p)$, then $(\forall t)(\exists q)(q \preceq p \setminus \{t\} \land \theta^n_q(q)).$

(b) $(\forall p)[\theta_n(p) \leftrightarrow \theta^n_\omega(p)]$ with respect to $A$.

In particular $\theta_n(o^A) \leftrightarrow \theta^n_\omega(o^A)$.

Proof

(a) Suppose that $\theta^n_\omega(p)$ is the case and let $t \in A_i$ be a new element. $w_1$ is a partition of $A_{i-1}$ so let $\text{sr}(w_1 \uplus t) = \{t_1,\ldots,t_k\}$ be the elements of the simple refinement of $w_1$ augmented with $t$. We have to show that there is $q$ such that $q \preceq p \setminus \{t\}$ and $\theta^n_q(q)$. The proof is similar to that of Claim 1 of theorem [8, 2.8], so we omit it.

(b) Trivially $(\forall p)[\theta_n(p) \Rightarrow \theta^n_\omega(p)]$, with respect to any $A$.

For the converse, suppose $p$ is a simple pair such that $\theta^n_\omega(p)$ holds. We show by induction on $n$ that $\theta^n_n(p) \Rightarrow \theta_n(p)$.

Let $n = 1$, and let $t \in A_i$ be a new element. We have to show that there is a $q \preceq p \setminus \{t\}$. Let $\text{sr}(w_1 \uplus t) = \{t_1,\ldots,t_k\}$, where $t_i$ are either singletons or infinite sets contained in the atoms of $w_1$. By assumption $p$ is $k$-simply augmented, so there are simple c.p.’s $q_1,\ldots,q_k$ such that $q_i \preceq p \setminus \{t_i\}$ for $2 \leq i \leq k$. Therefore $q \preceq p \setminus \{t_1,\ldots,t_k\}$, and hence $q \preceq p \setminus \{t\}$. Therefore $\theta^n_n(p) \Rightarrow \theta_1(p)$.

Suppose now that the claim is true for $n$. Let $t$ a new element and let again $\text{sr}(w_1 \uplus t) = \{t_1,\ldots,t_k\}$. Then as in (a) above we can show that there is a $q$ such that $q \preceq p \setminus \{t\}$ and $\theta^n_q(q)$. By the induction hypothesis, $\theta^n_n(q) \Rightarrow \theta_n(q)$ is true. Therefore $(\forall t)(\exists q)(q \preceq p \setminus \{t\} \land \theta_n(q))$. But that means that $\theta_{n+1}(p)$ holds. So $\theta^n_{n+1}(p) \Rightarrow \theta_{n+1}(p)$. This completes the proof. $\blacksquare$
It follows from Proposition 8 (b) that with respect to saturated models of TST\textsuperscript{c}, ω-simple extendibility is no weaker than the full ω-extendibility.

5. Augmentability

Let us return to the unfolded formulation (6) of θ\textsubscript{n}:

\[ θ_n(p) = (∀t_1)(∃q_1)(∀t_2)(∃q_2)⋯(∀t_n)(∃q_n) \]
\[ \left[ q_1 ≤ p \prec \{t_1\} ∧ \bigwedge_{i=2}^{n} q_i ≤ q_{i-1} \prec \{t_i\} \right]. \]

Since for every φ, (∃x)(∀y)φ ⇒ (∀y)(∃x)φ is logically valid, the preceding formula logically implies

\[ (∀t_1)(∀t_2)⋯(∀t_n)(∃q_1)(∃q_2)⋯(∃q_n) \]
\[ \left[ q_1 ≤ p \prec \{t_1\} ∧ \bigwedge_{i=2}^{n} q_i ≤ q_{i-1} \prec \{t_i\} \right]. \] (11)

Moreover, obviously

\[ (∃q_1)(∃q_2)⋯(∃q_n) \]
\[ [q_1 ≤ p \prec \{t_1\} ∧ \bigwedge_{i=2}^{n} q_i ≤ q_{i-1} \prec \{t_i\}] \]
\[ ⇓ \]
\[ (exists)\{q \leq p \prec \{t_1, …, t_n\}\}. \] (12)

From (11) and (12) we get

\[ (∀t_1)(∀t_2)⋯(∀t_n)(∃q) [q ≤ p \prec \{t_1, …, t_n\}]. \]

This formula is a natural weakening of θ\textsubscript{n}(p). We denote it by χ\textsubscript{n}(p), and call the property it expresses \textit{n-augmentability}. That is, we set for every \( n ≥ 1 \),

\[ χ_n(p) := (∀t_1)(∀t_2)⋯(∀t_n)(∃q) [q ≤ p \prec \{t_1, …, t_n\}], \] (13)

and

\[ χ_ω(p) := \bigwedge_n χ_n(p). \]

It follows from (6), (11) and (12) that for all \( n ≥ 2 \)

\[ θ_n(p) ⇒ χ_n(p), \] (14)

while

\[ θ_1(p) ⇔ χ_1(p). \] (15)
**Definition 13.** Let \( p = (p_1, p_2) \) be a c.p. over \( A \). We call \( p \) \( n \)-augmentable if \( \chi_n(p) \) holds.

\( p \) is \( \omega \)-augmentable if it is \( n \)-augmentable for all \( n \geq 1 \).

\( \chi_n(p) \) expresses also an extendibility property of \( p \) but of a different kind: It says that any \( n \) new elements can be adjoined to \( p \) to give an extension \( q \leq p \), but with no claim as to the extendibility capabilities of \( q \). (Observe how much lower is the logical complexity of \( \chi_n \) compared to that of \( \theta_n \).)

**Lemma 9.** If for each \( n \) there is \( A_n \models \text{TST} \) containing an \( n \)-augmentable pair, then there is a \( B \models \text{TST} \) containing an \( \omega \)-augmentable pair.

**Proof.** — The proof is again similar to that of 1 in [8], for showing the existence of \( \omega \)-extendible pairs. Namely, we consider the theory \( T = \text{TST} + \{ \chi_n(b) : n \in \mathbb{N} \} \), where \( c \) is a new constant. Then the result follows by compactness. See [8, Th. 2.8] for details. \( \square \)

Since the elements \( t_1, \ldots, t_n \) of formula (13) above can be distributed arbitrarily among \( A_1, A_2, A_3 \), so that \( n_1 \) of them belong to \( A_1 \), \( n_2 \) to \( A_2 \) and \( n_3 \) to \( A_3 \), where \( n_1 + n_2 + n_3 = n \), instead of “\( n \)-augmentable” we shall use the more suggestive term “\((n_1, n_2, n_3)\) -augmentable”. Moreover, instead of arbitrary subsets \( w_1 \subseteq A_1 \), \( w_2 \subseteq A_2 \), \( w_3 \subseteq A_3 \) with \( |w_i| = n_i \), \( i = 1, 2, 3 \), we can take \( w_i \) to be partitions of the corresponding underlying sets. And, finally, we can take \( w_i \) to be just simple partitions.

**Definition 14.** Let \( A \) be a model of \( \text{TST}_4 \), \( p \) be a simple pair over \( A \) and \( n_1, n_2, n_3 \in \mathbb{N} \). \( p \) is said to be \((n_1, n_2, n_3)\)-simply augmentable if for any simple partitions \( w_1 \subseteq A_1 \), \( w_2 \subseteq A_2 \), \( w_3 \subseteq A_3 \) with \( |w_i| = n_i \), there is a simple pair \( q \) over \( A \) such that \( q \leq p \sim w_1 \cup w_2 \cup w_3 \).

\( p \) is said to be \( \omega \)-simply augmentable if it is \((n_1, n_2, n_3)\)-simply augmentable for all \( n_1, n_2, n_3 \geq 1 \).

Let

\[ \chi^*(n_1, n_2, n_3)(p) := \text{“} p \text{ is } (n_1, n_2, n_3)\text{-simply augmentable”}, \]

and

\[ \chi^*_{\omega}(p) := \bigwedge_{n_1, n_2, n_3 \geq 1} \chi^*(n_1, n_2, n_3)(p). \]

Let us write \((n_1, n_2, n_3) \leq (l_1, l_2, l_3)\) if \( n_i \leq l_i \) for each \( i = 1, 2, 3 \).
Lemma 10. For any \( p, (n_1, n_2, n_3), (l_1, l_2, l_3), \)
\[(n_1, n_2, n_3) \subseteq (l_1, l_2, l_3) \land \chi^s_{(l_1, l_2, l_3)}(p) \Rightarrow \chi^s_{(n_1, n_2, n_3)}(p). \tag{16} \]

Proof. — Indeed, given \( p \) and \( (n_1, n_2, n_3) \subseteq (l_1, l_2, l_3) \) s.t. \( \chi^s_{(l_1, l_2, l_3)}(p) \), let \( w_i \subseteq A_i \) be simple partitions such that \( |w_i| = n_i \). Since each \( w_i \) contains at least an infinite set, and \( n_i \leq l_i \), we can refine \( w_i \) to a simple partition \( w_i' \) such that \( |w_i'| = l_i \). This is done either by adding new singletons that we subtract from an infinite set, or by splitting an infinite set into two infinite subsets (here we need the property of richness for \( A \)). Now each \( w_i' \), \( i = 1, 2, 3 \), forms a simple partition with \( |w_i'| = l_i \geq n_i \), so by \( \chi^s_{(l_1, l_2, l_3)}(p) \), there is a c.p. \( q \leq p \prec B(w_1') \cup B(w_2') \cup B(w_3') \). Since \( B(w_i') \supseteq B(w_i) \), we have \( q \leq p \prec B(w_1) \cup B(w_2) \cup B(w_3) \), and we are done.

Lemma 11. If for every \( (n_1, n_2, n_3) \) there is \( A \) and \( p \) over \( A \) such that \( \chi^s_{(n_1, n_2, n_3)}(p) \), then there is \( B \) and \( q \) over \( B \) such that \( \chi^s_{(n_1, n_2, n_3)}(q) \).

Proof (Sketch, details in [8, 2.8]). — By compactness again. Consider the theory
\[ T = \text{TST}^c + \{ \chi^s_{(n_1, n_2, n_3)}(b) : n_1, n_2, n_3 \geq 1 \}, \]
where \( b \) is a new constant. To show that \( T \) is finitely satisfiable, take a finite subset \( \Sigma = \{ \chi^s_{(k_1, l_1, m_1)}(c) : i = 1, \ldots, n \} \) of \( T \). Let \( k = \max\{k_i : i = 1, \ldots, n\} \), \( l = \max\{l_i : i = 1, \ldots, n\} \), \( m = \max\{m_i : i = 1, \ldots, n\} \). By assumption there are \( A \) and \( p \) such that \( \chi^s_{(k, l, m)}(p) \) holds with respect to \( A \).
Also \( (k_i, l_i, m_i) \leq (k, l, m) \) for all \( i = 1, \ldots, n \). So by (16), all \( \chi^s_{(k_i, l_i, m_i)}(p) \), \( i = 1, \ldots, n \), hold with respect to \( A \). Equivalently, \( T = \text{TST}^c + \Sigma \) is satisfied in an expansion \( (A, C) \) of \( A \).

Lemma 12. For every simple c.p. \( p \), \( \chi_\omega(p) \Leftrightarrow \chi^s_{(n_1, n_2, n_3)}(p) \) with respect to any model \( A \) of TST. In particular \( \chi_\omega(\sigma^A) \Leftrightarrow \chi^s_{(n_1, n_2, n_3)}(\sigma^A) \).

Proof. — Trivially, \( \chi_\omega(p) \Rightarrow \chi^s_{(n_1, n_2, n_3)}(p) \). For the converse, suppose \( p \) is a simple pair and \( \chi^s_{(n_1, n_2, n_3)}(p) \) holds with respect to \( A \). Let \( w_i \subseteq A_i \), \( i = 1, 2, 3 \), be any finite partitions. It suffices to show that there is a \( q \leq p \prec \bigcup_i w_i \). Note that \( q \leq p \prec \bigcup_i w_i \) iff \( q \leq p \prec \bigcup_i B(w_i) \). Let \( w_i' = \text{st}(w_i) \). By \( \chi^s_{(n_1, n_2, n_3)}(p) \) there is a \( q \) such that \( q \leq p \prec \bigcup_i B(w_i') \), and hence \( q \leq p \prec \bigcup_i B(w_i) \). This shows that \( \chi_{(n_1, n_2, n_3)}(p) \) for all \( (n_1, n_2, n_3) \). Therefore \( \chi_\omega(p) \).
It follows from Lemma 12 that in order to prove the existence of \( \omega \)-augmentable pairs, it suffices to restrict the search to \( n \)-simply augmentable pairs for each \( n \).

Despite the above reductions, the general problem of \((n_1, n_2, n_3)\)-simple augmentability for \( \sigma^4 \) is still very hard to tackle. Tractable cases of this seem to be the special subcases of \((n, 0, 0)\), \((0, n, 0)\) and \((0, 0, n)\)-simple augmentability.

5.1. \((n, 0, 0)\)- and \((0, n, 0)\)- simple augmentability

We always refer to an underlying model \( \mathcal{A} = (A_0, A_1, A_2, A_3) \) of TST over which coherent pairs and triples are considered. A triple of partitions \((w_1, w_2, w_3)\), \( w_i \subseteq A_i \), is said to be similar if \( w_1 \sim w_2 \sim w_3 \). The size of a triple \((w_1, w_2, w_3)\) is \( n \), if \(|w_1| = |w_2| = |w_3| = n\).

Given a simple partition \( w \), let \( \inf(w) \) and \( \sin(w) \) denote the sets of infinite sets and of singletons of \( w \), respectively. A partition \( w \) such that \(|\inf(w)| = m \) and \(|\sin(w)| = l \) is called an \((m, l)\)-partition. \((m, l)\) is called the index of \( w \) and we write \( \Ind(w) = (m, l) \). In that case \( m + l = n \) is the size of \( w \). Clearly, if \( w_1 \sim w_2 \sim w_3 \), then all \( w_i \) are of the same index and size.

**Lemma 13.** Let \( \mathcal{A} \) be a rich model and let \( w_1, w_2 \) be simple partitions of \( A_0, A_1 \) respectively such that \( p_1 : w_1 \sim w_2 \). Then there is partition \( w_3 \) of \( A_2 \) such that \( \Co(w_1, w_2, w_3) \).

**Proof.** — Since \( w_1 \sim w_2 \), \( \Ind(w_1) = \Ind(w_2) = (m, l) \). Let \( \inf(w_1) = (x_1, \ldots, x_m) \), \( \sin(w_1) = (x_{m+1}, \ldots, x_n) \), \( \inf(w_2) = (y_1, \ldots, y_m) \), \( \sin(w_2) = (y_{m+1}, \ldots, y_n) \) such that \( p_1(x_i) = y_i \) for all \( i = 1, \ldots, n \). \( p_1 \) extends to the sets \( X \in B(w_1) \) as we have seen in section 2. We have to find a partition \( w_3 = (z_1, \ldots, z_n) \) of \( A_2 \), such that \( z_1, \ldots, z_m \) are infinite while \( z_{m+1}, \ldots, z_n \) are singletons, and for all \( X \in B(w_1) \), and all \( 1 \leq i \leq n \),

\[
X \in y_i \iff p_1(X) \in z_i.
\]

We first define \( z_i \) for \( m + 1 \leq i \leq k \). Take such an \( i \). If there is a \( X \in B(w_1) \) such that \( X \in y_i \), then clearly \( y_i = \{X\} \), so let us put \( z_i = \{p_1(X)\} \). If on the contrary \( y_i \cap B(w_1) = \emptyset \), then we choose an arbitrary singleton \( z_i \) such that \( z_i \cap B(w_2) = \emptyset \). This way we have defined \( z_i \), for all \( m + 1 \leq i \leq n \).

Next we define the infinite sets \( z_i \), for \( 1 \leq i \leq m \). Let

\[
K = A_2 - \bigcup \{z_i : m + 1 \leq i \leq n\}.
\]
For each \( i = 1, \ldots, m \), let \( E_i = y_i \cap B(w_1) \), and let \( D_i = \{ p_1(X) : X \in y_i \cap B(w_1) \} \). Each \( D_i \) is a finite subset of the infinite set \( K \). Using the richness of \( A \) we can find a partition of \( K \) into \( m \) infinite subsets \( z_1, \ldots, z_m \) such that \( D_i \subset z_i \). This completes the definition of \( z_i \). Their choice clearly guarantees the truth of the equivalences \( X \in y_i \Leftrightarrow p_1(X) \in z_i \). Thus \( w_3 \) is as required. \( \square \)

**Corollary 14.** Let \( A \) be a rich model.

(a) For every simple partition \( w_1 \), there are \( w_2, w_3 \) such that 
\[ \text{Co}(w_1, w_2, w_3). \]

(b) For every simple partition \( w_2 \), there are \( w_1, w_3 \) such that 
\[ \text{Co}(w_1, w_2, w_3). \]

**Proof**

(a) Given \( w_1 \), pick an arbitrary \( w_2 \) such that \( w_1 \sim w_2 \). Then use Lemma 13 to find \( w_3 \) such that \( \text{Co}(w_1, w_2, w_3). \)

(b) Given \( w_2 \), pick an arbitrary \( w_1 \) such that \( w_1 \sim w_2 \). Then use again Lemma 13 to find \( w_3 \) such that \( \text{Co}(w_1, w_2, w_3). \) \( \square \)

The above immediately implies the following.

**Corollary 15.** Let \( A \) be rich. Then for all \( n \), \( \chi_{(n,0,0)}^{(\sigma A)}(\sigma A) \) and \( \chi_{(0,0,0)}^{(\sigma A)}(\sigma A) \) hold true with respect to \( A \).

Note that Corollary 14 is a strengthening of \( A_1 \)- and \( A_2 \)-extendibility of Theorem 4 (Theorem 3.6. of [8]).

It is of some interest to observe that, in contrast to Lemma 13, we have the following impossibility result:

**Lemma 16.** Let \( A \) be a rich model. Then:

(a) There are partitions \( w_2, w_3 \) and \( p_2 : w_2 \sim w_3 \) such that \( \text{Co}(w_1, w_2, w_3) \) for no partition \( w_1 \).

(b) There are partitions \( w_1, w_3 \) such that \( \text{Co}(w_1, w_2, w_3) \) for no partition \( w_2 \).
Proof

(a) We shall use the simplest kind of partitions, namely binary ones. Pick an infinite and cofinite set \( y_0 \in A_2 \) which is “consistent”, that is, \((\forall x)(x \in y_0 \Rightarrow -x \notin y_0)\), and consider the partition \( w_2 = (y_0, -y_0) \). Next take an infinite and cofinite \( z_0 \) such that \( \{y_0, -y_0\} \subseteq z_0 \), and let \( p_2(y_0) = z_0 \) and \( p_2(-y_0) = -z_0 \). Clearly \( p_2 : w_2 \rightarrow w_3 \). We claim that there is no \( w_3 \) such that \( \text{Co}(w_1, w_2, w_3) \). Suppose not and let \( \text{Co}(w_1, w_2, w_3) \) for some \( w_1 = (x, -x) \) and \( p_1(x) = y_0 \), \( p_1(-x) = -y_0 \). Then we must have \( x \in y_0 \Leftrightarrow y_0 \in z_0 \) and \( -x \in y_0 \Leftrightarrow -y_0 \in z_0 \). Since \( \{y_0, -y_0\} \subseteq z_0 \), we must have \( \{x, -x\} \subseteq y_0 \), which contradicts the consistency of \( y_0 \).

(b) We again use binary partitions. Pick an infinite and cofinite \( x_0 \) and set \( w_1 = (x_0, -x_0) \). Next set \( z_0 = \{y \in A_2 : x_0 \notin y\} \). Since \( z_0 \) is definable, it belongs to \( A_3 \) and is infinite and cofinite. Put \( w_3 = (z_0, -z_0) \). We claim that there is no \( w_2 = (y, -y) \) such that \( \text{Co}(w_1, w_2, w_3) \) under the obvious correspondences. Suppose not and let \( w_2 = (y, -y) \) be one such. Then it should be \( x_0 \in y \Leftrightarrow y \in z_0 \). But \( y \in z_0 \Leftrightarrow x_0 \notin y \), and hence \( x_0 \in y \Leftrightarrow x_0 \notin y \), a contradiction. \( \square \)

Since in Lemma 16 we use binary partitions, we think of this as an indication that the proof of \( A_1 \)-extendibility of \( \sigma^4 \) in Lemma 3.5 of [8], specifically case 3 of the proof, cannot be simplified significantly. Yet we guess that \( A_1 \)-extendibility can be strengthened to hold for an arbitrary number of elements instead of a single one. Equivalently, we guess that \((0, 0, 2)\)-augmentability can be strengthened to \((0, 0, n)\)-one, for all \( n \geq 2 \). For the time being this is still open. In the next section we offer a partial result and some discussion concerning this problem.

5.2. Remarks on \((0, 0, n)\)-augmentability

Recall that in order to prove \( 1 \)-extendibility of \( \sigma^4 \), one has to consider an arbitrary \( x \in A_1 \) (resp. \( y \in A_2 \), and \( z \in A_3 \)) and try to find corresponding elements \( y, z \) (resp. \( x, z \), and \( x, y \)) so that the triple of binary partitions \((x, -x), (y, -y), (z, -z)\) is coherent. But this obviously coincides with proving \((2, 0, 0)\)-augmentability (resp. \((0, 2, 0)\)- and \((0, 0, 2)\)-augmentability. Therefore \((2, 0, 0)\)-, \((0, 2, 0)\)- and \((0, 0, 2)\)-augmentability have already been settled by Lemma 3.6 of [8], where the above properties are called \( A_1 \)-, \( A_2 \)- and \( A_3 \)-extendibility, respectively. Thus, in view of Corollary 15 above, the only open and seemingly tractable problem of this type is \((0, 0, n)\)-simple augmentability for \( n \geq 3 \). As was the case with
We want to show that there exist enumerated partitions that was the case for such other choice is possible.

given $y$ properties, e.g. with asymmetry of the relation

Obviously the question of $(0, 0, n)$-simple-augmentability of $\sigma^4$, for $n \geq 3$, amounts to the following:

**Question 1.** Let $A$ be a sufficiently rich $A$ (e.g. $A$ is isomorphic to an elementary submodel of a full model). Let $w_3$ be any simple partition of $A_2$ with $|w_3| \geq 3$. Do there exist similar simple partitions $w_1$, and $w_2$ of $A_0, A_1$ respectively such that $\text{Co}(w_1, w_2, w_3)$?

Below we offer some remarks with respect to this question. We work over a fixed sufficiently rich model $A$ of TST$_4$. Let us fix a simple enumerated partition $w_3 = (z_1, \ldots, z_n)$ of $A_2$ with $|w_3| = n \geq 3$ and $\text{Ind}(w_3) = (m, n - m)$. In this enumeration we assume that the first $m$ sets $z_1, \ldots, z_m$ are the infinite ones while the next $n - m$ elements $z_{m+1}, \ldots, z_n$ are the singletons.

We want to show that there exist enumerated partitions $w_1 = (x_1, \ldots, x_n)$, $w_2 = (y_1, \ldots, y_n)$, with corresponding elements $x_i \mapsto y_i \mapsto z_i$, such that $\text{Co}(w_1, w_2, w_3)$.\footnote{In general, the construction “from left to right” is the easy one, while the construction “from right to left” is the hard one. The reason of this asymmetry is simply the strong asymmetry of the relation $x \in y$: In every reasonably rich structure (like a rich model of TST, given $x$, one can find $y$ such that $x \in y$ possessing almost any prescribed properties, e.g. with $y$ being finite, or cofinite, or infinite and cofinite. In contrast, given $y$, the prescribed choices for $x$ such that $x \in y$ are drastically restricted by the very extension of $y$. If e.g. $y$ is a set of singletons or a set of cofinite sets, obviously no other choice is possible.}

Recall from section 2, that given an enumerated partition $w_3 = (x_1, \ldots, x_n)$, we denote by $X_I$ the set $\bigcup\{x_i : i \in I\}$, and similarly for $Y_I, Z_I$. Now if $w_1$ varies, to avoid ambiguity, we should write $x_i^{w_1}$ and $X_I^{w_1}$ rather than $x_i$ and $X_I$, respectively. ($x_i^w$ of course means “the $i$-set of the sequence $w_1$.”)

In view of the relation (2) of Definition 4 (page 114), the fact that there exists $w_1, w_2$ such that $\text{Co}(w_1, w_2, w_3)$ has the following formulation:

$$\exists w_1 \exists w_2 (w_1 \sim w_2 \sim w_3 \land (\forall I \subseteq [n]) (\forall i \in [n]) (X_I^{w_1} \in y_i^{w_2} \Leftrightarrow Y_I^{w_2} \in z_i^{w_3})). \tag{17}$$

We have fixed only the following partial result.

\footnote{Note that it suffices to prove the statement not for each particular $n$, but for every sufficiently large $n$, i.e., for $w_3$ with $|w_3| \geq n_0$, where $n_0$ is any given number. If for that was the case for such $w_3$, that would hold also for $w'_3$ with all smaller cardinalities. Indeed, given $w'_3$ such that $|w'_3| < n_0$, just extend arbitrarily $w'_3$ to a finer partition $w_3$ such that $|w_3| \geq n_0, |w'_3|$. Then any coherent pair that captures $w_3$, captures also $w'_3$.}
Lemma 17. Suppose \( \text{Ind}(w_3) = (1, n - 1) \), i.e., the partition \( w_3 \) contains a unique infinite set, the rest being singletons. Then (17) is true, and hence Question 1 above is answered in the affirmative.

Proof. We argue by contradiction. Let the negation of (17)

\[
(\forall w_1)(\forall w_2)(w_1 \sim w_2 \sim w_3) \Rightarrow (\exists I \subseteq [n])(\exists i \in [n])(X^{w_1}_i \in y_i^{w_2} \iff Y^{w_2}_i \notin z_i^{w_3}).
\]  

be true. Let \( z_0 \) be the unique infinite set of \( w_3 \). Then \( z_0 \) is cofinite. Fix a partition \( w_1 \) of \( A_0 \) such that \( w_1 \sim w_3 \) whose unique infinite set is \( x_0 \) and let

\[ Y = \{ w_2 : w_2 \sim w_3 \land \forall y(\inf(w_2) = \{ y \} \Rightarrow B(w_1) \subseteq y) \}. \]

Since \( B(w_1) \) is finite, clearly \( Y \) is infinite. By (18),

\[
(\forall w_2 \in Y)(\exists I \subseteq [n])(\exists i \in [n])(X^{w_1}_i \in y_i^{w_2} \iff Y^{w_2}_i \notin z_i^{w_3}).
\]

But for every \( w_2 \in Y \), if \( y_0 \) is the element corresponding to \( x_0 \) and \( z_0 \), \( B(w_1) \subseteq y_0 \). Therefore (18) implies

\[ (\forall w_2 \in Y)(\exists I)(Y^{w_2} \notin z_0). \]

Equivalently,

\[ (\forall w_2 \in Y)(\exists Y \in B(w_2))(Y \notin z_0), \]

or

\[ (\forall w_2 \in Y)(B(w_2) \cap -z_0 \neq \varnothing). \]

The latter easily implies that \(-z_0\) must be infinite, which is a contradiction since \(-z_0\) is finite. \(\square\)

Towards proving in the affirmative the general statement of Question 1, we have tried to generalize the method used in the proof of \((0,0,2)\)-augmentability of \( \sigma^A \), in Lemma 3.5 of [8]. The nontrivial case of the proof was the one where \( \text{Ind}(w_3) = (2, 0) \), i.e., \( w_3 \) was a binary partition \((z, -z)\), consisting of two infinite sets. The proof was by contradiction again. Namely we assumed that (18) is true for all \( w_1 = (x, -x) \) and all \( w_2 = (y, -y) \). But then (18) should hold also for the permutations of the partitions \( w_1, w_2 \), that is, the partitions \((-x, x) \) \((-y, y) \). The exploitation of this fact led eventually to a contradiction (actually considering only the permutations of \( w_1 \) suffices).

Does this idea work in the case of an arbitrary simple partition \( w_3 \), with \(|w_3| \geq 3\)? For simplicity one may consider the case where \( \text{Ind}(w_3) = (n, 0) \)
consists of a disjunction of case (19) reduces to a conjunction of the form
through the specific consequences, unattainable and infeasible. Perhaps an approach partitions disjuncts! (All the previous discussion concerns a fixed particular pair of for reaching a contradiction, we shall face the monstrous number of formula into a disjunction of conjunctions in order to exploit all chances eventually a contradiction can emerge. But if we make a step ahead and these formulas can be controlled and manipulated so that (19) is a combinatorial statement involving three types of entities:
- elements of the set \([n]\), having cardinality \(n\),
- elements of the set \(\mathcal{P}([n])\) (or, actually, of \(\mathcal{P}^*([n]) = \mathcal{P}([n]) - \{\emptyset, [n]\}\)), having cardinality \(2^n\), and
- elements of the set \(S_n\), having cardinality \(n!\).

For \(n = 2\), we have \(|[2]| = |\mathcal{P}^*([2])| = |S_2| = 2\). In that case (especially if we take \(\sigma = id\)) (19) reduces to a Boolean combination of no more than 8 concrete equivalences of the form \(X_i^\sigma(w_1) \in y_i^\sigma(w_2) \Leftrightarrow Y_i^\sigma(w_2) \notin z_i^w\). As a consequence these formulas can be controlled and manipulated so that eventually a contradiction can emerge. But if we make a step ahead and take \(n = 3\), we have \(|[3]| = 3, |\mathcal{P}^*([3])| = 6\) and \(|S_3| = 6\). In such a case (19) reduces to a conjunction of \(|S_3 \times S_3| = 36\) clauses each of which consists of a disjunction of \(|\mathcal{P}^*([3]) \times [3]| = 18\) concrete equivalences of the form \(X_i^\sigma(w_1) \in y_i^\sigma(w_2) \Leftrightarrow Y_i^\sigma(w_2) \notin z_i^w\). If we attempt to turn this formula into a disjunction of conjunctions in order to exploit all chances for reaching a contradiction, we shall face the monstrous number of \(18^{36}\) disjuncts! (All the previous discussion concerns a fixed particular pair of partitions \(w_1, w_2\)) That makes any attempt to reduce (19) to a set of specific consequences, unattainable and infeasible. Perhaps an approach through the structural consequences of (19) could prove successful.

\[7\] We may reasonably assume that if this case is settled, then the general case of index \((m, l)\) can also be settled by easy adjustments.
6. Extendibility with no reference to TST

The discussion of the extendibility and augmentability properties of c.p.’s always takes place in ZFC but relatively to models of TST. In order to focus on the combinatorial character of these issues alone one can relax the role of particular models of TST by restricting oneself to the most natural and common of these, namely the full models

\[(A_0, \mathcal{P}(A_0), \mathcal{P}^2(A_0), \mathcal{P}^3(A_0))\]

of TST, which are just sequences of consecutive powersets. Given an infinite set \(A_0\) (preferably countable for simplicity) let \(A_1 = \mathcal{P}(A_0)\), \(A_2 = \mathcal{P}(A_1)\) and \(A_3 = \mathcal{P}(A_2)\). We call such a sequence \((A_0, A_1, A_2, A_3)\), a 4-staircase.

In order to be able to talk about similarity of partitions in the sets \(A_1, A_2, A_3\), we consider all infinite cardinalities as identical, denoted \(\infty\).

We denote this reduced cardinality of a set \(X\) by \(\|X\|\) and write \(\|X\| = |X| = n\) if \(X\) is finite with \(n\) elements, and \(\|X\| = \infty\) if \(X\) is infinite.

Replacing the ordinary notion of equipollence of sets \(|X| = |Y|\) by reduced equipollence \(\|X\| = \|Y\|\), coherent pairs can be formulated for any staircase \(\mathcal{A} = (A_0, A_1, A_2, A_3)\). A pair of functions \(p = (p_1, p_2)\) is a coherent pair if the definition given above for coherence holds for \(p\) with the relation \(\|X\| = \|Y\|\) in place of \(|X| = |Y|\). Namely

**Definition 15.** Let \(A_1, A_2\) be infinite sets and \(w_1, w_2\) be finite partitions of \(A_1, A_2\) respectively. We say that \(w_1\) and \(w_2\) are similar and write \(w_1 \sim w_2\), if there is a bijection \(p : w_1 \rightarrow w_2\) such that \(\|p(x)\| = \|x\|\) for every \(x \in w_1\). In that case we write \(p : w_1 \sim w_2\). If \(B_1, B_2\) are finite Boolean algebras, then \(B_1 \sim B_2\) if Atom\((B_1) \sim\) Atom\((B_2)\).

**Definition 16.** Let \(\mathcal{A} = (A_0, A_1, A_2, A_3)\) be a 4-staircase. A coherent pair over \(\mathcal{A}\) is a pair \(p = (p_1, p_2)\) of finite 1-1 mappings with the following properties:

(a) \(\text{dom}(p_1)\) is a finite Boolean subalgebra of \(A_1\), \(\text{rng}(p_1) = \text{dom}(p_2)\) is a finite Boolean subalgebra of \(A_2\), and \(\text{rng}(p_2)\) is a finite Boolean subalgebra of \(A_3\). We set \(u_1 = \text{dom}(p_1)\), \(u_2 = \text{rng}(p_1) = \text{dom}(p_2)\) and \(u_3 = \text{rng}(p_2)\).

(b) \(p_1 : u_1 \sim u_2\) and \(p_2 : u_2 \sim u_3\).
We believe that the study of extendibility properties of coherent pairs might constitute a serious research project towards the solution of NF consistency. But independently of that, extendibility questions are genuine combinatorial problems interesting in themselves. Of course dealing with staircases instead of general models of TST is a restriction rather than a generalization concerning the results one may obtain (in the sense that if some $\phi$ holds with respect to all staircases, it doesn’t follow that it holds for all models of TST). However, people working or just being interested in ordinary set theoretic combinatorics (e.g. partition calculus) can get interested in problems concerning coherent pairs more easily through the framework of staircases rather than through TST models. The purpose of using the term “4-staircase” instead of “full model of TST$_4$” is simply to disconnect the issue from the milieu of TST, its language, models etc, that might bother a combinatoricist.

**Bibliography**


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