Notions of symmetry in set theory with classes.

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Abstract

We adapt C. Freiling's axioms of symmetry [5] to models of set theory with classes by identifying small classes with sets getting thus a sequence of principles A^n, for $n \geq 2$, of increasing strength. Several equivalents of A^2 are given. A^2 is incompatible both with the foundation axiom and the antifoundation axioms $AFA^{\sim}$ considered in [1]. A hierarchy of symmetry degrees of preorderings (and of classes carrying such preorderings) is introduced and compared with A^n. Models are presented in which this hierarchy is strict. The main result of the paper is that (modulo some choice principles) a class X satisfies $\neg A^n$ iff it has symmetry degree $n-2$.

Key words. Proper class, non-foundation, symmetry axioms, m-symmetric total preordering.

1991 *Mathematics Subject Classification.* Primary: 03E70, 03E65; Secondary: 03E05.

1 Introduction

This paper deals with a classification of proper classes in set theories without foundation. If the universe is wellfounded the hierarchy simply collapses to a single level. Classes are firstly classified according to symmetry principles which are an adaptation of Freiling's axioms proposed in [5] for real numbers. The weakest of them, axiom A^2, has a simple and intuitive characterization: It holds iff there is no preordering of the universe whose initial segments are sets. Thus it is incompatible with foundation, but it is also shown to be incompatible with Aczel's antifoundation axioms $AFA^{\sim}$.

The case of the principles A^n, for $n > 2$, is more intriguing. Attempting to reduce them to more intuitive concepts, we are led to a hierarchy of total preorderings with respect to degrees of symmetry. Comparing the two hierarchies and showing that they do not collapse is the main content of the paper (sections 5 and 6) and the main result is that they capture precisely the same notion of symmetry.

C. Freiling [5] has proposed certain axioms for the continuum of the real numbers intended to express the symmetric behavior of small subsets, like the countable ones, the sets of cardinality less than the continuum, or the sets of measure zero. For each such class we have corresponding symmetry axioms. Typical is the following statement concerning countable subsets:

$$(\mathrm{A}^2_{\aleph_0}) \qquad (\forall f : \mathbb{R} \to \mathbb{R}_{\aleph_0})(\exists x, y)(x \notin f(y) \ \& \ y \notin f(x)).$$

For $n > 2$ it generalizes to

$$(\mathrm{A}^n_{\aleph_0}) \qquad (\forall f : \mathbb{R}_{\aleph_0} \to \mathbb{R}_{\aleph_0})(\exists X \in \mathbb{R}_n)(\forall x \in X)(x \notin f(X \backslash \{x\})),$$

as well as to

$$(\mathrm{A}^{\aleph_0}_{\aleph_0}) \qquad (\forall f : \mathbb{R}_{\aleph_0} \to \mathbb{R}_{\aleph_0})(\exists X \in \mathbb{R}_{\aleph_0})(\forall x \in X)(x \notin f(X \backslash \{x\})),$$

where $\mathbb{R}_n$ and $\mathbb{R}_{\aleph_0}$ are the sets of n-element subsets and countable subsets of $\mathbb{R}$ respectively.

The intuition behind $\mathrm{A}^2_{\aleph_0}$ is the following: Suppose we assign to each real number x a countable set of reals $f(x)$ (e.g. the rational multiples of x). Then if we throw two darts at $\mathbb{R}$, landing at x, y respectively, then the second dart will miss (with probability 1) the set $f(x)$. Then, by symmetry ("the real line does not know which dart is thrown first or second"), the first dart should also miss $f(y)$. The interesting thing is that, over ZFC, $\mathrm{A}^n_{\aleph_0} \Leftrightarrow 2^{\aleph_0} \geq \aleph_n$ for every $n \geq 2$.

Although these axioms can be given a very general formulation and applied to any second-order structure with respect to some appropriate class of small sets definable in the structure (see [7]), in this paper we shall concentrate on set theory with classes and shall identify small classes with sets. In section 2 we give various equivalents of A^2 in terms of (nonexistence of) preorderings of the universe. In section 3 we examine the connections of A^2 with the antifoundation axioms studied in [1]. In section 4 we show that

Fraenkel-Mostowski models with a proper class of reflexive sets are natural models of A^∞. Sections 5 and 6 are the main ones. There we introduce symmetry degrees for total preorderings and classes and show that there are models where the hierarchy is genuine. The main result says that a class X satisfies $\neg A^n$ iff it has symmetry degree $n-2$.

2 Symmetry and preorderings

It is well-known that sets are "small" classes and most of the axioms of set theory (pair, union, powerset, infinity, subset, replacement) express closure properties of the ideal *Set* of sets. In order to formulate Freiling's axioms with respect to this ideal we have to work in a set theory accommodating also classes. Such theories are GB (Gödel-Bernays) and KM (Kelley-Morse) of predicatively and impredicatively defined classes respectively. GB suffices for our purpose because in no place we need impredicative definitions. GBC is GB plus the AC (AC is the set form of the axiom of choice), while GBC^- is GBC minus the foundation axiom and similarly for GB^-. Our main theory in this paper will be GBC^- augmented in the last two sections with the choice scheme SSC and the maximal principle MP saying that every preordering has a maximal sub-wellordering.

We use both lowercase and uppercase letters x, y, z, X, Y, Z to denote sets or classes. The size of the letter is not a safe indication of the size of the class denoted. The rule is as follows: (a) Every lowercase letter denotes a set. (b) Every proper class is denoted by an uppercase letter. But uppercase letters (as well as the term "class") are ambiguous, ranging over either sets or proper classes. The letters F, G denote class-functions.

We shall also frequently talk about classes of classes. Such classes will always be *definable*, i.e., given by a formula $\phi(X)$ with one class variable and we feel free to write informally $\{X : \phi(X)\}$, although this is not an object of the universe. A class of classes is said to be *coded* (with code X) if it is of the form $\{X_{(x)} : x \in dom(X)\}$, where X is a class of pairs and $X_{(x)} = \{y : (x, y) \in X\}$. A coded class of classes will be also referred to as a *family* of classes and we often write $X_i, i \in I$, instead of $X_{(i)}, i \in dom(X)$.

V, On, Cn are the classes of all sets, ordinals and cardinals respectively. M, N denote models of GB or variants of it. For a definable class X and a model M, X^M is the corresponding element of M. For every class X, $S(X)$

is the class of subsets of X and $C(X)$ the class of all subclasses of X. If X is a set $S(X)$ is the usual powerset.

$\mathbb{N}$ is the set of nonnegative integers. For every $n \in \mathbb{N}$ and every class X, $[X]^n = \{x \in S(X) : |x| = n\}$ is the class of n-subsets of X and $[X]^\infty$ is the class of subsets of X of infinite cardinality.

In order to translate the principles $\mathrm{A}^n_{\aleph_0}$ into our context, just put V and $[V]^n$ in place of $\mathbb{R}$ and $\mathbb{R}_n$ respectively and note that $\mathbb{R}_{\aleph_0}$, the class of small subsets of $\mathbb{R}$, is translated to the class of small subclasses of V, i.e. the class of subsets of V, which is V again. Thus the translation of $\mathrm{A}^n_{\aleph_0}$ is the statement:

$$(\mathrm{A}^n): \quad (\forall F : [V]^{n-1} \to V)(\exists x \in [V]^n)(\forall y \in x)(y \notin F(x \backslash \{y\})).$$

For $n = 2$ the axiom is written:

$$(\mathrm{A}^2): \quad (\forall F : [V]^1 \to V)(\exists x, y)(x \neq y \ \&\ x \notin F(\{y\}) \ \&\ y \notin F(\{x\})).$$

We can also relativize A^n to any particular class X. This time however the class of small subclasses of X is $S(X)$ (the class of subsets of X) rather than X. Thus

$$\mathrm{A}^n(X): \quad (\forall F : [X]^{n-1} \to S(X))(\exists x \in [X]^n)(\forall y \in x)(y \notin F(x \backslash \{y\}))$$

and

$$\mathrm{A}^\infty(X): \quad (\forall F : [X]^\infty \to S(X))(\exists x \in [X]^\infty)(\forall y \in x)(y \notin F(x \backslash \{y\})).$$

So A^n is just $\mathrm{A}^n(V)$. We shall work mostly with $\neg\mathrm{A}^n(X)$, which is an existential formula, rather than $\mathrm{A}^n(X)$. Namely

$$\neg\mathrm{A}^n(X): \quad (\exists F : [X]^{n-1} \to S(X))(\forall x \in [X]^n)(\exists y \in x)(y \in F(x \backslash \{y\})).$$

We shall call a function F realizing $\neg\mathrm{A}^n(X)$ (resp. $\neg\mathrm{A}^\infty(X)$), $(n-1)$-*ary total on* X (resp. *infinitary total*). Thus $\neg\mathrm{A}^n(X)$ can be restated as follows:

$\neg\mathrm{A}^n(X)$: *There is an (n-1)-ary total function on X.*,

and similarly for $\neg\mathrm{A}^\infty(X)$.

Lemma 2.1 *(i) For every* n, $X \subseteq Y \ \&\ \mathrm{A}^n(X) \Rightarrow \mathrm{A}^n(Y)$.

(ii) $\mathrm{A}^n \iff (\exists X)A^n(X)$.

Proof. (i) Equivalently it suffices to check that $X \subseteq Y$ &$\neg \mathrm{A}^n(Y)$ imply $\neg \mathrm{A}^n(X)$. By $\neg \mathrm{A}^n(Y)$, there is an $(n-1)$-ary total function F on Y. Since $X \subseteq Y$, it follows $[X]^n \subseteq [Y]^n$ and $S(X) \subseteq S(Y)$. Thus putting for every $x \in [X]^{n-1}$ $G(x) = F(x) \cap X$, it is easy to check that G is an $(n-1)$-ary total function on X.

(ii) Since $\mathrm{A}^n \equiv \mathrm{A}^n(V)$, $\Rightarrow$ is obvious. The converse follows from (i). $\Box$

Concerning the relative strength of $\mathrm{A}^n(X)$, for $n \in \mathbb{N}$, we have the following:

Lemma 2.2 *For every class X and every $n \in \mathbb{N}$, $n \geq 2$,*
(i) $\mathrm{A}^{n+1}(X) \Rightarrow \mathrm{A}^n(X)$*, and*
(ii) $\mathrm{A}^{\infty}(X) \Rightarrow \mathrm{A}^n(X)$

Proof. (i) We show the contrapositive. Suppose $\neg \mathrm{A}^n(X)$ holds and let F be an $(n-1)$-ary total function on X. Define $G : [X]^n \to S(X)$ by putting $G(x) = \bigcup\{F(y) : y \subset x \ \& \ y \in [X]^{n-1}\}$. Clearly $G(x) \in S(X)$ and G is n-ary. To see that it is total, let $x \in [X]^{n+1}$. Take some $y \subset x$, $y \in [X]^n$. By the totality of F, there is $z \in y$ such that $z \in F(y \backslash \{z\})$. Since $y \backslash \{z\} \subset x \backslash \{z\}$, it follows that $z \in G(x \backslash \{z\})$. Hence G is n-ary total on X and $\mathrm{A}^{n+1}(X)$ fails.

(ii) Similarly, if F is $(n-1)$-ary total on X, define the ∞-ary G as follows: $G(x) = \bigcup\{F(y) : y \subseteq x \ \& \ y \in [X]^{n-1}\}$. By replacement $G(x) \in S(X)$ and as before we see that G is total. $\Box$

Definition 2.3 Let X be a class. A *total preordering* of X is a binary reflexive and transitive relation $\preceq \subseteq X \times X$ such that $x \preceq y \ \vee \ y \preceq x$, for all $x, y \in X$.

If $(X, \preceq)$ is a total preordering, for every $x \in$X, $\preceq^x$ denotes the initial segment of X determined by x, i.e., $\preceq^x = \{y \in X : y \preceq x\}$.

$(X, \preceq)$ is said to be *asymmetric* if $\preceq^x$ is a set for all $x \in X$.

Warning. Throughout the symbol $\preceq$, often with subscripts, denotes a total preordering. On the contrary, the symbol $\preceq$ or $\preceq_i$ with superscript an element, e.g. $\preceq_i^x$, used frequently below, denotes just a set-an initial segment of $(X, \preceq_i)$. Realizing that this may be visually misleading we beg the reader's understanding. Perhaps $\preceq_i^x$ is a bad notation, but the alternative ones would be worse!

Trivially every preordering *on a set* is asymmetric. The term "asymmetric" is used to indicate that for a proper class X, $\preceq$ splits X at every point x into two asymmetric parts, a set $\preceq^x$ and a coset $X \setminus \preceq^x$. Note that a wellordering of a proper class X need *not* be asymmetric. $(On, \leq)$, however, is asymmetric.

Theorem 2.4 (GB$^-$) *For every class X the following are equivalent.*
(i) $\neg \mathrm{A}^2(X)$.
(ii) There is a total asymmetric (t.a.) preordering on X.

Proof. (ii)$\Rightarrow$(i). Let $(X, \preceq)$ be a t.a. preordering. Then the function $F : [X]^1 \to S(X)$ such that $F(\{x\}) = \preceq^x$ is unary and total since for any $x, y \in X$, either $x \in F(\{y\})$ or $y \in F(\{x\})$, hence $\neg \mathrm{A}^2(X)$ holds.

(i)$\Rightarrow$(ii). Let $F : [X]^1 \to S(X)$ be a unary total function on X. Put $xRy := (x = y) \vee x \in F(\{y\})$. R is a binary reflexive total relation on X whose segments are sets. If $\preceq$ is the transitive closure of R, then, clearly, $\preceq$ is a total asymmetric preordering on X. $\Box$

Corollary 2.5 GB $\vdash \neg \mathrm{A}^2$, *or equivalently,* GB $\vdash (\forall X) \neg \mathrm{A}^2(X)$.

Proof. The axiom of foundation of GB says that the set universe V is wellfounded, or $V = \bigcup_{\alpha \in On} R_\alpha$, where R_α are the sets of the cumulative hierarchy. The ordering induced by the rank function is a total asymmetric preordering of V. The second claim follows from lemma 2.1 (ii). $\Box$

In the presence of AC, for every t.a. preordered class $(X, \preceq)$ and every $x \in X$, the segment $\preceq^x$ is assigned a cardinal $| \preceq^x |$. Thus $\preceq$ induces an ordering $\preceq_c$ on X, the *cardinal completion* of $\preceq$, defined by:

$$x \preceq_c y := | \preceq^x | \leq | \preceq^y |.$$

Obviously $\preceq \subseteq \preceq_c$ and $\preceq_c$ is total but we don't know if it is asymmetric. For every $x \in X$, if $y \in \preceq_c^x$, then $\preceq^y \subseteq \preceq_c^x$, i.e., $\preceq_c^x$ is an initial segment of $(X, \preceq)$ but we cannot be sure that it is a proper one. It may be the case that $X = \preceq_c^x$ for some x, which means that there is a cardinal κ such that $(\forall x \in X)(| \preceq^x | \leq \kappa)$.

Definition 2.6 A proper class X is said to be *normal* if for every $\kappa \in Cn$ there is a $x \subset X$ such that $|x| = \kappa$.

Clearly V, as well as every class containing a proper subclass of On, is normal. Also if V is wellfounded, every class is normal. Models of GBC$^-$ with non-normal classes are easy to construct. For example if M contains a proper class of urelements A and N is the submodel of M such that V^N contains the sets x built on A with finite support (i.e. $TC(x) \cap A$=finite) and classes the definable subclasses of V^N in M, then $A \in M$ and the only subsets of A in N are the finite ones (see [4] for details). It is unknown to us whether there is a model M containing a non-normal X which has a t.a. preordering. However we can eliminate non-normal classes if we add to GB$^-$ the following choice scheme:

(SC): $(\forall x)(\exists y)\phi(x, y) \Rightarrow (\exists F)(\forall x)\phi(x, F(x))$,
for every formula ϕ without class quantifiers.

Lemma 2.7 (GB$^-$+SC) *Every proper class is normal.*

Proof. Let X be a proper class. Then $(\forall x)(X \not\subseteq x)$, i.e., $(\forall x)(\exists y)(y \in X\backslash x)$. By SC there is an F such that $(\forall x)(F(x) \in X\backslash x)$. Fix some set x and define inductively $(y_\alpha), \alpha \in On$ as follows: $y_0 = F(x)$ and $y_\alpha = F(x \cup \{y_\beta : \beta < \alpha\})$. The elements y_α are all distinct and $\{y_\beta : \beta < \alpha\} \subseteq X$, hence X contains sets of any cardinality. $\Box$

The principle SC is stronger than AC, namely it implies the existence of a universal choice function F such that $F(x) \in x$ for every nonempty set x. However in the absence of foundation it is strictly weaker than the principle that there is a bijection between V and On (see [3]). For normal classes a further characterization of $\mathrm{A}^2(X)$ is possible.

Lemma 2.8 *If X is normal and $(X, \preceq)$ is a t.a. preordering, then $(X, \preceq_c)$ is also a t.a. preordering.*

Proof. It suffices to show the every $\preceq_c^x$ is a set. As mentioned above $\preceq_c^x$ is an initial segment of $(X, \preceq)$. By normality clearly $\preceq_c^x \neq X$. Thus $\preceq_c^x$ is bounded in $(X, \preceq)$. If $y \notin \preceq_c^x$, then $\preceq_c^x \subseteq \preceq^y$. Since the latter is a set, the claim follows. $\Box$

A class X is said to be *set-stratifiable* if $X = \bigcup_{\alpha\in On} x_\alpha$ for some family $(x_\alpha)_{\alpha\in On}$ of sets. The family $(x_\alpha)_{\alpha\in On}$ is called a *set-stratification* of X.

Given a preordering $\preceq$, $x \prec y$ means $x \preceq y$ and $y \not\preceq x$. $\preceq$ is said to be a *prewellordering* if $\prec$ is wellfounded. Equivalently this can be expressed as follows: On X consider the equivalence relation $x \sim y := x \preceq y$ & $y \preceq x$. Let $\hat{x}$ be the equivalence class of x, and $\hat{\preceq}$ be the induced total ordering on $\hat{X} = X/\sim$. Then $(X, \preceq)$ is an (asymmetric) prewellordering iff $(\hat{X}, \hat{\preceq})$ is an (asymmetric) wellordering. Note every two asymmetric wellorderings $(X, \preceq_1)$, $(Y, \preceq_2)$ with X, Y proper classes, are isomorphic. Hence $(X, \preceq) \cong (On, \leq)$ for every asymmetric wellordering $(X, \preceq)$.

Theorem 2.9 (GBC$^-$) *For every proper class X the following are equivalent:*

(i) X is normal and $\neg \mathrm{A}^2(X)$.

(ii) There is an asymmetric prewellordering on X.

(iii) X is set-stratifiable.

Proof. (i)⇒(ii). Let (i) hold. By theorem 2.4 there is a t.a. preordering $\preceq$ of X. By lemma 2.8 $(X, \preceq_c)$ is a t.a. preordering. But it is also a prewellordering since $x \prec_c y \Leftrightarrow |\preceq^x| < |\preceq^y|$.

(ii)⇒(iii). Let $(X, \preceq)$ be an asymmetric prewellordering. Then by the comments above $(\hat{X}, \hat{\preceq})$ is an asymmetric wellordering and there is an isomorphism $F : (On, \leq) \cong (\hat{X}, \hat{\preceq})$. Then $F(\alpha)$ are sets and $X = \bigcup_{\alpha\in On} F(\alpha)$.

(iii)⇒(i). Let $X = \bigcup_{\alpha\in On} x_\alpha$ be a set-stratification of X. Clearly the rank function induced by this stratification is a t.a. preordering of X. Since X is proper we may assume that $x_\alpha \subset x_\beta$ for $\alpha < \beta$. For every cardinal κ use a choice function f on $S(x_\kappa)$ with $dom(f) = \kappa$ and such that $f(\alpha) \in x_{\alpha+1} \backslash x_\alpha$. Then $rang(f) \subset X$ and $|rang(f)| = \kappa$. Hence X is normal. □

3 Symmetry and non-foundation

We have seen that A^2 implies that the set universe is unfounded but the converse is false. In this section we show that Aczel's antifoundation axioms AFA$^\sim$ are also incompatible with A^2.

As is well-known, to every set x there corresponds a directed graph G whose points are the elements of $TC(x)$ and the arrows $y \to z$ depict the relation $z \in y$. If x is wellfounded, so is G. Following P.Aczel [1], we call G a *picture* of x and x a *decoration* of G. Thus in ZF^- every set has a picture. If x is wellfounded, the picture is a unique (up to isomorphism) directed tree. But a non-wellfounded set may have class-many pictures. This is the case for example with "reflexive" sets $x = \{x\}$, if they exist.

P. Aczel [1] went also the other way around. He started from graphs and asked for sets decorating them. Throughout this section we follow the terminology and notation of [1]. We recall here some basic definitions and facts but the reader must consult Aczel's work for details. A *graph* is always a directed graph. An *accessible pointed graph*, or *apg* for short, is a graph with a distinguished node a and such that every other node is joined with a by a finite path. An apg is said to be *wellfounded* if it has no infinite or circular paths. If the nodes and edges of the graph form a class we call it *system*. The letter M ranges over systems. $a, b, x, y \in M$ means a, b, x, y are nodes of M. For $a, b \in M$ we write $a \to b$ for the fact that $(a, b) \in M$ is an edge. The universe V itself is a system with nodes the sets and edges the pairs (x, y) such that $y \in x$. Given M and $a \in M$, we set:

$a_M = \{b \in M : a \to b\}$ (the set of children nodes of a in M),

$Ma =$ the apg with point a and nodes and edges those of M lying on paths starting from a.

Thus every apg can be written in the form Ga where G is a graph and $a \in G$. A system M is *extensional* if

$$a_M = b_M \Rightarrow a = b.$$

A *decoration* of an apg Ga is a mapping $d : G \to V$ such that for any two nodes $a, b \in G$, $a \to b$ iff $d(a) \in d(b)$. The apg Ga is a *a picture* of a set x, if there is a decoration d of Ga such that $d(a) = x$. The decoration d of Ga is *injective* if it is 1-1. In this case Ga is called an *exact picture* of $d(a)$.

Let V_0 be the class of apg's. This can be seen as a system if we consider as edges the pairs (Ga, Gb) such that $a \to b$ is an edge of G. Let $\sim$ be a bisimulation on V_0. $\sim$ is said to be a regular bisimulation if:

(1) $\sim$ is an equivalence relation on V_0.

(2) $Ga \cong G'a' \Rightarrow Ga \sim G'a'$.

(3) $a_G = a'_G \Rightarrow Ga \sim Ga'$, for any $a, a' \in G$.

A system M is said to be $\sim$*-extensional* if $Ma \sim Mb \Rightarrow a = b$. Each regular bisimulation $\sim$ gives rise to an antifoundation axiom $\mathrm{AFA}^\sim$ which reads as follows:

AFA$^\sim$: *An apg is an exact picture if it is $\sim$-extensional.*

This is equivalent to the conjunction of the following two statements:

AFA$_1^\sim$: *Every $\sim$-extensional graph has an injective decoration.*

AFA$_2^\sim$: *V is $\sim$-extensional, i.e., $Vx \sim Vy \Rightarrow x = y$.*

We stop here the citation of notions and facts from Aczel's book and come to their connections with symmetry. We shall prove that for every regular bisimulation $\sim$, AFA$_2^\sim$ is incompatible with A^2. The key lemma is the following.

Lemma 3.1 *For every regular bisimulation $\sim$,* AF$_2^\sim$ *implies the following: If Ga is an exact picture and d_1, d_2 are two injective decorations of Ga, then $d_1(a) = d_2(b)$. That is, every exact picture is an exact picture of a unique set.*

Proof. Let d_1, d_2 be two injective decorations of Ga. Then the graphs $Vd_1(a)$, $Vd_2(a)$ are obviously the graphs $(TC(d_1(a)), \in)$, $(TC(d_2(a)), \in)$ of the transitive closures of $d_1(a), d_2(a)$ respectively. It is easy to see that they are isomorphic. Indeed define an isomorphism π between them as follows: If i is a node of Ga, put $\pi(d_1(i)) = d_2(i)$. Since d_1, d_2 are injective, clearly this is an isomorphism. Hence $Vd_1(a) \cong Vd_2(a)$. By condition (2) of regular bisimulations, $Vd_1(a) \sim Vd_2(a)$. Then by AFA$_2^\sim$, $d_1(a) = d_2(a)$. $\Box$

Now for any exact picture Ga, let

$$ID(Ga) = \{x : (\exists d)(d \text{ is an injective decoration of } Ga \ \&\ d(a) = x)\}.$$

Theorem 3.2 (GBC$^-$) *If for every exact picture Ga, $ID(Ga)$ is a set, then* $\neg\mathrm{A}^2$.

Proof. Suppose the hypothesis holds. For every cardinal κ let Γ_κ be the class of exact pictures whose nodes form a subset of κ. Clearly Γ_κ is a set since its elements are binary relations on $\kappa \times \kappa$. Let also $S_\kappa = \bigcup\{ID(Ga) : Ga \in \Gamma_\kappa\}$. Since by assumption each $ID(Ga)$ is a set, so is S_κ, for every κ. On the other hand for every set x, there is an exact picture Ga and an injective decoration d of Ga such that $d(a) = x$. (Indeed it suffices to consider any apg and its decoration d by the elements of the transitive closure of x and then identify the nodes i, j for which $d(i) = d(j)$.) That is, $x \in ID(Ga)$. If $|Ga| \leq \kappa$, clearly we can take Ga to be in Γ_κ, hence $x \in S_\kappa$. It follows that $V = \bigcup_{\kappa \in Cn} S_\kappa$. By AC, Cn is a subclass of On, hence S_κ yield a set-stratification of V. By theorem 2.9, $\neg \mathrm{A}^2$. □

Corollary 3.3 *For every regular bisimulation* $\sim$, $\mathrm{GBC}^- + \mathrm{AFA}_2^\sim \vdash \neg \mathrm{A}^2$.

Proof. Lemma 3.1 says that for every regular bisimulation, $\mathrm{AFA}_2^\sim \Rightarrow |ID(Ga)| = 1$ for every exact picture Ga. Thus the claim follows immediately from theorem 3.2. □

Instances of the axioms $\mathrm{AFA}^\sim$ are the axioms AFA, FAFA and SAFA (due to Aczel, Finsler and Scott, respectively) for the special regular bisimulations $\equiv$, $\cong^*$ and $\cong^t$, respectively (see [1]).

Especially $\equiv$ is the relation:

$$x \equiv y \Leftrightarrow \text{ there is an apg that is a picture of both } x \text{ and } y,$$

and

AFA: *Every apg has a unique decoration.*

In the opposite direction of $\mathrm{AFA}^\sim$ is Boffa's axiom BAFA ([1], §5). This axiom is the conjunction of the following statements:

BA_1: *An apg is an exact picture iff it is extensional.*

BA_2: *If* $f : (x, \in) \cong (y, \in)$, *where* x, y *are transitive sets and* $x' \supseteq x$ *is also transitive, then* f *can be extended to* $f' : (x', \in) \cong (y', \in)$ *for some transitive* $y' \supseteq y$.

An immediate consequence of BA_1 is the following:

Lemma 3.4 *If* BA_1 *holds, then there are class-many reflexive sets* $x = \{x\}$.

Proof. For every cardinal κ consider the apg having point m, nodes n_α, $\alpha < \kappa$, and edges $m \to n_\alpha$, and $n_\alpha \to n_\alpha$ for all $\alpha < \kappa$. This is an extensional graph, and by BA_1 it has an injective decoration d. If $d(n_\alpha) = a_\alpha$, then $a_\alpha = \{a_\alpha\}$ and $a_\alpha \neq a_\beta$, for every $\alpha < \kappa$. Thus for every κ there are κ distinct reflexive sets. $\square$

Aczel proves that if the real world satisfies $|V| = |On|$, there is a unique model (up to isomorphism) $M \subseteq V$ of ZFC$^-$+BAFA. This is shown by a back and forth argument using the enumeration of V and using BAFA for extending small isomorphisms to larger and larger ones. Given this we shall prove that this unique model of ZFC$^-$+BAFA satisfies also A^∞.

Theorem 3.5 *Let $|V| = |On|$ and let $M \subseteq V$ be the model of* ZFC$^-$+BAFA*. Then $(M, Def(M)) \models A^\infty$.*

Lemma 3.6 *Let M be the above model of* BAFA*, and let A be the proper class of reflexive sets of M (whose existence follows from 3.4). Then every automorphism π of A can be extended to an automorphism $\bar{\pi}$ of M.*

Proof. Let $\pi : A \to A$ be a permutation of the class of reflexive sets. For each $x \in V$, let $supp(x) = TC(x) \cap A$. $supp(x)$ is trivially transitive and $\pi{\restriction}supp(x) : (supp(x), \in) \cong (\pi'' supp(x), \in)$. By BAFA, $\pi{\restriction}supp(x)$ can be extended to a mapping $f : (TC(x), \in) \cong (TC(x'), \in)$. Taking an enumeration of M by On and using this fact, we can construct by back and forth an automorphism $\bar{\pi}$ extending π. $\square$

Proof of Theorem 3.5. Suppose F is a definable ∞-ary total function on M defined with parameters $c_1, \ldots, c_k$ and let B be an infinite subset of the class A of reflexive sets such that $B \cap (\bigcup_{i\leq n} supp(c_i)) = \emptyset$. There is a $b \in B$ such that $b \in F(B\backslash\{b\})$. Let $D = A\backslash(\bigcup_{i\leq n} supp(c_i) \cup B)$. D is a proper class and for every $d \in D$ consider the permutation of A interchanging b and d and fixing the rest elements. By the previous lemma this permutation extends to an automorphism f of M that fixes F and $B\backslash\{b\}$. Then $d \in F(B\backslash\{b\})$ for all $d \in D$, i.e., $D \subseteq F(B\backslash\{b\})$. This contradicts the fact that the latter is a set. $\square$

4 Models of full symmetry

It is easy to construct models of $\mathrm{GBC}^-+\mathrm{A}^\infty$, i.e. with the greatest degree of symmetry. It suffices to take models of GBC^- with a proper class A of urelements (or atoms), which can be taken to be reflexive sets $a = \{a\}$ (see e.g. [2]).

We start with a ground model N of GBC^- containing a proper class of atoms A, and let $W(A)$ be the cumulative hierarchy of sets of V built on A. Namely, let

$$W_0(A) = A \cup \{\emptyset\},\ W_\alpha = S(\bigcup_{\beta<\alpha} W_\beta),\ W(A) = \bigcup_{\alpha\in On} W_\alpha(A),$$

where recall that $S(X)$ is the class of subsets of X. Let $M = Def(W(A))$, where $Def(W(A))$ is the class of definable classes of $W(A)$. Each set being definable (with parameters), $W(A) = V^M$ and $A \in M\backslash V^M$. Clearly $M \models \mathrm{GB}^-$ and $M \models \mathrm{GBC}^-$ provided the initial ground model N satisfies AC.

Note that M is almost wellfounded, i.e. every set x is assigned a rank $r(x) \in On$ with $r(a) = 0$ for every atom $a \in A$. For every $x \in W(A)$, let $supp(x) = TC(x) \cap A$ be the *support* of x. Let $Aut(A)$ be the class of permutations of A belonging to M. Every $\pi \in Aut^M(A)$ is extended, using the rank function, to an automorphism of M in the obvious way which we denote again π, and $\pi(X) = \pi''X$.

Theorem 4.1 $M \models \mathrm{A}^\infty$.

Proof. Assume the contrary, i.e. there is an ∞-ary total function F on $W(A)$ defined over $W(A)$ by a formula with parameters $c_1, \ldots, c_k$. Let $C = \bigcup_{i\leq k} supp(c_i)$. Take an infinite set $B \subseteq A$ such that $B \cap C = \emptyset$. By the totality of F there is $b \in B$ such that $b \in F(B\backslash\{b\})$. Now $A\backslash(B \cup C)$ is a proper class and for every $c \in A\backslash(B\cup C)$ the permutation π exchanging only b and c is definable, thus $\pi \in M$. The corresponding automorphism fixes $B\backslash\{b\}$ and C pointwise, hence it fixes the parameters c_i and so it fixes F. Thus $\pi(b) = c \in F(B\backslash\{b\})$ for every $c \in A\backslash(B \cup C)$ which contradicts the fact that $F^{B\backslash\{b\}}$ is a set. $\Box$

For a class $X \in M$, let $supp(X) = \bigcup\{supp(x) : x \in X\}$. The above theorem can be generalized in a straightforward way as follows:

Theorem 4.2 *Let M be as above and $X \in M$. Then*
(i) If supp$(X) \in V^M$ then $M \models \neg \mathrm{A}^2(X)$.
(ii) If supp$(X) \notin V^M$ then $M \models \mathrm{A}^\infty(X)$.

5 Degrees of symmetry

In section 2 we characterized the principle $\mathrm{A}^2(X)$ in terms of t.a. preorderings of X. These preorderings have the greatest degree of asymmetry (or the smallest degree of symmetry) since they split X at every point into a set and a coset. Assigning to t.a. preorderings *symmetry degree* 0, we can go on and define inductively preorderings of growing symmetry degrees n, for $n \in \mathbb{N}$. We shall show that the hierarchy of symmetry degrees reflects precisely the hierarchy of the principles A^n.

The definition below can in fact be given along all ordinals but we shall confine ourselves to finite ones.

Definition 5.1 Let $(X, \preceq)$ be a total preordering. For every $m \in \mathbb{N}$, the property *"$(X, \preceq)$ is m-symmetric"*, will be defined inductively. The class X will be called also *m-symmetric* if there is a $\preceq$ such that $(X, \preceq)$ is an m-symmetric total preordering.

(a) $(X, \preceq)$ is 0-symmetric if $\preceq$ is asymmetric, i.e., for every $x \in X$, $\preceq^x$ is a set.

(b) $(X, \preceq)$ is $(m+1)$-symmetric if for every $x \in X$, $\preceq^x$ is m-symmetric.

For every $m \in \mathbb{N}$ let $\mathcal{S}_m$ be the class of m-symmetric classes. Let also $\mathcal{S} = \bigcup\{\mathcal{S}_n : n \in \mathbb{N}\}$. For a model M, $\mathcal{S}_m^M$ is the usual relativization of $\mathcal{S}_m$ to M. More generally for any class X, we put

$$\mathcal{S}_m^X = \mathcal{S}_m \cap C(X) = \{Y \in \mathcal{S}_m : Y \subseteq X\}.$$

m-symmetric preorderings in fact generalize initial ordinals in two ways: First by relaxing wellorderings to total preorderings and second by referring to classes rather than sets. For example, if $M \models \mathrm{GBC}^-$ is such that $On^M = \alpha$, and a class $X \in M$ has a wellordering $\leq$ in M of order-type α, then $(X, \leq)$ is 0-symmetric, since every segment $\leq^x$ has cardinality less than α, therefore $\leq^x \in V^M$. If $(X, \leq)$ has order-type α^+, then X is 1-symmetric, because for every x $| \leq^x | \leq \alpha$, hence it can be ordered by a wellordering of type α or β

for $\beta \in V^M$ and thus it is 0-symmetric. And so on. This fact will be used in the existence theorem 5.4 below.

Our aim in this section is to correlate the symmetric degrees with the axioms of symmetry. The full correlation needs two rather strong choice principles, namely the Strong Scheme of Choice (SSC), which is a strengthening of SC mentioned in section 2 (see [3] for the relative strength of this principle), and a Maximal Principle (MP).

(SSC) $\quad (\forall x)(\exists Y)\phi(x,Y) \Rightarrow (\exists Y)(\forall x)\phi(x,Y_{(x)})$,
for every formula ϕ without class quantifiers.

(MP) *For every preordering R there is a maximal wellordering $T \subseteq R$.*

Note that SSC is necessary when we treat families of m-symmetric classes in order to choose total preorderings for the classes of the family. Namely, if $(X_i)_{i\in I}$ is a family such that $X_i \in \mathcal{S}_m$, then SSC enables one to have a family $(\preceq_i)_{i\in I}$ such that for every $i \in I$, $(X_i, \preceq_i)$ is an m-symmetric preordering.

Main Theorem (GB$^-$+SSC+MP) *For every class X and every $m \geq 0$, $X \in \mathcal{S}_m \Leftrightarrow \neg \mathrm{A}^{m+2}(X)$. In particular $V \in \mathcal{S}_m \Leftrightarrow \neg \mathrm{A}^{m+2}$.*

In this section we shall prove direction $\Rightarrow$ in GBC$^-$, as well as $\Leftarrow$ for $m \leq 1$ in GB$^-$+SSC. In the next section we shall prove the full $\Leftarrow$ in GB$^-$+SSC+MP.

The next lemma contains some easy consequences of definition 5.1.

Lemma 5.2 *(i) $V \subseteq \mathcal{S}_0$ and $On \in \mathcal{S}_0$.*
(ii) $m < k \Rightarrow \mathcal{S}_m \subseteq \mathcal{S}_k$.
(iii) Every $\mathcal{S}_m$ is closed under subsets, i.e., $X \in \mathcal{S}_m \,\&Y \subseteq X \;\Rightarrow Y \in \mathcal{S}_m$.
(iv) $V^M \in \mathcal{S}_m^M \iff \mathcal{S}_m^M = M$.
(v) $\neg \mathrm{A}^2(X) \iff X \in \mathcal{S}_0$.
(vi) $M \models \neg \mathrm{A}^2 \iff \mathcal{S}_0^M = M$.
(vii) $X \in \mathcal{S}_m^X \iff X \in \mathcal{S}_m \Rightarrow \mathcal{S}_m^X = \mathcal{S}_{m+1}^X$.

Proof. (i) For every set x, $(x, =)$ is a trivial 0-symmetric total preordering, and so is the natural ordering of On.

(ii) Let $(X, \preceq) \in \mathcal{S}_0$. For every $x \in X$, $\preceq^x$ is a set hence 0-symmetric by (i). Thus $X \in \mathcal{S}_1$. Then use induction.

(iii) If $(X, \preceq) \in \mathcal{S}_0$ and $Y \subseteq X$, then the restriction of $\preceq$ to Y is 0-symmetric. Then use induction again.

(iv) follows from (iii).

(v) By the preceding section $\mathrm{A}^2(X)$ holds iff there is a t.a. preordering $\preceq$ on X, and these are precisely the 0-symmetric orderings.

(vi) follows from (iv) and (v).

(vii) Trivial. $\square$

Lemma 5.3 *(i) If for some $m > 0$, $\mathcal{S}_{m-1} \subset \mathcal{S}_m$, then for every $i < m$ $\mathcal{S}_{i-1} \subset \mathcal{S}_i$.*

(ii) Therefore either $\mathcal{S}_m \subset \mathcal{S}_{m+1}$ for every $m \geq 0$, or $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{m-1} \subset \mathcal{S}_m = \mathcal{S}$ for some m.

Proof. (ii) follows immediately from (i). Suppose $(X, \preceq) \in \mathcal{S}_m \backslash \mathcal{S}_{m-1}$. For every $x \in X$ $\preceq^x \in \mathcal{S}_{m-1}$. If for all x, $\preceq^x \in \mathcal{S}_{m-2}$, X would belong to $\mathcal{S}_{m-1}$. Thus for some x, $\preceq^x \in \mathcal{S}_{m-1} \backslash \mathcal{S}_{m-2}$. It follows that $\mathcal{S}_{m-2} \subset \mathcal{S}_{m-1}$. Let $X_1 = \preceq^x$ and $\preceq_1$ its $(m-1)$- symmetric preordering. Working similarly with $(X_1, \preceq_1)$ as before we see that $\mathcal{S}_{m-3} \subset \mathcal{S}_{m-2}$ and so on for all $i < m$. $\square$

First we must make sure that the hierarchy $\mathcal{S}_m$ does not collapse in general. In fact (using a large cardinal hypothesis) we can find models satisfying any one of the cases mentioned in lemma 5.3.

Theorem 5.4 *If there is a model N of* ZFC *containing an inaccessible cardinal, then:*

(i) For every $m \in \mathbb{N}$ there is $M \models \mathrm{GBC}^-$ such that

$$\mathcal{S}_0^M \subset \mathcal{S}_1^M \subset \cdots \subset \mathcal{S}_{m-1}^M \subset \mathcal{S}_m^M = M.$$

(ii) There is $M \models \mathrm{GBC}^-$ such that $\mathcal{S}_i^M \subset \mathcal{S}_{i+1}^M$ for all $i \in \mathbb{N}$.

Proof. Let N be a model of ZFC+GCH containing an inaccessible cardinal $\kappa = \omega_\kappa$.

(i) Given $m \in \mathbb{N}$, consider the cardinal $\lambda = \omega_{\kappa+m}$. As in [2], Chapter III, we produce a set of reflexive sets of size λ considering the permutation F of N defined as follows:

$$F(x) = \begin{cases} \{x\} \text{ if } x \in \lambda\backslash\{1\}, \\ y \text{ if } x = \{y\} \ \& \ y \in \lambda\backslash\{1\} \\ x \text{ otherwise.} \end{cases}$$

Let $\in_F$ be the relation defined by $x \in_F y$ iff $F(x) \in y$. Let

$$A = \{\{\alpha\} : \alpha \in \lambda\backslash\{1\}\}.$$

$(N, \in_F)$ satisfies the same axioms as $(N, \in)$ except foundation, that is

$$(N, \in_F) \models \text{ZFC}^- + \text{GCH} + \kappa \text{ is inaccessible.}$$

Moreover $(N, \in_F) \models a = \{a\}$ for every $a \in A$ and

$$(N, \in_F) \models A \text{ has a wellordering of order-type } \lambda.$$

We are working in $(N, \in_F)$. Let $W(A)$ be the cumulative hierarchy of sets constructed from A as in section 4 and let $H_\kappa(A) = \{x \in W(A) : |TC(x)| < \kappa\}$ be the sets of $W(A)$ of hereditary cardinality less than κ. Since κ is strongly inaccessible $H_\kappa(A) \models \text{ZFC}^-$. Let $M = H_\kappa(A) \cup S(H_\kappa(A))$ with $V^M = H_\kappa(A)$. It is not hard to see that M is as required. Indeed $M \models \text{GBC}^-$, $On^M = \kappa$, and if $X \in M$ and $\leq$ is a wellordering of X in N, then $\leq \in M$.

Claim. For every $i \leq m$, $\mathcal{S}_i^M = \{X \in M : N \models |X| \leq \omega_{\kappa+i}\}$.

Proof of the claim. By induction on i. Let $i = 0$. Let $X \in M$ and $N \models |X| = \omega_\kappa = \kappa$. There is a wellordering $\leq \in M$ of order-type κ. Then $X \in \mathcal{S}_0^M$ because for every $x \in X$, the initial segment $\leq^x$ has cardinality $< \kappa$, hence $\leq^x \in H_\kappa(A) = V^M$. Therefore

$$\{X \in M : N \models |X| \leq \omega_\kappa\} \subseteq \mathcal{S}_0^M.$$

Conversely, if $(X, \preceq)$ is a preordering in M such that $\preceq^x \in V^M$, $(X, \preceq)$ is a set in N and we can find by choice a cofinal wellordering of type $\leq \kappa$, hence $N \models |X| \leq \kappa = \omega_\kappa$. Thus also

$$\mathcal{S}_0^M \subseteq \{X \in M : N \models |X| \leq \omega_\kappa\}.$$

For the induction step observe that if $N \models |X| = \omega_{\kappa+i}$, then taking a wellordering $\leq$ of X of order-type $\omega_{\kappa+i}$, every segment $\leq^x$ has cardinality $\leq \omega_{\kappa+i-1}$, hence a wellordering of this order-type, so we use the induction hypothesis to prove the claim.

It follows that for every $0 < i \leq m$ and every $X \in M$ such that $N \models |X| = \omega_{k+i}$, $X \in \mathcal{S}_i^M \backslash \mathcal{S}_{i-1}^M$. On the other hand $V^M \in \mathcal{S}_m^M$, hence $\mathcal{S}_m^M = M$.

(ii) We work as before except that we take now $\lambda > \omega_{\kappa+m}$ for every $m \in \mathbb{N}$, say $\lambda = \omega_{\kappa+\omega}$. □

If $(X, \preceq)$ is 0-symmetric then for any two distinct $x_1, x_2 \in X$, either $x_2 \preceq x_1$ and $\preceq^{x_1}$ is a set or $x_1 \preceq x_2$ and $\preceq^{x_2}$ is a set. For $m \in \mathbb{N}$ and $m > 0$ this can be generalized as follows:

Lemma 5.5 *Let $(X, \preceq)$ be m-symmetric for $m \in \mathbb{N}$ and $m > 0$. Then the following condition holds:*

(*) For every multiset $u \subseteq X$ with $|u| = m+2$ (i.e., an element $x \in u$ may have finitely many occurrences and $m+2$ is the sum of all occurrences of elements of u) there is an enumeration $x_1, \ldots, x_{m+2}$ of u and total preorderings $\preceq_1, \ldots, \preceq_m$, such that:

(a) $(\preceq_1^x, \preceq_1)$ is $(m-1)$-symmetric and $x_2 \preceq x_1$.

(b) For every $i < m+1$,
$(\preceq_{i-1}^{x_i}, \preceq_i)$ is $(m-i)$-symmetric and $x_{i+1} \preceq_{i-1} x_i$.

d) $\preceq_m^{x_{m+1}}$ is a set and $x_{m+2} \in \preceq_m^{x_{m+1}}$.

Proof. By induction on m. Suppose $(X, \preceq)$ is 1-symmetric and let $u \subseteq X$ be a multiset with $|u| = 3$. Since $\preceq$ is total there is $x_1 \in u$ such that $u \backslash \{x_1\} \preceq_* x_1$ (where $w \preceq_* x$ means that $z \preceq x$ for all $z \in w$). Then $\preceq^{x_1}$ is 0-symmetric, i.e. there is $\preceq_1$ such that $(\preceq^{x_1}, \preceq_1)$ is 0-symmetric. $u \backslash \{x_1\} \subseteq \preceq^{x_1}$ and let $x_2 \in u \backslash \{x_1\}$ such that $u \backslash \{x_1, x_2\} \preceq_{1*} x_2$. Then $\preceq_1^{x_2}$ is a set and if $u \backslash \{x_1, x_2\} = \{x_3\}$, $x_3 \in \preceq_1^{x_2}$. Thus the enumeration x_1, x_2, x_3 of u and $\preceq_1$ satisfy (*).

Suppose that every m-symmetric preordering satisfies (*). Let $(X, \preceq)$ be $(m+1)$-symmetric and $u \subseteq X$ be a multiset with $|u| = m+3$. Let $x_1 \in u$ be such that $u \backslash \{x_1\} \preceq_* x_1$. Then there is $\preceq_1$ such that $(\preceq^{x_1}, \preceq_1)$ is m-symmetric and $u \backslash \{x_1\} \subseteq \preceq^{x_1}$. Since $|u \backslash \{x_1\}| = m+2$, by the induction hypothesis there is an enumeration $x_2, x_3, \ldots, x_{m+3}$ of the elements of $u \backslash \{x_1\}$ and preorderings $\preceq_2, \ldots, \preceq_{m+1}$ satisfying (*). Then clearly the sequences $x_1, \ldots, x_{m+3}$ and $\preceq_1, \ldots, \preceq_{m+1}$ also satisfy (*). □

The preceding lemma says that in an m-symmetric preordering $(X, \preceq)$, given any multiset $u \subseteq X$ with at least $m+2$ elements, there are preorderings

$\preceq_1, \dots, \preceq_{m+1}$ and an arrangement $x_1, \dots, x_{m+2}$ of these elements that can be used as stairs of a "ladder" to go down and hit a set. We abbreviate the two sequences, of elements and of preorderings, by a common one of length $m+2$ writing

$$x_{m+2} \preceq_m x_{m+1} \preceq_{m-1} x_m \preceq_{m-2} x_{m-1} \preceq_{m-3} \cdots \preceq_1 x_2 \preceq x_1.$$

The $(m+1)$-subsequence

$$(**) \qquad x_{m+1} \preceq_{m-1} x_m \preceq_{m-2} x_{m-1} \preceq_{m-3} \cdots \preceq_1 x_2 \preceq x_1,$$

resulting from the first one by deleting its last element, and for which the bottom element x_{m+1} defines a set in the preordering $\preceq_m$, will be called an $(m+1)$-*ladder of* u *in* X or just a *ladder*. The letters ξ, ζ denote ladders. For every multiset u, let $lad_X(u)$ be the set of ladders of u. For every ladder ξ let $gr(\xi)$ be the *ground* set determined by the bottom element of ξ, i.e. if ξ is the sequence (**),

$$gr(\xi) = \preceq_m^{x_{m+1}} .$$

Finally, for every m-symmetric X and every multiset $u \subseteq X$, with $|u| \geq m+1$, let

$$gr_X(u) = \bigcup \{gr(\xi) : \xi \in lad_X(u)\}.$$

Clearly for every X, u $gr_X(u)$ is a set.

Using ladders and ground sets we can establish a first connection between the hierarchy of symmetric classes and that of symmetric principles $\mathrm{A}^n(X)$.

Lemma 5.6 *Let X be m-symmetric. For every set $u \subseteq X$ with $|u| \geq m+2$, there is a $x \in u$ such that $x \in gr_X(u \backslash \{x\})$.*

Proof. Take a set $w \subseteq u$ with $|w| = m+2$. By lemma 5.5 w has an enumeration $x_1, \dots, x_{m+2}$ so that

$$x_{m+2} \preceq_m x_{m+1} \preceq_{m-1} x_m \preceq_{m-2} x_{m-1} \preceq_{m-3} \cdots \preceq_1 x_2 \preceq x_1,$$

for certain preorderings $\preceq_i$, hence

$$x_{m+1} \preceq_{m-1} x_m \preceq_{m-2} x_{m-1} \preceq_{m-3} \cdots \preceq_1 x_2 \preceq x_1$$

is a ladder ξ for $u \backslash \{x_{m+2}\}$. Moreover, by definition, $x_{m+2} \in gr_X(\xi)$. Hence $x_{m+2} \in gr_X(u \backslash \{x_{m+2}\})$. $\Box$

The next theorem gives the direction $\Rightarrow$ of the Main Theorem.

Theorem 5.7 (GBC^-) *For every class X, $X \in \mathcal{S}_m \Rightarrow \neg\mathrm{A}^{m+2}(X)$. In particular*

$$V \in \mathcal{S}_m \Rightarrow \neg\mathrm{A}^{m+2}.$$

Proof. Let $X \in \mathcal{S}_m$. Recall that $\neg\mathrm{A}^{m+2}(X)$ holds if there is a function $F : [X]^{m+1} \to S(X)$ which is total, i.e., for every $u \in [X]^{m+2}$ there is $x \in u$ such that $x \in F(u\backslash\{x\})$. Now if we take $F : [X]^{m+1} \to S(X)$ such that $F(u) = gr_X(u)$, by lemma 5.6, F is total, so we are done. □

By lemma 5.2, the converse of 5.7(v) holds for $m = 0$. We can see that it holds also for $m = 1$. First we prove some closure properties for $\mathcal{S}_0$.

Recall that, by theorem 2.9, for normal X, $X \in \mathcal{S}_0$ iff X is set-stratifiable. Recall also (lemma 2.7) that if we assume SC all proper classes are normal. Thus we easily see that:

Lemma 5.8 (GB^-+SC) *(i) $X \in \mathcal{S}_0 \Leftrightarrow X$ is set-stratifiable $\Leftrightarrow |X| = |On|$, (where $|X| = |Y|$ means that there is a bijection between X and Y).*

(ii) Every proper class X contains a proper subclass $Y \in \mathcal{S}_0$.

Lemma 5.9 (GB^-+SSC) *Let $(X_i)_{i\in I}$ be a family such that $I, X_i \in \mathcal{S}_0$. Then $X = \bigcup_{i\in I} X_i \in \mathcal{S}_0$.*

Proof. By 5.8, $|X_i| = |I| = |On|$. Thus I can be identified with On and using SSC we can find a family of bijections $F_\alpha : On \to X_\alpha$. Thus $X_\alpha = \{x_{\alpha\beta} : \beta \in On\}$. Then obviously $|\bigcup_\alpha X_\alpha| = |On|$, hence $\bigcup_\alpha X_\alpha \in \mathcal{S}_0$. □

Corollary 5.10 (GB^-+SSC) *If $(X, \preceq) \in \mathcal{S}_1$ and X contains a cofinal subclass Y such that $Y \in \mathcal{S}_0$, then $X \in \mathcal{S}_0$.*

Proof. Since Y is cofinal in $(X, \preceq)$, $X = \bigcup_{y\in Y} \preceq^y$ and $\preceq^y \in \mathcal{S}_0$. Hence if $Y \in \mathcal{S}_0$ the claim follows from the preceding lemma. □

Corollary 5.11 (GB^-+SSC) *Let R be a binary relation such that for every x the initial segment $R^x = \{y : yRx\} \in \mathcal{S}_0$. If $\bar{R}$ is the reflexive and transitive closure of R then $\bar{R}^x \in \mathcal{S}_0$ for all x.*

Proof. Obviously $\bar{R} = \bigcup_{n\in\mathbb{N}} R_n$ where
$R_0^x = R^x \cup x$, and
$R_{n+1}^x = \bigcup\{R_n^y : y \in R_n^x\}$.
It follows inductively using lemma 5.9 that for all n, x $R_n^x \in \mathcal{S}_0$. Hence also is $\bar{R}^x = \bigcup\{R_n^x : n \in \mathbb{N}\} \in \mathcal{S}_0$. □

We generalize the notion of n-ary function on a class X to that of an *n-ary mapping* on X by allowing its values to be *subclasses* of X instead just subsets. We write $F : [X]^n \to C(X)$ for n-ary mappings (recall that $C(X)$ is the class of subclasses of X). As before F is said to be *total* if for every $u \in [X]^{n+1}$ there is $x \in u$ such that $x \in F(u\backslash\{x\})$.

Theorem 5.12 (GB$^-$+SSC) *For every class X, $X \in \mathcal{S}_1 \Leftrightarrow \neg\mathrm{A}^3(X)$. In particular*

$$V \in \mathcal{S}_1 \Leftrightarrow \neg\mathrm{A}^3.$$

Proof. $\Rightarrow$ follows from 5.7. We assume X is a proper class, otherwise the claim holds trivially. Suppose $\neg\mathrm{A}^3(X)$ holds and let $F : [X]^2 \to S(X)$ be a binary total function on X. Take a proper 0-symmetric $E \subseteq X$ and consider the unary mapping $F_E : [X]^1 \to C(X)$ defined by

$$F_E(\{x\}) = \bigcup\{F(\{x, e\}) : e \in E \ \&\ e \neq x\}.$$

Since $F(\{x, e\})$ are sets (hence 0-symmetric), and E is 0-symmetric, it follows from lemma 5.9 that $F_E(\{x\})$ is 0-symmetric for every x. Moreover F_E is also total. Indeed assume there are $x_1 \neq x_2 \in X$ such that $x_1 \notin F_E(\{x_2\})$ and $x_2 \notin F_E(\{x_1\})$. It follows that

$$(\forall e \in E\backslash\{x_1, x_2\})(x_1 \notin F(\{x_2, e\}) \ \&\ x_2 \notin F(\{x_1, e\})).$$

By the totality of F, $(\forall e \in E\backslash\{x_1, x_2\})(e \in F(\{x_1, x_2\}))$, or $E\backslash\{x_1, x_2\} \subseteq F(\{x_1, x_2\})$, which is a contradiction since $F(\{x_1, x_2\})$ is a set and $E\backslash\{x_1, x_2\}$ is proper. Thus the relation

$$xRy \iff x \in F_E(y)$$

on X is total. If $\preceq$ is the transitive and reflexive closure of R, then $\preceq$ is a preordering and by corollary 5.11, for every $x \in X$, $\preceq^x$ is a normal 0-symmetric class. Therefore $\preceq$ is 1-symmetric. □

The general implication $\neg A^{m+2}(X) \Rightarrow X \in \mathcal{S}_m$ is open without further assumptions about $\mathcal{S}_m$. The first such assumption concerns the closure of $\mathcal{S}_m$ under unions of classes of elements of $\mathcal{S}_m$ indexed by a class of $\mathcal{S}_m$ (i.e. the condition analogous to that of lemma 5.9 for $\mathcal{S}_0$).

Definition 5.13 We say that $\mathcal{S}_m$ is *closed* if for any family $(X_i)_{i\in I}$ of classes such that $X_i \in \mathcal{S}_m$ and $I \in \mathcal{S}_m$, $\bigcup_{i\in I} X_i \in \mathcal{S}_m$.

In GB$^-$+SSC we do not know even whether $\mathcal{S}_1$ is closed. The other assumption concerns the collapsing of $\mathcal{S}_i$. The best we can prove for the time being is the next result. Recall that $\mathcal{S}_m^X = \mathcal{S}_m \cap C(X)$.

Theorem 5.14 (GB$^-$+SSC) *Let X be a proper class and let $m \geq 2$ such that:*

(a) $\mathcal{S}_{m-1}^X \backslash \mathcal{S}_{m-2}^X \neq \emptyset$.

(b) $\mathcal{S}_i$ is closed for all $i \leq m-1$.

Then $\neg A^{m+2}(X) \Rightarrow X \in \mathcal{S}_m$.

Proof. Let $\neg A^{m+2}(X)$ hold and $F : [X]^{m+1} \to S(X)$ be a total function. By (a) we can show as in lemma 5.3 that $\mathcal{S}_i^X \backslash \mathcal{S}_{i-1}^X \neq \emptyset$ for all $i \leq m-1$. So for every $i \leq m-1$ let $E_i \in \mathcal{S}_i^X \backslash \mathcal{S}_{i-1}^X$ and define $F_i : [X]^{m-i} \to C(X)$ inductively as follows:

$F_0(u) = \bigcup\{F(u \cup \{e\}) : e \in E_0 \backslash u\}$,

$F_i(u) = \bigcup\{F_{i-1}(u \cup \{e\}) : e \in E_i \backslash u\}$.

The following claims are easily proved by induction on i:

Claim 1. For every $i \leq m-1$, and for every $u \in dom(F_i)$, $F_i(u) \in \mathcal{S}_i$.
By induction on i and using the fact that $\mathcal{S}_i$ are closed.

Claim 2. Every F_i is total.
Suppose F_{i-1} is total while F_i is not. Then for some u such that $|u| = m-i+1$, $(\forall x \in u)(x \notin F_i(u\backslash\{x\}))$. Equivalently

$$(\forall x \in u)(\forall e \in E_i\backslash u)(x \notin F_{i-1}(u \cup \{e\}\backslash\{x\})).$$

By the totality of F_{i-1}, $E_i\backslash u \subseteq F_{i-1}(u)$. But $F_{i-1}(u) \in \mathcal{S}_{i-1}$ by claim 1, while $(E_i\backslash u) \in \mathcal{S}_i\backslash\mathcal{S}_{i-1}$, which is a contradiction.

It follows that the relation $xRy \iff x \in F_{m-1}(\{y\})$ on X is total and $R^x \in \mathcal{S}_{m-1}$ for every x. If $\preceq$ is the reflexive and transitive closure of R, then as in corollary 5.11 and using the fact that $\mathcal{S}_{m-1}$ is closed we see that the segments of $\preceq$ are also $(m-1)$-symmetric, hence $(X, \preceq)$ is an m-symmetric preordering. □

6 Wellorderings and asymmetry

We have already seen in theorem 5.4 that the prototypes of m-symmetric preorderings are initial wellorderings.

Given two classes X, Y we write $|X| \le |Y|$ (resp. $|X| = |Y|$) if there is an injection (resp. bijection) of X into (resp. onto) Y. Schröder-Bernstein theorem (that holds also for classes) gives $|X| \le |Y| \ \& \ |Y| \le |X| \ \Rightarrow |X| = |Y|$. We write $|X| < |Y|$ if $|X| \le |Y|$ and $|Y| \not\le |X|$.

A total ordering T on a class X is a *wellordering* if every subclass of X has a T-least element. The letters T, U, T_1, T_2 will range over wellorderings. By some abuse of language we identify T with $Field(T)$ and write $x \in T$, $|T|$ instead of $x \in Field(T)$ and $|Field(T)|$ respectively. If $x \in T$ we write T^x for the initial segment $\{y : yTx\}$. Any two wellorderings T_1, T_2 are comparable in GB^-, i.e., there is a 1-1 order-preserving mapping such that either $F : T_1 \cong T_2$, or $F : T_1 \cong T_2^x$ or $F : T_2 \cong T_1^y$. We write $T_1 \lhd T_2$, $T_2 \lhd T_1$ in the last two cases respectively.

Let $\mathcal{W}$ be the class of wellorderings. Define also the classes $\mathcal{W}_m$ inductively as follows:

$\mathcal{W}_0 = \{T \in \mathcal{W} : |T| \le |On|\}$.

$\mathcal{W}_{m+1} = \{T \in \mathcal{W} : (\forall x \in T)(T^x \in \mathcal{W}_m)\}$.

Clearly $\mathcal{W}_m \subseteq \mathcal{W}_{m+1}$.

Lemma 6.1 *(i)* $T \in \mathcal{W}_m \ \& \ U \lhd T \Rightarrow U \in \mathcal{W}_m$.

(ii) For every m, $\mathcal{W}_m = W \cap \mathcal{S}_m$.

(iii) If $\mathcal{W}_m \subset \mathcal{W}_{m+1}$, *then* $\mathcal{S}_m \subset \mathcal{S}_{m+1}$.

(iv) $\mathcal{W}_m = \mathcal{W} \iff \mathcal{W}_m = \mathcal{W}_{m+1}$.

Proof. (i) and (ii) are shown by easy induction on m. (iii) follows immediately from (ii).

(iv) One direction is trivial. For the other, let $T \in \mathcal{W} \backslash \mathcal{W}_m$. Suppose for every $x \in T$, $T^x \in \mathcal{W}_m$. Then, by definition, $T \in \mathcal{W}_{m+1}$ and the claim holds. Suppose now that there is $x \in T$ such that $T^x \notin \mathcal{W}_m$. Let x_0 be the least such x. Then clearly $T^{x_0} \in \mathcal{S}_{m+1} \backslash \mathcal{S}_m$. □

Let R be a preordering and T be a wellordering such that $T \subseteq R$, i.e., $xTy \Rightarrow xRy$. T is said to be *maximal* in R if there is no wellordering $U \subseteq R$ such that $T \lhd U$. Recall that MP is the following principle

(MP): *For every preordering R there is a maximal wellordering $T \subseteq R$.*

Note that MP follows from the principle "V has a wellordering". The latter in consistent with SSC+GB$^-$. For instance in the model $M = H_\kappa(A) \cup S(H_\kappa(A))$ of theorem 5.4, pick a wellordering $\leq$ of $H_\kappa(A)$ and let $RA(H_\kappa(A), \leq)$ be the ramified analytical hierarchy constructed inside M. Then the model M' whose sets are the elements of $H_\kappa(A)$ and whose classes are the elements of $RA(H_\kappa(A), \leq)$ is a model of GB$^-$+SSC+"V has a wellordering" (see [6]).

Using SSC+MP we can prove that every $\mathcal{S}_m$ is closed.

Lemma 6.2 (GB$^-$+MP+SSC) *For every $m \geq 0$*
(i) $X_1, X_2 \in \mathcal{S}_m \Rightarrow X_1 \times X_2 \in \mathcal{S}_m$.
(ii) $\mathcal{S}_m$ is closed.

Proof. (i) By induction on m. For $m = 0$ this is clear. Suppose it holds for $m-1$ and let $X_1, X_2 \in \mathcal{S}_m$. Let $\preceq_1, \preceq_2$ be m-symmetric preorderings for X_1, X_2 respectively. By MP there are maximal (hence cofinal) wellorderings $T_1 \subseteq \preceq_1$ and $T_2 \subseteq \preceq_2$. For $x \in X_i$, $i = 1, 2$, let $r_i(x)$ = biggest initial segment of T_i not exceeding x. Define $\preceq$ on $X_1 \times X_2$ as follows:

$$(x_1, x_2) \preceq (y_1, y_2) \iff max(r_1(x_1), r_2(x_2)) \leq max(r_1(y_1), r_2(y_2)),$$

where at the right-hand side we compare wellorderings. Now for every (x_1, x_2) such that $max(r_1(x_1), r_2(x_2)) = U$, we have $\preceq^{(x_1,x_2)} = \preceq_1^{a_1} \times \preceq_2^{a_2}$, where $a_i \in T_i$ such that $T^{a_i} \cong U$. But $\preceq_i^{a_i} \in \mathcal{S}_{m-1}$, and by the induction hypothesis $\preceq_1^{a_1} \times \preceq_2^{a_2} \in \mathcal{S}_{m-1}$. Hence $\preceq$ is m-symmetric

(ii) (Sketch) Suppose again that the claim holds for $m-1$. Let $(X_i)_{i \in I}$ be a family of $\mathcal{S}_m$-classes coded by the $\mathcal{S}_m$-class I. Using SSC we can find a coded class $\preceq_i$ of m-symmetric preorderings for them and let $\preceq$ be an m-symmetric preordering for I. By MP every X_i contains a maximal (hence cofinal) wellordering $T \subseteq \preceq_i$ and obviously $T \in \mathcal{S}_m$. Using SSC we can choose a coded family $T_i, i \in I$, of such wellorderings. Let also T be cofinal in $\preceq$. For every $i \in I$ and every $x \in X_i$, let

$$r(x, i) = \text{biggest initial segment of } T_i \text{ not exceeding } x.$$

For every $x \in \bigcup_i X_i$, $W(x) = \{r(x,i) : x \in X_i\}$ is a coded class of wellorderings and we can again choose by SSC for every x an element $U_x \in W(x)$ of least length. Let also

$$S_x = \text{biggest initial segment of } T \text{ not exceeding a } j \text{ such that } x \in X_j.$$

That is S_x is the biggest segment of T below every index j such that $x \in X_j$. Clearly, for every x, U_x and S_x belong to $\mathcal{S}_{m-1}$.

Define the preordering $\preceq'$ on $\bigcup_i X_i$ as follows:

$$x \preceq' y := max\{U_x, S_x\} \leq max\{U_y, S_y\}.$$

Using (i) and the induction hypothesis it is easy to see that this is an m-symmetric preordering. □

The following lemma is crucial for the proof of the main theorem.

Lemma 6.3 (GBC$^-$+MP) *Suppose that $\mathcal{S}_m$ contains a coded cofinal subclass with respect to $\subseteq$, i.e., there is K such that*

$$(\dagger) \quad (\forall x \in dom(K))(K_{(x)} \in \mathcal{S}_m) \ \& \ (\forall X \in \mathcal{S}_m)(\exists x \in dom(K))(X \subseteq K_{(x)}).$$

Then either $V \in \mathcal{S}_m$ or $\mathcal{S}_{m+1} \backslash \mathcal{S}_m \neq \emptyset$.

Proof. Suppose (†) holds and let $L = dom(K)$. Let R be the relation on L defined by

$$xRy \iff K_{(x)} \subseteq K_{(y)}.$$

By MP there is a maximal wellordering $T \subseteq R$. Let $Z = \bigcup\{K_{(x)} : x \in T\}$.

Assume $T \in \mathcal{W}_m$. Then $T \in \mathcal{S}_m$ and since $\mathcal{S}_m$ is closed, $Z \in \mathcal{S}_m$. Then Z is a $\subseteq$-maximal m-symmetric class, but this obviously happens only if $Z = V$. therefore $V \in \mathcal{S}_m$.

Assume $T \notin \mathcal{W}_m$. Then by 6.1 (iv), $\mathcal{W}_{m+1} \backslash \mathcal{W}_m \neq \emptyset$, hence by 6.1 (iii), $\mathcal{S}_{m+1} \backslash \mathcal{S}_m \neq \emptyset$. □

In fact the preceding lemma also holds if relativized to any class X and the proof is quite the same. We showed it for V for reasons of transparency. Thus more generally we have:

Lemma 6.4 (GBC$^-$+MP) *Let X be a class and suppose that $\mathcal{S}_m^X$ contains a coded cofinal subclass with respect to $\subseteq$. Then either $X \in \mathcal{S}_m$ or $\mathcal{S}_{m+1}^X \backslash \mathcal{S}_m^X \neq \emptyset$.*

Theorem 6.5 (GB$^-$+MP+SSC) *For every class X and every $m \geq 0$, $\neg \mathrm{A}^{m+2}(X) \Rightarrow X \in \mathcal{S}_m$. In particular $\neg \mathrm{A}^{m+2} \Rightarrow V \in \mathcal{S}_m$.*

Proof. The proof of the first implication is based on lemma 6.4 precisely in the same way that the proof of the second one is based on lemma 6.3. So for simplicity we give the proof of the second implication the other being similar.

The implication has been proved for $m \leq 1$ (lemma 5.2 and theorem 5.12). It has also been shown in theorem 5.14 under the conditions (a) that $\mathcal{S}_i$ are closed and (b) that $\mathcal{S}_{m-2} \subset \mathcal{S}_{m-1}$. By lemma 6.2 (a) holds in the presence of SSC and MP. So it remains to prove it when $\mathcal{S}_{m-2} = \mathcal{S}_{m-1}$.

Assume $\neg \mathrm{A}^{m+2}$, $m \geq 2$,s and $\mathcal{S}_{m-2} = \mathcal{S}_{m-1}$. Let i be the least integer such that $\mathcal{S}_i = \mathcal{S}_{i+1}$. Then $i \leq m-2$ and

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_i = \mathcal{S}_{i+1} = \mathcal{S}.$$

Fix a total $(m+1)$-ary total function $F : [V]^{m+1} \to V$.

Case 1. $i = 0$. Then $\mathcal{S}_0 = \mathcal{S}_1$. Pick a proper $E_0 \in \mathcal{S}_0$ and put as in the proof of 5.14 $F_0(u) = \bigcup\{F(u \cup \{e\}) : e \in E_0 \backslash u\}$ for $u \in [V]^m$. Then F_0 is an m-ary total function, and for all $u \in [V]^m$, $F_0(u) \in \mathcal{S}_0$. Hence the family $\{F_0(u) : u \in [V]^m\}$ is a coded subclass of $\mathcal{S}_0$.

Subcase 1a. Suppose $\mathcal{S}_0$ satisfies (†) of lemma 6.3. Then either $V \in \mathcal{S}_0$ or $\mathcal{S}_0 \subset \mathcal{S}_1$. Since the latter is false by assumption we get $V \in \mathcal{S}_0$, hence also $V \in \mathcal{S}_m$. Thus the claim holds.

Subcase 1b. Let $\mathcal{S}_0$ not satisfy (†). Then the family $\{F_0(u) : u \in [V]^m\}$ is not cofinal in $(\mathcal{S}_0, \subseteq)$, i.e., there is $E_1 \in \mathcal{S}_0$ such that

$$(\forall u \in [V]^m)(E_1 \not\subseteq F_0(u)). \tag{1}$$

Define $F_1 : [V]^{m-1} \to C(V)$, by putting $F_1(u) = \bigcup\{F_0(u \cup \{e\}) : e \in E_1 \backslash u\}$. Using the totality of F_0 and (1) we easily see that F_1 is also total. (In fact (1) implies that $(\forall u \in [V]^m)((E_1 \backslash u) \not\subseteq F_0(u))$ since we may assume that $u \subseteq F(u)$.) Moreover $F_1(u) \in \mathcal{S}_0$ by the closure of $\mathcal{S}_0$ under unions. The family $\{F_1(u) : u \in [V]^{m-1}\}$ is again a coded subclass of $\mathcal{S}_0$, so by the negation of (†) it is not cofinal in $\mathcal{S}_0$. Pick as before a class $E_2 \in \mathcal{S}_0$ omitting all $F_1(u)$ and define similarly F_2 which will be total. Finally after $m-1$ steps we find a total $F_{m-1} : [V]^1 \to C(V)$ such that $F_{m-1}(\{x\}) \in \mathcal{S}_0$. The relation $xRy \iff x \in F_{m-1}(\{y\})$ is total and can be extended to a total preordering $\preceq$ with $\preceq^x \in \mathcal{S}_0$. Thus $V \in \mathcal{S}_1$. Hence $V \in S_m$ since $m \geq 2$, and the implication is true.

Case 2. $i > 0$. This case is treated as in the proof of theorem 5.14 until we reach i and then we work as in case 1. That is we pick $E_k \in \mathcal{S}_k \backslash \mathcal{S}_{k-1}$ for $k \leq i$ and define the total functions $F_0, \ldots, F_i$, with $F_i : [V]^{m-i} \to C(V)$ and $F_i(u) \in \mathcal{S}_i$. Then we have again:

Subcase 2a. $\mathcal{S}_i$ satisfies (†). By lemma 6.3, either $V \in \mathcal{S}_i$ or $\mathcal{S}_i \subset \mathcal{S}_{i+1}$. By our assumption the latter is false, hence $V \in \mathcal{S}_i$, consequently $V \in \mathcal{S}_m$.

Subcase 2b. (†) fails for $\mathcal{S}_i$. Then the family $\{F_i(u) : u \in dom(F_i)\}$, which is a coded subclass of $\mathcal{S}_i$, cannot be cofinal in $\mathcal{S}_i$, hence there is $E_{i+1} \in \mathcal{S}_i$ such that $(\forall u \in dom(F_i))(E_{i+1} \not\subseteq F_i(u))$. Using E_{i+1} we find a total $F_{i+1} : [V]^{m-i-1} \to C(V)$ with $F_{i+1}(u) \in \mathcal{S}_i$ by the closure property of $\mathcal{S}_i$, and proceeding along $m-i-1$ steps we find again a total mapping $F_{m-1} : [V]^1 \to C(V)$ with $F_{m-1}(\{x\}) \in \mathcal{S}_i$. This produces a total preordering $\preceq$ on V with $\preceq^x \in \mathcal{S}_i$. That means that $V \in \mathcal{S}_{i+1} \subseteq \mathcal{S}_m$ (recall that $i \leq m-2$) so we are done. This completes the proof. □

Proof of the Main Theorem. Immediate consequence of theorems 5.7 and 6.5. □

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