Sets with dependent elements: Elaborating on Castoriadis' notion of 'magma'

Athanassios Tzouvaras
Depart. of Mathematics, Univ. of Thessaloniki
540 06 Thessaloniki, Greece
E-mail:tzouvara@auth.gr

1 Introduction

In his seminal work [2], C. Castoriadis[1] is concerned, among many other things, with the way collections of things are presented to (or created by) human mind. In particular, he aims at making us rethink of the "self-evident" belief of western rationalistic tradition that a collection needs always to be a collection of distinct, definite and ontologically independent elements. (Recall Cantor's well-known definition: "By an aggregate we understand any collection into a whole of definite and separate objects of our intuition or our thought".) His favorite (counter) example is the "totality of meanings" of a natural language, or the "totality of one's memories". Since the elements of these totalities are ny no means definite, separate and independent, one could hardly classify them as sets of the Cantorian universe. For instance, if a is a particular meaning or memory, then the non-definiteness of a in isolation would mean that the object $\{a\}$, consisted only of a, would hardly exist as the Cantorian tradition requires. Yet these collections are still there and need to be comprehended in the framework of a theory. Castoriadis seems to

[1] Cornelius Castoriadis (1922-1997) has been a prominent social and political thinker of our time. He spent most of his life in Paris. From 1979 till his death he was Director of Studies at the École des Hautes Études en Sciences Sociales (EHESS). [2] is widely considered his main work. Although a humanitarian philosopher by training, he had got an impressive and solid background in science, especially in economics, mathematics and theoretical physics.

believe that the specific Cantorian construing of collections (and the dropping of all the rest as non-existing) is rather accidental and due to the early adoption by the western thought of what he calls “setist-identitist logic” (roughly, the two-valued classical logic interacting with naive set theory of predicate extensions).

> “For twenty five centuries, the Greek-western thought is being developed, expanded and refined upon the following thesis: To be means to be something determined, and to say means to say something determined; and of course, to say the truth means to determine the saying and what is said in terms of the being or to determine the being in terms of the saying, and the eventual verification of the congruence of these two. This result which relies on the demands of a certain aspect of saying, was neither accidental nor indispensable; it was the proclaiming of Logos (ratio) as the institutionalized thought by the West. (...) I call the logic mentioned above *identitist* (*identitaire*), as well as *setist* (*ensembliste*), being aware of the terminological abuse of language, for reasons which will be clear immediately. (...) The most advanced and wealthy reach of the identitist logic is the working out of mathematics. (...) We are interested here in the elementary logical principles of set theory, because the latter abbreviate, clarify and point out vividly what was always subject to this logic.” ([2], p. 319-321)

Castoriadis proposes the verb “setify” (ensembliser) to signify the act of “creating sets” out of something pre-existent and rather undifferentiated. This kind of undifferentiated reality out of which the setist-identitist logic creates sets, classes, objects and properties, is, roughly, what he calls *magma*.

> “We aim to reveal the way of being of what is given, before the human mind imposes upon it the setist-identitist logic; this, which is offered in that way of being, is what we call *magma*. Obviously, giving a typical definition of the notion, is out of question. Nevertheless the following statement might not be quite useless:
>
> Magma is that from which we can extract (or in which we can construct) setist organizations in indefinite number, but which

can never be recovered (ideally) by a finite or infinite combination of any of these organizations." ([2], p. 479)

As said above, Castoriadis' standard example of a magma is the "totality of meanings" of a natural language, or the "totality of one's representations (represéntation, Vorstellung)". Concerning such a totality

> "We must think of a multiplicity which is not one in the traditional sense of the word, but which we point out as one, and which is not a multiplicity in the sense that we could enumerate, actually or potentially, its 'elements', yet we can point out within it terms not altogether confused. (...) And we must think of the acts of identitist logic as multiple simultaneous dissections which transform and activate those ingredients, those terms, into distinguished and determined elements; which organize the set, the be-in, be-over, be-near in a system of determined and determining relations; which differentiate what is thus discerned into 'objects' and 'properties', and use this differentiation to create 'sets' and 'classes'."([2], p. 480)

Objects and properties of the ordinary world seem to be the outcome of the ability of our mind for separation, partitioning and individuation. Sets and classes are also products of this very mental mechanism. Castoriadis, mentions times and again Cantor's well-known "definition" of set: "A set is a collection into a whole of definite and separate objects of our intuition or our thought. These objects are called 'elements' of the set." Any specification of a set is clearly an act of separation and individuation. When we say "let X be the set of all x such that...", we focus on a specific part of reality, we individualize it and cut it off as a separate object by an act of saying, i.e., a linguistic term (a formula).

On the other hand, the main characteristic of magma that distinguishes it from the ordinary sets and classes seems to be the fact that its 'elements' are not fully determined, nor fully distinguishable and separable from other elements. As a consequence, we can't construct new magmas or submagmas by the acts of saying, i.e., by the help of linguistic properties.

> "As a magma the (collection of) meanings of the language are not elements of a set that are subject to determinacy as a

condition and manner of being. A meaning is unboundedly determinable, without being determined. It can always be pointed out, be temporarily assigned an identitist element (as in naming), be 'something' as a departure in an open series of successive determinations. But these determinations can never completely exhaust it. Moreover, they can enforce us, and actually they always do it, to return to the 'something' of the departure and set it as 'something else '. " ([2], p. 483)

Quite analogous is John Searle's position about mental states and the content of our consciousness in general:

> "One has conscious states such as pains and thoughts only as a part of living a conscious life, and each state has the identity it has only in relation to other such states. My thought, for example, about a ski race I ran long ago, is only that very thought because of its position in a complex network of other thoughts, experiences, and memories. My mental states are internally related to each other in the sense that in order for a mental state to be that state with that character it has to stand in certain relation to the real world. " ([8], p. 42)

The above speculations about magma are rather vague. However in [3] Castoriadis devotes a whole chapter to the subject entitled "The logic of magmas and the problem of autonomy". After recalling the definition of magma already given in [2], (cited above) Castoriadis says:

> "I note by the way that Jean-Pierre Dupuy observed that the 'definition' given above is not satisfactory, because it applies equally well to what in mathematics has been named 'class' in order to avoid Russell's paradox, something which is not possible unless by means of a *regressus ad infinitum*.
>
> Instead however to comment on this 'definition', I shall attempt to shed some light on other aspects of the idea of magma, exploring the ways (and the no-way-outs) of a more 'formal' language.
>
> For this purpose we need to introduce a primitive term/relation: The term/relation of *signifying*, or *pointing out* (repérage), which

> is both unary and binary. So I assume that the reader understands unambiguously the expressions: 'signifying x', 'x signifies y' and 'signifying x in y'. ('signifying a dog', 'the collar signifies the dog', 'signifying the dog in the field'.) Using this term/relation, I 'define' magma through the following properties:
>
> (M1) If M is a magma, we can signify in M an indefinite number of sets.
>
> (M2) If M is a magma, we can signify in M magmas different from M.
>
> (M3) If M is a magma, there is no partitioning of M into magmas.
>
> (M4) If M is a magma, every analysis of M into sets leaves a magma as a residue.
>
> (M5) If something is not a magma, then it is a set or nothing."
> ([3], p. 309-310)

Unfortunately, the above relation of "signifying" is quite obscure. We can't tell precisely what M1 and M2 mean. However by subsequent explications we can say r that M1 most likely means: For every magma M there are sets x such that $x \subseteq M$. While M2 means: For every magma M there is a magma $N \neq M$ such that $N \subseteq M$. If we interpret M1 this way, then we easily see that M1 and M4 are contradictory. This is proved by Castoriadis himself (p. 312): Given a magma M, let N be the union of all sets contained in M. By M4, $M - N$ is a magma. But then, by M1, there is a set x such that $x \subseteq M - N$. This contradicts the fact that N is the union of all sets contained in M.

Castoriadis says that the crucial property is M3.

> "It expresses the impossibility of applying the scheme/operator of *separation* (my comment: He apparently means the separation scheme of set theory) – and, the most important, that this scheme is not significant in the present situation. Within the magma of my representations, I cannot strictly separate those that 'refer to my family' from those that do not. (In other words: among the representations which, at first sight, 'do not refer to my family' there is the first link of at least one coherent chain of representations leading to 'my family'. This means that a representation is

not a 'distinct and well defined object', but that it is everything that is connected with.) In the domain of meanings carried by the present French language, I cannot strictly separate those that refer to mathematics in any way, from those that do not." ([3], p. 311)

To sum up: The notion of magma as described above is rather obsure, and the properties M1-M5 are most probably unformalizable if not inconsistent. Yet the idea that deserves further elaboration, and which will be discussed in the rest of this paper, is the following: Certain objects come up not in isolation, but in unbreakable *chains* with other dependent objects, so that one cannot add or subtract one element without adding or subtracting the elements depending on it. Hence, besides the fundamental relation $\in$ between objects and collections, we may assume that objects are equipped from the very beginning with another fundamental *relation of dependence*, denoted, say, by $b \triangleleft a$ ("b depends on a"). Then "legitimate collections" (i.e., collections "seen" by our mind) are only those x, which when they contain a they contain also every object b such that $b \triangleleft a$. Such collections will be the *sets* of our universe. Further, given a property $\phi(x)$, the "extension" of $\phi(x)$ is not in general a set. However we can find sets x, y that are best approximations of the extension of $\phi(x)$ from within and from without.

It is these aspects of Castoriadis' magma-theoretic discussion that we shall attempt to formalize below.

2 Similarities with H. Skala's set theory

H. Skala [9], guided by a totally different motivation, was led to a certain system of set theory whose main axioms bear a striking similarity with M3 of the preceding section. Her purpose was to avoid Russell's paradox without completely rejecting the full comprehension scheme. She wanted just to block the way leading from full comprehension to Russell's paradox. To obtain this she postulates the following:

(A1) Extensionality.

(A2) For for every formula $\phi(x)$ of set theory there is a *least set* a such that $\forall x(\phi(x) \Rightarrow x \in a)$.

(A3) For every formula $\phi(x)$ there is a *greatest* set b such that $\forall x(x \in b \Rightarrow \phi(x))$.

The idea behind (A2) and (II) of the preceding section is roughly the same, with the exception that Skala considers a and b to be a *sets*, i.e., a first class objects.

Skala calls a and b above the *least upper set* and the *greatest lower set* of ϕ, denoted $\text{lus}\phi(x)$ and $\text{gls}\phi(x)$ respectively. A predicate $\phi(x)$ is called *definite* if $\text{lus}\phi(x) = \text{gls}\phi(x)$, in which case (and only then) one may write $\{x : \phi(x)\}$ for the extension of $\phi(x)$. For example it is easily seen that for every set a, the predicate "$x \in a$" is definite. So are the predicates $x \neq x$ and $x = x$, whence the existence of the empty set $\emptyset$ and the universal set U.

Russell's paradox is blocked by the fact that the predicate "$x \notin x$" is not definite, i.e., if $r_1 = \text{gls}(x \notin x)$ and $r_2 = \text{lus}(x \notin x)$, then it is easily seen that $r_1 \neq r_2$. (By the definition of r_1, r_2, for every x, $x \in r_1 \Rightarrow x \notin x \Rightarrow x \in r_2$. Therefore $r_1 \notin r_1$ while $r_2 \in r_2$).

Finally, in order to guarantee the existence of intersection, union and complement Skala assumes the following:

(A4.1) If $\phi(x)$ and $\psi(x)$ are definite, then so is $\phi(x) \vee \psi(x)$.

(A4.2) If $\phi(x)$ is definite, then so is $\neg\phi(x)$.

The system A1-A4.2 is referred to as Skala' set theory.

The price to be paid for this paradox elimination is that not all sets have singletons, i.e., $\{a\}$ does not exist for every a. Given a non-definite predicate ϕ, the set $\text{lus}\phi(x) - \text{gls}\phi(x)$ is said to be the *vagueness set* of ϕ. It is easy to see that for every $a \in \text{lus}\phi(x) - \text{gls}\phi(x)$, $\{a\}$ does not exist (equivalently, the predicate "$x = a$" is not definite). By the same token one can see that "$x \subseteq a$" is not definite for such a, i.e., the powerset of a does not exist.

A set a such that $\{a\}$ exists is said to be *normal*. The predicate "x is normal" is proved to be definite, hence the set U_0 of normal sets exists. A set a is *fully definite* if for every $\phi(x)$ the predicate $x \in a \wedge \phi(x)$ is definite. It is proved that U_0 is fully definite. It is shown that a is fully definite iff $a \subseteq U_0$.

Later Manakos [7] observed that Skala's axioms A2, A3 above are equivalent to a "general union" and a "general intersection" axioms formulated as follows:

(GU) For every formula $\phi(x)$ there is a set y such that $y = \bigcup\{x : \phi(x)\}$. (Formally: $\exists y \forall x(x \in y \Leftrightarrow \exists z(\phi(z) \wedge x \in z))$.)

(GI) For every formula $\phi(x)$ there is a set y such that $y = \bigcap\{x : \phi(x)\}$. (Formally: $\exists y \forall x(x \in y \Leftrightarrow \forall z(\phi(z) \Rightarrow x \in z))$.)

Manakos calls the system A1+GU+GI "UI" (from "union-intersection") and shows that UI is equivalent to Skala's A1-A4.1. Similarly UI+A4.2 is equivalent to A1-A4.2. By A3 (or GI), for every set a the collection $\bigcap\{x : a \in x\}$ is a set and we denote it by $\widehat{a}$. $\widehat{a}$ is the smallest set containing a, i.e., $\forall x(a \in x \Rightarrow \widehat{a} \subseteq x)$. Define the relation

$$b \lhd a := \forall x(a \in x \Rightarrow b \in x),$$

or equivalently, $\widehat{b} \subseteq \widehat{a}$.

If we accept A4.2, i.e., if every set has a complement, then $b \lhd a$ implies $a \lhd b$, hence $\lhd$ is an equivalence relation with equivalence classes the sets $\widehat{a}$, called in [7] *molecules*. When "$x = a$" is definite, $\widehat{a} = \{a\}$, but when "$x = a$" is not definite $\{a\} \subset \widehat{a}$. It is easy to see that the molecules are atoms of the boolean algebra of sets, i.e., they are inseparable.

Perhaps more interesting is the case where the relation $\lhd$ is not symmetric, i.e., when we drop existence of complements. In this case one can postulate that a collection is a set iff it is closed under $\lhd$. For instance Manakos shows in [7] that the axiom scheme

$$\forall x \forall y(\phi(x) \ \& \ y \lhd x \Rightarrow \phi(y)) \Rightarrow \exists y \forall x(x \in y \Leftrightarrow \phi(y)),$$

is equivalent (modulo extensionality) to (GU)+(GI).

An obvious shortcoming of Skala's theory, however, is that we do not see how to construct some basic classical objects like natural numbers and ordinal numbers. This shorcoming will be eliminated in section 4.

3 Dependence relations as preorderings

Castoriadis examples of magmas, namely, the collection of meanings of the French language, the collection of one's mental representations (or memories) etc., are in fact examples of collections whose (certain) members present a certain *dependence* to other members. We find this notion of ontological dependence interesting, and in this paper we shall focus on that and its interaction to membersip. Formally speaking, Castoriadis introduces into the set universe a second *primitive* relation besides $\lhd$ (dependence), and asks how $\lhd$ affects $\in$, i.e., the set formation.

The dependence has the form of "pointing to" and is in general one-way. For instance, if Anna used to wear a specific perfume X, then the smell of

X always "points" to Anna, i.e., it recalls to my mind Anna's picture. The converse is not true. The mental picture of Anna does not (necessarily) recall the smell of her perfume. Similarly, hearing a song may point, in one's mind, to one's birthplace, while the converse may not happen. When we think of our birthplace we do not think of everything they point to it. In general, in the fields of meanings, memories, etc., the dependence is directed from simple objects to complex ones and from indifferent to familiar. When x points to y, then x acts as a signal or a reminder of y. x is something simple while y is complex and y is familiar, while x would be indifferent out of this connection with y.

Saying that $\lhd$ is primitive we mean that $\lhd$ comes from outside, from the fact that things are the way they are. In the preceding section we denoted by $\lhd$ a concrete dependence relation defined by $b \lhd a := (\forall x)(a \in x \Rightarrow b \in x)$. I.e., we defined dependence *in terms of sethood*. However if dependence is really primitive, while sethood (i.e., which collections are "legitimate") is a secondary property imposed to things by our mind, then the definition should run the opposite direction. Namely we should define sets in terms of $\lhd$ rather than the othger way around.

Since we want to have $\lhd$ as a primitive relation, the question is: What should be the general characteristics of a dependence relation? Obviously (a) every object a depends on itself, and (b) if a depends on b and b depends on c, then a depends on c. So *reflexivity* and *transitivity* (and only these) seem to be the properties necessary for a binary relation to be construed as dependence. Such a relation is usually called *preordering*. So henceforth we shall identify dependence relations with preorderings. Every preordering $\lhd$ produces an equivalence relation $\sim$, where $a \sim b$ iff $b \lhd a$ and $a \lhd b$. Thus $a \sim b$ if a and b are interdependent.

4 Formalizing dependence in a set universe

To formalize the above considerations, we need to work in the language $L_d = \{\in, \lhd\}$, where $\lhd$ is a binary relation symbol intended to be interpreted as dependence. Our set theory will contain urelements, so, as usual, L_d will be two-sorted, with variables $a, b, c, \ldots$ for urelements and $x, y, z, \ldots$ for sets. Variables $s, t, r, \ldots$ will denoted either sets or urelements. Below we shall introduce step by step certain axioms in the language L_d, with the necessary motivation when needed. For sets of urelements the axioms are essentially those of Manakos [7]. However we are going to extend $\lhd$ over the cumulative hierarcy build on the set of urelements.

(D0) *Extensionality*:

$$(\forall x)(\forall y)(x = y \Leftrightarrow (\forall t)(t \in x \Leftrightarrow t \in y).$$

(D1) There is a set containing exactly the urelements.

$$(\exists x)(\forall t)[t \in x) \Leftrightarrow (\exists a)(a = t)].$$

Because of (D0) this set is unique and we denote it by A. We shall refer to A as the *set of urelements*.

(D2) $\lhd$ is a (partial) preordering (i.e. a reflexive and transitive relation) on urelements.

(D3) *Definability of* $\lhd$ *in terms of* $\in$: For all $a, b \in A$,

$$b \lhd a \ \Leftrightarrow \ (\forall x)(a \in x \Leftrightarrow b \in x).$$

Direction "$\Rightarrow$" of the above equivalence is motivated by the fact that every set should be closed under dependent elements. Direction "$\Leftarrow$" on the other hand is necessary in order to separate independent elements.

The following two axioms are identical to Manakos' (GU) and (GI) (see [7]).

(D4) *General Union*: For every formula $\phi(y)$ of L_d (with a single free set variable y),

$$(\exists x)(\forall s)[s \in x \Leftrightarrow (\exists y)(\phi(y) \ \& \ s \in y)].$$

We denote this set by $\bigcup \phi(y)$.

(D5) *General Intersection*: For every formula $\phi(y)$ of L_d (with a single free set variable y),

$$(\exists x)(\forall s)[s \in x \Leftrightarrow (\forall y)(\phi(y) \Rightarrow s \in y)].$$

We denote this set by $\bigcap \phi(y)$.

Already, by the help of the preceding axioms we can prove the existence of some sets other than A. Let $\phi(x) : x \neq x$. Then $\bigcup \phi(y) = \emptyset$ and $\bigcap \phi(y) = V$.

Now for every $a \in A$, let $\sigma_a(x) : a \in x$ and let $\widehat{a} = \bigcap \sigma_a(x)$. By (D5) $\widehat{a}$ is a set and by (D3) $\widehat{a} = \{b : b \lhd a\}$. Also $b \lhd a \iff \widehat{b} \subseteq \widehat{a}$. Further, if x_0, x_1 are given sets, then applying (D4) and (D5) to the predicates $x = x_0 \vee x = x_1$, we define the sets $x_0 \cup x_1$ and $x_0 \cap x_1$.

We are going now to consider the cumulative hierarchy built on A and extend $\lhd$ over this hierarchy. Given any preordering $\preceq$ whose domain contains either sets or urelements or both, define its "shifting" $\preceq^+$ *exclusively on sets* as follows:

$$x \preceq^+ y \iff (x = \emptyset \ \& \ y = \emptyset) \vee [y \neq \emptyset \ \& \ (\forall t \in x)(\forall s \in y)(t \preceq s)].$$

Lemma 4.1 *For all sets x, x', y, if $x \subseteq y$ and $x' \lhd^+ x$, then $x' \subseteq y$.*

Proof. Let $x \subseteq y$ and $x' \lhd^+ x$. Let $t \in x'$. Then x is not empty ant if $s \in x$, then $t \rhd s$. But $s \in y$ and hence, by D3, $t \in y$. Thus $x' \subseteq y$. $\dashv$

Given a class X, let $\mathcal{P}^*(X)$ denote the classes of *nonempty* subsets of X. Then, let $V_0(M) = M$, $V_{\alpha+1} = V_\alpha \cup \mathcal{P}^*(V_\alpha(M))$, and $V(M) = \bigcup_\alpha V_\alpha(M)$.

5 Shiftings of preorderings

Since dependence relations $\lhd$ are preorderings, we shall use more standard notation for them. Namely we shall write $a \preceq b$ instead of $b \lhd a$. Given a preordering $\preceq$ on a class X, $\preceq$ induces a relation $\preceq^+$ on the class $\mathcal{P}(X)$ of all subsets of X in a natural way as follows:

$$x \preceq^+ y := (\forall t \in x)(\exists s \in y)(t \preceq s). \tag{1}$$

Similarly, if $\equiv$ is an equivalence relation on X, $\equiv$ induces an equivalence $\equiv^+$ on $\mathcal{P}(X)$ as follows:

$$x \equiv^+ y := (\forall t \in x)(\exists s \in y)(t \equiv s) \ \& \ (\forall s \in y)(\exists t \in x)(t \equiv s). \tag{2}$$

$\preceq^+$ and $\equiv^+$ are said to be the *exponential shiftings* (or just the *shiftings*) of the relations $\preceq$ and $\equiv$ respectively. In fact $\preceq^+$ is defined here for the first time (as far as we know) and imitates the definition of $\equiv^+$ which is well-known in the bibliography. $\equiv^+$ is met in such diverse places in the bibliography, as the the definition of a *bisimulation relation* (see [1]) and uniform spaces and Vietoris topology see [4], [5]). For every preordering $\preceq$, let $eq(\preceq)$ denote the equivalence generated by $\preceq$, i.e.,

$$(x, y) \in eq(\preceq) \iff x \preceq y \ \&\ y \preceq x.$$

The following is easy to check.

Lemma 5.1 *(i) $\preceq^+$ is a preordering on $\mathcal{P}(X)$.*

(ii) If $\preceq$ is a total preordering, i.e., $t \preceq s$ or $s \preceq t$ for every $s, t \in X$, then so is $\preceq^+$.

Recall from [1] that if X is a transitive class, a relation $\equiv$ on X is said to be a *bisimulation* if $\equiv = \equiv^+$. The smallest bisimulation on X is $=$. This is also the greatest one if X is well-founded. If X is non-well-founded, the greatest bisimulation can be different from $=$. This is the greatest fixed point of the positive operator $\Gamma : \mathcal{P}(X \times X) \to \mathcal{P}(X \times X)$, defined by

$$\Gamma(R) = \{(x, y) : (\forall t \in x)(\exists s \in y)((t, s) \in R) \ \&\ (\forall s \in y)(\exists t \in x)((t, s) \in R)\}.$$

Now if $\preceq$ is a preordering, $\preceq$ is "half" of the equivalence $eq(\preceq)$ it generates. However $\preceq^+$ is not half" of the equivalence $eq(\preceq)^+$, that is $eq(\preceq^+)$ does *not* coincide in general with $eq(\preceq)^+$. We have

$$x \ eq(\preceq^+) \ y := (\forall t \in x)(\exists s \in y)(t \preceq s) \ \&\ (\forall s \in y)(\exists t \in x)(s \preceq t),$$

while

$$x \ eq(\preceq)^+ y := (\forall t \in x)(\exists s \in y)(t \preceq s \ \&\ s \preceq t) \ \&\ (\forall s \in y)(\exists t \in x)(s \preceq t \ \&\ t \preceq s).$$

So the obvious connection of the two relations is $eq(\preceq)^+ \subseteq eq(\preceq^+)$.

Our main concern is to study preorderings of *urelements* and their interaction with membership. However as a guiding tool let us fisrt consider preorderings of pure sets.

Given a transitive class X, does there exist a preordering $\preceq$ on X such that $\preceq=\preceq^+$? The answer is Yes. If $\preceq_{\mathrm{IND}}$ is such a preordering, then we must have, for all $x, y \in X$,

$$x \preceq_{\mathrm{IND}} y \iff (\forall t \in x)(\exists s \in y)(t \preceq_{\mathrm{IND}} s), \tag{3}$$

that is, $\preceq_{\mathrm{IND}}$ must be a fixed point of the positive operator

$$\Delta(R) = \{(x, y) : (\forall t \in x)(\exists s \in y)((t, s) \in R)\}. \tag{4}$$

Proposition 5.2 *If X is a transitive and well-founded class, there is a unique preordering $\preceq_{\mathrm{MIN}}$ on X satisfying (3). Namely for all $x, y \in X$,*

$$x \preceq_{\mathrm{MIN}} y \iff \rho(x) \leq \rho(y), \tag{5}$$

where $\rho(x)$ is the ordinal rank of x in the usual cumulative hierarchy of X.

Proof. Let $\preceq_{\mathrm{MIN}}$ satisfy (3). We prove (5) by induction on the pairs of ordinals $(\rho(x), \rho(y))$ ordered by the canonical well-ordering (see [6], p.20). Let $x, y \in X$ and assume

$$t \preceq_{\mathrm{MIN}} s \iff \rho(t) \leq \rho(s), \tag{6}$$

holds for all t, s such that $(\rho(t), \rho(s)) < (\rho(x), \rho(y))$. We prove the equivalence (5).

"$\Rightarrow$": Let $x \preceq_{\mathrm{MIN}} y$. If $x = \emptyset$, then $\rho(x) = 0$, so $\rho(x) \leq \rho(y)$ and the implication holds. So let $x \neq \emptyset$. Let $t \in x$. By (3), there is $s \in y$ such that $t \preceq_{\mathrm{MIN}} s$. By the induction hypothesis (6), $\rho(t) \leq \rho(s)$. That means that $\sup\{\rho(t) : t \in x\} \leq \sup\{\rho(s) : s \in y\}$, hence $\rho(x) \leq \rho(y)$ and the implication "$\Rightarrow$" is proved.

"$\Leftarrow$": Let $\rho(x) \leq \rho(y)$. If $\rho(x) = 0$, then $x = \emptyset$ and it is easy to infer from (3) that $\emptyset \preceq_{\mathrm{MIN}} y$ for every $y \in X$. So the implication holds. Let $x \neq \emptyset$. Let $t \in x$. Since $\rho(x) \leq \rho(y)$, clearly there must be an $s \in y$ such that $\rho(t) \leq \rho(s)$. By (6), $t \preceq_{\mathrm{MIN}} s$. Therefore $(\forall t \in x)(\exists s \in y)(t \preceq_{\mathrm{MIN}} s)$. It follows from (3) that $x \preceq_{\mathrm{MIN}} y$, and the implication "$\Leftarrow$" is proved. $\dashv$

If X is non-well-founded, then the preordering of proposition 5.2 is the *least* fixed point of Δ – that is why we denoted by $\preceq_{\mathrm{MIN}}$. Besides there is a a greatest fixed-point for the operator Δ above, i.e., a greatest relation $\preceq_{\mathrm{IND}}$

satisfying (3). We denote this by $\preceq_{\text{MAX}}$. As Aczel proves in [1] (theorems 6.4 and 6.5), $\preceq_{\text{MAX}}$ will be the union of all relations R such that $R \subseteq \Delta(R)$, i.e.,

$$\preceq_{\text{MAX}} = \bigcup\{R : R \subseteq \Delta(R)\}. \tag{7}$$

It is not clear at first sight that any fixed point of (4) must be a preordering. However for $\preceq_{\text{MAX}}$ this is the case. the case

Proposition 5.3 $\preceq_{\text{MAX}}$ *is a preordering on* X*.*

Proof. First note that the equality $=$ on X is a preordering and moreover $=\subseteq \Delta(=)$, hence $=\subseteq\preceq_{\text{MAX}}$. This shows that $\preceq_{\text{MAX}}$ is reflexive relation. To show transitivity we first show the following

Claim. If $R \subseteq \Delta(R)$ and $S \subseteq \Delta(S)$, then $R \circ S \subseteq \Delta(R \circ S)$, (where $R \circ S = \{(x, y) : \exists z((x, z) \in R \;\&\; (z, y) \in S)\}$).

Proof of the Claim. By assumption $R \subseteq \Delta(R)$ and $S \subseteq \Delta(S)$. Hence $R \circ S \subseteq \Delta(R) \circ \Delta(S)$. So it suffices to show that $\Delta(R) \circ \Delta(S) \subseteq \Delta(R \circ S)$. Let $(x, y) \in \Delta(R) \circ \Delta(S)$. Then there is z such that $(x, z) \in \Delta(R)$ and $(z, y) \in \Delta(S)$. These two yield, respectively,

$$(\forall t \in x)(\exists u \in z)((t, u) \in R)$$

and

$$(\forall u \in z)(\exists s \in y)((u, s) \in S).$$

The last two formulas obviously imply

$$(\forall t \in x)(\exists s \in y)((t, s) \in R \circ S),$$

therefore $(x, y) \in \Delta(R \circ S)$. So indeed $\Delta(R) \circ \Delta(S) \subseteq \Delta(R \circ S)$ and this proves the Claim.

Now let $x \preceq_{\text{MAX}} y$ and $y \preceq_{\text{MAX}} z$. Then there are R, S such that $(x, y) \in R \subseteq \Delta(R)$ and $(y, z) \in S \subseteq \Delta(S)$. Then $(x, y) \in R \circ S$ and, by the Claim, $R \circ S \subseteq \Delta(R \circ S)$, hence $(x, z) \in R \circ S \subseteq\preceq_{\text{MAX}}$, i.e., $x \preceq_{\text{MAX}} z$. This proves the transitivity of $\preceq_{\text{MAX}}$. ⊣

Note that since by Proposition 5.2 $\preceq_{\text{MIN}}$ is identical to the preordering induced by the rank function on X, $\preceq_{\text{MAX}}$ is in fact the *generalization of the rank function in the case* that X is non-well-founded.

Lemma 5.4 *(i) If y is a non-well-founded set, then for every set x, $x \preceq_{\mathrm{MAX}} y$.*

(ii) If x is a well-founded set and y is non-well-founded, and $\prec_{\mathrm{IND}}$ is any fixed point of Δ, then $y \not\preceq_{\mathrm{IND}} x$.

Proof. (i) Since y is non-well-founded, there is an infinite sequence $(t_n)_n$, $n \geq 0$, such that $t_0 = y$ and for all n, $t_{n+1} \in t_n$. Let x be any set. If $x = \emptyset$, the claim is obvious. So let $x \neq \emptyset$. Let $x_0 = x$ and $x_{n+1} = \bigcup x_n$, for every n. Define the relations R_n, $n \geq 0$, inductively as follows:

$R_0 = \{(x_0, t_0)\}$,

$R_{n+1} = \{(u, t_{n+1}) : u \in x_n\}$.

It is easy to check that for every n, $R_n \subseteq \Delta(R_{n+1})$. Indeed, let $(u, v) \in R_n$. Then $v = t_n$, while $u \in x_{n-1}$. Is $s \in u$, then $s \in x_n$, hence $(s, t_{n+1}) \in R_{n+1}$, where $t_{n+1} \in t_n$. Therefore $(u, v) \in \Delta(R_{n+1})$. Let $R = \bigcup_n R_n$. Let $(t, s) \in R$. Then $(t, s) \in R_n$ for some n, hence $(t, s) \in \Delta(R_{n+1})$, therefore $(t, s) \in \Delta(R)$, since Δ is a monotone operator. So $(x, y) \in R \subseteq \Delta(R)$ which means, by the definition of $\preceq_{\mathrm{MAX}}$, that $x \preceq_{\mathrm{MAX}} y$.

(ii) Let again $(t_n)_n$, $n \geq 0$ be an infinite sequence such that $t_0 = y$ and for all n, $t_{n+1} \in t_n$. Suppose $y \preceq_{\mathrm{IND}} x$. Then clearly, by (3), there must be a sequence $(s_n)_n$ such that $s_0 = x$, $s_{n+1} \in s_n$ and $t_n \preceq_{\mathrm{IND}} s_n$. But $(s_n)_n$ must be finite and if its length is $n+1$, then $s_n = \emptyset$. So $t_n \preceq_{\mathrm{IND}} \emptyset$. But the latter is clearly false (unless $t_n = \emptyset$, which is not the case), since it entails that $(\forall u \in t_n)(\exists v)(v \in \emptyset \ \& \ u \preceq_{\mathrm{IND}} v)$. $\dashv$

As an immediate corollary of the previous proposition we get:

Corollary 5.5 *$\preceq_{\mathrm{MAX}}$ is a total preordering for any class X. Namely: If x, y are well-founded, then $x \preceq_{\mathrm{MAX}} y$ iff $\rho(x) \leq \rho(y)$. If x is well-founded and y is non-well-founded, then $x \prec_{\mathrm{MAX}} y$. If both x, y are non-well-founded, then $(x, y) \in eq(\preceq_{\mathrm{MAX}})$.*

It follows also from the previous corollary that $eq(\preceq_{\mathrm{MAX}})$ is *not* a bisimulation. Because otherwise it would be included in the greatest bisimulation $\equiv_{\mathrm{MAX}}$ of X, where $x \equiv_{MAX} y$ iff the sets x, y decorate the same graph (see [1]). In that case we would have $x \equiv_{MAX} y$ for any two non-well-founded sets, which is false. For instance $x = \{x\}$ and $y = \{\emptyset, y\}$ are equivalent with respect to $eq(\preceq_{\mathrm{MAX}})$ but not with respect to $\equiv_{\mathrm{MAX}}$.

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