Typicality à la Russell in set theory

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Abstract

We adjust the notion of typicality originated with Russell, which was introduced and studied in a previous paper for general first-order structures, to make it expressible in the language of set theory. The adopted definition of the class NT of nontypical sets comes out as a natural strengthening of Russell's initial definition, which employs properties of small (minority) extensions, when the latter are restricted to the various levels V_ζ of V. This strengthening leads to defining NT as the class of sets that belong to some countable ordinal definable set. It follows that OD $\subseteq$ NT and hence HOD $\subseteq$ HNT. It is proved that the class HNT of hereditarily nontypical sets is an inner model of ZF. Moreover the (relative) consistency of $V \neq$ NT is established, by showing that in many forcing extensions $M[G]$ the generic set G is a typical element of $M[G]$, a fact which is fully in accord with the intuitive meaning of typicality. In particular it is consistent that there exist continuum many typical reals. In addition it follows from a result of Kanovei and Lyubetsky that HOD $\neq$ HNT is also relatively consistent. In particular it is consistent that $\mathcal{P}(\omega)\cap \mathrm{OD} \subsetneq \mathcal{P}(\omega)\cap \mathrm{NT}$. However many questions remain open, among them the consistency of HOD $\neq$ HNT $\neq V$, HOD $=$ HNT $\neq V$ and HOD $\neq$ HNT $= V$.

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1 Introduction

Inspired by B. Russell's definition of "typical Englishman" in [8], we introduced in [9] a rigorous definition of *typical property* over a first-order structure $\mathfrak{M} = (M, \ldots)$ (expressed in the language L of $\mathfrak{M}$), and of *typical element* of $\mathfrak{M}$. The definition of typical element exhibits clear similarities with the well-known definition of Martin-Löf random real and its variants so it could be conceived as an alternative notion of randomness. Moreover, allowing parameters in the formulas involved leads to a notion of *relative typicality* "object a is typical with respect to objects $\overline{b}$", or "$\overline{b}$-typical", denoted $\mathsf{Tp}(a, \overline{b})$. In this form, typicality exhibits also clear similarities with van Lambalgen's relative randomness notion $R(x, \overline{y})$, and $\mathsf{Tp}(a, \overline{b})$ was shown to satisfy most of the randomness axioms proposed in [6] and other papers of the same author. Let us first recall some basic definitions from [9].

Russell's inspiring definition of typical Englishman [8, p. 89] is as follows:

> *A typical Englishman is one who possesses all the properties possessed by a majority of Englishmen.*

Without the distinction between object language and metalanguage, the definition is obviously circular, so practically useless, but it becomes natural and sound as soon as we make the aforementioned distinction. This is easily made by the help of elementary tools of model theory.

Throughout our background theory is ZFC. Let L be a first-order language, $\mathfrak{M} = (M, \ldots)$ an infinite L-structure and $A \subseteq M$. $L(A)$ denotes L augmented with parameters from A. By some abuse of language we refer also to $L(A)$ as the set of formulas of $L(A)$. By a property of $L(A)$ we mean a formula $\phi(x) \in L(A)$ with one free variable. A *majority set* is any $X \subseteq M$ that contains strictly more elements than its complement, i.e., $|X| > |M \backslash X|$. (Accordingly $M \backslash X$ is then a *minority set*). Note that $|X| > |M \backslash X|$ is equivalent to $|M \backslash X| < |M|$ (which in particular implies that $|X| = |M|$). Let

$$\mathsf{mj}(M) = \{X \subseteq M : |X| > |M \backslash X|\} = \{X : |M \backslash X| < |M|\}$$

be the class of majority subsets of M. $\mathsf{mj}(M)$ is a filter on M that extends the Fréchet filter of cofinite subsets of M.

Given an L-structure $\mathfrak{M}$ and a property $\phi(x, \overline{y})$ of L and parameters $\overline{b} \in M$, $\phi(\mathfrak{M}, \overline{b})$ denotes the extension of $\phi(x, \overline{b})$ in $\mathfrak{M}$, i.e.,

$$\phi(\mathfrak{M}, \overline{b}) = \{a \in M : \mathfrak{M} \models \phi(a, \overline{b})\}.$$

Definition 1.1 A property $\phi(x, \overline{y})$ of L is said to be *A-typical over $\mathfrak{M}$*, for some $A \subseteq M$, if for every $\overline{b} \in A$, $\phi(\mathfrak{M}, \overline{b}) \in \mathsf{mj}(M)$. In particular $\phi(x)$ is *typical* if it is $\emptyset$-typical. An element $a \in M$ is said to be *A-typical* if it satisfies every A-typical property over $\mathfrak{M}$.

If A is finite we write it in the form $\overline{a} = \langle a_1, \ldots, a_n \rangle$, and say $\overline{a}$-typical instead of A-typical. Given a structure $\mathfrak{M}$, the notation $\mathsf{Tp}(a, \overline{b})$ means "a is $\overline{b}$-typical" over $\mathfrak{M}$. $\mathsf{Tp}(a)$ means a is "$\emptyset$-typical", or just "typical".

The following existence results for typical elements were shown in [9].

Theorem 1.2 *If $\mathfrak{M}$ is κ-saturated, for some $\kappa \geq \aleph_0$, then for every $A \subseteq M$ such that $|A| < \kappa$, M contains A-typical elements.*

Theorem 1.3 *Let $\mathfrak{M}$ be an L-structure, for a countable L, and let $A \subseteq M$ be a set of parameters such that $cf(|M|) > \max(\aleph_0, |A|)$. Then $\mathfrak{M}$ contains $|M|$ A-typical elements, while only $< |M|$ non-A-typical ones.*

Applying Theorem 1.3 to the standard model of full second-order arithmetic $\mathcal{R} = (\omega, \mathcal{P}(\omega), +, \cdot, <, \in, 0, 1)$ we have the following immediate consequence.

Theorem 1.4 *For every $A \subseteq \mathcal{P}(\omega)$ such that $|A| \leq \aleph_0$, there exist $2^{\aleph_0}$ A-typical reals, while only $< 2^{\aleph_0}$ non-A-typical ones. More precisely: For every finite (or even countable) tuple $\overline{b}$ of reals, $|\{x : \mathsf{Tp}(x, \overline{b})\}| = 2^{\aleph_0}$, while $|\{x : \neg \mathsf{Tp}(x, \overline{b})\}| < 2^{\aleph_0}$.*

2 Typicality in ZFC

The definitions and facts given in the Introduction refer to what might be called "external typicality" with respect to a given first-order structure $\mathfrak{M}$, because the cardinality measure used to tell which properties are typical and which are not, is external with respect to $\mathfrak{M}$. In [9] (Remark 2.15) we already observed that such a concept is suitable for all first-order structures except for models of set theory, because every such model possesses its own internal cardinality, which is of course preferable over the external one, and we stressed "the challenge to find a notion of typicality suitable for models of set theory." It is the purpose of this paper to examine ways for implementing Russell's notion of typicality naturally in any model of ZFC. This amounts to finding a typicality notion expressible in the language of set theory. The basic problem has to do with the implementation of Russell's majority/minority criterion concerning the size of extensions of properties in the universe V of ZFC.

We have argued in [9] that typical objects behave much like *random* entities, which is equivalent to saying that nontypical objects are expected to be "special" entities, of the kind of objects by which we build "familiar" structures. For instance all definable elements of a structure are nontypical according to the definitions of the previous section (referring to external typicality), and clearly the same must be true for any reasonable notion of typicality in ZFC. That is to say, every set definable without parameters in the universe V should be nontypical. To simplify notation, henceforth we shall identify a structure $\mathfrak{M}$ with its domain M. Given a set or class structure M, we denote by $\mathrm{Df}(M)$ the set or class of first-order definable (without parameters) elements of M (i.e., $a \in \mathrm{Df}(M)$ if there is a formula $\phi(x)$ such that a is the unique element of M that satisfies $\phi(x)$). In particular, $\mathrm{Df}(V)$ is the class of all sets definable without parameters in the universe $(V, \in)$. By Tarski's undefinability of truth, $\mathrm{Df}(M)$ is not a definable subclass of M (in particular, the fact that a set belongs to $\mathrm{Df}(V)$ is not expressible by a formula of set theory). On the other hand it is of some interest to note (although we are not going to use this fact below) that $\mathrm{Df}(V)$ is an elementary subclass of the inner model HOD of hereditarily definable sets of ZFC, i.e., $\mathrm{Df}(V) \preccurlyeq \mathrm{HOD}$ (see e.g. [2, Thm. 4]).

Let TP, NT denote the complementary classes of *typical* and *nontypical* sets of V, respectively, the definitions of which are being sought. It follows from the foregoing discussion that the definition of NT should satisfy the following requirement:

$$\mathrm{Df}(V) \subseteq \mathrm{NT}. \tag{1}$$

Further, as already mentioned above, the definition of NT should be expressible in the language of set theory. Recall from the previous section that given a first structure M and a formula $\phi(x)$ (or $\phi(x, \overline{b})$, with $\overline{b} \in M$), $\phi(M)$ (resp. $\phi(M, \overline{b})$) denotes the extension of $\phi(x)$ in M. The same notation will be used in set theory even when M is a proper class. So $\phi(V)$ is the class $\{x : (V, \in) \models \phi(x)\}$, which is commonly written $\{x : \phi(x)\}$. Thus a second requirement for the definition of NT is that there is a first-order property $\sigma(x)$ (without parameters) such that:

$$\mathrm{NT} = \sigma(V). \tag{2}$$

Apart from requirements (1) and (2), the definition should be nontrivial, e.g. ZFC should not prove that there are no typical elements (i.e., $V = \mathrm{NT}$), or that NT coincides with some already known subclass of V.

Below we examine three options for the definition of NT. All of them are attempts to implement the Russell' majority/minority criterion for extensions of properties of a set a. The first option leads to $V = \mathrm{NT}$ and thus, inevitably, is rejected as trivial. The second one lies, in a sense, to the opposite end as it makes $V \neq \mathrm{NT}$ provable in ZFC. A consequence of this is that $\mathrm{Df}(V) \nsubseteq \mathrm{NT}$, i.e., the refutation of condition (1), is consistent. This is a sufficient reason to reject this option as well. The third option leads to two sub-options, one of which, happily, offers a definition for NT according to which its subclass HNT (of hereditarily nontypical sets) turns out to be a *new* inner model of ZF.

Since dealing with extensions $\phi(V)$ over the entire V leads to notions non-expressible in the language of set theory, we must be confined to extensions of ϕ's in the segments $V_\zeta = \{x : rank(x) < \zeta\}$, $\zeta \in Ord$, of V, i.e., to sets $\phi(V_\zeta) = \{x : (V_\zeta, \in) \models \phi(x)\}$. (We shall write simply $V_\zeta \models \phi$ instead of $(V_\zeta, \in) \models \phi$.) Working in $(V_\zeta, \in)$, we often use ordinal parameters from V_ζ, i.e., tuples $\overline{\theta} < \zeta$, but, as will be explained later, the collection of sets defined *over all* $(V_\zeta, \in)$ using ordinal parameters is no different than the collection of sets defined in the same structures with no parameters at all. So we can safely dispense, at least for the moment, with ordinal parameters.

Before going on let us recall the formal definition of Russell's typicality and fix some notation. Given a structure M, let

$$\mathrm{TP}(M) = \{a \in M : \mathsf{Tp}(a)\}, \quad \mathrm{NT}(M) = \{a \in M : \neg\mathsf{Tp}(a)\}$$

be the classes of typical and nontypical elements of M, respectively. It follows from Definition 1.1 that

$$\mathrm{NT}(M) = \{a \in M : (\exists\phi)(a \in \phi(M) \ \wedge \ |\phi(M)| < |M|)\}. \tag{3}$$

We come to the three options for the definition of $\mathrm{NT}(V) := \mathrm{NT}$.

Option 1. Given that the definition of Russell's typicality does not apply directly to the entire V, a natural alternative is to define it with respect to the set-approximations V_ζ, $\zeta \in Ord$, of V. Namely, we say that a set a is nontypical if and only if it is so *with respect to some* V_ζ to which it belongs (that is, for some $\zeta > rank(a)$), i.e., we set

$$\mathrm{NT} := \bigcup_{\zeta > \omega} \mathrm{NT}(V_\zeta), \tag{4}$$

where $\mathrm{NT}(V_\zeta)$ is defined as in (3), that is

$$\mathrm{NT}(V_\zeta) = \{a \in V_\zeta : (\exists\phi)(a \in \phi(V_\zeta) \wedge |\phi(V_\zeta)| < |V_\zeta|)\}.$$

It is easy to see that definition (4) satisfies both conditions (1) and (2), but, unfortunately, is trivial in the sense that it implies that every set is nontypical, i.e., $\mathrm{NT} = V$. Indeed, take any $a \in V$ and pick some ζ such that $a \in V_\zeta$. Then clearly ζ is definable in $V_{\zeta+1}$ (as its greatest ordinal), i.e., $\zeta \in \mathrm{Df}(V_{\zeta+1})$, and thus $V_\zeta \in \mathrm{Df}(V_{\zeta+1})$ (as the set of elements of rank $< \zeta$). Thus there is $\phi(x)$ such that $V_\zeta = \phi(V_{\zeta+1})$, hence $a \in \phi(V_{\zeta+1})$. Moreover $|\phi(V_{\zeta+1})| = |V_\zeta| < |V_{\zeta+1}|$. Therefore $a \in \mathrm{NT}(V_{\zeta+1})$, and hence $a \in \mathrm{NT}$. It follows that no typical sets exist, or $\mathrm{TP} = \emptyset$.

This fact would be normal and acceptable if it concerned only *some model* of ZFC. But being a theorem of ZFC trivializes NT, indicating that its definition is too generous. So next we shall attempt to restrict it somehow, and widen, accordingly, the class TP.

Option 2. We saw above that if $rank(a) = \zeta$, i.e., a first occurs in $V_{\zeta+1}$, then already $a \in \mathrm{NT}(V_{\zeta+2})$ and so $a \in \mathrm{NT}$ according to the definition in Option 1. Thus a possible way to restrict NT and avoid $\mathrm{NT} = V$ might be to decide the typicality of a set a, not in *any level* V_ζ with $\zeta > rank(a)$, but rather at the first level of its occurrence, i.e., at $V_{rank(a)+1}$. That would lead to replacing definition (4) of NT with the following:

$$\mathrm{NT}_0 := \{a : a \in \mathrm{NT}(V_{rank(a)+1})\}. \tag{5}$$

Then for every a with $rank(a) = \zeta$, $a \in \mathrm{NT}_0$ if and only if there is $\phi(x)$ such that $a \in \phi(V_{\zeta+1})$ and $|\phi(V_{\zeta+1})| < |V_{\zeta+1}|$.

It turns out, however, that definition (5) is now too restrictive for the class of nontypical sets (and accordingly too generous for the class of typical sets) and results in the (consistency of the) failure of condition (1). This will be a consequence of the following result. (Note that $\mathrm{NT}(V_\zeta)$ and $\mathrm{TP}(V_\zeta)$ keep the meaning specified in definition (3).)

Lemma 2.1 (ZFC) *(i) For every $\zeta \geq \omega$, $|\mathrm{NT}(V_{\zeta+1})| < |V_{\zeta+1}|$, while $|\mathrm{TP}(V_{\zeta+1})| = |V_{\zeta+1}|$.*

(ii) Moreover, for every $\zeta \geq \omega$, $|\mathrm{TP}(V_{\zeta+1}) \cap (V_{\zeta+1} \backslash V_\zeta)| = |V_{\zeta+1}|$.

Proof. (i) The proof is easy and quite similar to the proof of Theorem 1.3 above (see [9], Theorem 2.9). It follows simply from two facts: first, there are countably many properties $\phi(x)$ of L without parameters, and second, the cofinality of $|V_{\zeta+1}| = 2^{|V_\zeta|}$ is always uncountable. Just observe that by definition

$$\mathrm{NT}(V_{\zeta+1}) = \bigcup\{\phi(V_{\zeta+1}) : \phi \in L, \text{ and } |\phi(V_{\zeta+1})| < |V_{\zeta+1})|\}.$$

The sets $\phi(V_{\zeta+1})$ on the right-hand side are countably many and each with cardinality $< |V_{\zeta+1}|$, so, since $\mathrm{cf}(|V_{\zeta+1}|) > \aleph_0$, $|\mathrm{NT}(V_{\zeta+1})| < |V_{\zeta+1}|$. Consequently $|\mathrm{TP}(V_{\zeta+1})| = |V_{\zeta+1} \backslash \mathrm{NT}(V_{\zeta+1})| = |V_{\zeta+1}|$.

(ii) We have

$$\mathrm{TP}(V_{\zeta+1}) = (\mathrm{TP}(V_{\zeta+1}) \cap V_\zeta) \cup (\mathrm{TP}(V_{\zeta+1}) \cap (V_{\zeta+1} \backslash V_\zeta)).$$

Since $|\mathrm{TP}(V_{\zeta+1}) \cap V_\zeta| \leq |V_\zeta|$ while, by (i), $|\mathrm{TP}(V_{\zeta+1})| = |V_{\zeta+1}|$, the claim follows. $\dashv$

Corollary 2.2 ZFC *proves that* $V \neq \mathrm{NT}_0$. *Specifically, for every successor level* $V_{\zeta+1}$, $|V_{\zeta+1} \cap \mathrm{TP}_0| = |V_{\zeta+1}|$ *(where* $\mathrm{TP}_0 = V \backslash \mathrm{NT}_0$*).*

Proof. The elements of $V_{\zeta+1} \backslash V_\zeta$ are exactly those of rank ζ, so by the definition of TP_0, clearly $V_{\zeta+1} \cap \mathrm{TP}_0 = (V_{\zeta+1} \backslash V_\zeta) \cap \mathrm{TP}(V_{\zeta+1})$. Thus the claim follows from Lemma 2.1 (ii). $\dashv$

Corollary 2.3 $\mathrm{Df}(V) \not\subseteq \mathrm{NT}_0$ *is consistent with* ZFC. *Therefore condition (1) is not provable.*

Proof. It is known that if ZFC is consistent, then it has models all elements of which are definable, i.e., $\mathrm{Df}(V) = V$ holds there (see [2] where the existence of such models, called pointwise definable, is proved). Let $M \models \mathrm{ZFC}$ be such a model. Then, $M \models V \neq \mathrm{NT}_0$, since Corollary 2.2 is provable in ZFC, or equivalently $M \models V \not\subseteq \mathrm{NT}_0$. But since also in M $\mathrm{Df}(V) = V$, it follows that in this model $\mathrm{Df}(V) \not\subseteq \mathrm{NT}_0$. $\dashv$

To sum up: replacing definition (4) with definition (5) leads to the opposite end of the spectrum, narrowing too much the class of nontypical sets and leading to the non-provability of condition (1). Therefore some intermediate solution is needed, and this is examined in the next option.

Option 3. Our next attempt to restrict NT of Option 1 is by requiring a nontypical element of V_ζ to be caught in the extension $\phi(V_\zeta)$ of some property ϕ with cardinality not just $< |V_\zeta|$ but rather $< \kappa$, for some κ with $\aleph_0 \leq \kappa \leq |V_\zeta|$.

To be a little bit more general, given a structure M and a cardinal κ with $\aleph_0 \leq \kappa \leq |M|$, the definition (3) of $\mathrm{NT}(M)$ can be refined as follows:

$$\mathrm{NT}_\kappa(M) = \{a \in M : (\exists\phi)(a \in \phi(M) \ \wedge \ |\phi(M)| < \kappa)\}. \tag{6}$$

(In this notation $\mathrm{NT}(M) = \mathrm{NT}_{|M|}(M)$.)

Fact 2.4 *For any M and any $\aleph_0 \leq \kappa < \lambda \leq |M|$, $\mathrm{NT}_\kappa(M) \subseteq \mathrm{NT}_\lambda(M)$. In particular, $\mathrm{NT}_{\aleph_0}(M)$ is the set of algebraic elements of M.*

Proof. The first claim follows immediately from definition (6). Concerning the second claim, recall that an element $a \in M$ is *algebraic* in M if there is a formula $\phi(x)$ in the language of M without parameters such that $\phi(M)$ is finite and $a \in \phi(M)$. (In particular every element of M which is definable without parameters is algebraic.) Therefore a is algebraic if and only if for some ϕ, $a \in \phi(M)$ and $|\phi(M)| < \aleph_0$, i.e., if and only if $a \in \mathrm{NT}_{\aleph_0}(M)$. $\dashv$

It follows that the class of nontypical elements of M is *minimized*, becoming identical to the class of algebraic elements, if we set $\mathrm{NT}(M) := \mathrm{NT}_{\aleph_0}(M)$ (when accordingly the class of typical elements is maximized). For $M = (V_\zeta, \in)$ and for κ such that $\aleph_0 \leq \kappa \leq |V_\zeta|$, we have in particular:

$$\mathrm{NT}_\kappa(V_\zeta) = \{a \in V_\zeta : (\exists \phi)(a \in \phi(V_\zeta) \ \wedge \ |\phi(V_\zeta)| < \kappa)\}. \tag{7}$$

Applying Fact 2.4 to the structures V_ζ, we get:

Fact 2.5 *For any $\zeta > \omega$ and any $\aleph_0 \leq \kappa < \lambda \leq |V_\zeta|$, $\mathrm{NT}_\kappa(V_\zeta) \subseteq \mathrm{NT}_\lambda(V_\zeta)$). In particular, $\mathrm{NT}_{\aleph_0}(V_\zeta)$ is the set of algebraic elements of the structure $(V_\zeta, \in)$.*

The definition of $\mathrm{NT}_\kappa(V_\zeta)$ generalizes the definition of NT locally but it is not clear if it applies to the class NT itself. Recall that for any ordinal $\alpha \geq 0$, $|V_{\omega+\alpha}| = \beth_\alpha$, where $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ and $\beth_\alpha = \sup\{\beth_\beta : \beta < \alpha\}$, for a limit α. (Under GCH, $\beth_\alpha = \aleph_\alpha$, for all α.)

From the perspective of this paper the dichotomy of sets into typical and nontypical concerns exclusively infinite sets, or finite sets containing infinite sets etc, while the elements of V_ω, being hereditarily finite, are definable and hence nontypical with respect to any definition of NT. So the ordinal ζ in (7) must range beyond ω, i.e., $\zeta > \omega$. Consequently for each infinite cardinal κ it is natural to set

$$\mathrm{NT}_\kappa = \bigcup_{\zeta > \omega} \mathrm{NT}_\kappa(V_\zeta)$$

as a generalization of (4). The question is: what should be the range of κ? If $|V_\zeta| < \kappa$, for some ζ, then by (7) clearly $V_\zeta = \mathrm{NT}_\kappa(V_\zeta)$, and hence $V_\zeta \subseteq \mathrm{NT}_\kappa$. But given that $\zeta > \omega$, this seems quite unnatural. For in that case a whole segment of V that strictly extends V_ω should consist exclusively of nontypical sets. In particular we would have $\mathcal{P}(\omega) \subseteq V_{\omega+1} \subseteq \mathrm{NT}_\kappa$, i.e., all reals would be nontypical, contrary to the fact that typicality was introduced in [9] (rather successfully) as a parallel notion of randomness, with a large amount of reals to be proved typical (with respect to the "external" notion of typicality employed there).

The only way to avoid this unnatural situation is to restrict ourselves to those κ for which $\kappa \leq |V_\zeta|$ for all $\zeta > \omega$, or equivalently to those κ for which $\kappa \leq |V_{\omega+1}|$. Given that $|V_{\omega+1}| = 2^{\aleph_0}$, and the least possible value of the latter is $\aleph_1$, we conclude that the only acceptable values for κ are $\aleph_0$ and $\aleph_1$, and therefore we should consider only the classes

$$\mathrm{NT}_{\aleph_0} = \bigcup_{\zeta>\omega} \mathrm{NT}_{\aleph_0}(V_\zeta), \quad \mathrm{NT}_{\aleph_1} = \bigcup_{\zeta>\omega} \mathrm{NT}_{\aleph_1}(V_\zeta). \tag{8}$$

So $\mathrm{NT}_{\aleph_0}$ and $\mathrm{NT}_{\aleph_1}$, with $\mathrm{NT}_{\aleph_0} \subseteq \mathrm{NT}_{\aleph_1}$, are the two natural candidate definitions for the class NT of nontypical sets. Both are expressible in the language of set theory, so they satisfy condition (2), and we shall see below that $\mathrm{OD} \subseteq \mathrm{NT}_{\aleph_0} \subseteq \mathrm{NT}_{\aleph_1}$, so in particular $\mathrm{Df}(V) \subseteq \mathrm{NT}_{\aleph_0} \subseteq \mathrm{NT}_{\aleph_1}$, i.e., they satisfy also condition (1).

Let us first deal with $\mathrm{NT}_{\aleph_0}$. By Fact 2.5, and using Reflection, it follows that $\mathrm{NT}_{\aleph_0}$ is the class of algebraic elements of V, i.e., those belonging to finite sets which are definable in V with ordinal parameters, i.e. to finite OD sets. Actually this class, as well as the class $\mathrm{HNT}_{\aleph_0} = \{x : TC(\{x\}) \subseteq \mathrm{NT}_{\aleph_0}\}$ of hereditarily nontypical sets, is not new. It has already come up and been investigated by J.D. Hamkins and C. Leahy in [3], through a different motivation and under the name class of "ordinal algebraic" sets, denoted OA. The only difference between the definitions of OA and $\mathrm{NT}_{\aleph_0}$ is that while $\mathrm{NT}_{\aleph_0} = \bigcup_{\zeta>\omega} \mathrm{NT}_{\aleph_0}(V_\zeta)$, OA is defined in [3] as the class of sets which are algebraic in the structures V_ζ by the extra help of ordinal parameters. This is equivalent to saying that $\mathrm{OA} = \bigcup_{\zeta>\omega} \mathrm{NT}^*_{\aleph_0}(V_\zeta)$, where $\mathrm{NT}^*_{\aleph_0}(V_\zeta)$ denotes the set of algebraic elements of the structure $(V_\zeta, \in, \zeta)$, which is $(V_\zeta, \in)$ endowed with ordinal parameters $< \zeta$.

Now it is well-known from [7], that although for each particular V_ζ definability in $(V_\zeta, \in, \zeta)$ and definability in $(V_\zeta, \in)$ do not coincide, the *totality* of sets definable in some $(V_\zeta, \in, \zeta)$ is no different than the totality of sets definable in some $(V_\zeta, \in)$. This is due to the following key fact (see the Extended Reflection Principle in [7]).

Lemma 2.6 *For any ordinals $\theta_1, \ldots, \theta_n$, there is an ordinal $\eta > \theta_1, \ldots, \theta_n$ such that $\theta_1, \ldots, \theta_n \in \mathrm{Df}(V_\eta)$.*

It is because of this Lemma that the class OD of ordinal definable sets can be defined just as $\bigcup_{\zeta\in On}\mathrm{Df}(V_\zeta)$, although we very often allow ordinal parameters in the definitions, i.e., we practically deal with the sets in $\bigcup_{\zeta\in On}\mathrm{Df}(V_\zeta,\zeta)$. For the same reason the following holds.

Lemma 2.7 $\mathrm{OA}=\mathrm{NT}_{\aleph_0}$.

Proof. We have to show that $\bigcup_{\zeta>\omega}\mathrm{NT}^*_{\aleph_0}(V_\zeta)=\bigcup_{\zeta>\omega}\mathrm{NT}_{\aleph_0}(V_\zeta)$. Since trivially for every $\zeta>\omega$, $\mathrm{NT}_{\aleph_0}(V_\zeta)\subseteq\mathrm{NT}^*_{\aleph_0}(V_\zeta)$, one inclusion is obvious. For the converse, let $a\in\mathrm{NT}^*_{\aleph_0}(V_\zeta)$, for some ζ. Then there are $\phi(x,\overline{y})$ and $\overline{\theta}<\zeta$ such that $a\in\phi(V_\zeta,\overline{\theta})$ and $|\phi(V_\zeta,\overline{\theta})|<\aleph_0$. By Lemma 2.6, there is η such that $\zeta,\overline{\theta}\in\mathrm{Df}(V_\eta)$. Then $V_\zeta\in V_\eta$ and by the absoluteness of satisfaction relation we have that for all $x\in V_\zeta$,

$$V_\zeta\models\phi(x,\overline{\theta})\Leftrightarrow V_\eta\models(V_\zeta\models\phi(x,\overline{\theta})).$$

Since $\zeta,\overline{\theta}$ are definable in V_η, the formula $\psi(x,\zeta,\phi,\overline{\theta}):=(V_\zeta\models\phi(x,\overline{\theta}))$ defines $\phi(V_\zeta,\overline{\theta})$ in V_η without parameters, i.e., $\phi(V_\zeta,\overline{\theta})=\psi(V_\eta)$. Since $a\in\psi(V_\eta)$ and $|\psi(V_\eta)|=|\phi(V_\zeta,\overline{\theta})|<\aleph_0$, it follows that $a\in\mathrm{NT}_{\aleph_0}(V_\eta)$. This proves that $\bigcup_{\zeta>\omega}\mathrm{NT}^*_{\aleph_0}(V_\zeta)\subseteq\bigcup_{\zeta>\omega}\mathrm{NT}_{\aleph_0}(V_\zeta)$. $\dashv$

In view of Lemma 2.6 and its impact on the definition of OD, the following simple characterizations of the classes $\mathrm{NT}_{\aleph_0}$ and $\mathrm{NT}_{\aleph_1}$ come out easily.

Lemma 2.8 *(i) A set belongs to* $\mathrm{NT}_{\aleph_0}$ *iff it belongs to some finite* OD *set.*
(ii) A set belongs to $\mathrm{NT}_{\aleph_1}$ *iff it belongs to some countable* OD *set.*

Proof. Let us sketch (ii). Let $a\in\mathrm{NT}_{\aleph_1}$. Then there are ζ and $\phi(x)$ such that $a\in\phi(V_\zeta)$ and $|\phi(V_\zeta)|<\aleph_1$, i.e., $|\phi(V_\zeta)|\leq\aleph_0$. But clearly $\phi(V_\zeta)\in\mathrm{OD}$. Conversely let $A\in\mathrm{OD}$, $a\in A$ and $|A|\leq\aleph_0$. Then $A\in\mathrm{Df}(V_\zeta)$ for some ζ. Since $A\subseteq V_\zeta$, clearly there is ϕ such that $A=\phi(V_\zeta)$. Then $a\in\phi(V_\zeta)$ and $|\phi(V_\zeta)|\leq\aleph_0$, therefore $a\in\mathrm{NT}_{\aleph_1}(V_\zeta)\subseteq\mathrm{NT}_{\aleph_1}$. $\dashv$

It follows immediately from (i) above that $\mathrm{OD}\subseteq\mathrm{OA}=\mathrm{NT}_{\aleph_0}$. However an interesting fact established in [3] is that the class HOA of hereditarily ordinal algebraic sets coincides with the class HOD of hereditarily ordinal definable sets. Let $\mathrm{HNT}_{\aleph_i}$ denote the hereditary subclasses of $\mathrm{NT}_{\aleph_i}$, for $i=0,1$ respectively. By 2.7, $\mathrm{HOA}=\mathrm{HNT}_{\aleph_0}$.

Theorem 2.9 ([3]) $\mathrm{HOA}=\mathrm{HOD}$. *Therefore also* $\mathrm{HNT}_{\aleph_0}=\mathrm{HOD}$.

(On the other hand it is known that $\mathrm{NT}_{\aleph_0} \neq \mathrm{OD}$ is consistent. Specifically it was proved in [1] that there is a generic extension of $\mathbf{L}$ containing a pair $\{X, Y\}$ which is OD, while neither X nor Y is OD. Thus $X, Y \in \mathrm{NT}_{\aleph_0} \backslash \mathrm{OD}$.)

By the preceding theorem, $\mathrm{NT}_{\aleph_0}$ loses some of its interest as a candidate class for the definition of NT, since its hereditary subclass collapses to the familiar inner model HOD. So if one is looking for a really *new* inner model of ZF which strictly exceeds HOD, I think the only option left is to identify NT with the class $\mathrm{NT}_{\aleph_1}$. It turns out that the subclass $\mathrm{HNT}_{\aleph_1}$ of the latter is indeed a new inner model of ZF.

Theorem 2.10 ZFC *proves that* $\mathrm{HNT}_{\aleph_1}$ *is an inner model of* ZF *such that* $\mathrm{HOD} \subseteq \mathrm{HNT}_{\aleph_1}$.

Proof. We work in ZFC. That $\mathrm{HOD} \subseteq \mathrm{HNT}_{\aleph_1}$ follows from the fact that already $\mathrm{HOD} \subseteq \mathrm{HNT}_{\aleph_0}$ (actually $\mathrm{HOD} = \mathrm{HNT}_{\aleph_0}$ by 2.9) and $\mathrm{HNT}_{\aleph_0} \subseteq \mathrm{HNT}_{\aleph_1}$.

Let us write for simplicity HNT throughout this proof instead of $\mathrm{HNT}_{\aleph_1}$. Extensionality holds in HNT because of the transitivity of the latter, and Foundation is true trivially because is true in the underlying universe V. Also Infinity holds trivially since $\omega \in \mathrm{HOD} \subseteq \mathrm{HNT}$. So it remains to prove Pairing, Union, Powerset and Replacement. The proof is based on the characterization given in Lemma 2.8 (ii), that $a \in \mathrm{HNT} = \mathrm{HNT}_{\aleph_1}$ if and only if there is $A \in \mathrm{OD}$ such that $a \in A$ and $|A| \leq \aleph_0$.

Pairing. Let $a, b \in \mathrm{HNT}$. We have to show that $\{a, b\} \in \mathrm{NT}$. By assumption there are $A, B \in \mathrm{OD}$ such that $a \in A$, $b \in B$ and $|A|, |B| \leq \aleph_0$. Let $C = \{\{x, y\} : x \in A, y \in B\}$. Clearly $C \in \mathrm{OD}$, $|C| \leq \aleph_0$ and $\{a, b\} \in C$.

Union. Let $a \in \mathrm{HNT}$. It suffices to see that $\cup a \in \mathrm{NT}$. Let $a \in A$, where $A \in \mathrm{OD}$ such that $|A| \leq \aleph_0$. Let $B = \{\cup x : x \in A\}$. Clearly $B \in \mathrm{OD}$, $|B| \leq \aleph_0$ and $\cup a \in B$.

Powerset. Let $a \in \mathrm{HNT}$. It suffices to show that $\mathcal{P}^{\mathrm{HNT}}(a) = \mathcal{P}(a) \cap \mathrm{HNT}$ belongs to NT. Let $a \in A$ for some $A \in \mathrm{OD}$ with $|A| \leq \aleph_0$. For every x, $\mathcal{P}(x) \cap \mathrm{HNT}$ is a set (by Separation in V), and (by Replacement) so is also $B = \{\mathcal{P}(x) \cap \mathrm{HNT} : x \in A\}$. Moreover it is easy to check that $B \in \mathrm{OD}$. Since obviously $|B| \leq |A| \leq \aleph_0$ and $\mathcal{P}(a) \cap \mathrm{HNT} \in B$, we are done.

Replacement. Let $a \in \mathrm{HNT}$ and let $\phi(x, y, b_1, \ldots, b_n)$ be a formula with parameters $b_i \in \mathrm{HNT}$, such that $\mathrm{HNT} \models (\forall x \in a)(\exists! y)\phi(x, y, b_1, \ldots, b_n)$, or equivalently, $(\forall x \in a)(\exists! y)\phi^{\mathrm{HNT}}(x, y, b_1, \ldots, b_n)$, where ϕ^{HNT} is the usual relativization of ϕ to the class HNT. ϕ^{HNT} defines a functional relation on a, so let us write $F^{\mathrm{HNT}}_{\phi(\bar{b})}(x) = y$ instead of $\phi^{\mathrm{HNT}}(x, y, b_1, \dots, b_n)$, where $\bar{b} = \langle b_1, \dots, b_n\rangle$. If for some tuple $\langle b_1, \dots, b_n\rangle$, $\phi(x, y, b_1, \dots, b_n)$ does not define a function, we set $F^{\mathrm{HNT}}_{\phi(\bar{b})}(x) = \emptyset$.

Under this notation we have to show that for the given $a, \bar{b} \in \mathrm{HNT}$, the set $c = F^{\mathrm{HNT}}_{\phi(\bar{b})}[a] = \{F^{\mathrm{HNT}}_{\phi(\bar{b})}(x) : x \in a\}$ is an element of HNT. By our assumption there are countable $A, B_1, \dots, B_n \in \mathrm{OD}$ such that $a \in A$ and $b_i \in B_i$, for $i = 1, \dots, n$. Since $F^{\mathrm{HNT}}_{\phi(\bar{b})}$ is a function within HNT, for each $x \in a$, $F^{\mathrm{HNT}}_{\phi(\bar{b})}(x) \in \mathrm{HNT}$, therefore $c \subseteq \mathrm{HNT}$. So it suffices to show that $c \in \mathrm{NT}$, i.e., $c \in C$ for some countable $C \in \mathrm{OD}$. Let

$$C = \{F^{\mathrm{HNT}}_{\phi(\overline{w})}[z] : z \in A, w_1 \in B_1, \dots, w_n \in B_n\}.$$

Using Reflection and the fact that A, B_i are in OD, it is not hard to see that $C \in \mathrm{OD}$. Since $a \in A$ and $b_i \in B_i$, we have that $c = F^{\mathrm{HNT}}_{\phi(\bar{b})}[a]$ belongs to C. Moreover, since the variables z and w_i range over the countable sets A and B_i, respectively, it follows that

$$|C| \leq |A \times B_1 \times \cdots \times B_n| \leq \aleph_0.$$

Thus C is a countable OD set containing c and we are done. $\dashv$

We do not know if AC holds in $\mathrm{HNT}_{\aleph_1}$ or not. We only know that we cannot prove in ZFC that AC fails in $\mathrm{HNT}_{\aleph_1}$. Because if ZFC is consistent, then so is $\mathrm{ZFC} + V = \mathrm{HOD}$. But the latter theory implies $\mathrm{HOD} = \mathrm{HNT}_{\aleph_1}$ and HOD satisfies AC, so $\mathrm{ZFC} + AC^{\mathrm{HNT}_{\aleph_1}}$ is consistent. The latter would be also a consequence of the consistency of $\mathrm{ZFC} + V = \mathrm{HNT}_{\aleph_1}$ alone, no matter whether $\mathrm{HOD} = \mathrm{HNT}_{\aleph_1}$ or $\mathrm{HOD} \neq \mathrm{HNT}_{\aleph_1}$.

In general, in view of the inclusions $\mathrm{HOD} \subseteq \mathrm{HNT}_{\aleph_1} \subseteq V$, the questions regarding the consistency of the various mutual relationships among these three classes arise naturally. The simplest such relationship is of course $\mathrm{HOD} = \mathrm{HNT}_{\aleph_1} = V$ and follows from $V = \mathrm{HOD}$, whose consistency is well-known.[1] Concerning the other ones we have only two partial answers.

Perhaps the most urgent question to answer is the *existence* itself of typical sets, i.e., the consistency of $V \neq \mathrm{NT}_{\aleph_1}$. For if we do not know whether V can be separated from $\mathrm{NT}_{\aleph_1}$, the definition of $\mathrm{TP}_{\aleph_1}$ is vacuous. Fortunately

[1] Note by the way that, as is the case with the classes HOD and OD (described in [7, p. 276]), the following equivalences hold: $\mathrm{HNT}_{\aleph_1} = \mathrm{NT}_{\aleph_1} \Leftrightarrow V = \mathrm{HNT}_{\aleph_1} \Leftrightarrow V = \mathrm{NT}_{\aleph_1}$. The last equivalence, as well as $\Leftarrow$ of the first equivalence are obvious. Concerning $\Rightarrow$ of the first equivalence, assume $\mathrm{HNT}_{\aleph_1} = \mathrm{NT}_{\aleph_1}$. For every $\alpha \in Ord$, clearly $V_\alpha \in \mathrm{OD} \subseteq \mathrm{NT}_{\aleph_1}$, so $V_\alpha \in \mathrm{HNT}_{\aleph_1}$, whence $V_\alpha \subseteq \mathrm{HNT}_{\aleph_1}$ and therefore $V = \mathrm{HNT}_{\aleph_1}$.

this question can be affirmatively settled by a lot of forcing notions which have rich sets of automorphisms. More specifically the following holds.

Theorem 2.11 *(i) Let $M \models \mathrm{ZFC}$, $\mathbb{P} \in M$ be a forcing notion, and $G \subseteq \mathbb{P}$ be M-generic. For any $p \in \mathbb{P}$, let $A_p = \mathrm{Aut}^M_{\{p\}}(\mathbb{P})$ be the set of automorphisms of $\mathbb{P}$ in M which fix p. Assume further that for every $p \in G$, the set $\{\pi''G : \pi \in A_p\}$ is uncountable in $M[G]$. Then G is typical in $M[G]$, and hence $M[G] \models V \neq \mathrm{NT}_{\aleph_1}$.*

(ii) In particular, there are M and G such that $M[G] \models |\mathcal{P}(\omega) \cap \mathrm{TP}_{\aleph_1}| = 2^{\aleph_0}$ (where $\mathrm{TP}_{\aleph_1} = V \backslash \mathrm{NT}_{\aleph_1}$), i.e. $M[G]$ contains continuum many typical reals.

Proof. (i) First note that the set A_p belongs to M, and hence to $M[G]$, so $\{\pi''G : \pi \in A_p\}$ is also an element of $M[G]$. Next it is known that there is an abundance of forcing notions $\mathbb{P}$ satisfying the requirement of the theorem. For example such is the poset $\mathbb{P}$ of finite functions p with $dom(p) \subset \omega$ and $rng(p) \subseteq \{0,1\}$, ordered by reverse inclusion, which adds a single Cohen real. The automorphisms of $\mathbb{P}$ are induced by the permutations $\pi : \omega \to \omega$, and are defined as follows: for every $p \in \mathbb{P}$, $dom(\pi(p)) = \pi[dom(p)]$ and $\pi(p)(\pi(n)) = p(n)$. Then for every generic G and $p \in G$, $\{\pi''G : \pi \in A_p\}$ is uncountable in $M[G]$.

To verify the claim of the theorem, let $\mathbb{P} \in M$ and G be as stated. We have to show, according to Lemma 2.8 (ii), that for every $A \in \mathrm{OD}^{M[G]}$ such that $G \in A$, $M[G] \models |A| > \aleph_0$. Pick any $A \in \mathrm{OD}^{M[G]}$ containing G. Then there are a formula $\phi(x, y_1, \ldots, y_n)$ and ordinals $\alpha_1, \ldots, \alpha_n$ such that $A = \{x : M[G] \models \phi(x, \alpha_1, \ldots, \alpha_n)\}$ and $M[G] \models \phi(G, \alpha_1, \ldots, \alpha_n)$. The key fact here is that all generic subsets G of $\mathbb{P}$ have a "common" $\mathbb{P}$-name, sometimes called "canonical name", namely $\Gamma = \{\langle \check{q}, q\rangle : q \in \mathbb{P}\}$. If t^G denotes the G-interpretation of a $\mathbb{P}$-name t into $M[G]$, then for every generic G, $\Gamma^G = G$. Now $M[G] \models \phi(G, \alpha_1, \ldots, \alpha_n)$ means that there is some $p \in G$ such that

$$p \Vdash \phi(\Gamma, \check{\alpha}_1, \ldots, \check{\alpha}_n),$$

where Γ is the canonical name. Fix such a $p \in G$. It is well-known that for every automorphism π and generic G, $\pi''G$ is generic too and moreover $M[G] = M[\pi''G]$. But then for every $\pi \in A_p$, $p \in \pi''G$, so $p \Vdash \phi(\Gamma, \check{\alpha}_1, \ldots, \check{\alpha}_n)$ implies also that $M[\pi''G] \models \phi(\Gamma^{\pi''G}, \alpha_1, \ldots, \alpha_n)$, or $M[\pi''G] \models \phi(\pi''G, \alpha_1, \ldots, \alpha_n)$. Therefore for every $\pi \in A_p$,

$$M[G] = M[\pi''G] \models \phi(\pi''G, \alpha_1, \ldots, \alpha_n).$$

It follows that the set A, which is the extension of ϕ in $M[G]$, has as subset the set $\{\pi''G : \pi \in A_p\}$ which is uncountable in $M[G]$ by assumption, and thus also $M[G] \models |A| > \aleph_0$.

(ii) Just take $\mathbb{P}$ to be the forcing notion mentioned in the beginning of the proof for adding a Cohen real, so, essentially, $G \in \mathcal{P}(\omega)$. Then $M \models |Aut(\mathbb{P})| = 2^\omega$ and it is a folklore fact that there are continuum many images $\pi(G)$ of G in $M[G]$ all of which are generic sets, hence typical. So $M[G] \models |\mathcal{P}(\omega) \cap \mathrm{TP}_{\aleph_1}| = 2^{\aleph_0}$. ⊣

One of the referees informed me that the fact shown in Theorem 2.11, i.e., that a generic set which satisfies the given conditions does not belong to $\mathrm{HNT}_{\aleph_1}$, has been established also in [5] for Cohen and Solovay-random extensions by a different and more complex argument.

Nevertheless, not all generic sets are typical. This is a side consequence of the following relevant result of V.G. Kanovei and V.A. Lyubetsky in [4], which implies that the classes HOD and $\mathrm{HNT}_{\aleph_1}$ can be separated.

Theorem 2.12 ([4, Theorem 4]) *There is a generic extension* $\mathbf{L}[(x_n)_{n<\omega}]$ *of the constructible universe* $\mathbf{L}$ *by a sequence of reals* $x_n \in 2^\omega$*, in which it is true that* $\{x_n : n < \omega\}$ *is a countable* Π^1_2 *set with no* OD *elements.*

In the proof of this theorem the sets x_n are added to $\mathbf{L}$ *generically*, so, in contrast to Theorem 2.11, generics are used here, essentially, to show the existence of *nontypical* sets which are not ordinal definable. Namely, the following holds.

Corollary 2.13 *If* ZF *is consistent, then so is* $\mathrm{ZFC} + \mathrm{HOD} \neq \mathrm{HNT}_{\aleph_1}$*. In particular it is consistent that* $\mathcal{P}(\omega) \cap \mathrm{OD} \subsetneq \mathcal{P}(\omega) \cap \mathrm{NT}_{\aleph_1}$*.*

Proof. If $X = \{x_n : n < \omega\}$ is the set of Theorem 2.12, clearly $X \in \mathrm{OD}$ and X is countable. Therefore every x_n belongs to $\mathrm{NT}_{\aleph_1}$. Moreover every x_n is a real, so $x_n \in \mathrm{HNT}_{\aleph_1}$. On the other hand, since $x_n \notin \mathrm{OD}$, it follows that $x_n \in \mathrm{HNT}_{\aleph_1} \backslash \mathrm{HOD}$. By the same token, $x_n \in (\mathcal{P}(\omega) \cap \mathrm{NT}_{\aleph_1}) \backslash (\mathcal{P}(\omega) \cap \mathrm{OD})$. ⊣

Question 2.14 *Are the following consistent with* ZFC*?*

1) $\mathrm{HOD} \neq \mathrm{HNT}_{\aleph_1} \neq V$*,*

2) $\mathrm{HOD} = \mathrm{HNT}_{\aleph_1} \neq V$*,*

3) $\mathrm{HOD} \neq \mathrm{HNT}_{\aleph_1} = V$*.*

Question 2.15 *Is it consistent with* ZFC *that AC fails in* $\mathrm{HNT}_{\aleph_1}$*?*

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