Propositional superposition logic

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Abstract

We extend classical Propositional Logic (PL) by adding a new primitive binary connective $\varphi|\psi$, intended to represent the "superposition" of sentences φ and ψ, an operation motivated by the corresponding notion of quantum mechanics, but not intended to capture all aspects of the latter as they appear in physics. To interpret the new connective, we extend the classical Boolean semantics by employing models of the form $\langle M, f \rangle$, where M is an ordinary two-valued assignment for the sentences of PL and f is a choice function for all pairs of classical sentences. In the new semantics $\varphi|\psi$ is strictly interpolated between $\varphi \wedge \psi$ and $\varphi \vee \psi$. By imposing several constraints on the choice functions we obtain corresponding notions of logical consequence relations and corresponding systems of tautologies, with respect to which | satisfies some natural algebraic properties such as associativity, closedness under logical equivalence and distributivity over its dual connective. Thus various systems of Propositional Superposition Logic (PLS) arise as extensions of PL. Axiomatizations for these systems of tautologies are presented and soundness is shown for all of them. Completeness is proved for the weakest of these systems. For the other systems completeness holds if and only if every consistent set of sentences is extendible to a consistent and complete one, a condition whose truth is closely related to the validity of the deduction theorem.

1   Introduction

In this paper we present an extension of classical Propositional Logic (PL) (more precisely, an array of extensions of increasing strength), obtained by adding a new binary logical operation, called "superposition", together with a new semantics extending the standard one, inspired and motivated by the corresponding notion of quantum mechanics. That the notion of superposition is central in quantum mechanics is rather well-known. For the sake of completeness let us outline briefly the core of the idea.

A quantum system A, for example an electron or a photon, can be only in finitely many possible "states" (or rather "pure states", see [3]) with respect to a certain physical magnitude Q such as spin, charge, etc. Suppose for simplicity that A can be only in two possible states, say "spin up" and "spin down". We know that whenever the spin of A is measured, the outcome will necessarily be either "spin up" or "spin down" but one cannot predict it in advance precisely, except only with a certain degree of probability. While unobserved, A is thought to be at some kind of a mixture or composition of these states, called *superposition of states.* But as soon as the system A is scanned and measured, the superposition breaks down, or "collapses" according to the jargon of quantum mechanics, to one of the constituent states. So in a sense, the states "spin up" and "spin down" co-exist and at the same time exclude each other.

From its very beginning quantum mechanics had developed an effective and flexible formalism to represent the states of a system (see [3] for a brief overview of the subject), namely as vectors of a Hilbert space. For example the pure states "spin up" and "spin down" are represented by vectors $\vec{u}_0$, $\vec{u}_1$, respectively. Then the "principle of superposition" says that for any complex numbers c_0, c_1 such that $|c_0|^2 + |c_1|^2 = 1$, the linear combination $c_0\vec{u}_0 + c_1\vec{u}_1$ is also a legitimate state of the system A. Moreover, $|c_0|^2$, $|c_1|^2$ represent the probabilities for A to be in state $\vec{u}_0$ or $\vec{u}_1$, respectively, when measured. This treatment of superposition as a linear combination of vectors is mainly due to P.A.M. Dirac[1], who considered the principle of superposition as one of the most fundamental properties of quantum mechanics.

Later on a new approach to quantum mechanics through *quantum logics* was developed by the work of G. Birkhoff- J. von Neumann[2], G. Mackey[3]

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[1] P.A.M. Dirac, *The Principles of Quantum Mechanics,* Oxford U.P., 1958.

[2] G. Birkhoff and J. von Neumann, The logic of quantum mechanics, *Ann. Math.* **37** (1936), 823-843.

[3] G. Mackey, *The Mathematical Foundations of Quantum Mechanics,* W. A. Benjamin, 1963.

and others. Here the emphasis was in the formalization of *non-distributivity,* another characteristic phenomenon of quantum mechanics, and it was not clear whether and how non-distributivity and superposition were related to each other. As S. Gudder says[4], the problem arose to find a formulation of the principle of superposition in the quantum logic approach, roughly equivalent to Dirac's formulation in the vector-space approach. Various such formulations of superposition can be found in the literature.[5] Note that all versions of quantum logic are weaker than classical logic, since they lack the distributivity law.

An important source of inspiration for the present work has been E. Schrödinger's 1935 paper [11] containing the "cat paradox", in which the author shows, by his famous thought experiment, how superposition of quantum states might (in principle) be transformed into superposition of *macroscopic situations.* Although Schrödinger refers to the experiment with ridicule, as a "serious misgiving arising if one notices that the uncertainty affects macroscopically tangible and visible things", it is perhaps the first hint towards thinking that the phenomenon could be conceived in a much broader sense, even in contexts different from the original one. And it is this general, abstract and purely *logical content* of superposition that we are interested in and deal with in this paper.[6]

In particular, the purpose of the paper is to offer a simple interpretation of superposition not by means of a variant of quantum logic, but rather by an *extension of classical logic.* The interpretation is absolutely within classical reasoning and common sense, since we do not drop any law of classical logic, but only augment them by new ones concerning the superposition operation. The ingredient that makes it possible to go beyond classical tautologies is the use at each truth evaluation of a *choice function* acting upon pairs of sentences, a tool originating in set-theory rather than logic.

Let φ_0, φ_1 denote the statements "A is at state $\vec{u}_0$" and "A is at state $\vec{u}_1$", respectively, and let $\varphi_0|\varphi_1$ denote the statement "A is at the superposition of states $\vec{u}_0$ and $\vec{u}_1$". φ_0, φ_1 are ordinary statements, so they can be assigned ordinary truth values. But what about the truth values of $\varphi_0|\varphi_1$? Clearly

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[4]S.P. Gudder, A superposition principle in physics, *J. Math. Phys.* **11** (1970), no. 3, 1037-1040.

[5]See for example S.P.Gudder above and S. Pulmannová, A superposition principle in quantum logics, *Commun. Math. Phys.* **49** (1976), no. 3, 47-51.

[6]As already said earlier, whether the logical content of superposition, as isolated here, bears actual *connections* with and/or *applications* to the existing systems of quantum mechanics and quantum logic is not known at present. Some further comments on this issue are made in the last section.

the operation $\varphi_0|\varphi_1$ cannot be expressed in classical logic, that is, $\varphi_0|\varphi_1$ cannot be logically equivalent to a Boolean combination $S(\varphi_0, \varphi_1)$ of φ_0, φ_1.[7] However, an intriguing feature of $\varphi_0|\varphi_1$ is that it has points in common *both* with classical conjunction and classical disjunction. In a sense it is a "mixture" of $\varphi_0 \wedge \varphi_1$ and $\varphi_0 \vee \varphi_1$, or a property between them, since it bears a conjunctive as well as a disjunctive component. Indeed, $\varphi_0|\varphi_1$ means on the one hand that the properties φ_0 and φ_1 hold *simultaneously* (at least partly) during the non-measurement phase, which is clearly a conjunctive component of $\varphi_0|\varphi_1$, and on the other, at any particular collapse of the superposed states during a measurement, $\varphi_0|\varphi_1$ reduces to either φ_0 *or* φ_1, which is a disjunctive component of the operation. The interpretation of $\varphi_0|\varphi_1$ given below justifies in fact this meaning of $\varphi_0|\varphi_1$ as "something between $\varphi_0 \wedge \varphi_1$ and $\varphi_0 \vee \varphi_1$".

Let us consider a propositional language $L = \{p_0, p_1, \ldots\} \cup \{\wedge, \vee, \neg\}$, where p_i are symbols of atomic propositions, whose interpretations are usual two-valued truth assignments $M : Sen(L) \to \{0, 1\}$ respecting the classical truth tables. Let us extend L to $L_s = L \cup \{|\}$, where $|$ is a new primitive binary connective. For any sentences φ, ψ of L_s, $\varphi|\psi$ denotes the superposition of φ and ψ. Then an interpretation for the sentences of L_s can be given by the help of a truth assignment M for the sentences of L, together with a *collapsing mapping* c from the sentences of L_s into those of L. The mapping c is intended to represent the collapsing of the superposed $\varphi|\psi$ to one of its components.

The basic idea is that the collapse of the composite state $c_0\vec{u}_0 + c_1\vec{u}_1$ to one of the sates $\vec{u}_0$, $\vec{u}_1$ can be seen, *from the point of view of pure logic,* just as a (more or less random) choice from the set of possible outcomes $\{\vec{u}_0, \vec{u}_1\}$. This is because from the point of view of pure logic probabilities are irrelevant or, which amounts to the same thing, the states $\vec{u}_0$ and $\vec{u}_1$ are considered equiprobable. In such a case the superposition of $\vec{u}_0$ and $\vec{u}_1$ is unique and the outcome of the collapse can be decided by a coin tossing or, more strictly, by a *choice function* acting on pairs of observable states, which in our case coincide with pairs of sentences of L. This of course constitutes a major deviation from the standard treatment of superposition, according to which there is not just one superposition of $\vec{u}_0$ and $\vec{u}_1$ but infinitely many, actually as many as the number of linear combinations $c_0\vec{u}_0 + c_1\vec{u}_1$, for $|c_0|^2 + |c_1|^2 = 1$. So the logic presented here is hardly the logic of

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[7]As is well-known there exist precisely 16 classical binary operations $S(\varphi_0, \varphi_1)$, definable in terms of $\wedge$, $\vee$ and $\neg$ (including $\wedge$, $\vee$ themselves, and also $\to$, $\leftrightarrow$, their negations, as well as other trivial ones), none of which can express the logical content of $|$.

superposition as this concept is currently used and understood in physics today. It is rather the logic of superposition, when the latter is understood as the "logical extract" of the corresponding physics concept. Whether it could eventually have applications to the field of quantum mechanics we don't know.

The elementary requirements for a collapsing map c are the following: (a) it must be the identity on classical sentences, that is, $\mathsf{c}(\varphi) = \varphi$ for every L-sentence φ. (b) It must commute with the standard connectives $\wedge$, $\vee$ and $\neg$, that is, $\mathsf{c}(\varphi \wedge \psi) = \mathsf{c}(\varphi) \wedge \mathsf{c}(\psi)$, $\mathsf{c}(\varphi \vee \psi) = \mathsf{c}(\varphi) \vee \mathsf{c}(\psi)$ and $\mathsf{c}(\neg\varphi) = \neg\mathsf{c}(\varphi)$. (c) $\mathsf{c}(\varphi|\psi)$ must be *some* of the sentences $\mathsf{c}(\varphi)$, $\mathsf{c}(\psi)$, which is chosen by the help of a choice function f for pairs of classical sentences, that is,

$$\mathsf{c}(\varphi|\psi) = f(\{\mathsf{c}(\varphi), \mathsf{c}(\psi)\}).$$

Since every sentence of L_s is built from atomic sentences all of which belong to the initial classical language L, it follows that c is *fully determined* by the choice function f, and below we shall write $\mathsf{c} = \overline{f}$. Therefore choice functions f for pairs of sentences of L are the cornerstone of the new semantics.

Given a truth assignment M for L and a choice function f for L, a sentence φ of L_s is *true in M under the choice function f*, denoted $\langle M, f \rangle \models_s \varphi$, if and only if $\overline{f}(\varphi)$ is (classically) true in M. That is:

$$\langle M, f \rangle \models_s \varphi \text{ iff } M \models \overline{f}(\varphi).$$

Since $\overline{f}$ is generated by f, special conditions on f induce special properties for $\overline{f}$ that in turn affect the properties of $\models_s$. Such a condition is needed, for instance, in order for $|$ to be associative.

The above truth concept $\models_s$ extends the classical one and induces the notions of s-logical consequence, $\varphi \models_s \psi$, and s-logical equivalence $\sim_s$, which generalize the corresponding standard relations $\models$ and $\sim$. A nice feature of the new semantics is that for all sentences φ, ψ,

$$\varphi \wedge \psi \models_s \varphi|\psi \models_s \varphi \vee \psi, \tag{1}$$

where the relations $\models_s$ in both places are strict, that is, they cannot in general be reversed (see Theorem 2.8 below). It means that $\varphi|\psi$ is *strictly interpolated* between $\varphi \wedge \psi$ and $\varphi \vee \psi$, a fact that in some sense makes precise the above expressed intuition that $\varphi|\psi$ is a "mixture" of $\varphi \wedge \psi$ and $\varphi \vee \psi$. In particular,

$$\varphi \wedge \neg\varphi \models_s \varphi|\neg\varphi \models_s \varphi \vee \neg\varphi,$$

which means that the superposition of two contradictory situations, like those in Schrödinger's cat experiment [11] mentioned above, is neither a contradiction nor a paradox at all (see Corollary 2.9 below).

Another nice feature of the semantics is that in order for $|$ to be associative with respect to a structure $\langle M, f \rangle$, that is, $\langle M, f \rangle \models_s \varphi | (\psi | \sigma) \leftrightarrow (\varphi | \psi) | \sigma$, it is necessary and sufficient for f to coincide with the function $\min_<$ induced by a total ordering $<$ of the set of sentences of L. Such an $f = \min_<$ picks from each pair $\{\alpha, \beta\}$ not a "random" element but the *least* one with respect to $<$. This kind of choice functions will be the dominant one throughout the paper.

No knowledge of quantum mechanics or quantum logic is required for reading this paper. The only prerequisite is just knowledge of basic Propositional Logic (PL), namely its semantics, axiomatization and soundness and completeness theorems, as well as some elementary set-theoretic facts concerning choice functions for sets of finite sets, total orderings etc. For example [4] is one of the many logic texts that contain the necessary material. Nevertheless, some familiarity with non-classical logics, their axiomatization and their semantics, would be highly helpful. Also for the subject of choice functions and choice principles the reader may consult [7].

Finally, I should mention some other current treatments of superposition from a logical point of view, although one can hardly find to them points of overlapping and convergence with the present one. Such logical approaches are contained in [2], [8] and [1], to mention the most recent ones. The main difference of theses approaches from the present one is that they are all based on some non-classical logical system, while our point of departure is the solid ground of classical propositional logic. For instance [2] relies heavily on paraconsistent logic that allows one to accommodate contradictions without collapsing the system. In fact superposition is captured in [2] as a "contradictory situation": if a quantum system S is in the state of superposition of the states s_1 and s_2, this is expressed by the help of a two-place predicate K and the conjunction of axioms $K(S, s_1)$, $\neg K(S, s_1)$, $K(S, s_2)$ and $\neg K(S, s_2)$. (Here the negation $\neg$ is "weak" and the conjunction of these claims is not catastrophic.) Analogously, [8] uses a version of modal logic in an enriched language that, besides $\neg$, $\wedge$, $\vee$ and $\diamond$ (possibility operator), contains a binary connective $\star$ for the superposition operation and a unary connective M for "measurement has been done". Also a Kripke semantics is used, and the basic idea, as I understood it, is to avoid the contradiction arising e.g. from Schrödinger's cat, by "splitting" it, after the measurement, between two different possible worlds, one containing the cat alive and one containing the cat dead. Finally [1] is more syntactically oriented. It treats

superposition syntactically by employing a version of sequent calculus called "basic logic" (developed in [10]), which encompasses aspects of linear logic and quantum logic.

Summary of Contents. Section 2 contains the semantics of $|$ based on choice functions for pairs of sentences of L. More specifically, in subsection 2.1 we give the basic definitions of the new semantics and the corresponding notions of logical consequence $\models_s$ and logical equivalence $\sim_s$. The models for the sentences of L_s are structures of the form $\langle M, f \rangle$, where M is a truth assignment to sentences of L and f is an arbitrary choice function for L. We prove the basic facts, among which that $\varphi \wedge \psi \models_s \varphi | \psi \models_s \varphi \vee \psi \models_s$. The properties of $|$ supported by such general structures are only $\varphi | \varphi \leftrightarrow \varphi$ (idempotence) and $\varphi | \psi \leftrightarrow \psi | \varphi$ (commutativity). In order to obtain further properties for $|$ we need to impose additional conditions on the choice functions employed which entail more and more refined truth notions. In general if $\mathcal{F}$ is the set of all choice functions for L, for any nonempty $X \subseteq \mathcal{F}$ the relations $\models_X$, of X-logical consequence, and $\sim_X$, of X-logical equivalence, are naturally defined by employing models $\langle M, f \rangle$ with $f \in X$ (rather than $f \in \mathcal{F}$). For each such $X \subseteq \mathcal{F}$ the set of X-tautologies $Taut(X) = \{\varphi : \models_X \varphi\}$ is defined.

In the next subsections of §2 we focus on certain natural such subclasses $X \subseteq \mathcal{F}$ and the corresponding truth notions.

In subsection 2.2 we introduce the class $Asso$ of *associative* choice functions and a simple and elegant characterization of them is given, as the functions $\min_<$ with respect to total orderings $<$ of the set of L-sentences. The term comes from the fact that if $f \in Asso$, then $|$ is associative with respect to every structure $\langle M, f \rangle$. A kind of converse holds also: If $|$ is associative with respect to $\langle M, f \rangle$, then f is "essentially associative".

In subsection 2.3 we introduce the class Reg of *regular* choice functions, as well as the finer class $Reg^* = Reg \cap Asso$. Regularity guarantees that the truth relation $\models_{Reg}$, as well as $\models_{Reg^*}$, is "logically closed", that is, for any subsentence σ of φ and any $\sigma' \sim_{Reg} \sigma$, $\varphi[\sigma'/\sigma]$ and φ are equivalent in $\langle M, f \rangle$, with $f \in Reg$.

In subsection 2.4 we introduce the even finer class Dec of $\neg$-*decreasing* regular associative functions, that is, $Dec \subset Reg^*$. A total ordering of $Sen(L) <$ is $\neg$-decreasing if and only if for all $\alpha, \beta, \alpha < \beta \Leftrightarrow \neg\beta < \neg\alpha$. f is $\neg$-decreasing if and only if $f = \min_<$ for some $\neg$-decreasing total ordering $<$. The existence of $\neg$-decreasing regular total orderings of $Sen(L)$ is shown and a syntactic characterization of $\neg$-decreasingness is given.

In subsection 2.5 we consider the dual connective $\varphi \circ \psi := \neg(\neg\varphi | \neg\psi)$ of

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| and show that it commutes with | if and only the choice functions involved are ¬-decreasing.

Section 3 is devoted to the axiomatization of Propositional Superposition Logic(s) (PLS). In the general section we give axiomatizations for the logics based on the sets of choice functions $\mathcal{F}$, Reg, Reg^* and Dec. In general for every set $X \subseteq \mathcal{F}$ of choice functions and every set $K \subseteq Taut(X)$ of tautologies with respect to the truth notion $\models_X$, a logic $\mathrm{PLS}(X, K)$ is defined, whose axioms are those of PL plus K and its semantics is the relation $\models_X$. Within $\mathrm{PLS}(X, K)$ K-consistency is defined and Soundness Theorem is proved for every logic $\mathrm{PLS}(X, K)$ with $K \subseteq Taut(X)$. Next we introduce specific axiomatizations (by finitely many schemes of axioms) K_0, K_1, K_2, K_3 for the truth relations defined by the classes $\mathcal{F}$, Reg, Reg^* and Dec, respectively. The logics $\mathrm{PLS}(\mathcal{F}, K_0)$, $\mathrm{PLS}(Reg, K_1)$, $\mathrm{PLS}(Reg^*, K_2)$, $\mathrm{PLS}(Dec, K_3)$ are sound as a consequence of the previous general fact. There exists an essential difference between the axiomatization of $\mathcal{F}$, and those of the rest systems Reg, Reg^* and Dec. The difference consists in that K_1-K_3 contain an extra inference rule (besides Modus Ponens) because of which the truth of the Deduction Theorem (DT) is open. This has serious effects on the completeness of the systems based on K_1-K_3. So we split the examination of completeness for $\mathrm{PLS}(\mathcal{F}, K_0)$ on the one hand and for the rest systems on the other.

In subsection 3.1 we prove the (unconditional) completeness of the system $\mathrm{PLS}(\mathcal{F}, K_0)$.

In subsection 3.2 we examine completeness for the logics $\mathrm{PLS}(Reg, K_1)$, $\mathrm{PLS}(Reg^*, K_2)$ and $\mathrm{PLS}(Dec, K_3)$. The possible failure of DT makes it necessary to distinguish between two forms of completeness, CT1 and CT2, which in the lack of DT need not be equivalent. CT1 implies CT2 but the converse is open. Concerning the systems K_1-K_3, we are seeking to prove CT2 rather than CT1. We show that these systems are *conditionally complete* in the sense that each of these systems is CT2-complete if and only if each K_i satisfies a certain extendibility property $cext(K_i)$ saying that every K_i-consistent set of sentences is extended to a K_i-consistent and complete set. This property is trivial for formal systems K satisfying DT, but is open for systems for which DT is open. Assuming that $cext(K_i)$ is true, the proofs of CT2-completeness for the above logics are all variations of the proof of completeness of $\mathrm{PLS}(\mathcal{F}, K_0)$. On the other hand failure of $cext(K_i)$ implies the failure of CT2-completeness of the corresponding system.

In general, the proof of (CT2-)completeness of a logic $\mathrm{PLS}(X, K)$, with $K \subset Taut(X)$, goes roughly as follows: start with a K-consistent and complete set $\Sigma \subset Sen(L_s)$. To prove it is X-verifiable, pick $\Sigma_1 = \Sigma \cap Sen(L)$.

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Then Σ_1 is a consistent and complete set in the sense of PL. So by completeness of the latter there exists a two-valued assignment M such that $M \models \Sigma_1$. Then in order to prove the X-verifiability of Σ, it suffices to define a choice function f such that $f \in X$ and $\langle M, f \rangle \models \Sigma$.

Finally in section 4 we describe briefly two goals for future research, namely, (1) the goal to find alternative semantics for the logics PLS, and (2) to develop a superposition extension of first-order logic (FOL) with an appropriate semantics and complete axiomatization.

2 Semantics of superposition propositional logic based on choice functions

2.1 Definitions and basic facts

Let us fix a propositional language $L = \{p_0, p_1, \ldots\} \cup \{\neg, \wedge\}$, where p_i are symbols of atomic sentences. The other connectives $\vee$, $\rightarrow$, $\leftrightarrow$ are defined as usual in terms of the previous ones.[8] Let $Sen(L)$ denote the set of sentences of L. Throughout M will denote some *truth assignment* for the sentences of L, that is, a mapping $M : Sen(L) \rightarrow \{0, 1\}$ that is defined according to the standard truth tables. For a given $\alpha \in Sen(L)$ we shall use the notation $M \models \alpha$, $M \models \neg\alpha$ instead of $M(\alpha) = 1$ and $M(\alpha) = 0$, respectively, for practical reasons. Namely, below we shall frequently refer to the truth of sentences denoted $\overline{f}(\varphi)$, so it would be more convenient to write $M \models \overline{f}(\varphi)$ than $M(\overline{f}(\varphi)) = 1$.

Let $L_s = L \cup \{|\}$, where $|$ is a new primitive binary logical connective. The set of atomic sentences of L_s are identical to those of L, while the set of sentences of L_s, $Sen(L_s)$, is recursively defined along the obvious steps: If $\varphi, \psi \in Sen(L_s)$, then $\varphi \wedge \psi$, $\varphi|\psi$, $\neg\varphi$ belong to $Sen(L_s)$.

Basic notational convention. To keep track of whether we refer, at each particular moment, to sentences of L or L_s, throughout the letters φ, ψ, σ will denote general sentences of L_s, while the letters α, β, γ will denote sentences of L only. Also we often refer to sentences of L as "classical".

Throughout given a set A we let

$$[A]^2 = \{\{a, b\} : a, b \in A\}$$

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[8]The functions $\overline{f}$ defined below are going to respect classical connectives, and hence classical equivalences, so it makes no difference if, e.g., we define $\varphi \rightarrow \psi$ as $\neg(\varphi \wedge \neg\psi)$ or $\neg(\neg\psi \wedge \phi)$.

denote the set of all 2-element and 1-element subsets of A. We refer to the elements of $[A]^2$ as *pairs* of elements of A. A *choice function* for $[A]^2$ is as usual a mapping $f : [A]^2 \to A$ such that $f(\{a,b\}) \in \{a,b\}$ for every $\{a,b\} \in [A]^2$. To save brackets we write $f(a,b)$ instead of $f(\{a,b\})$. So in particular $f(a,b) = f(b,a)$ and $f(a,a) = a$.[9]

Definition 2.1 Given the language L, a choice function for $[Sen(L)]^2$, the set of pairs of sentences of L, will be referred to as a *choice function for L*.

Let
$$\mathcal{F}(L) = \{f : \ f \text{ is a choice function for } L\}.$$

Throughout we shall write more simply $\mathcal{F}$ instead of $\mathcal{F}(L)$. Below the letters f, g will range over elements of $\mathcal{F}$ unless otherwise stated. In particular for all $\alpha, \beta \in Sen(L)$, we write $f(\alpha, \beta)$ instead of $f(\{a,b\})$ so $f(\alpha, \beta) = f(\beta, \alpha)$ and $f(\alpha, \alpha) = \alpha$.

Definition 2.2 Let f be a choice function for L. Then f generates a *collapsing function* $\overline{f} : Sen(L_s) \to Sen(L)$ defined inductively as follows:
  (i) $\overline{f}(\alpha) = \alpha$ for every $\alpha \in Sen(L)$.
  (ii) $\overline{f}(\varphi \wedge \psi) = \overline{f}(\varphi) \wedge \overline{f}(\psi)$.
  (iii) $\overline{f}(\neg \varphi) = \neg \, \overline{f}(\varphi)$.
  (iv) $\overline{f}(\varphi | \psi) = f(\overline{f}(\varphi), \overline{f}(\psi))$.

Remarks 2.3 (i) Since the connectives $\vee$ and $\to$ are defined in terms of $\neg$ and $\wedge$, $\overline{f}$ commutes also with respect to them, that is,
$\overline{f}(\varphi \vee \psi) = \overline{f}(\varphi) \vee \overline{f}(\psi)$.
$\overline{f}(\varphi \to \psi) = \overline{f}(\varphi) \to \overline{f}(\psi)$.
  (ii) The crucial clause of the definition is of course (iv). It says that for any sentences φ, ψ, $\overline{f}(\varphi | \psi)$ is a *choice* from the set $\{\overline{f}(\varphi), \overline{f}(\psi)\}$. In particular, for classical sentences α, β we have

$$\overline{f}(\alpha | \beta) = f(\alpha, \beta), \tag{2}$$

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[9]The claim of the existence of a choice function for the set $[A]^2$, for every set A, is a weak form of the axiom of choice (AC), denoted C_2 in [7]. In general for every $n \in \mathbb{N}$, C_n denotes the principle that every set of n-element sets has a choice function. The interested reader may consult [7, section 7.4] for various interrelations between such principles, as well as with the Axiom of Choice for Finite Sets (saying that every nonempty set of nonempty finite sets has a choice function). See also Remark 2.16 below.

Definition 2.4 (Main Truth definition). Let M be a truth assignment for L, f a choice function for L and $\overline{f} : Sen(L_s) \to Sen(L)$ be the corresponding collapsing function. The *truth relation* $\models_s$ *between the pair* $\langle M, f \rangle$ *and a sentence* φ *of* L_s is defined as follows:

$$\langle M, f \rangle \models_s \varphi \text{ iff } M \models \overline{f}(\varphi).$$

More generally, for a set $\Sigma \subset Sen(L_s)$ we write $\langle M, f \rangle \models_s \Sigma$, if $\langle M, f \rangle \models_s \varphi$ for every $\varphi \in \Sigma$.

The following facts are easy consequences of the preceding definitions.

Fact 2.5 *(i) The truth relation $\models_s$ extends the Boolean one $\models$, that is for every $\alpha \in Sen(L)$, and every $\langle M, f \rangle$, $\langle M, f \rangle \models_s \alpha \Leftrightarrow M \models \alpha$.*

(ii) $\models_s$ is a bivalent notion of truth, that is for every $\langle M, f \rangle$ and every sentence φ, either $\langle M, f \rangle \models_s \varphi$ or $\langle M, f \rangle \models_s \neg\varphi$.

(iii) For every sentence φ of L_s, every structure M and every collapsing function $\overline{f}$, $\langle M, f \rangle \models_s \varphi|\varphi$ if and only if $\langle M, f \rangle \models_s \varphi$.

(iv) For all $\varphi, \psi \in Sen(L_s)$, M and f, $\langle M, f \rangle \models_s \varphi|\psi$ if and only if $\langle M, f \rangle \models_s \psi|\varphi$.

Proof. (i) Immediate from the fact that by clause (i) of 2.2, $\overline{f}(\alpha) = \alpha$ for every sentence $\alpha \in Sen(L)$. Thus $\langle M, f \rangle \models_s \alpha$ if and only if $M \models \alpha$.

(ii) Let $\langle M, f \rangle \not\models_s \varphi$. Then $M \not\models \overline{f}(\varphi)$, that is, $M \models \neg\overline{f}(\varphi)$. By clause (iii) of 2.2, $\neg\overline{f}(\varphi) = \overline{f}(\neg\varphi)$, so $\langle M, f \rangle \not\models_s \varphi$ implies $M \models \overline{f}(\neg\varphi)$, or $\langle M, f \rangle \models_s \neg\varphi$.

(iii): By definition 2.1, $f(\alpha, \alpha) = \alpha$, for every α. Therefore $\langle M, f \rangle \models_s \varphi|\varphi \Leftrightarrow \mathcal{M} \models f(\overline{f}(\varphi), \overline{f}(\varphi)) \Leftrightarrow \mathcal{M} \models \overline{f}(\varphi) \Leftrightarrow \langle M, f \rangle \models_s \varphi$.

(iv): By 2.1 again $f(\alpha, \beta) = f(\beta, \alpha)$. So $\langle M, f \rangle \models_s \varphi|\psi \Leftrightarrow M \models f(\overline{f}(\varphi), \overline{f}(\psi)) \Leftrightarrow M \models f(\overline{f}(\psi), \overline{f}(\varphi)) \Leftrightarrow \langle M, f \rangle \models_s \psi|\varphi.$ $\dashv$

Let $\Sigma \models \alpha$, $\alpha \models \beta$, (for $\Sigma \subset Sen(L)$), and $\alpha \sim \beta$ denote the classical logical consequence and logical equivalence relations, respectively, for classical sentences. These are extended to the relations $\varphi \models_s \psi$, $\Sigma \models_s \varphi$ (for $\Sigma \subset Sen(L_s)$), and $\varphi \sim_s \psi$ for L_s-sentences as follows.

Definition 2.6 Let $\Sigma \subset Sen(L)$, $\varphi, \psi \in Sen(L_s)$. We say that φ is an *s-logical consequence* of Σ, denoted $\Sigma \models_s \varphi$, if for every structure $\langle M, f \rangle$, $\langle M, f \rangle \models_s \Sigma$ implies $\langle M, f \rangle \models_s \varphi$. In particular we write $\varphi \models_s \psi$ instead of $\{\varphi\} \models_s \psi$. We say that φ and ψ are *s-logically equivalent*, denoted $\varphi \sim_s \psi$, if for every $\langle M, f \rangle$,

$$\langle M, f \rangle \models_s \varphi \Leftrightarrow \langle M, f \rangle \models_s \psi.$$

Finally, φ is an *s-tautology,* denoted $\models_s \varphi$, if $\langle M, f\rangle \models_s \varphi$ for every $\langle M, f\rangle$.

Fact 2.7 *(i) $\varphi \models_s \psi$ if and only if $\models_s \varphi \to \psi$.*

(ii) $\varphi \sim_s \psi$ if and only if $\models_s \varphi \leftrightarrow \psi$.

(iii) For $\alpha, \beta \in Sen(L)$, $\alpha \models_s \beta$ if and only if $\alpha \models \beta$ and $\alpha \sim_s \beta$ if and only if $\alpha \sim \beta$.

(iv) $\varphi \sim_s \psi$ if and only if for all choice functions f, $\overline{f}(\varphi) \sim \overline{f}(\psi)$.

(v) Let $\alpha(p_1, \ldots, p_n)$ be a sentence of L, made up by the atomic sentences $p_1, \ldots, p_n$, let $\psi_1, \ldots, \psi_n$ be any sentences of L_s and let $\alpha(\psi_1, \ldots, \psi_n)$ be the sentence resulting from α if we replace each p_i by ψ_i. Then:

$$\models \alpha(p_1, \ldots, p_n) \Rightarrow \models_s \alpha(\psi_1, \ldots, \psi_n).$$

(vi) For all φ, ψ, $\varphi|\varphi \sim_s \varphi$ and $\varphi|\psi \sim_s \psi|\varphi$.

(vii) If $\varphi \sim_s \psi$, then $\varphi|\psi \sim_s \varphi$.

Proof. (i): Let $\varphi \models_s \psi$. It means that for every $\langle M, f\rangle$, $\langle M, f\rangle \models_s \varphi$ implies $\langle M, f\rangle \models_s \psi$. Equivalently, $M \models \overline{f}(\varphi)$ implies $M \models \overline{f}(\psi)$, or $M \models (\overline{f}(\varphi) \to \overline{f}(\psi))$, or $M \models \overline{f}(\varphi \to \psi)$. It means that for every M and every $\overline{f}$, $\langle M, f\rangle \models_s \varphi \to \psi$. Thus $\models_s \varphi \to \psi$. The converse is similar.

(ii) and (iii) follow from (i).

(iv): Note that $\varphi \sim_s \psi$ holds if and only if for all $\langle M, f\rangle$, $\langle M, f\rangle \models_s \varphi$ if and only if $\langle M, f\rangle \models_s \psi$, or equivalently, $M \models \overline{f}(\varphi)$ if and only if $M \models \overline{f}(\psi)$. But this means that for every f, $\overline{f}(\varphi) \sim \overline{f}(\psi)$.

(v): Suppose $\models \alpha(p_1, \ldots, p_n)$. For any choice function f, clearly

$$\overline{f}(\alpha(\psi_1, \ldots, \psi_n)) = \alpha(\overline{f}(\psi_1), \ldots, \overline{f}(\psi_n)),$$

since α is classical and $\overline{f}$ commutes with standard connectives. Moreover $\models \alpha(\overline{f}(\psi_1), \ldots, \overline{f}\psi_n)$, since by assumption $\models \alpha(p_1, \ldots, p_n)$ and $\overline{f}(\psi_i)$ are standard sentences. Thus $M \models \alpha(\overline{f}(\psi_1), \ldots, \overline{f}(\psi_n))$, for every M, or $M \models \overline{f}(\alpha(\psi_1, \ldots, \psi_n))$. It means that $\langle M, f\rangle \models_s \alpha(\psi_1, \ldots, \psi_n)$ for every structure $\langle M, f\rangle$, so $\models_s \alpha(\psi_1, \ldots, \psi_n)$.

(vi) This follows immediately from clauses (iii) and (iv) of Fact 2.5.

(vii) Let $\varphi \sim_s \psi$ and let $\langle M, f\rangle \models \varphi|\psi$. Then $M \models f(\overline{f}(\varphi), \overline{f}(\psi))$. By clause (iv) above, $\overline{f}(\varphi) \sim \overline{f}(\psi)$ since $\varphi \sim_s \psi$. Therefore whatever the choice of f would be between $\overline{f}(\varphi)$ and $\overline{f}(\psi)$, we shall have $M \models \overline{f}(\varphi)$. Thus $\langle M, f\rangle \models_s \varphi$. $\dashv$

The following *interpolation property* of the new semantics is perhaps the most striking one. Notice that it holds for the *general choice functions,* not requiring any of the additional conditions to be considered in the subsequent sections.

Theorem 2.8 *For all $\varphi, \psi \in Sen(L_s)$,*

$$\varphi \wedge \psi \models_s \varphi|\psi \models_s \varphi \vee \psi,$$

while in general

$$\varphi \vee \psi \not\models_s \varphi|\psi \not\models_s \varphi \wedge \psi.$$

Proof. Assume $\langle M, f \rangle \models_s \varphi \wedge \psi$. Then $M \models \overline{f}(\varphi) \wedge \overline{f}(\psi)$, that is, $M \models \overline{f}(\varphi)$ and $M \models \overline{f}(\psi)$. But then, whatever f would choose from $\{\overline{f}(\varphi), \overline{f}(\psi)\}$, it would be true in M, that is, $M \models f(\overline{f}(\varphi), \overline{f}(\psi))$. This exactly means that $\langle M, f \rangle \models_s \varphi|\psi$. Therefore $\varphi \wedge \psi \models_s \varphi|\psi$.

On the other hand, if $\langle M, f \rangle \models_s \varphi|\psi$ then $M \models f(\overline{f}(\varphi), \overline{f}(\psi))$. If $f(\overline{f}(\varphi), \overline{f}(\psi)) = \overline{f}(\varphi)$, then $M \models \overline{f}(\varphi)$. If $f(\overline{f}(\varphi), \overline{f}(\psi)) = \overline{f}(\psi)$, then $M \models \overline{f}(\psi)$. So either $M \models \overline{f}(\varphi)$ or $M \models \overline{f}(\psi)$. Therefore $M \models \overline{f}(\varphi) \vee \overline{f}(\psi)$. But clearly $\overline{f}(\varphi) \vee \overline{f}(\psi) = \overline{f}(\varphi \vee \psi)$, since $\overline{f}$ commutes with all standard connectives. Thus $M \models \overline{f}(\varphi \vee \psi)$, or equivalently, $\langle M, f \rangle \models_s \varphi \vee \psi$. Therefore $\varphi|\psi \models_s \varphi \vee \psi$.

To see that the converse relations are false, pick $\alpha \in Sen(L)$ and a truth assignment M such that $M \models \alpha$. Pick also a choice function for L such that $f(\alpha, \neg\alpha) = \neg\alpha$. Since $\overline{f}(\alpha \vee \neg\alpha) = \alpha \vee \neg\alpha$, $\langle M, f \rangle \models_s \alpha \vee \neg\alpha$. On the other hand, $M \not\models \neg\alpha$ implies $M \not\models f(\alpha, \neg\alpha)$, thus $\langle M, f \rangle \not\models_s \alpha|\neg\alpha$. Therefore $\alpha \vee \neg\alpha \not\models_s \alpha|\neg\alpha$. Similarly, if M, α are as before, but we take a choice function g such that $g(\alpha, \neg\alpha) = \alpha$, then $\langle M, g \rangle \models_s \alpha|\neg\alpha$, while $\langle M, g \rangle \not\models_s \alpha \wedge \neg\alpha$. So $\alpha|\neg\alpha \not\models_s \alpha \wedge \neg\alpha$. $\dashv$

Corollary 2.9 *If α is neither a tautology nor a contradiction, then $\alpha|\neg\alpha$ is neither an s-tautology nor an s-contradiction.*

Proof. If α is as stated, then by the proof of Theorem 2.8 $\alpha|\neg\alpha$ is strictly interpolated between $\alpha \wedge \neg\alpha$ and $\alpha \vee \neg\alpha$. $\dashv$

In the semantics $\models_s$ used above, arbitrary choice functions for L are allowed to participate. This practically means that for any pair $\{\alpha, \beta\}$, f may pick an element from $\{\alpha, \beta\}$ quite randomly, e.g. by tossing a coin. However, if we want $\models_s$ to support additional properties of $|$, we must refine $\models_s$ by imposing extra conditions to the choice functions. Such a refinement can be defined in a general manner as follows.

Definition 2.10 For every $\emptyset \neq X \subseteq \mathcal{F}$, define the *X-logical consequence relation* $\models_X$ and the *X-logical equivalence relation* $\sim_X$ as follows: $\Sigma \models_X \varphi$

if and only if for every truth assignment M for L and every $f \in X$,

$$\langle M, f \rangle \models_s \Sigma \;\Rightarrow\; \langle M, f \rangle \models_s \varphi.$$

Also $\varphi \sim_X \psi$ if and only if $\varphi \models_X \psi$ and $\psi \models_X \varphi$.

[The purpose of condition $X \neq \emptyset$ is to block trivialities. For if $X = \emptyset$, we vacuously have $\varphi \models_\emptyset \psi$ and $\varphi \sim_\emptyset \psi$ for all $\varphi, \psi \in Sen(L_s)$. So all sets X, $Y \subseteq \mathcal{F}$ referred to below are assumed to be $\neq \emptyset$.]

Using the above notation, the relations $\models_s$ and $\sim_s$ are alternatively written $\models_\mathcal{F}$, $\sim_\mathcal{F}$, respectively.

The following simple fact reduces $\sim_X$ to the standard $\sim$.

Lemma 2.11 *For every $X \subseteq \mathcal{F}$, and any $\varphi, \psi \in Sen(L_s)$,*

$$\varphi \sim_X \psi \;\Leftrightarrow\; (\forall f \in X)(\overline{f}(\varphi) \sim \overline{f}(\psi)).$$

Proof. By definition, $\varphi \sim_X \psi$ if for every M and every $f \in X$,

$$\langle M, f \rangle \models_s \varphi \;\Leftrightarrow\; \langle M, f \rangle \models_s \psi,$$

or, equivalently, if for all M and $f \in X$,

$$M \models \overline{f}(\varphi) \;\Leftrightarrow\; M \models \overline{f}(\psi).$$

The latter is true if and only if for all $f \in X$, $\overline{f}(\varphi) \sim \overline{f}(\psi)$. $\quad\dashv$

The next properties are easy to verify.

Fact 2.12 *For every $X, Y \subseteq \mathcal{F}$:*
    (i) $\varphi \models_X \psi$ if and only if $\models_X \varphi \to \psi$.
    (ii) $\varphi \sim_X \psi$ if and only if $\models_X \varphi \leftrightarrow \psi$.
    (iii) If $X \subseteq Y$, then $\models_Y \subseteq \models_X$ and $\sim_Y \subseteq \sim_X$.
    (iv) The restriction of $\sim_X$ to classical sentences coincides with $\sim$, that is, for all $\alpha, \beta \in Sen(L)$,

$$\alpha \sim_X \beta \;\Leftrightarrow\; \alpha \sim \beta.$$

Before closing this section I should give credit to [6] for some notions introduced above. It was not until one of the referees drew my attention to [6], when I learned (with surprise) that the notion of choice function for pairs of formulas, and, essentially, the germ of the satisfaction relation defined in 2.4 above, were not entirely new but had already been defined independently with some striking similarities in the style of presentation. In fact in Example 3.24.14, p. 479, of [6] we read:

"By a *pair selection function* on a set U we mean a function f such that for all $a, b \in U$, $f(\{a, b\}) \in \{a, b\}$. We write $f(a, b)$ for '$f(\{a, b\}$' and include the possibility that $a = b$ in which case $f(a, b) = a = b$. (...) A pair selection function is accordingly a commutative idempotent binary operation which is in addition a *quasi-projection* or a *conservative* operation, meaning that its value for a given pair of arguments is always one of those arguments. For the current application consider f as a pair selection function on the set of formulas of the language generated from the actual stock of propositional variables with the aid of the binary connective $\circ$. Consider the gcr (=generalized consequence relation) determined by the class of all valuations v satisfying the condition that for some pair selection function f we have: For all formulas A, B, $v(A \circ B) = v(f(A, B))$. Then, if $\succ$ denotes this gcr, it satisfies the rules: (I) $A, B \succ A \circ B$, (II) $A \circ B \succ A, B$ and (IV) $A \circ B \succ B \circ A$."

Note that rules (I) and (II) are essentially the "interpolation property" of Theorem 2.8, while rule (IV) is the commutativity property (vi) of Fact 2.7.

In the next section we consider a first natural subclass $X \subset \mathcal{F}$, the class of *associative* choice functions. These are precisely the functions that support the associativity property of the connective $|$. Clearly associativity is a highly desirable property from an *algebraic* point of view. However, as one of the referees interestingly observed at this point, we must distinguish between what is algebraically desirable and what is quantum mechanically desirable, i.e., close to the real behavior of a quantum system. In his view, classes of choice functions with not very attractive and smooth properties might also deserve to be isolated and scrutinized.

2.2 Associative choice functions

By clause (vi) of Fact 2.7, $\varphi|\varphi \sim_s \varphi$ and $\varphi|\psi \sim_s \psi|\varphi$. These two properties, idempotence and commutativity up to logical equivalence, are in accord with the intended intuitive meaning of the operation $|$. Another desirable property that is in accord with the meaning of $|$ is *associativity*, that is, the logical equivalence of $(\varphi|\psi)|\sigma$ and $\varphi|(\psi|\sigma)$. Is it true with respect to $\sim_s$? The answer is: not in general. In order to ensure it we need to impose a certain condition on the choice functions. The specific condition does not depend on the nature of elements of $Sen(L)$, so we prove it below in a general setting.

Let A be an infinite set and let $f : [A]^2 \to A$ be a choice for pairs of elements of A. One might extend it to the set $[A]^3$, of nonempty sets with at most 3 elements, by setting, say,

$$f(a, b, c) := f(\{a, b, c\}) = f(f(a, b), c).$$

But this does not guarantee that $f(a, b, c) = f(b, c, a) = f(c, b, a)$, etc, as it would be obviously required, unless $f(f(a, b), c) = f(a, f(b, c))$ for all a, b, c. This is exactly the required condition.

Definition 2.13 Let f be a choice function for $[A]^2$. f is said to be *associative* if for all $a, b, c \in A$,

$$f(f(a, b), c) = f(a, f(b, c)).^{10}$$

We show below that associative choice functions on $[A]^2$ are, essentially, the functions $\min_<$, where $<$ is a total ordering of A. I do not know if the next theorem is new or a known result. In any case I couldn't find a proof in the current bibliography.

Theorem 2.14 *(i) If $<$ is a total ordering on A, then the mapping $\min_<(a, b)$ from $[A]^2$ into A is associative.*
(ii) Conversely, if $f : [A]^2 \to A$ is an associative choice function, then it defines a total ordering $<$ on A such that for all $a, b \in A$, $f(a, b) = \min_<(a, b)$.

Proof. (i) Let $<$ be a total ordering of A. Let $Fin(A)$ denote the set of all nonempty finite subsets of A and let $\min_<$ be the function picking the $<$-least element of x for every $x \in Fin(A)$. Let us write min instead of $\min_<$. Obviously min is a choice function for $Fin(A)$. In particular, for all $a, b, c \in A$,

$$\min(a, b, c) = \min(\min(a, b), c) = \min(a, \min(b, c)).$$

Thus min restricted to $[A]^2$ is associative.
(ii) Let $f : [A]^2 \to A$ be an associative choice function. Define the relation $<$ on A as follows: For any $a, b \in A$, let $a < b$ if and only if $a \neq b$ and $f(a, b) = a$. Obviously $<$ is total and anti-reflexive (that is, $a \not< a$ for every $a \in A$). Thus in order for $<$ to be a total ordering it suffices to be also

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[10]If we write $a \star b$ instead of $f(a, b)$, then the condition $f(f(a, b), c) = f(a, f(b, c))$ is rewritten $(a \star b) \star c = a \star (b \star c)$, which justifies the term "associative".

transitive. Let $a < b$ and $b < c$. We show that $a < c$. By the assumptions, we have $a \neq b$, $f(a, b) = a$, $b \neq c$ and $f(b, c) = b$. It follows from them that $a \neq c$, for otherwise $b = f(b, c) = f(b, a) = f(a, b) = a$, a contradiction. It remains to show that $f(a, c) = a$. By associativity and commutativity of f, $f(a, f(b, c)) = f(b, f(a, c))$. Since $f(a, f(b, c)) = f(a, b) = a$, it follows that $f(b, f(a, c)) = a$ too. If $f(a, c) = c$, then we would have $f(b, f(a, c)) = f(b, c) = b \neq a$, a contradiction. Therefore $f(a, c) = a$ and we are done. Thus $<$ is a total ordering of A, and by definition $f(a, b) = \min_<(a, b)$, for all $a, b \in A$. $\dashv$

As an immediate corollary of Theorem 2.14 we obtain the following.

Corollary 2.15 *If $f : [A]^2 \to A$ is an associative choice function, then it defines uniquely a total ordering $<$ of A such that $f(a, b) = \min_<(a, b)$. Therefore f extends uniquely to the choice function $f^+ : Fin(A) \to A$, such that for every $x \in Fin(A)$, $f^+(x) = \min_<(x)$. Thus $f = f^+{\restriction}[A]^2$.*

In view of the preceding Corollary, we can without serious loss of precision identify an associative choice function f on $[A]^2$ with the generated choice function f^+ on the entire $Fin(A)$, and write $f = \min_<$ instead of $f^+ = \min_<$, where $<$ is the ordering defined by f.

Remark 2.16 From a set-theoretical point of view, the existence of an associative function is a much stronger statement than the existence of a simple choice function for $[A]^2$. As noticed in footnote 9 the latter is identical to the principle C_2. On the other hand, it follows from 2.14 and 2.15 that the existence of an associative choice function for $[A]^2$, for every set A, is equivalent to the existence of a total ordering on A, i.e., to the *Ordering Principle* saying that "Every set can be totally ordered", which is much stronger than C_2. Specifically, it was shown in [9] that the Ordering Principle is strictly stronger than the Axiom of Choice for Finite Sets. The latter is in turn strictly stronger than the conjunction of all axioms C_n, for $n \geq 2$ (see [7, Theorem 7.11]).

Let us now return to the set $Sen(L)$ of sentences of L. In particular a choice function f for L is said to be *associative* if for all $\alpha, \beta, \gamma \in Sen(L)$,

$$f(f(\alpha, \beta), \gamma) = f(\alpha, f(\beta, \gamma)). \tag{3}$$

We often call also the pair $\langle M, f \rangle$ *associative* if f is associative. As an immediate consequence of Theorem 2.14, Corollary 2.15 and the comments following the latter we have the following.

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Corollary 2.17 *A choice function f for L is associative if and only if there is a total ordering $<$ of $Sen(L)$ such that $f = \min_<$.*

Let
$$Asso = \{f \in \mathcal{F} : f \text{ is associative }\},$$
denote the set of all associative choice functions for L. We shall also abbreviate the logical consequence relation $\models_{Asso}$ and the logical equivalence relation $\sim_{Asso}$ that are induced by $Asso$ (see definition 2.10 of the previous section), by $\models_{Asso}$ and $\sim_{Asso}$, respectively. From the general facts 2.12 (iii) we immediately obtain the following.

Fact 2.18 *For all φ, ψ, Σ,*
   (i) $\Sigma \models_s \varphi \;\Rightarrow\; \Sigma \models_{Asso} \varphi$.
   (ii) $\varphi \sim_s \psi \;\Rightarrow\; \varphi \sim_{Asso} \psi$.

It is easy to verify that the arrows in the preceding Fact cannot in general be reversible. The main consequence of associativity is the following.

Theorem 2.19 *Let $X \subseteq \mathcal{F}$ be a class of choice functions. If $X \subseteq Asso$, then $|$ is associative with respect to the truth notion $\models_X$, that is, for all φ, ψ, σ, $\varphi|(\psi|\sigma) \sim_X (\varphi|\psi)|\sigma$.*

   Proof. Let $X \subseteq Asso$. It suffices to show that for every M and every $f \in X$, and any sentences φ, ψ, σ of L_s, $\langle M, f \rangle$,

$$\langle M, f \rangle \models_s (\varphi|\psi)|\sigma \text{ iff } \langle M, f \rangle \models_s \varphi|(\psi|\sigma).$$

Fix some M and some $f \in X$. By definition we have:

$$\langle M, f \rangle \models_s (\varphi|\psi)|\sigma \Leftrightarrow M \models f(\overline{f}(\varphi|\psi), \overline{f}(\sigma)) \Leftrightarrow M \models f(f(\overline{f}(\varphi), \overline{f}(\psi)), \overline{f}(\sigma)).$$

By assumption $f \in Asso$, so by the associativity property (3)

$$f(f(\overline{f}(\varphi), \overline{f}(\psi)), \overline{f}(\sigma)) = f(\overline{f}(\varphi), f(\overline{f}(\psi), \overline{f}(\sigma))).$$

Thus,

$$\langle M, f \rangle \models_s (\varphi|\psi)|\sigma \Leftrightarrow M \models f(\overline{f}(\varphi), \overline{f}(\psi|\sigma)) \Leftrightarrow \langle M, f \rangle \models_s \varphi|(\psi|\sigma).$$

$$\dashv$$

If we slightly weaken the property of associativity, the converse of 2.19 holds too.

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Definition 2.20 Let us call a choice function for L *essentially associative,* if (3) holds with $\sim$ in place of $=$, that is, for all $\alpha, \beta, \gamma \in Sen(L)$,

$$f(f(\alpha, \beta), \gamma) \sim f(\alpha, f(\beta, \gamma)). \tag{4}$$

Let $Asso'$ denote the class of essentially associative choice functions.

Theorem 2.21 *If $X \subseteq \mathcal{F}$ and $\mid$ is associative with respect to $\models_X$, then $X \subseteq Asso'$.*

Proof. Let $X \not\subseteq Asso'$. We have to show that $\mid$ is not associative with respect to $\models_X$. Pick $f \in X - Asso'$. It suffices to find M and α, β, γ in $Sen(L)$ such that

$$\langle M, f \rangle \models_s (\alpha|\beta)|\gamma \not\Leftrightarrow \langle M, f \rangle \models_s \alpha|(\beta|\gamma).$$

Since f is not essentially associative, there are α, β, γ in $Sen(L)$, such that $f(f(\alpha, \beta), \gamma) \not\sim f(\alpha, f(\beta, \gamma))$. Without loss of generality we may assume that

$$f(f(\alpha, \beta), \gamma) = \gamma \not\sim \alpha = f(\alpha, f(\beta, \gamma)).$$

Since $\alpha \not\sim \gamma$ there is M such that $M \models \alpha \wedge \neg\gamma$ or $M \models \neg\alpha \wedge \gamma$. Without loss of generality assume that the first is the case. Then $M \models \alpha = f(\alpha, f(\beta, \gamma))$, so $M \models \overline{f}(\alpha|(\beta|\gamma))$, therefore $\langle M, f \rangle \models_s \alpha|(\beta|\gamma)$. On the other hand $M \not\models \gamma = f(f(\alpha, \beta), \gamma)$, which implies $\langle M, f \rangle \not\models_s (\alpha|\beta)|\gamma$. This proves the theorem. $\dashv$

Obviously $Asso \subseteq Asso'$. Are the two classes distinct? The answer is yes, but the functions in $Asso' - Asso$ behave *non-associatively* only on sentences α, β, γ such that $\alpha \sim \beta \sim \gamma$. To be precise, let us say that a triple of sentences α, β, γ *witnesses non-associativity* of f, if $f(f(\alpha, \beta), \gamma) \neq f(\alpha, f(\beta, \gamma))$, or $f(f(\alpha, \beta), \gamma) \neq f(\beta, f(\alpha, \gamma))$, or $f(f(\alpha, \gamma), \beta) \neq f(\alpha, f(\beta, \gamma))$. Then the following holds.

Lemma 2.22 *(i) $Asso \subsetneq Asso'$.*

(ii) If $f \in Asso'$ and α, β, γ are sentences such that $\alpha \not\sim \beta$, $\beta \not\sim \gamma$ and $\alpha \not\sim \gamma$, then f is associative on α, β, γ, i.e., $f(f(\alpha, \beta), \gamma) = f(\alpha, f(\beta, \gamma)) = f(f(\alpha, \gamma), \beta)$.

(iii) If $f \in Asso' - Asso$, and α, β, γ witness the non-associativity of f, then α, β, γ are all distinct, and besides $f(\alpha, \beta)$, $f(\alpha, \gamma)$, $f(\beta, \gamma)$ are all distinct.

(iv) Therefore, if $f \in Asso' - Asso$ and α, β, γ witness the non-associativity of f, then $\alpha \sim \beta \sim \gamma$.

(v) Further, if $f \in Asso' - Asso$, then f is associative on every triple α, β, γ such that $\alpha \sim \beta \not\sim \gamma$.

Proof. (i) Let $\alpha \sim \beta \sim \gamma$, while all α, β, γ are distinct. Let $f \in \mathcal{F}$ be such that $f(\alpha, \beta) = \beta$, $f(\alpha, \gamma) = \alpha$, $f(\beta, \gamma) = \gamma$. Then obviously all $f(f(\alpha, \beta), \gamma)$, $f(f(\alpha, \gamma), \beta)$, $f(f(\beta, \gamma), \alpha)$ are equivalent, so $f \in Asso'$. On the other hand, for example, $f(f(\alpha, \beta), \gamma) \neq f(\alpha, f(\beta, \gamma))$, so $f \notin Asso$.

(ii) If $\alpha \not\sim \beta$, $\beta \not\sim \gamma$, $\alpha \not\sim \gamma$, then it cannot be $f(f(\alpha, \beta), \gamma) \sim f(\alpha, f(\beta, \gamma))$ unless $f(f(\alpha, \beta), \gamma) = f(\alpha, f(\beta, \gamma))$.

(iii) It is easy to see that for every $f \in \mathcal{F}$ and every α, β, $f(f(\alpha, \beta), \alpha) = f(f(\alpha, \beta), \beta)$. This shows that if any two elements of a triple α, β, γ are equal, this triple cannot witness the non-associativity of any function. Let $f \in Asso' - Asso$ and suppose α, β, γ witness the non-associativity of f. We have just seen that they are all distinct. We show that $f(\alpha, \beta)$, $f(\alpha, \gamma)$, $f(\beta, \gamma)$ are distinct too. Indeed assume that two of the values $f(\alpha, \beta)$, $f(\alpha, \gamma)$, $f(\beta, \gamma)$ are identical. It will follow that

$$f(f(\alpha, \beta), \gamma) = f(\alpha, f(\beta, \gamma)) = f(f(\alpha, \gamma), \beta), \tag{5}$$

which contradicts the fact that α, β, γ witness the non-associativity of f. Assume without loss of generality that $f(\alpha, \beta) = f(\beta, \gamma)$. Since f is a choice function and α, β, γ are distinct, necessarily $f(\alpha, \beta) = f(\beta, \gamma) = \beta$. Therefore

$$f(f(\alpha, \beta), \gamma) = f(\beta, \gamma) = f(\alpha, \beta) = f(\alpha, f(\beta, \gamma)) = \beta.$$

So two members of (5) are equal. As to the third one, observe that $f(\alpha, \gamma)$ is either α or γ. In both cases $f(f(\alpha, \gamma), \beta) = \beta$, as required.

(iv) Let $f \in Asso' - Asso$ and let α, β, γ witness the non-associativity of f. By (iii) above, $f(\alpha, \beta)$, $f(\alpha, \gamma)$, $f(\beta, \gamma)$ take up all the values α, β, γ, and therefore so do $f(f(\alpha, \beta), \gamma)$, $f(\alpha, f(\beta, \gamma))$, $f(f(\alpha, \gamma), \beta)$. But since $f \in Asso'$, the latter are all logically equivalent. Therefore $\alpha \sim \beta \sim \gamma$.

(v) If $f \in Asso' - Asso$, $\alpha \sim \beta \not\sim \gamma$ and f were not associative on α, β, γ, the latter triple would witness the non-associativity of f, so, by (iv), $\alpha \sim \beta \sim \gamma$. A contradiction. $\dashv$

It follows from the preceding Lemma that every $f \in Asso'$ defines essentially an associative choice function (and hence a total ordering) for the set of pairs of elements of $Sen(L)/\sim = \{[\alpha] : \alpha \in Sen\}$ rather than $Sen(L)$.

By Facts 2.5, 2.18 and Theorem 2.19, we obtain the following.

Corollary 2.23 *The operation $|$ is idempotent, commutative and associative with respect to $\sim_{Asso}$. That is:*

(i) $\varphi|\varphi \sim_{Asso} \varphi$.

(ii) $\varphi|\psi \sim_{Asso} \psi|\varphi.$
(iii) $\varphi|(\psi|\sigma) \sim_{Asso} (\varphi|\psi)|\sigma.$

It follows that *when confined to truth in associative structures,* one can drop parentheses from $(\varphi|\psi)|\sigma$ and write simply $\varphi|\psi|\sigma$ (as with the case of $\wedge$ and $\vee$ of in classical PL), and more generally $\varphi_1|\cdots|\varphi_n$ for any sentences φ_i of L_s.

Moreover, in view of Theorem 2.14 and Corollary 2.23, when the choice function f is associative, then $f = \min_<$ for a total ordering $<$ of $Sen(L)$. Namely, the following is proved by an easy induction:

Corollary 2.24 *Let $\langle M, f \rangle$ be associative, with $f = \min_<$ for a total ordering $<$ of $Sen(L)$. Then for every $n \in \mathbb{N}$ and any $\{\varphi_1, \ldots, \varphi_n\} \subset L_s$,*

$$\langle M, f \rangle \models_s \varphi_1|\cdots|\varphi_n \text{ iff } M \models f(\overline{f}(\varphi_1), \ldots, \overline{f}(\varphi_n)) \text{ iff } M \models \min_<(\overline{f}(\varphi_1), \ldots, \overline{f}(\varphi_n)),$$

where $f(\sigma_1, \ldots, \sigma_n)$ abbreviates $f(\{\sigma_1, \ldots, \sigma_n\})$. In particular, for classical sentences $\alpha_1, \ldots, \alpha_n$,

$$\langle M, f \rangle \models_s \alpha_1|\cdots|\alpha_n \quad \text{iff } M \models \min_<(\alpha_1, \ldots, \alpha_n).$$

2.3 Regularity

For every $\varphi \in Sen(L)$, let $Sub(\varphi)$ denote the set of sub-sentences of φ. Given $\varphi, \sigma \in Sub(\varphi)$ and any σ', let $\varphi[\sigma'/\sigma]$ denote the result of replacing σ by σ' throughout φ.

Definition 2.25 For $X \subseteq \mathcal{F}$, $\sim_X$ is said to be *logically closed* if for all φ, $\sigma \in Sub(\varphi)$ and σ',

$$\sigma \sim_X \sigma' \;\Rightarrow\; \varphi \sim_X \varphi[\sigma'/\sigma].$$

Classical logical equivalence $\sim$ is logically closed of course, but $\sim_s$ and $\sim_{Asso}$ are not in general. The question is what further condition on X is required in order for $\sim_X$ to be logically closed. This is the condition of *regularity* introduced below.

Regularity is a condition independent from associativity, yet compatible with it. So it is reasonable to introduce it independently from associativity.

Definition 2.26 A choice function f for L is said to be *regular* if for all α, α', β,

$$\alpha \sim \alpha' \;\Rightarrow\; f(\alpha, \beta) \sim f(\alpha', \beta).$$

The following properties are immediate consequences of the definition.

Fact 2.27 *Let f be regular. Then for all α, α', β, β':*
(i) If $\alpha \sim \alpha' \not\sim \beta \sim \beta'$ and $f(\alpha, \beta) = \alpha$ then $f(\alpha', \beta') = \alpha'$, while if $f(\alpha, \beta) = \beta$ then $f(\alpha', \beta') = \beta'$.
(ii) If $\alpha \sim \alpha' \sim \beta \sim \beta'$, $f(\alpha, \beta)$ and $f(\alpha', \beta')$ can be any element of the sets $\{\alpha, \beta\}$, $\{\alpha', \beta'\}$, respectively.

Let
$$Reg = \{f \in \mathcal{F} : \ f \text{ is regular}\}$$

denote the set of regular choice functions, and let $\models_{Reg}$, $\sim_{Reg}$ abbreviate the relations $\models_{Reg}$, $\sim_{Reg}$, respectively. Regularity not only guarantees that $\sim_{Reg}$ is logically closed, but also the converse is true.

Theorem 2.28 $\sim_X$ *is logically closed if and only if $X \subseteq Reg$.*

Proof. "$\Leftarrow$" : We assume $X \subseteq Reg$ and show that $\sim_X$ is logically closed. For $\sigma \in Sub(\varphi)$, $\varphi[\sigma'/\sigma]$ is defined by induction on the length of φ as usual, so we prove
$$\sigma \sim_X \sigma' \ \Rightarrow \ \varphi[\sigma'/\sigma] \sim_X \varphi \qquad (6)$$

along the steps of the definition of $\varphi[\sigma'/\sigma]$. Actually regularity is needed only for the treatment of case $\varphi = \varphi_1|\varphi_2$, since $\overline{f}$ commutes with standard connectives, so let us treat this step of the induction only. That is, let $\varphi = \varphi_1|\varphi_2$, so
$$\varphi[\sigma'/\sigma] = \varphi_1[\sigma'/\sigma]|\varphi_2[\sigma'/\sigma],$$

and assume the claim holds for φ_1, φ_2. Let us set for readability, $\varphi' = \varphi[\sigma'/\sigma]$, $\varphi_1' = \varphi_1[\sigma'/\sigma]$, $\varphi_2' = \varphi_2[\sigma'/\sigma]$. Then $\varphi' = \varphi_1'|\varphi_2'$, and by the induction assumption $\varphi_1' \sim_X \varphi_1$, $\varphi_2' \sim_X \varphi_2$. We have to show that $\varphi_1'|\varphi_2' \sim_X \varphi_1|\varphi_2$. Pick $f \in X$. By our assumption $f \in Reg$. Our induction assumptions become
$$\overline{f}(\varphi_1') \sim \overline{f}(\varphi_1) \text{ and } \overline{f}(\varphi_2') \sim \overline{f}(\varphi_2), \qquad (7)$$
and it suffices to show that $\overline{f}(\varphi_1'|\varphi_2') \sim \overline{f}(\varphi_1|\varphi_2)$, or equivalently,
$$f(\overline{f}(\varphi_1'), \overline{f}(\varphi_2')) \sim f(\overline{f}(\varphi_1), \overline{f}(\varphi_2)). \qquad (8)$$

But since $f \in Reg$, in view of Fact 2.27, (8) follows immediately from (7). This completes the proof of direction $\Leftarrow$.
"$\Rightarrow$": Suppose $X \not\subseteq Reg$. We have to show that $\sim_X$ is not logically closed. Pick $f \in X - Reg$. Since f is not regular, there exist α, α' and β in

$Sen(L)$ such that $\alpha \sim \alpha'$ and $f(\alpha,\beta) \not\sim f(\alpha',\beta)$. In particular this implies that $\alpha \not\sim \beta$. Moreover, either $f(\alpha,\beta) = \alpha$ and $f(\alpha',\beta) = \beta$, or $f(\alpha,\beta) = \beta$ and $f(\alpha',\beta) = \alpha'$.

Assume $f(\alpha,\beta) = \alpha$ and $f(\alpha',\beta) = \beta$, the other case being treated similarly. Since $\alpha \not\sim \beta$ there exists a truth assignment M for L such that $M \models \alpha \wedge \neg\beta$ or $M \models \neg\alpha \wedge \beta$. Without loss of generality assume $M \models \alpha \wedge \neg\beta$. Then $M \models f(\alpha,\beta)$ which means $\langle M, f \rangle \models \alpha|\beta$. On the other hand, since $f(\alpha',\beta) = \beta$ and $M \models \neg\beta$, we have $M \models \neg f(\alpha',\beta)$, that is, $M \not\models f(\alpha',\beta)$, which means $\langle M, f \rangle \not\models \alpha'|\beta$. Therefore for some M and some $f \in X$, $\langle M, f \rangle \models \alpha|\beta$ and $\langle M, f \rangle \not\models \alpha'|\beta$. Thus $\alpha'|\beta \not\sim_X \alpha|\beta$, while $\alpha' \sim \alpha$, and hence $\alpha' \sim_X \alpha$. It follows that $\sim_X$ is not logically closed. $\dashv$

In general if $X \subseteq Y \subseteq \mathcal{F}$ and $\sim_X$ is logically closed, it doesn't seem likely that one can infer that $\sim_Y$ is logically closed (or vice-versa). Yet we have the following generalization of 2.28.

Corollary 2.29 *If $X \subseteq Reg \subseteq Y \subseteq \mathcal{F}$, then for all φ, $\sigma \in Sub(\varphi)$ and σ',*

$$\sigma \sim_Y \sigma' \;\Rightarrow\; \varphi[\sigma'/\sigma] \sim_X \varphi.$$

Proof. By Fact 2.12 (iii), $X \subseteq Y$ implies $\sim_Y \subseteq \sim_X$. So given $X \subseteq Reg \subseteq Y$, we have $\sim_Y \subseteq \sim_{Reg} \subseteq \sim_X$. Thus if $\sigma \sim_Y \sigma'$, then $\sigma \sim_{Reg} \sigma'$. By theorem 2.28, this implies $\varphi[\sigma'/\sigma] \sim_{Reg} \varphi$, and therefore $\varphi[\sigma'/\sigma] \sim_X \varphi$. $\dashv$

Using the axiom of choice one can easily construct regular choice functions for L. For every $\alpha \in Sen(L)$, let $[\alpha]$ denote the $\sim$-equivalence class of α, i.e.,

$$[\alpha] = \{\beta : \beta \sim \alpha\}.$$

Proposition 2.30 (AC) *There exist regular choice functions for L.*

Proof. Using AC, pick a representative ξ_α from each equivalence class $[\alpha]$ and let $D = \{\xi_\alpha : \alpha \in Sen(L)\}$. Every $\alpha \in Sen(L)$ is logically equivalent to $\xi_\alpha \in D$. Let $f_0 : [D]^2 \to D$ be an arbitrary choice function for all pairs of elements of D. Then f_0 extends to a regular choice function f for L, defined as follows:

$f(\alpha,\beta) = \alpha$, if $\alpha \not\sim \beta$ and $f_0(\xi_\alpha, \xi_\beta) = \xi_\alpha$. If $\alpha \sim \beta$, we define $f(\alpha,\beta)$ arbitrarily (to be precise, by setting $f(\alpha,\beta) = g(\alpha,\beta)$, where g is a choice function on all pairs $\{\alpha,\beta\}$ of sentences such that $\alpha \sim \beta$). $\dashv$

Next we come to *associative regular* choice functions.

Definition 2.31 A total ordering $<$ of $Sen(L)$ is said to be *regular* if for all α, β,

$$\alpha \not\sim \beta \ \& \ \alpha < \beta \ \Rightarrow \ [\alpha] < [\beta],$$

(where $[\alpha] < [\beta]$ means that for all $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$, $\alpha' < \beta'$).

The following is an immediate consequence of the preceding definitions.

Fact 2.32 *Let $<$ be a total ordering of $Sen(L)$. Then $<$ is regular if and only if the corresponding associative choice function $f = \min_<$ is regular.*

Thus the following simple construction of regular total orderings for $Sen(L)$ supplements Proposition 2.30 above.

Proposition 2.33 (AC) *(i) There exist regular total orderings of $Sen(L)$.*
(ii) Moreover, for any set $A \subset Sen(L)$ of pairwise inequivalent sentences, and any partial ordering R of A, there is a regular total ordering $<$ of $Sen(L)$ such that $R \subseteq <$.

Proof. (i) Let $Sen(L)/\sim$ be the set of equivalence classes $[\alpha]$, $\alpha \in Sen(L)$. For each $[\alpha] \in Sen(L)/\sim$ pick by AC a total ordering $<_{[\alpha]}$ of $[\alpha]$. Pick also a total ordering $<_1$ of $Sen(L)/\sim$. These orderings generate a regular total ordering $<$ of $Sen(L)$ defined by:
$\alpha < \beta$ if and only if $\alpha \not\sim \beta$ and $[\alpha] <_1 [\beta]$, or $\alpha \sim \beta$ and $\alpha <_{[\alpha]} \beta$.
(ii) Since the elements of A are pairwise inequivalent, we can think of A as a subset of $Sen(L)/\sim$. Since R is already a partial ordering of A, it suffices to pick (by the help of AC) the total ordering $<_1$ of the preceding case so that $R \subset <_1$. $\qquad \dashv$

Both associativity and regularity are indispensable for a reasonable notion of truth $\models_X$ that captures the behavior of $|$. This is because on the one hand without associativity one would have to face unmanageable complexity caused by incomparable sentences of the form $((\alpha|\beta)|\gamma)|\delta$, $(\alpha|\beta)|(\gamma|\delta)$, $\alpha|((\beta|\gamma)|\delta)$, etc. On the other hand regularity entails logical closeness, without which one cannot establish even that the sentences, for example, $\alpha|\beta$ and $\alpha|\neg\neg\beta$ are essentially identical. Thus a natural class of choice functions to work with is

$$Reg^* = Reg \cap Asso.$$

We abbreviate the corresponding semantic notions $\models_{Reg^*}$, $\sim_{Reg^*}$, by $\models_{Reg^*}$ and $\sim_{Reg^*}$, respectively.

Note that in view of regularity (and only in view of that) we can write, for example, $\varphi|\top$ and $\varphi|\bot$, where $\top$ and $\bot$ denote the classes of classical tautologies and contradictions, respectively.

The next question is how the standard connectives act on $|$ and vice-versa. Specifically we shall examine whether:

(a) $\neg$ can commute with $|$,

(b) $\wedge$ and $\vee$ can distribute over $|$,

(c) $|$ can distribute over $\wedge$ and $\vee$.

We shall see that all three questions are answered in the negative with respect to the truth relations $\models_{Reg^*}$.

Concerning the first question one can construct a choice function f such that for every truth assignment M and any sentences φ, ψ,

$$\langle M, f \rangle \models_s \neg(\varphi|\psi) \leftrightarrow \neg\varphi|\neg\psi.$$

For that it suffices to define f so that $\overline{f}(\neg\varphi|\neg\psi) = \overline{f}(\neg(\varphi|\psi))$, or equivalently

$$f(\neg\overline{f}(\varphi), \neg\overline{f}(\psi)) = \neg f(\overline{f}(\varphi), \overline{f}(\psi)).$$

This can be done by defining $f(\alpha, \beta)$ by induction along an ordering of the pairs $\langle r(\alpha), r(\beta) \rangle$, where $r(\alpha)$ is the usual rank of α.

Nevertheless, such an f supporting $\neg(\varphi|\psi) \leftrightarrow \neg\varphi|\neg\psi$ would serve just as a counterexample or a curiosity, and could by no means characterize a natural class of functions. Specifically it is easily seen that such an f cannot be regular.

Fact 2.34 *If f is regular then for every M there are φ, ψ such that $\langle M, f \rangle \not\models \neg(\varphi|\psi) \leftrightarrow \neg\varphi|\neg\psi$. Thus for regular f, the scheme $\neg(\varphi|\psi) \leftrightarrow \neg\varphi|\neg\psi$ is always false in $\langle M, f \rangle$.*

Proof. Let f be regular. Then for every α, $f(\neg\alpha, \neg\neg\alpha) \sim f(\neg\alpha, \alpha) = f(\alpha, \neg\alpha)$, therefore $\neg f(\alpha, \neg\alpha) \leftrightarrow f(\neg\alpha, \neg\neg\alpha)$ is a contradiction. Hence for every M, $M \not\models \neg f(\alpha, \neg\alpha) \leftrightarrow f(\neg\alpha, \neg\neg\alpha)$, which means that $M \not\models \overline{f}(\neg(\alpha|\neg\alpha) \leftrightarrow \neg\alpha|\neg\neg\alpha)$, or $\langle M, f \rangle \not\models \neg(\alpha|\neg\alpha) \leftrightarrow \neg\alpha|\neg\neg\alpha$. $\dashv$

Concerning question (b) above the answer is negative with respect to the semantics $\models_X$ for any $X \subseteq Reg^*$. Let us give some definitions with the purpose to prove later that they are void.

Definition 2.35 Let $<$ be a regular total ordering of $Sen(L)$. $<$ is said to be:

(a) $\wedge$-*monotonic, if for all* $\alpha, \beta, \gamma \in Sen(L)$ *such that* $\alpha \wedge \gamma \not\sim \beta \wedge \gamma$,

$$\alpha < \beta \;\Leftrightarrow\; \alpha \wedge \gamma < \beta \wedge \gamma.$$

(b) $\vee$-*monotonic, if for all* $\alpha, \beta, \gamma \in Sen(L)$ *such that* $\alpha \vee \gamma \not\sim \beta \vee \gamma$,

$$\alpha < \beta \;\Leftrightarrow\; \alpha \vee \gamma < \beta \vee \gamma.$$

Accordingly, a choice function $f \in Reg^*$ is said to be $\wedge$-*monotonic* (resp. $\vee$-*monotonic*) if $f = \min_<$ and $<$ is $\wedge$-monotonic (resp. $\vee$-monotonic).

Lemma 2.36 *(i) If $<$ is $\wedge$-monotonic, then*

$$\min(\alpha \wedge \gamma, \beta \wedge \gamma) \sim \gamma \wedge \min(\alpha, \beta).$$

(ii) If $<$ is $\vee$-monotonic, then

$$\min(\alpha \vee \gamma, \beta \vee \gamma) \sim \gamma \vee \min(\alpha, \beta).$$

Proof. (i) If $\alpha \wedge \gamma \sim \beta \wedge \gamma$ then obviously $\min(\alpha \wedge \gamma, \beta \wedge \gamma) \sim \gamma \wedge \min(\alpha, \beta)$. So assume $\alpha \wedge \gamma \not\sim \beta \wedge \gamma$. Then also $\alpha \not\sim \beta$. Without loss of generality suppose $\alpha < \beta$, so $\min(\alpha, \beta) = \alpha$. By $\wedge$-monotonicity, $\alpha \wedge \gamma < \beta \wedge \gamma$, so $\min(\alpha \wedge \gamma, \beta \wedge \gamma) = \alpha \wedge \gamma = \gamma \wedge \min(\alpha, \gamma)$.

(ii) Similar. $\dashv$

It is easy to give syntactic characterizations of $\wedge$- and $\vee$-monotonicity. The proof of the following is left to the reader.

Lemma 2.37 *Let $f \in Reg^*$. Then:*
(i) f is $\wedge$-monotonic if and only if for all M and all $\varphi, \psi, \sigma \in Sen(L_s)$:

$$\langle M, f \rangle \models \varphi \wedge (\psi | \sigma) \leftrightarrow (\varphi \wedge \psi) | (\varphi \wedge \sigma).$$

(ii) f is $\vee$-monotonic if and only if for all M and all $\varphi, \psi, \sigma \in Sen(L_s)$:

$$\langle M, f \rangle \models \varphi \vee (\psi | \sigma) \leftrightarrow (\varphi \vee \psi) | (\varphi \vee \sigma).$$

It follows from the previous Lemma that $\wedge$- and $\vee$-monotonicity are exactly the conditions under which $\wedge$ and $\vee$, respectively, distribute over $|$. However we can easily see by a counterexample that there are no $\wedge$-monotonic or $\vee$-monotonic regular functions (or orderings).

Proposition 2.38 *There is no regular total ordering $<$ of $Sen(L)$ which is $\wedge$-monotonic or $\vee$-monotonic. Consequently there is no $X \subseteq Reg^*$ such that the schemes $\varphi \wedge (\psi | \sigma) \leftrightarrow (\varphi \wedge \psi) | (\varphi \wedge \sigma)$ and $\varphi \vee (\psi | \sigma) \leftrightarrow (\varphi \vee \psi) | (\varphi \vee \sigma)$, are not $\models_X$-tautologies.*

Proof. Suppose there is a regular total ordering $<$ of $Sen(L)$ which is $\wedge$-monotonic. Let p, q r be atomic sentences such that $p < q < r$. Consider the formula $\alpha = p \wedge r \wedge \neg q$. Then by $\wedge$- monotonicity $p < q$ implies $p \wedge r \wedge \neg q < q \wedge r \wedge \neg q$, or by regularity $\alpha < \bot$. For the same reason $q < r$ implies $p \wedge q \wedge \neg q < p \wedge r \wedge \neg q$, or $\bot < \alpha$, a contradiction. Working with $\beta = p \vee r \vee \neg q$ we similarly show that is no regular total ordering $<$ which is $\vee$-monotonic. $\dashv$

Having settled the question about the distributivity of $\wedge$ and $\vee$ over $|$, we come to the converse question, whether $|$ can distribute over $\wedge$ and/or $\vee$ for some class X of choice functions such that $X \subseteq Reg^*$. The answer is "no" again with respect to Reg^*. Namely:

Proposition 2.39 *There is no regular total ordering $<$ of $Sen(L)$ such that if $f = \min_<$, then for every M, $\langle M, f \rangle$ satisfies the scheme*

$$(*) \quad \varphi|(\psi \wedge \sigma) \leftrightarrow (\varphi|\psi) \wedge (\varphi|\sigma).$$

Consequently there is no $X \subseteq Reg^*$ such that (*) is a $\models_X$-tautology. Similarly for the dual scheme

$$(**) \quad \varphi|(\psi \vee \sigma) \leftrightarrow (\varphi|\psi) \vee (\varphi|\sigma).$$

Proof. Towards reaching a contradiction assume that there is a regular total ordering $<$ of $Sen(L)$ such that if $f = \min_<$, then (*) is true in all models $\langle M, f \rangle$. Fix some atomic sentence p of L. By regularity we have either $p < \bot$ or $\bot < p$. We examine below some consequences of each of these cases.

(i) Let $p < \bot$. Pick some $q \neq p$ and an M such that $M \models p \wedge q$. Then

$$\langle M, f \rangle \models_s p|(q \wedge \neg q) \leftrightarrow (p|q) \wedge (p|\neg q),$$

or

$$\langle M, f \rangle \models_s p|\bot \leftrightarrow (p|q) \wedge (p|\neg q). \tag{9}$$

Since $p < \bot$ and $M \models p$, the left-hand side of the equivalence in (9) is true in $\langle M, f \rangle$. Thus so is the right-hand side of the equivalence. Since $M \models p \wedge q$, the conjunct $p|q$ is true, while the truth of the conjunct $p|\neg q$ necessarily implies $p < \neg q$, since $M \models q$. Then pick N such that $N \models p \wedge \neg q$. We have also

$$\langle N, f \rangle \models_s p|\bot \leftrightarrow (p|q) \wedge (p|\neg q). \tag{10}$$

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Again the left-hand side of the equivalence in (10) is true in $\langle N, f \rangle$. So the right-hand side is true too. Since $N \models p \wedge \neg q$, the conjunct $p|\neg q$ holds. In order for the conjunct $p|q$ to hold too we must have $p < q$, since $N \models \neg q$. Summing up the above two facts we conclude that if the letters p, q range over atomic sentences, then

$$(\forall p \neq q)(p < \bot \ \Rightarrow p < q \ \& \ p < \neg q). \tag{11}$$

(ii) Let now $\bot < p$. Pick again some $q \neq p$ and an M such that $M \models p \wedge q$. Then (9) holds again, but now the left-hand side of the equivalence in (9) is false $\langle M, f \rangle$. Thus so is the right-hand side, which, since $M \models p$, necessarily implies $\neg q < p$. Then pick N such that $N \models p \wedge \neg q$. (10) holds again with the left-hand side of the equivalence being false. The right-hand side is false too and this holds only if $q < p$, since $N \models \neg q$. Therefore from these two facts we conclude that

$$(\forall p \neq q)(\bot < p \ \Rightarrow q < p \ \& \ \neg q < p). \tag{12}$$

Now since there are at least three distinct atoms p, q, r and $p, q, r, \bot$ are linearly ordered by $<$, then at least two of them lie on the left of $\bot$, or on the right of $\bot$. That is, there are p, q such that $p, q < \bot$ or $\bot < p, q$. If $p, q < \bot$, (11) implies that $p < q$, $p < \neg q$, $q < p$ and $q < \neg p$, a contradiction. If $\bot < p, q$, then (12) implies that $q < p$, $\neg q < p$, $p < q$ and $\neg p < q$, a contradiction again. This completes the proof that (*) cannot be a $\models_X$-tautology for any $X \subseteq Reg^*$. Concerning the scheme (**) we consider the instances

$$p|(q \vee \neg q) \leftrightarrow (p|q) \vee (q|\neg q),$$

i.e.,

$$p|\top \leftrightarrow (p|q) \vee (q|\neg q),$$

for atomic sentences p, q, and argue analogously as before, by examining the cases $p < \top$ and $\top < p$. $\quad\dashv$

2.4 ¬-decreasingness

There is still the question of how $\neg$ behaves with respect to $|$. As we saw in Fact 2.34, $\neg$ cannot commute with $|$ in models $\langle M, f \rangle$ with regular f. Equivalently, if $f \in Reg^*$ and $f = \min_<$, $\neg$ cannot be "increasing", that is, cannot satisfy $\alpha < \beta \Leftrightarrow \neg\alpha < \neg\beta$, for all α, β. However it can be "decreasing", and this turns out to be a useful property.

Definition 2.40 $<$ is said to be $\neg$-*decreasing,* if for all $\alpha, \beta \in Sen(L)$ such that $\alpha \not\sim \beta$,

$$\alpha < \beta \Leftrightarrow \neg\beta < \neg\alpha.$$

Accordingly a choice function $f \in Reg^*$ is said to be $\neg$-*decreasing* if $f = \min_<$ and $<$ is $\neg$-decreasing.

Lemma 2.41 $<$ *is* $\neg$-*decreasing if and only if for all* $\alpha \not\sim \beta$,

$$\neg\min(\neg\alpha, \neg\beta) \sim \max(\alpha, \beta).$$

Proof. Let $<$ be $\neg$-decreasing. Then for any $\alpha \not\sim \beta$, $\alpha < \beta \Leftrightarrow \neg\beta < \neg\alpha$, so, $\min(\neg\alpha, \neg\beta) = \neg\max(\alpha, \beta)$, hence $\neg\min(\neg\alpha, \neg\beta) \sim \max(\alpha, \beta)$.

Conversely, suppose $<$ is not $\neg$-decreasing. Then there are α, β such that $\alpha \not\sim \beta$, $\alpha < \beta$ and $\neg\alpha < \neg\beta$. But then $\neg\min(\neg\alpha, \neg\beta) = \neg\neg\alpha \not\sim \beta = \max(\alpha, \beta)$. $\qquad\qquad\dashv$

We can also give a syntactic characterization of $\neg$-decreasingness.

Theorem 2.42 *Let* $f \in Reg^*$. *Then* f *is* $\neg$-*decreasing if and only if for every* M *and any* φ, ψ,

$$\langle M, f \rangle \models_s \varphi \wedge \neg\psi \rightarrow (\varphi|\psi \leftrightarrow \neg\varphi|\neg\psi).$$

Proof. "$\Rightarrow$": Let f be $\neg$-decreasing and $f = \min_<$. Let M and φ, ψ such that $\langle M, f \rangle \models_s \varphi \wedge \neg\psi$, that is, $M \models \overline{f}(\varphi) \wedge \neg\overline{f}(\psi)$. It suffices to show that $\langle M, f \rangle \models_s (\varphi|\psi \leftrightarrow \neg\varphi|\neg\psi)$, or equivalently,

$$M \models f(\overline{f}(\varphi), \overline{f}(\psi)) \leftrightarrow f(\neg\overline{f}(\varphi), \neg\overline{f}(\psi)).$$

If we set $\overline{f}(\varphi) = \alpha$ and $\overline{f}(\psi) = \beta$, the above amount to assuming that $M \models \alpha \wedge \neg\beta$ and concluding that $M \models f(\alpha, \beta) \leftrightarrow f(\neg\alpha, \neg\beta)$, or

$$M \models \min(\alpha, \beta) \leftrightarrow \min(\neg\alpha, \neg\beta).$$

But since $M \models \alpha \wedge \neg\beta$, $M \models \min(\alpha, \beta)$ implies $\min(\alpha, \beta) = \alpha$. Then, since $<$ is $\neg$-decreasing, $\min(\neg\alpha, \neg\beta) = \neg\beta$. Therefore $M \models \min(\neg\alpha, \neg\beta)$. So $M \models \min(\alpha, \beta) \rightarrow \min(\neg\alpha, \neg\beta)$. The converse is similar.

"$\Leftarrow$": Let f be non-$\neg$-decreasing, with $f = \min_<$, and let $\alpha \not\sim \beta$ such that $\alpha < \beta$ and $\neg\alpha < \neg\beta$. Without loss of generality there is M such that $M \models \neg\alpha \wedge \beta$. Then $M \not\models \min(\alpha, \beta)$, thus $\langle M, f \rangle \not\models_s \alpha|\beta$, while

$M \models \min(\neg\alpha, \neg\beta)$, or $\langle M, f \rangle \models_s \neg\alpha|\neg\beta$. So $\langle M, f \rangle \not\models (\alpha|\beta \leftrightarrow \neg\alpha|\neg\beta)$, and therefore

$$\langle M, f \rangle \not\models \neg\alpha \wedge \beta \rightarrow (\alpha|\beta \leftrightarrow \neg\alpha|\neg\beta).$$

Therefore $\langle M, f \rangle$ does not satisfy the scheme $\varphi \wedge \neg\psi \rightarrow (\varphi|\psi \leftrightarrow \neg\varphi|\neg\psi)$. $\dashv$

Next let us make sure that $\neg$-decreasing total orderings exist.

Theorem 2.43 *There exist regular $\neg$-decreasing total orderings of $Sen(L)$, and hence regular $\neg$-decreasing choice functions for L.*

Proof. There is a general method for constructing regular and $\neg$-decreasing total orderings of $Sen(L)$ that makes use of the Axiom of Choice. This is the following. Let $P = \{\{[\alpha], [\neg\alpha]\} : \alpha \in Sen(L)\}$. Pick by AC a choice function F for P, and let $A = \bigcup F"P$ and $B = Sen(L) - A$. Both A, B are $\sim$-saturated, that is, $\alpha \in A \Rightarrow [\alpha] \subset A$, and similarly for B. As in the proof of Proposition 2.33 pick a regular total ordering $<_1$ of A. By the definition of A, B, clearly for every $\alpha \in Sen(L)$,

$$\alpha \in A \Leftrightarrow \neg\alpha \in B,$$

so $<_1$ induces a regular total ordering $<_2$ of B by setting

$$\alpha <_2 \beta \Leftrightarrow \neg\beta <_1 \neg\alpha.$$

Then define $<$ of $Sen(L)$ as follows: $\alpha < \beta$ if and only if :
   $\alpha \in A$ and $\beta \in B$, or
   $\alpha, \beta \in A$ and $\alpha <_1 \beta$, or
   $\alpha, \beta \in B$ and $\alpha <_2 \beta$.
It is easy to verify that $<$ is a total regular and $\neg$-decreasing ordering of $Sen(L)$. $\dashv$

We can further show that every regular $\neg$-decreasing total ordering is constructed by the general method of Theorem 2.43. Let us give some definitions. A set $X \subset Sen(L)$ is said to be *selective* if of every pair of opposite sentences $\{\alpha, \neg\alpha\}$ X contains exactly one. Recall also that X is $\sim$-*saturated* if for every α, $\alpha \in X \Rightarrow [\alpha] \subset X$. Note that familiar examples of selective and $\sim$-saturated sets are the consistent and complete sets $\Sigma \subset Sen(L)$ (as well as their complements $Sen(L) - \Sigma$). However not every selective and $\sim$-saturated set is of this kind. For instance the sets A, B in the proof of 2.43 are selective and $\sim$-saturated forming a partition of $Sen(L)$.

Moreover A is an initial segment and B is a final segment of $\langle Sen(L), < \rangle$. We shall see that such a partition exists for every regular $\neg$-decreasing total ordering.

Proposition 2.44 *Let $<$ be a regular $\neg$-decreasing total ordering of $Sen(L)$. Then $Sen(L)$ splits into two $\sim$-saturated sets I and J which are selective, hence*

$$\alpha \in I \Leftrightarrow \neg\alpha \in J,$$

and $I < J$, that is, I is an initial and J a final segment of $<$.

Proof. Let $<$ be a regular and $\neg$-decreasing total ordering of $Sen(L)$. Let us call an initial segment of $<$ *weakly selective* if from every pair $\{\alpha, \neg\alpha\}$, I contains *at most one* element. We first claim that there are weakly selective initial segments of $<$. Observe that if for some α, $(\forall\beta)(\alpha < \beta \vee \alpha < \neg\beta)$ is true, then the initial segment $\{\beta : \beta \leq \alpha\}$ is weakly selective. So if, towards a contradiction, we assume that no weakly selective initial segment exists, then

$$\forall\alpha\exists\beta(\beta \leq \alpha \wedge \neg\beta \leq \alpha). \tag{13}$$

Assume (13) holds and fix some α. Pick β such that $\beta \leq \alpha$ and $\neg\beta \leq \alpha$. By $\neg$-decreasingness, $\neg\alpha \leq \neg\beta$. Therefore $\neg\alpha < \alpha$. Now apply (13) to $\neg\alpha$ to find γ such that $\gamma \leq \neg\alpha$ and $\neg\gamma \leq \neg\alpha$. By $\neg$-decreasingness, $\neg\neg\alpha \leq \neg\gamma$. Thus $\neg\neg\alpha \leq \neg\alpha$ and by regularity, $\alpha \leq \neg\alpha$. But this contradicts $\neg\alpha < \alpha$.

So there exist weakly selective initial segments of $<$. Taking the union of all such initial segments, we find a greatest weakly selective initial segment I. It is easy to see that I is selective, i.e., from each pair $\{\alpha, \neg\alpha\}$ it contains *exactly one* element. Indeed, assume the contrary. Then there is α, such that either $I < \alpha < \neg\alpha$, or $I < \neg\alpha < \alpha$. Assume the first is the case, the other being similar. But then there is β such that $I < \{\beta, \neg\beta\} < \alpha < \neg\alpha$, because otherwise the segment $\{\gamma : \gamma \leq \alpha\}$ would be a weakly selective segment greater than I, contrary to the maximality of I. Now $\{\beta, \neg\beta\} < \alpha < \neg\alpha$ implies that $\beta < \alpha$ and $\neg\beta < \neg\alpha$, which contradicts the $\neg$-decreasingness of $<$.

Further, let $J = \{\neg\alpha : \alpha \in I\}$. Then J is a greatest selective final segment, $I < J$ and $I \cap J = \emptyset$. To show that I (and hence J) is $\sim$-saturated, let $\alpha \in I$. Assume first that α is not the greatest element of I, so there is $\beta \in I$ such that $\alpha < \beta$. By regularity, $[\alpha] < \beta$. Hence $[\alpha] \subset I$. Next assume that α is the greatest element of I. Then necessarily α is the greatest element of $[\alpha]$ too, otherwise $I \cup [\alpha] \supsetneq I$ and $I \cup [\alpha]$ is selective, contrary to the maximality of I. Thus again I is $\sim$-saturated.

It remains to show that $I \cup J = Sen(L)$. Assume $\alpha \notin I$. Since I is selective, $\neg\alpha \in I$, therefore $\neg\neg\alpha \in J$. Since J is $\sim$-saturated, $\alpha \in J$. Thus $I \cup J = Sen(L)$. $\dashv$

So regular $\neg$-decreasing functions constitute a natural class of choice functions stronger than Reg^*. Let

$$Dec = \{f \in Reg^* : f \text{ is } \neg\text{-decreasing}\}.$$

We abbreviate the corresponding semantic notions $\models_{Dec}$ and $\sim_{Dec}$, by $\models_{Dec}$ and $\sim_{Dec}$, respectively.

2.5 The dual connective

Every binary or unary logical operation when combined with negation produces a *dual* one. The dual of $|$ is

$$\varphi \circ \psi := \neg(\neg\varphi|\neg\psi)$$

for all $\varphi, \psi \in Sen(L_s)$.

A natural question is whether each of the operations $|$ and $\circ$ distributes over its dual with respect to a truth relation $\models_X$, that is, whether there is a class of functions X such that

$$\models_X \varphi \circ (\psi|\sigma) \leftrightarrow (\varphi \circ \psi)|(\varphi \circ \sigma) \tag{14}$$

and

$$\models_X \varphi|(\psi \circ \sigma) \leftrightarrow (\varphi|\psi) \circ (\varphi|\sigma) \tag{15}$$

are X-tautologies. (14), (15) are dual and equivalent to each other, since taking the negations of both sides of (14) one obtains (15), and vice-versa.

Proposition 2.45 *There exist α, β, γ, M and $f \in Reg^*$ (f non-$\neg$-decreasing) such that*

$$\langle M, f \rangle \not\models_s \alpha \circ (\beta|\gamma) \leftrightarrow (\alpha \circ \beta)|(\alpha \circ \gamma).$$

Proof. Pick α, β, γ such that $\alpha \not\models \gamma$, $\gamma \not\sim \neg\alpha$, and M such that $M \models \alpha \wedge \neg\gamma$. Then we can easily find a regular total ordering of $Sen(L)$ such that $\neg\alpha < \neg\beta$, $\neg\gamma < \neg\alpha$, $\gamma < \alpha$ and $\beta < \gamma$. Let $f = \min_< = \min$. By definition, $\alpha \circ (\beta|\gamma) = \neg(\neg\alpha|\neg(\beta|\gamma))$, and $(\alpha \circ \beta)|(\alpha \circ \gamma) = \neg(\neg\alpha|\neg\beta)|\neg(\neg\alpha|\neg\gamma))$. Therefore

$$\overline{f}(\alpha \circ (\beta|\gamma)) = \overline{f}(\neg(\neg\alpha|\neg(\beta|\gamma)) = \neg\overline{f}(\neg\alpha|\neg(\beta|\gamma)) = \neg f(\neg\alpha, \neg f(\beta, \gamma)) =$$

$$\neg \min(\neg\alpha, \neg\min(\beta, \gamma)) = \neg\min(\neg\alpha, \neg\beta) = \neg\neg\alpha.$$

On the other hand,

$$\overline{f}((\alpha\circ\beta)|(\alpha\circ\gamma)) = \overline{f}(\neg(\neg\alpha|\neg\beta)|\neg(\neg\alpha|\neg\gamma)) = f(\neg f(\neg\alpha, \neg\beta), \neg f(\neg\alpha, \neg\gamma)) =$$

$$\min(\neg\min(\neg\alpha, \neg\beta), \neg\min(\neg\alpha, \neg\gamma)) = \min(\neg\neg\alpha, \neg\neg\gamma) = \neg\neg\gamma,$$

where the last equation is due to the fact that $\min(\alpha, \gamma) = \gamma$ and $<$ is regular. Thus $M \models \overline{f}(\alpha \circ (\beta|\gamma))$ and $M \not\models \overline{f}((\alpha \circ \beta)|(\alpha \circ \gamma))$. Therefore $\langle M, f \rangle \models_s \alpha \circ (\beta|\gamma)$ and $\langle M, f \rangle \not\models_s (\alpha \circ \beta)|(\alpha \circ \gamma)$. $\dashv$

Note that in the preceding counterexample we have $\gamma < \alpha$ and $\neg\gamma < \neg\alpha$, so the ordering $<$ is not $\neg$-decreasing. We see next that if f is $\neg$-decreasing, then in $\langle M, f \rangle$ | and $\circ$ do distribute over each other.

Proposition 2.46 *If $f \in Reg^*$ is $\neg$-decreasing, then for all M, φ, ψ, σ,*

$$\langle M, f \rangle \models_s \varphi \circ (\psi|\sigma) \leftrightarrow (\varphi \circ \psi)|(\varphi \circ \sigma),$$

and

$$\langle M, f \rangle \models_s \varphi|(\psi \circ \sigma) \leftrightarrow (\varphi|\psi) \circ (\varphi|\sigma).$$

Proof. The above equivalences are dual to each other, so it suffices to show the first of them. Specifically it suffices to prove that if $f \in Reg^*$ and f is $\neg$-decreasing, then

$$\overline{f}(\varphi \circ (\psi|\sigma)) \sim \overline{f}((\varphi \circ \psi)|(\varphi \circ \sigma)).$$

Fix such an f and let $<$ be the regular, $\neg$-decreasing ordering such that $f = \min_< = \min$. If we set $\overline{f}(\varphi) = \alpha$, $\overline{f}(\psi) = \beta$, $\overline{f}(\sigma) = \gamma$, express $\circ$ in terms of | and replace f with $<$, the above equivalence is written:

$$\neg\min(\neg\alpha, \neg\min(\beta, \gamma)) \sim \min(\neg\min(\neg\alpha, \neg\beta), \neg\min(\neg\alpha, \neg\gamma)). \qquad (16)$$

If $\alpha \sim \beta \sim \gamma$, obviously (16) is true. Assume $\alpha \sim \beta$ and $\alpha \not\sim \gamma$. Then, by regularity, (16) becomes

$$\neg\min(\neg\alpha, \neg\min(\alpha, \gamma)) \sim \min(\alpha, \neg\min(\neg\alpha, \neg\gamma)).$$

To verify it we consider the cases $\alpha < \gamma$ and $\gamma < \alpha$. E.g. let $\alpha < \gamma$. By $\neg$-decreasingness, $\neg\gamma < \neg\alpha$, so both sides of the above relation are $\sim$ to α. Similarly if $\gamma < \alpha$.

So it remains to prove (16) when $\alpha \not\sim \beta$ and $\alpha \not\sim \gamma$. Then, by Lemma 2.41, $\neg \min(\neg\alpha, \neg\beta) \sim \max(\alpha, \beta)$, so (16) is written

$$\max(\alpha, \min(\beta, \gamma)) \sim \min(\max(\alpha, \beta), \max(\alpha, \gamma)). \tag{17}$$

We don't know if there is some more elegant direct (that is, not-by-cases) proof of (17). So we verify it by cases.

Case 1. Assume $\alpha \leq \min(\beta, \gamma)$. Then $\max(\alpha, \min(\beta, \gamma)) = \min(\beta, \gamma)$. Besides $\alpha \leq \min(\beta, \gamma)$ implies $\max(\alpha, \beta) = \beta$ and $\max(\alpha, \gamma) = \gamma$. Therefore both sides of (17) are $\sim$ to $\min(\beta, \gamma)$.

Case 2. Assume $\min(\beta, \gamma) < \alpha$. Then $\max(\alpha, \min(\beta, \gamma)) = \alpha$. To decide the right-hand side of (17), suppose $\beta \leq \gamma$ so we have the following subcases.

(2a) $\beta < \alpha \leq \gamma$: Then $\max(\alpha, \beta) = \alpha$, $\max(\alpha, \gamma) = \gamma$, therefore, $\min(\max(\alpha, \beta), \max(\alpha, \gamma)) = \alpha$, thus (17) holds.

(2b) $\beta \leq \gamma < \alpha$: Then $\max(\alpha, \beta) = \max(\alpha, \gamma) = \alpha$. So

$$\min(\max(\alpha, \beta), \max(\alpha, \gamma)) = \alpha,$$

thus (17) holds again.

Case 3. Assume $\min(\beta, \gamma) < \alpha$, so $\max(\alpha, \min(\beta, \gamma)) = \alpha$, but suppose now $\gamma < \beta$. Then we have the subcases:

(3a) $\gamma < \alpha \leq \beta$: Then $\max(\alpha, \beta) = \beta$ and $\max(\alpha, \gamma) = \alpha$. Thus $\min(\max(\alpha, \beta), \max(\alpha, \gamma)) = \alpha$, that is, (17) holds.

(3b) $\gamma < \beta \leq \alpha$: Then $\max(\alpha, \beta) = \alpha$ and $\max(\alpha, \gamma) = \alpha$. So

$$\min(\max(\alpha, \beta), \max(\alpha, \gamma)) = \alpha.$$

This completes the proof of the Proposition. $\dashv$.

Corollary 2.47 *The schemes*

$$\varphi \circ (\psi|\sigma) \leftrightarrow (\varphi \circ \psi)|(\varphi \circ \sigma) \tag{18}$$

(or its dual) and

$$\varphi \wedge \psi \rightarrow (\varphi|\psi \leftrightarrow \neg\varphi|\neg\psi) \tag{19}$$

are equivalent and each one of them is a syntactic characterization of the regular $\neg$-decreasing orderings (and the corresponding choice functions).

Proof. The equivalence of (18) and (19), as *schemes*, follows from Propositions 2.45, 2.46, as well as from Lemma 2.41 by which (19) characterizes the regular $\neg$-decreasing orderings. $\dashv$

Interchanging $|$ and $\circ$ inside a sentence gives rise to a duality of sentences of L_s, that is, a mapping $\varphi \mapsto \varphi^d$ defined inductively as follows:

$\varphi^d = \varphi$, for classical φ,
$(\varphi \wedge \psi)^d = \varphi^d \wedge \psi^d$,
$(\neg\varphi)^d = \neg\varphi^d$,
$(\varphi|\psi)^d = \varphi^d \circ \psi^d$.

By the help of dual orderings $<^d$ and dual choice functions f^d, one can without much effort establish the following "Duality Theorem" which is the analogue of Boolean Duality:

Theorem 2.48 *For every $\varphi \in Fml(L_s)$, $\models_{Reg^*} \varphi$ if and only if $\models_{Reg^*} \varphi^d$.*

3 Axiomatization. Soundness and completeness results

A *Propositional Superposition Logic* (PLS for short) will consist as usual of two parts, a *syntactic* one, i.e., a formal system K, consisted of axiom-schemes and inference rules, and a *semantical* one, consisted essentially of a set $X \subseteq \mathcal{F}$ of choice functions over $Sen(L)$.[11] Let us start with the latter.

The semantical part X induces the truth relation $\models_X$, that is, the class of structures $\langle M, f \rangle$, where $M : Sen(L) \to \{0, 1\}$ and $f \in X$, with respect to which the X-tautologies and X-logical consequence are defined. For every $X \subseteq \mathcal{F}$ let

$$Taut(X) = \{\varphi \in Sen(L_s) : \models_X \varphi\}$$

be the set of tautologies of L_s with respect to $\models_X$. Let also $Taut$ be the set of classical tautologies. Then for any $X, Y \subseteq \mathcal{F}$,

$$X \subseteq Y \;\Rightarrow\; Taut \subseteq Taut(Y) \subseteq Taut(X).$$

The following simple fact will be used later but has also an obvious interest in itself.

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[11] More or less the same is true for every logical system, e.g. PL. Although we often identify PL with the set of its logical axioms and the inference rule of Modus Ponens, tacitly we think of it as a set of axiom-schemes Ax(PL) and the inference rule *MP* on the ones hand, and the natural Boolean semantics on the other. Specifically Ax(PL) will consist of the following schemes:

(i) $\alpha \to (\beta \to \alpha)$
(ii) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
(iii) $(\neg\alpha \to \neg\beta) \to ((\neg\alpha \to \beta) \to \alpha)$.

Lemma 3.1 *For every $X \subseteq \mathcal{F}$, the set $Taut(X)$ is decidable (i.e., computable).*

Proof. By the definition of $\models_s$, $\varphi \in Taut(X)$ if and only if $(\forall f \in X)(\overline{f}(\varphi) \in Taut)$ (where $Taut$ is the set of tautologies of PL), i.e.,

$$\varphi \in Taut(X) \ \Leftrightarrow \ \{\overline{f}(\varphi) : f \in X\} \subset Taut.$$

Now given φ and f, the collapse $\overline{f}(\varphi)$ results from φ by inductively replacing each subformula $\psi_1 | \psi_2$ of φ with either $\overline{f}(\psi_1)$ or $\overline{f}(\psi_2)$. So clearly for every φ, the set of all possible collapses $\{\overline{f}(\varphi) : f \in X\}$ is finite. Therefore, since $Taut$ is decidable, it is decidable whether $\{\overline{f}(\varphi) : f \in X\} \subset Taut$. $\qquad \dashv$

In particular we are interested in the sets

$$Taut(\mathcal{F}) \subseteq Taut(Reg) \subseteq Taut(Reg^*) \subseteq Taut(Dec), \qquad (20)$$

(as well as in $Taut(Asso) \subseteq Taut(Reg^*)$) corresponding to the truth relations considered above. It follows from 3.1 that these sets are decidable. The question is whether each of these sets of tautologies is axiomatizable by a recursive set of axioms and inference rules. We shall see that the answer is yes.

Let us come to the formal system K. Every K consists of a set of axioms $\mathsf{Ax}(K)$ and a set of inference rules $\mathsf{IR}(K)$. Also the axioms and rules of K extend the axioms and rules of PL, i.e.,

$$\mathsf{Ax}(K) = \mathsf{Ax}(\mathrm{PL}) + \{S_i : i \leq n\}, \text{ and } MP \in \mathsf{IR}(K),$$

where S_i will be some schemes considered below expressing basic properties of $|$. Given $X \subseteq \mathcal{F}$ in order to axiomatize $Taut(X)$ by a formal system K, clearly it is necessary for the axioms of K to be X-tautologies, i.e.,

$$\mathsf{Ax}(K) \subseteq Taut(X).$$

For any such $X \subseteq \mathcal{F}$ and K, we have a *logic* that extends PL, called *Propositional Superposition Logic w.r.t. to X and K,* denoted

$$\mathrm{PLS}(X, K).$$

Given a formal system K as above and $\Sigma \cup \{\varphi\} \subset Sen(L)$, a (Hilbert-style) *$K$-proof of φ from Σ* is defined just as a proof in PL (mutatis mutandis), that is, as a sequence of sentences $\sigma_1, \ldots, \sigma_n$ such that $\sigma_n = \varphi$ and

each σ_i either belongs to Σ or belongs to $\mathsf{Ax}(K)$, or is derived from previous ones by the inference rules in $\mathsf{IR}(K)$. We denote by

$$\Sigma \vdash_K \varphi$$

the fact that there is a K-proof of φ from Σ. Especially for classical sentences, i.e., $\Sigma \cup \{\alpha\} \subseteq Sen(L)$, it is clear that

$$\Sigma \vdash_{PL} \alpha \ \Leftrightarrow \ \Sigma \vdash_K \alpha,$$

where $\vdash_{PL}$ denotes provability in PL. Σ is said to be K-consistent, if $\Sigma \not\vdash_K \bot$. Again for $\Sigma \subset Sen(L)$,

$$\Sigma \text{ is } K\text{-consistent} \ \Leftrightarrow \ \Sigma \text{ is consistent.}$$

Recall that a formal system K (or its proof relation $\vdash_K$) satisfies the Deduction Theorem (DT) if for all Σ, φ and ψ,

$$\Sigma \cup \{\varphi\} \vdash_K \psi \ \Rightarrow \ \Sigma \vdash_K \varphi \to \psi. \tag{21}$$

It is well-known that if the only inference rule of K is MP (and perhaps also the Generalization Rule), then DT holds for $\vdash_K$. But in systems with additional inference rules DT often fails. Below we shall consider formal systems K augmented with an additional inference rule. So we shall need to examine the validity of DT later.

Definition 3.2 A set $\Sigma \subset Sen(L_s)$ is said to be X-satisfiable if for some truth assignment M for L and some $f \in X$, $\langle M, f \rangle \models_s \Sigma$.

As is well-known the Soundness and Completeness Theorems of a logic have two distinct formulations, which are not always equivalent, depending on the semantics and the validity of Deduction Theorem. For the logic $PLS(X, K)$ these two forms, ST1 and ST2 for Soundness and CT1 and CT2 for Completeness, are the following:

(ST1) $\qquad\qquad\qquad \Sigma \vdash_K \varphi \ \Rightarrow \ \Sigma \models_X \varphi,$

(ST2) $\qquad\qquad \Sigma \text{ is } X\text{-satisfiable} \ \Rightarrow \ \Sigma \text{ is } K\text{-consistent}$

(CT1) $\qquad\qquad\qquad \Sigma \models_X \varphi \ \Rightarrow \ \Sigma \vdash_K \varphi,$

(CT2) $\qquad\qquad \Sigma \text{ is } K\text{-consistent} \ \Rightarrow \ \Sigma \text{ is } X\text{-satisfiable.}$

Concerning the relationship between ST1 and ST2 and between CT1 and CT2 for $PLS(X, K)$ the following holds.

Fact 3.3 *(i) For every X*

$$\Sigma \not\models_X \varphi \;\Rightarrow\; \Sigma \cup \{\neg\varphi\} \text{ is } X\text{-satisfiable.} \tag{22}$$

As a consequence, (ST1) $\Leftrightarrow$ (ST2) *holds for every* $\mathrm{PLS}(X, K)$.

(ii) (CT1) $\Rightarrow$ (CT2) *holds for every* $\mathrm{PLS}(X, K)$. *If* $\vdash_K$ *satisfies DT, then the converse holds too, i.e.,* (CT1) $\Leftrightarrow$ (CT2).

Proof. (i) (22) follows immediately from the definition of $\models_X$ and the fact that the truth is bivalent. Now (ST1) $\Rightarrow$ (ST2) is straightforward. For the converse assume ST2 and $\Sigma \not\models_X \varphi$. By (22) $\Sigma \cup \{\neg\varphi\}$ is X-satisfiable. By ST2, $\Sigma \cup \{\neg\varphi\}$ is K-consistent, therefore $\Sigma \not\vdash_K \varphi$.

(ii) (CT1) $\Rightarrow$ (CT2) is also straightforward. For the converse assume CT2, DT and $\Sigma \not\vdash_K \varphi$. It is well-known that by DT the latter is equivalent to the K-consistency of $\Sigma \cup \{\neg\varphi\}$. By CT2, $\Sigma \cup \{\neg\varphi\}$ is X-satisfiable. Therefore $\Sigma \not\models_X \varphi$. $\dashv$

In view of Fact 3.3 (i) we do not need to distinguish any more between ST1 and ST2, and can refer simply to "sound" logics.

However the distinction between CT1 and CT2 remains. This is also exemplified by considering the semantic analogue of DT. Given a class $X \subseteq \mathcal{F}$, let us call the implication:

$$\Sigma \cup \{\varphi\} \models_X \psi \;\Rightarrow\; \Sigma \models_X \varphi \to \psi \tag{23}$$

Semantic Deduction Theorem for X (or, briefly, SDT). Here is a relationship between DT and SDT via CT1.

Fact 3.4 *For every* $X \subseteq \mathcal{F}$, *SDT for X is true. This implies that if the logic* $\mathrm{PLS}(X, K)$ *is sound and satisfies CT1, then K satisfies DT.*

Proof. That SDT holds for every $X \subseteq \mathcal{F}$ is an easily verified consequence of the semantics $\models_X$. Now assume that $\mathrm{PLS}(X, K)$ is sound (i.e., satisfies (equivalently) both ST1 and ST2), satisfies CT1, and $\Sigma \cup \{\varphi\} \vdash_K \psi$. By ST1 it follows that $\Sigma \cup \{\varphi\} \models_X \psi$. By SDT (23) we have $\Sigma \models_X \varphi \to \psi$. Then CT1 implies $\Sigma \vdash_K \varphi \to \psi$, as required. $\dashv$

Next we give a list of specific axiom-schemes (referred also to simply as *axioms*) about $|$, certain nested groups of which are going to axiomatize the truth relations $\models_{\mathcal{F}}$, $\models_{Reg}$, $\models_{Reg^*}$ and $\models_{Dec}$ considered in the previous sections.

(S_1) $\varphi \wedge \psi \rightarrow \varphi | \psi$
(S_2) $\varphi | \psi \rightarrow \varphi \vee \psi$
(S_3) $\varphi | \psi \rightarrow \psi | \varphi$
(S_4) $(\varphi | \psi) | \sigma \rightarrow \varphi | (\psi | \sigma)$
(S_5) $\varphi \wedge \neg \psi \rightarrow (\varphi | \psi \leftrightarrow \neg \varphi | \neg \psi)$

We shall split the axiomatization of the four basic truth relations considered in the previous section in two parts. We shall consider first the basic truth relation $\models_{\mathcal{F}}$ relying on the entire class of functions $\mathcal{F}$, and then we shall consider the rest stricter relations $\models_{Reg}$, $\models_{Reg^*}$ and $\models_{Dec}$. The reason is that the relation $\models_{\mathcal{F}}$ can be axiomatized by a formal system having MP as the only inference rule, while the rest systems require formal systems augmented with a second rule. The latter requirement makes these systems considerably more complicated.

3.1 Axiomatizing the truth relation $\models_{\mathcal{F}}$

In this section we deal with the relation $\models_{\mathcal{F}}$ and show that it can be soundly and completely axiomatized by the first three axioms S_1-S_3 cited above and Modus Ponens (MP). We call this formal system K_0. Namely

$$\mathsf{Ax}(K_0) = \mathsf{Ax}(\mathrm{PL}) + \{S_1, S_2, S_3\} \text{ and } \mathsf{IR}(K_0) = \{MP\}. \tag{24}$$

Observe that S_1 and S_2, combined with the axioms of PL, prove (in K_0) $\varphi | \varphi \leftrightarrow \varphi$.

It is easy to see that the logic $\mathrm{PLS}(\mathcal{F}, K_0)$ is sound. Namely we have the following more general fact.

Theorem 3.5 *Let $X \subseteq \mathcal{F}$. If K is a system such that $\mathsf{Ax}(K) \subset Taut(X)$ and $\mathsf{IR}(K) = \{MP\}$, then $\mathrm{PLS}(X, K)$ is sound.*

Proof. Let X, K be as stated and $\Sigma \vdash_K \varphi$. Let $\varphi_1, \ldots, \varphi_n$, where $\varphi_n = \varphi$, be a K-proof of φ. As usual we show that $\Sigma \models_X \varphi_i$, for every $1 \leq i \leq n$, by induction on i. Given i, suppose the claim holds for all $j < i$, and let $\langle M, f \rangle \models_s \Sigma$, for some assignment M and $f \in X$. We show that $\langle M, f \rangle \models_s \varphi_i$. If $\varphi_i \in \Sigma$ this is obvious. If $\varphi_i \in \mathsf{Ax}(K)$, then $\langle M, f \rangle \models_s \varphi_i$, because by assumption $\mathsf{Ax}(K) \subset Taut(X)$ and $f \in X$. Otherwise, since MP is the only inference rule of K, φ_i follows by MP from sentences φ_j, $\varphi_k = (\varphi_j \rightarrow \varphi_i)$, for some $j, k < i$. By the induction assumption, $\langle M, f \rangle \models_s \varphi_j$ and $\langle M, f \rangle \models_s \varphi_k$. Therefore $\langle M, f \rangle \models_s \varphi_i$. $\dashv$

Corollary 3.6 *The logic $\mathrm{PLS}(\mathcal{F}, K_0)$ is sound.*

Proof. By Theorem 2.8 and Fact 2.5 (iv), S_1, S_2, S_3 are schemes that hold in $\langle M, f \rangle$ for all $f \in \mathcal{F}$, therefore $\mathsf{Ax}(K_0) \subset Taut(\mathcal{F})$. So the claim follows from 3.5. ⊣

Completeness of $\mathrm{PLS}(\mathcal{F}, K_0)$ We come to the completeness of the logic $\mathrm{PLS}(\mathcal{F}, K_0)$. As usual, a set $\Sigma \subseteq Sen(L_s)$ is said to be complete if for every $\varphi \in Sen(L_s)$, $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$. If Σ is K-consistent and complete, then for every $\varphi \in Sen(L_s)$, $\varphi \in \Sigma \Leftrightarrow \neg\varphi \notin \Sigma$. Moreover if $\Sigma \vdash_K \varphi$, then $\varphi \in \Sigma$.

Before coming to the logics introduced in the previous subsection, we shall give a general satisfiability criterion. Fix a class $X \subseteq \mathcal{F}$ of choice functions and a set of axioms $K \subseteq Taut(X)$. Let Σ be a K-consistent and complete set of sentences of L_s and let $\Sigma_1 = \Sigma \cap Sen(L)$ be the subset of Σ that contains the classical sentences of Σ. Then clearly Σ_1 is a consistent and complete set of sentences of L. By the Completeness Theorem of PL, there exists a truth assignment M for L such that, for every $\alpha \in Sen(L)$

$$\alpha \in \Sigma_1 \ \Leftrightarrow \ M \models \alpha. \tag{25}$$

Given Σ, Σ_1, M satisfying (25), and a set $X \subseteq \mathcal{F}$ of choice functions, the question is under what conditions M can be paired with a function $f \in X$ such that $\langle M, f \rangle \models_s \Sigma$. Below we give a simple characterization of this fact which is the key characterization of X-satisfiability.

Lemma 3.7 *Let $X \subseteq \mathcal{F}$ and $K \subset Taut(X)$. Let also Σ be a K-consistent and complete set of sentences of L_s and let $\Sigma_1 = \Sigma \cap Sen(L)$ and M satisfy (25). Then for every $f \in X$, $\langle M, f \rangle \models \Sigma$ if and only if for every $\varphi \in Sen(L_s)$,*

$$\varphi \in \Sigma \ \Rightarrow \overline{f}(\varphi) \in \Sigma. \tag{26}$$

(Actually (26) is equivalent to

$$\varphi \in \Sigma \ \Leftrightarrow \overline{f}(\varphi) \in \Sigma,$$

but the other direction follows from (26), the consistency and completeness of Σ and the fact that $\overline{f}(\neg\varphi) = \neg\overline{f}(\varphi)$.)

Proof. Pick an $f \in X$ and suppose $\langle M, f \rangle \models_s \Sigma$. Then by the completeness of Σ and the definition of $\models_s$, for every $\varphi \in Sen(L_s)$,

$$\varphi \in \Sigma \ \Leftrightarrow \ \langle M, f \rangle \models_s \varphi \Leftrightarrow M \models \overline{f}(\varphi).$$

Now by (25), $M \models \overline{f}(\varphi) \Rightarrow \overline{f}(\varphi) \in \Sigma_1 \subset \Sigma$. Therefore $\varphi \in \Sigma \ \Rightarrow \overline{f}(\varphi) \in \Sigma$. Thus (26) holds.

Conversely, suppose (26) is true. To show that $\langle M, f \rangle \models_s \Sigma$, pick some $\varphi \in \Sigma$. By (26) $\overline{f}(\varphi) \in \Sigma$. Then $\overline{f}(\varphi) \in \Sigma_1$ since $\overline{f}(\varphi)$ is classical, so by (25) $M \models \overline{f}(\varphi)$. This means that $\langle M, f \rangle \models_s \varphi$, as required. ⊣

We come next to the completeness of $\mathrm{PLS}(\mathcal{F}, K_0)$. The essential step of the proof is the following Lemma.

Lemma 3.8 *Let Σ be a K_0-consistent and complete set of sentences of L_s. Then Σ is $\mathcal{F}$-satisfiable.*

Proof. Let Σ be K_0-consistent and complete. Then for any $\varphi, \psi \in Sen(L_s)$, the possible subsets of Σ whose elements are $\varphi|\psi$, φ, ψ or their negations are the following:

(a1) $\{\varphi|\psi, \varphi, \psi\} \subset \Sigma$
(a2) $\{\varphi|\psi, \varphi, \neg\psi\} \subset \Sigma$
(a3) $\{\varphi|\psi, \neg\varphi, \psi\} \subset \Sigma$
(a4) $\{\neg(\varphi|\psi), \neg\varphi, \neg\psi\} \subset \Sigma$
(a5) $\{\neg(\varphi|\psi), \varphi, \neg\psi\} \subset \Sigma$
(a6) $\{\neg(\varphi|\psi), \neg\varphi, \psi\} \subset \Sigma$

The remaining cases,
(a7) $\{\varphi|\psi, \neg\varphi, \neg\psi\} \subset \Sigma$
(a8) $\{\neg(\varphi|\psi), \varphi, \psi\} \subset \Sigma$
are impossible because they contradict K_0-consistency and completeness of Σ. Indeed, in case (a7) we have $\neg\varphi \wedge \neg\psi \in \Sigma$. Also $\varphi|\psi \in \Sigma$, so by S_2 and completeness, $\varphi \vee \psi \in \Sigma$, a contradiction. In case (a8) $\varphi \wedge \psi \in \Sigma$. Also $\neg(\varphi|\psi) \in \Sigma$, so by S_1 and completeness $\neg(\varphi \wedge \psi) \in \Sigma$, a contradiction.

Given a pair $\{\alpha, \beta\}$ we say that "$\{\alpha, \beta\}$ satisfies (ai)" if for $\varphi = \alpha$ and $\psi = \beta$, the corresponding case (ai) above, for $1 \leq i \leq 6$, holds. We define a choice function g for L as follows:

$$g(\alpha, \beta) = \begin{cases} (i)\ \alpha, \text{ if } \{\alpha, \beta\} \text{ satisfies (a2) or (a6)} \\ (ii)\ \beta, \text{ if } \{\alpha, \beta\} \text{ satisfies (a3) or (a5)} \\ (iii)\ \text{any of the } \alpha, \beta, \text{ if } \{\alpha, \beta\} \text{ satisfies (a1) or (a4).} \end{cases} \quad (27)$$

Claim. $\overline{g}$ satisfies the implication (26) of the previous Lemma.

Proof of the Claim. We prove (26) by induction on the length of φ. For $\varphi = \alpha \in Sen(L)$, $\overline{g}(\alpha) = \alpha$, so (26) holds trivially. Similarly the induction steps for $\wedge$ and $\neg$ follow immediately from the fact that $\overline{g}$ commutes with

these connectives and the completeness of Σ. So the only nontrivial step of the induction is that for $\varphi|\psi$. It suffices to assume

$$\varphi \in \Sigma \;\Rightarrow\; \overline{g}(\varphi) \in \Sigma, \tag{28}$$

$$\psi \in \Sigma \;\Rightarrow\; \overline{g}(\psi) \in \Sigma, \tag{29}$$

and prove

$$\varphi|\psi \in \Sigma \;\Rightarrow\; \overline{g}(\varphi|\psi) \in \Sigma. \tag{30}$$

Assume $\varphi|\psi \in \Sigma$. Then the only possible combinations of φ, ψ and their negations that can belong to Σ are those of cases (a1), (a2) and (a3) above. To prove (30) it suffices to check that $\overline{g}(\varphi|\psi) \in \Sigma$ in each of these cases. Note that $\overline{g}(\varphi|\psi) = g(\overline{g}(\varphi), \overline{g}(\psi)) = g(\alpha, \beta)$, where $\overline{g}(\varphi) = \alpha$ and $\overline{g}(\psi) = \beta$ are sentences of L, so (27) applies.

Case (a1): Then $\varphi \in \Sigma$ and $\psi \in \Sigma$. By (28) and (29), $\overline{g}(\varphi) \in \Sigma$ and $\overline{g}(\varphi) \in \Sigma$. By definition (27), $\overline{g}(\varphi|\psi) = g(\overline{g}(\varphi), \overline{g}(\psi))$ can be either $\overline{g}(\varphi)$ or $\overline{g}(\psi)$. So in either case $\overline{g}(\varphi|\psi) \in \Sigma$.

Case (a2): Then $\varphi \in \Sigma$ and $\neg\psi \in \Sigma$. By (28) and (29), $\overline{g}(\varphi) \in \Sigma$, $\overline{g}(\psi) \notin \Sigma$. Also by (27), $\overline{g}(\varphi|\psi) = g(\overline{g}(\varphi), \overline{g}(\psi)) = \overline{g}(\varphi)$, thus $\overline{g}(\varphi|\psi) \in \Sigma$.

Case (a3): Then $\neg\varphi \in \Sigma$, $\psi \in \Sigma$. By (28) and (29), $\overline{g}(\varphi) \notin \Sigma$, $\overline{g}(\psi) \in \Sigma$. By (27), $\overline{g}(\varphi|\psi) = g(\overline{g}(\varphi), \overline{g}(\psi)) = \overline{g}(\psi)$, thus $\overline{g}(\varphi|\psi) \in \Sigma$. This completes the proof of the Claim.

It follows that condition (26) is true, so by Lemma 3.7, if $M \models \Sigma_1$ where $\Sigma_1 = \Sigma \cap Sen(L)$, then $\langle M, g \rangle \models \Sigma$, therefore Σ is $\mathcal{F}$-satisfiable. $\quad\dashv$

Let us remark here that, since $\vdash_{K_0}$ satisfies the Deduction Theorem, by Fact 3.3 the two forms of completeness theorem CT1 and CT2 are equivalent for $\mathrm{PLS}(\mathcal{F}, K_0)$. So it is indifferent which one we are going to prove for the system $\mathrm{PLS}(\mathcal{F}, K_0)$.

Theorem 3.9 (Completeness of $\mathrm{PLS}(\mathcal{F}, K_0)$) *The logic $\mathrm{PLS}(\mathcal{F}, K_0)$ is complete. That is, if Σ is K_0-consistent, then Σ is $\mathcal{F}$-satisfiable.*

Proof. Let Σ be K_0-consistent. Extend Σ to a K_0-consistent and complete $\Sigma^* \supseteq \Sigma$. By Lemma 3.8, Σ^* is $\mathcal{F}$-satisfiable. Therefore so is Σ. $\quad\dashv$

Corollary 3.10 *The set $\{\varphi : \vdash_{K_0} \varphi\}$ is decidable.*

Proof. By the soundness and completeness of $\mathrm{PLS}(\mathcal{F}, K_0)$, $\{\varphi : \vdash_{K_0} \varphi\} = Taut(\mathcal{F})$. But $Taut(\mathcal{F})$ is decidable by Lemma 3.1. $\quad\dashv$

3.2 Axiomatizing the truth relations for the classes Reg, Reg^* and Dec

The next systems, K_1-K_3, are intended to capture in addition the semantic property of regularity considered in section 2.3. We need to define K_1 so that if $\vdash_{K_1} \varphi$ then $\varphi \in Taut(Reg)$, and vice-versa (if possible). Specifically, if $\alpha \sim \alpha'$, we need K_1 to prove, for every β, $\alpha|\beta \leftrightarrow \alpha'|\beta$, i.e.,

$$\vdash_{K_1} (\alpha|\beta \leftrightarrow \alpha'|\beta).$$

This cannot be captured by an axiom-scheme, since no scheme can express the relation $\sim$ of logical equivalence. It can be captured however by a new *inference rule.* Roughly we need a rule guaranteeing that if φ, ψ are logically equivalent, then φ and ψ can be interchanged *salva veritate* in expressions containing $|$, that is one entailing $\varphi|\sigma \leftrightarrow \psi|\sigma$, for every σ.[12] This is the following rule denoted SV (for salva veritate):

$$(SV) \qquad from \;\; \varphi \leftrightarrow \psi \;\; infer \; \varphi|\sigma \leftrightarrow \psi|\sigma,$$

$$if \; \varphi \leftrightarrow \psi \; is \; provable \; in \; K_0.$$

We see that SV is a "conditional rule", applied under constraints, much like the Generalization Rule of first-order logic (from $\varphi(x)$ infer $\forall x \varphi(x)$, if x is not free in the premises, and also the Necessitation Rule of modal logic (from φ infer $\Box\varphi$, if $\vdash \varphi$). It follows that SV operates according to the following:

Fact 3.11 *Let K be a formal system such that $SV \in \mathsf{IR}(K)$. If $\vdash_{K_0} (\varphi \leftrightarrow \psi)$, then $\vdash_K (\varphi|\sigma \leftrightarrow \psi|\sigma)$ for every σ.*

Note that since, according to Corollary 3.10, it is decidable, given φ, whether $\vdash_{K_0} \varphi$, it is decidable, given a recursive set of sentences Σ, whether a sequence of sentences $\varphi_1, \ldots, \varphi_n$ is a proof (in K) from Σ or not.

Theorem 3.12 *Let $X \subseteq Reg$. If K is a system such that $\mathsf{Ax}(K) \subset Taut(X)$ and $\mathsf{IR}(K) = \{MP, SV\}$, then $\mathrm{PLS}(X, K)$ is sound.*

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[12]Of course substitution of logically equivalent sentences salva veritate holds also in classical logic, that is, if $\alpha \sim \alpha'$ and α is a subformula of β, then $\beta[\alpha] \sim \beta[\alpha']$, where $\beta[\alpha']$ is the result of replacing α with α' within β. This however is a simple consequence of the compositional semantics of classical logic. In contrast the choice semantics of PLS is by no means compositional.

Proof. Let $X \subseteq Reg$, $\mathsf{Ax}(K) \subset Taut(X)$ and $\mathsf{IR}(K) = \{MP, SV\}$, and let $\Sigma \vdash_K \varphi$. Let $\varphi_1, \ldots, \varphi_n$, where $\varphi_n = \varphi$, be a K-proof of φ. We show, by induction on i, that for all $i = 1, \ldots, n$, $\Sigma \models_X \varphi_i$. Let $\langle M, f \rangle \models_s \Sigma$, with $f \in X$. The proof that $\langle M, f \rangle \models_s \varphi_i$ goes exactly as in the proof of Theorem 3.5, except of the case where φ_i follows from a sentence φ_j, for $j < i$, by SV. It means that $\varphi_i = (\sigma|\tau \leftrightarrow \rho|\tau)$ while $\varphi_j = (\sigma \leftrightarrow \rho)$, where $\vdash_{K_0} (\sigma \leftrightarrow \rho)$. Now K_0 is a system satisfying the conditions of 3.5 above for $X = \mathcal{F}$, so $\models_{\mathcal{F}} (\sigma \leftrightarrow \rho)$. It means that for every assignment N and every $g \in \mathcal{F}$, $\langle N, g \rangle \models_s (\sigma \leftrightarrow \rho)$, i.e., $N \models \overline{g}(\sigma) \leftrightarrow \overline{g}(\rho)$, that is, $\overline{g}(\sigma) \leftrightarrow \overline{g}(\rho)$ is a classical tautology, or $\overline{g}(\sigma) \sim \overline{g}(\rho)$, for every $g \in \mathcal{F}$. In particular, $\overline{f}(\sigma) \sim \overline{f}(\rho)$. Now since $X \subseteq Reg$, $f \in X$ implies f is regular. Therefore $\overline{f}(\sigma) \sim \overline{f}(\rho)$ implies that $f(\overline{f}(\sigma), \overline{f}(\tau)) \sim f(\overline{f}(\rho), \overline{f}(\tau))$, or $\overline{f}(\sigma|\tau) \sim \overline{f}(\rho|\tau)$, therefore $M \models \overline{f}(\sigma|\tau) \leftrightarrow \overline{f}(\rho|\tau)$, or $\langle M, f \rangle \models_s (\sigma|\tau \leftrightarrow \rho|\tau)$, i.e., $\langle M, f \rangle \models_s \varphi_i$, as required. This completes the proof. $\dashv$

We define next the systems K_1-K_3 as follows:

$$\mathsf{Ax}(K_1) = \mathsf{Ax}(K_0) = \{S_1, S_2, S_3\}, \qquad \mathsf{IR}(K_1) = \{MP, SV\}, \qquad (31)$$

$$\mathsf{Ax}(K_2) = \mathsf{Ax}(K_1) + S_4, \qquad \mathsf{IR}(K_2) = \{MP, SV\}, \qquad (32)$$

$$\mathsf{Ax}(K_3) = \mathsf{Ax}(K_2) + S_5, \qquad \mathsf{IR}(K_3) = \{MP, SV\}. \qquad (33)$$

Theorem 3.13 (Soundness) *The logics* $\mathrm{PLS}(Reg, K_1)$, $\mathrm{PLS}(Reg^*, K_2)$ *and* $\mathrm{PLS}(Dec, K_3)$ *are sound.*

Proof. This follows essentially from the general soundness Theorem 3.12. We have $Dec \subseteq Reg^* \subseteq Reg$, so all these classes of choice functions satisfy the condition $X \subseteq Reg$ of 3.12. Also $\mathsf{IR}(K_i) = \{MP, SV\}$, for $i = 1, 2, 3$. By Corollary 3.6, $\mathsf{Ax}(K_0) \subset Taut(\mathcal{F})$, and $Taut(\mathcal{F}) \subseteq Taut(Reg) \subseteq Taut(Reg^*) \subseteq Taut(Dec)$. Since $\mathsf{Ax}(K_1) = \mathsf{Ax}(K_0)$, we have $\mathsf{Ax}(K_1) \subseteq Taut(Reg)$, so it follows that $\mathrm{PLS}(Reg, K_1)$ is sound. Next $\mathsf{Ax}(K_2) = \mathsf{Ax}(K_0) + S_4$, so to see that $\mathrm{PLS}(Reg^*, K_2)$ is sound, it suffices to see that $S_4 \in Taut(Reg^*)$. But by Theorem 2.19 $S_4 \in Taut(Asso) \subset Taut(Reg^*)$. Therefore $\mathsf{Ax}(K_2) \subseteq Taut(Reg^*)$, and we are done. Finally $\mathsf{Ax}(K_3) = \mathsf{Ax}(K_2) + S_5$ and by Theorem 2.42, the scheme S_5 characterizes $\neg$-decreasingness, thus $S_5 \in Taut(Dec)$. So $\mathsf{Ax}(K_3) \subset Taut(Dec)$ and by 3.12 $\mathrm{PLS}(Dec, K_3)$ is sound. $\dashv$

The following Lemma will be essential for the completeness of the aforementioned logics, proved in the next section.

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Lemma 3.14 *If $\Sigma \subset Sen(L_s)$ is closed with respect to $\vdash_{K_i}$, for some $i = 1, 2, 3$, and α, α' are sentences of L such that $\alpha \sim \alpha'$, then for every β, $(\alpha|\beta \leftrightarrow \alpha'|\beta) \in \Sigma$.*

Proof. Let $\alpha \sim \alpha'$. Then $\vdash_{PL} \alpha \leftrightarrow \alpha'$, hence also $\vdash_{K_0} \alpha \leftrightarrow \alpha'$. By $SV \in \mathsf{IR}(K_i)$ it follows that for every β, $\vdash_{K_i} \alpha|\beta \leftrightarrow \alpha'|\beta$. Therefore $(\alpha|\beta \leftrightarrow \alpha'|\beta) \in \Sigma$ since Σ is $\vdash_{K_i}$-closed. $\dashv$

Question 3.15 *Do the formal systems K_1-K_3 satisfy the Deduction Theorem (DT)?*

We guess that the answer to this question is negative but we do not have a proof. The standard way to prove DT for $\vdash_{K_i}$ is to assume $\Sigma \cup \{\varphi\} \vdash_{K_i} \psi$, pick a proof $\psi_1, \ldots, \psi_n$ of ψ, with $\psi_n = \psi$, and show that $\Sigma \vdash_{K_i} \varphi \rightarrow \psi_i$, for every $i = 1, \ldots, n$, by induction on i. The only crucial step of the induction is the one concerning the rule SV, i.e., to show that for any σ, σ', τ, if $\Sigma \vdash_{K_i} \varphi \rightarrow (\sigma \leftrightarrow \sigma')$, and $\vdash_{K_0} (\sigma \leftrightarrow \sigma')$, then $\Sigma \vdash_{K_i} \varphi \rightarrow (\sigma|\tau \leftrightarrow \sigma'|\tau)$. Now clearly

$$\vdash_{PL} (\sigma \leftrightarrow \sigma') \rightarrow (\varphi \rightarrow (\sigma \leftrightarrow \sigma')),$$

so also

$$\vdash_{K_0} (\sigma \leftrightarrow \sigma') \rightarrow (\varphi \rightarrow (\sigma \leftrightarrow \sigma')).$$

This combined with $\vdash_{K_0} (\sigma \leftrightarrow \sigma')$ and MP gives

$$\vdash_{K_0} \varphi \rightarrow (\sigma \leftrightarrow \sigma')),$$

and hence, by PL again,

$$\vdash_{K_0} (\varphi \rightarrow \sigma) \leftrightarrow (\varphi \rightarrow \sigma').$$

By SV it follows that

$$\Sigma \vdash_{K_i} ((\varphi \rightarrow \sigma)|\tau) \leftrightarrow ((\varphi \rightarrow \sigma')|\tau).$$

However it is not clear if and how one can get from the latter the required derivation $\Sigma \vdash_{K_i} \varphi \rightarrow (\sigma|\tau \leftrightarrow \sigma'|\tau)$.

It follows from the preceding discussion that DT is open for the formal systems K_i, $i = 1, 2, 3$. Now by Fact 3.4, if DT fails for K_i then necessarily CT1 fails for the logics $\mathrm{PLS}(Reg, K_1)$, $\mathrm{PLS}(Reg^*, K_2)$ and $\mathrm{PLS}(Dec, K_3)$. This means that CT1 is also open for the preceding logics. (In connection with the status of DT note that, surprisingly enough, the question about

the validity of this theorem remains essentially unsettled even for a logical theory as old as modal logic, see [5].)

Completeness We come to the completeness of the aforementioned logics based on the systems K_1-K_3. First in view of the open status of DT for the systems K_1-K_3 and Fact 3.3 (ii), we cannot identify the two forms of completeness CT1 and CT2 for these systems. We only know that (CT1)$\Rightarrow$(CT2). So we can hope to prove CT2 for K_1-K_3.

There is however another serious side-effect of the lack of DT. This is that we don't know whether every consistent set of sentences can be extended to a consistent and *complete* set. Clearly every consistent set Σ can be extended (e.g. by Zorn's Lemma) to a *maximal* consistent set $\Sigma' \supseteq \Sigma$. But maximality of Σ' cannot guarantee completeness without DT (while the converse is true). For, theoretically, Σ' may be maximal consistent and yet there is a φ such that $\varphi \notin \Sigma'$ and $\neg\varphi \notin \Sigma'$, in which case $\Sigma' \cup \{\varphi\}$ and $\Sigma' \cup \{\neg\varphi\}$ are both inconsistent. That looks strange but we don't see how it could be proved false without DT. This property of extendibility of a consistent set to a consistent and complete one, for a formal system K, plays a crucial role in the proof of completeness of K (with respect to a given semantics), so we isolate it as property of K denoted $cext(K)$. Namely we set

$(cext(K))$ *Every K-consistent set of sentences can be extended to*

         a K-consistent and complete set.

In view of the unknown truth-value of $cext(K_i)$, for $i = 1, 2, 3$, we shall prove only *conditional* versions of CT2-completeness for these systems. Actually it is shown that CT2-completeness is *equivalent* to $cext(K_i)$.

Theorem 3.16 (Conditional CT2-completeness for PLS(Reg, K_1)) *The logic* PLS(Reg, K_1) *is* CT2-*complete if and only if $cext(K_1)$ is true.*

Proof. One direction is easy. Assume $cext(K_1)$ is false. Then there is a maximal K_1-consistent set of sentences Σ non-extendible to a K_1-consistent and complete one. It means that there is a sentence φ such that both $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are K_1-inconsistent. But then Σ is not Reg-satisfiable. For if there are M and $f \in Reg$ such that $\langle M, f \rangle \models_s \Sigma$, then $\langle M, f \rangle$ satisfies also either φ or $\neg\varphi$. Thus either $\Sigma \cup \{\varphi\}$ or $\Sigma \cup \{\neg\varphi\}$ is Reg-satisfiable. But this is a contradiction since both $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are inconsistent and by Theorem 3.5 PLS(Reg, K_1) is sound. Therefore Σ is consistent and not Reg-satisfiable, so PLS(Reg, K_1) is not CT2-complete.

We come to the main direction of the equivalence assuming $cext(K_1)$ is true. Then given a K_1-consistent set Σ, we may assume without loss of generality that it is also complete. We have to find M and $g \in Reg$ such that $\langle M, g \rangle \models_s \Sigma$. It turns out that the main argument of Lemma 3.8, concerning the definition of the choice function g, works also, with the necessary adjustments, for the other logics defined in the previous section. Namely it suffices to find a choice function $g \in Reg$ such that $\langle M, g \rangle \models \Sigma$, where M is a model of $\Sigma_1 = \Sigma \cap Sen(L)$. The definition of g follows exactly the pattern of definition of g in the proof of Lemma 3.8, except that we need now to take care so that g be regular. Recall that g is regular if for all α, α', β,

$$\alpha' \sim \alpha \;\Rightarrow\; g(\alpha', \beta) \sim g(\alpha, \beta).$$

In (27) g is defined by three clauses: (i) (a2) or (a6), (ii) (a3) or (a5), (iii) (a1) or (a4).

Claim. The regularity constraint is satisfied whenever g is defined by clauses (i) and (ii) above.

Proof of Claim. Pick α, α', β such that $\alpha \sim \alpha'$. We prove the Claim for the case that $g(\alpha, \beta)$ is defined according to clause (i)-(a2). All other cases are verified similarly. That $g(\alpha, \beta)$ is defined by case (i)-(a2) of (27) means that $\alpha|\beta \in \Sigma$, $\alpha \in \Sigma$, $\neg\beta \in \Sigma$ and $g(\alpha, \beta) = \alpha$. It suffices to see that necessarily $g(\alpha', \beta) = \alpha' \sim g(\alpha, \beta)$.

Since Σ is complete, it is closed with respect to $\vdash_{K_1}$, so by Lemma 3.14, $\alpha \sim \alpha'$ implies that $(\alpha|\beta \leftrightarrow \alpha'|\beta) \in \Sigma$. Also by assumption, $\alpha|\beta \in \Sigma$, hence $\alpha'|\beta \in \Sigma$. Moreover $\alpha' \in \Sigma$, since $\alpha \in \Sigma$, and $\neg\beta \in \Sigma$. Therefore case (i)-(a2) occurs too for $\alpha'|\beta$, α' and β. So, by (27), $g(\alpha', \beta) = \alpha'$, therefore $g(\alpha', \beta) \sim g(\alpha, \beta)$. This proves the Claim.

It follows from the Claim that if we define g according to (27), regularity is guaranteed unless $g(\alpha, \beta)$ is given by clause (iii), that is, unless (a1) or (a4) is the case. In such a case either both α, β belong to Σ, or both $\neg\alpha$, $\neg\beta$ belong to Σ, and (27) allows $g(\alpha, \beta)$ to be *any* of the elements α, β. So at this point we must intervene by a new condition that will guarantee regularity. This is done as follows.

Pick, as in the proof of Proposition 2.30, from each $\sim$-equivalence class $[\alpha]$, a representative $\xi_\alpha \in [\alpha]$. Recall that, by completeness, the set $\Sigma_1 = \Sigma \cap Sen(L)$ as well as its complement $\Sigma_2 = Sen(L) - \Sigma_1$ are saturated with respect to $\sim$, that is, for every α, either $[\alpha] \subset \Sigma_1$ or $[\alpha] \subset \Sigma_2$. Let $D_1 = \{\xi_\alpha : \alpha \in \Sigma_1\}$, $D_2 = \{\xi_\alpha : \alpha \in \Sigma_2\}$. Let $[D_i]^2$ be the set of pairs of elements of D_i, for $i = 1, 2$, and pick an arbitrary choice function $g_0 :$

$[D_1]^2 \cup [D_2]^2 \to D_1 \cup D_2$. Then it suffices to define g by slightly revising definition (27) as follows:

$$g(\alpha, \beta) = \begin{cases} (i) \ \alpha, \ \text{if } \{\alpha, \beta\}, \ \text{satisfies (a2) or (a6)} \\ (ii) \ \beta, \ \text{if } \{\alpha, \beta\} \ \text{satisfies (a3) or (a5)} \\ (iii) \ \sim g_0(\xi_\alpha, \xi_\beta), \ \text{if } \{\alpha, \beta\} \ \text{satisfies (a1) or (a4).} \end{cases} \tag{34}$$

(The third clause is just a shorthand for: $g(\alpha, \beta) = \alpha$ if $g_0(\xi_\alpha, \xi_\beta) = \xi_\alpha$, and $g(\alpha, \beta) = \beta$ if $g_0(\xi_\alpha, \xi_\beta) = \xi_\beta$. See the similar formulation in the proof of 2.30.) In view of the Claim and the specific definition of g by (34), it follows immediately that if $\alpha \sim \alpha'$ then for every β, $g(\alpha, \beta) \sim g(\alpha', \beta)$. So g is regular. Further, exactly as in Lemma 3.8 it follows that $\langle M, g \rangle \models_s \Sigma$. This completes the proof. $\dashv$

Next we come to the logic PLS(Reg^*, K_2). The difference of K_2-consistency from K_1-consistency is that, as a result of axiom S_4, if Σ is K_2-consistent and $\varphi|(\psi|\sigma) \in \Sigma$, then $(\varphi|\psi)|\sigma \in \Sigma$, or more simply $\varphi|\psi|\sigma \in \Sigma$. Let us outline this difference by an example.

Example 3.17 *Let*

$$\Sigma = \{\alpha, \neg\beta, \neg\gamma, \alpha|\beta, \neg(\alpha|\gamma), \alpha|(\beta|\gamma)\},$$

where α, β, γ are pairwise inequivalent and $\alpha \wedge \neg\beta \wedge \neg\gamma$ is satisfiable. Then Σ is Reg-satisfiable, hence K_1-consistent, but is not K_2-consistent. In particular Σ is not Asso-satisfiable.

Proof. By hypothesis there is a truth assignment M such that $M \models \alpha \wedge \neg\beta \wedge \neg\gamma$. Pick a (partial) choice function for L such that $f(\alpha, \beta) = \alpha$, $f(\alpha, \gamma) = \gamma$ and $f(\beta, \gamma) = \beta$. Since α, β, γ are pairwise inequivalent, it is easy to see that f extends to a regular choice function for the entire L. Then $\overline{f}(\alpha|\beta) = \alpha$, $\overline{f}(\alpha|\gamma) = \gamma$ and $\overline{f}(\beta|\gamma) = \beta$. So $\overline{f}(\neg(\alpha|\gamma)) = \neg\gamma$. It follows that $\langle M, f \rangle \models_s \{\alpha|\beta, \neg(\alpha|\gamma)\}$. Moreover $\overline{f}(\alpha|(\beta|\gamma)) = f(\alpha, f(\beta, \gamma)) = f(\alpha, \beta) = \alpha$, which means that $\langle M, f \rangle \models_s \alpha|(\beta|\gamma)$ too. Thus $\langle M, f \rangle \models_s \Sigma$, so Σ is *Reg*-satisfiable.

Now in view of axiom S_4 of K_2, since $\alpha|(\beta|\gamma) \in \Sigma$ it follows that $\Sigma \vdash_{K_2} \alpha|\beta|\gamma$. By S_2 and S_3, the latter implies $\Sigma \vdash_{K_2} (\alpha|\gamma) \vee \beta$. On the other hand $\neg(\alpha|\gamma) \in \Sigma$ and $\neg\beta \in \Sigma$, so $\Sigma \vdash_{K_2} \neg(\alpha|\gamma) \wedge \neg\beta$, or $\Sigma \vdash_{K_2} \neg((\alpha|\gamma) \vee \beta)$. Thus $\Sigma \vdash_{K_2} \bot$, so it is K_2-inconsistent.

Finally assume that Σ is satisfied in $\langle N, f \rangle$, for some assignment N and some associative f. Let $f = \min_< = \min$ for some total ordering

$<$ of $Sen(L)$. Now $\langle N, f\rangle \models_s \{\alpha, \neg\beta, \alpha|\beta\}$ implies $\min(\alpha, \beta) = \alpha$, i.e., $\alpha < \beta$, while $\langle N, f\rangle \models_s \{\alpha, \neg\gamma, \neg(\alpha|\gamma)\}$ implies $\min(\alpha, \gamma) = \gamma$, so $\gamma < \alpha$. Therefore $\gamma < \alpha < \beta$. On the other hand, $\langle N, f\rangle \models_s \alpha|(\beta|\gamma)$ implies $N \models \min(\alpha, \min(\beta, \gamma)) = \min(\alpha, \beta, \gamma)$, therefore $\min(\alpha, \beta, \gamma) = \alpha$ since $N \models \neg\beta \wedge \neg\gamma$. Thus $\alpha < \gamma$, a contradiction. $\dashv$

Theorem 3.18 (Conditional CT2-completeness for $\mathrm{PLS}(Reg^*, K_2)$) *The logic* $\mathrm{PLS}(Reg^*, K_2)$ *is* CT2-*complete if and only if* $cext(K_2)$ *is true.*

Proof. One direction of the equivalence is proved exactly as the corresponding direction of Theorem 3.16. So let us come to the other direction assuming $cext(K_2)$ is true. Let Σ be a K_2-consistent set, so we may assume again that Σ is also complete. We must construct a regular and associative choice function g such that $\langle M, g\rangle \models \Sigma$, where $M \models \Sigma_1$. As already remarked, $\alpha|(\beta|\gamma) \in \Sigma$ implies $(\alpha|\beta)|\gamma \in \Sigma$. We shall define $\overline{g}$ basically as in definition (27) of Lemma 3.8, except that now we want g to induce a regular total ordering of $Sen(L)$. So let h be a partial choice function for L such that

$$dom(h) = \{\{\alpha, \beta\} : \{\alpha, \beta\} \text{ satisfies some of the cases (a2), (a3), (a5) and}$$

$$\text{(a6) of Lemma 3.8}\},$$

and

$$h(\alpha, \beta) = \left\{ \begin{array}{ll} (i)\ \alpha, & \text{if } \{\alpha, \beta\} \text{ satisfies (a2) or (a6)}, \\ (ii)\ \beta, & \text{if } \{\alpha, \beta\} \text{ satisfies (a3) or (a5)}. \end{array} \right. \tag{35}$$

Claim 1. For any α, β, γ, whenever two of the $h(\alpha, h(\beta, \gamma))$, $h(\beta, h(\alpha, \gamma))$, $h(\gamma, h(\alpha, \beta))$ are defined, they are equal.

Proof of Claim 1. Pick some α, β, γ. Then at least two of them belong either to Σ or to its complement. Without loss of generality assume that $\alpha \in \Sigma$, $\beta \notin \Sigma$, $\gamma \notin \Sigma$. Then in view of K_2-consistency and completeness of Σ,

$$A = \{\alpha, \neg\beta, \neg\gamma, \neg(\beta|\gamma)\} \subset \Sigma.$$

Also by K_2-consistency and completeness we can identify $\alpha|(\beta|\gamma)$ and $(\alpha|\beta)|\gamma$, with respect to their containment to Σ, and there are two options: either $\alpha|\beta|\gamma \in \Sigma$ or $\neg(\alpha|\beta|\gamma) \in \Sigma$. We consider now the combinations of the sentences $\alpha|\beta|\gamma$, $\alpha|\beta$, $\alpha|\gamma$ and their negations that can belong to Σ together with the elements of A. We write these combinations in the form of sets B_i. It is easy to see that the only sets B_i of this kind such that $A \cup B_i \subset \Sigma$, are the following:

$B_1 = \{\alpha|\beta|\gamma, \alpha|\beta, \alpha|\gamma\}$
$B_2 = \{\neg(\alpha|\beta|\gamma), \alpha|\beta, \alpha|\gamma\}$
$B_3 = \{\neg(\alpha|\beta|\gamma), \alpha|\beta, \neg(\alpha|\gamma)\}$
$B_4 = \{\neg(\alpha|\beta|\gamma), \neg(\alpha|\beta), \alpha|\gamma\}$
$B_5 = \{\neg(\alpha|\beta|\gamma), \neg(\alpha|\beta), \neg(\alpha|\gamma)\}$

The remaining sets:
$B_1' = \{\alpha|\beta|\gamma, \alpha|\beta, \neg(\alpha|\gamma)\}$
$B_2' = \{\alpha|\beta|\gamma, \neg(\alpha|\beta), \alpha|\gamma\}$
$B_3' = \{\alpha|\beta|\gamma, \neg(\alpha|\beta), \neg(\alpha|\gamma)\}$
cannot be included in Σ jointly with A. [For instance assume

$$A \cup \{\alpha|\beta|\gamma, \alpha|\beta, \neg(\alpha|\gamma)\} = \{\alpha, \neg\beta, \neg\gamma, \neg(\beta|\gamma), \alpha|\beta|\gamma, \alpha|\beta, \neg(\alpha|\gamma)\} \subset \Sigma.$$

Then $\alpha|\beta|\gamma$ is written $\beta|\alpha|\gamma$ so by S_2, it implies $\beta \vee (\alpha|\gamma) \in \Sigma$. By completeness, either $\beta \in \Sigma$ or $\alpha|\gamma \in \Sigma$. But already $\neg\beta$ and $\neg(\alpha|\gamma)$ are in Σ, a contradiction.]

Since $\beta, \gamma \notin \Sigma$, $h(\beta, \gamma)$, and hence $h(\alpha, h(\beta, \gamma))$ are not defined by (i) or (ii) of (27). So it suffices to verify that $h(\beta, h(\alpha, \gamma)) = h(\gamma, h(\alpha, \beta))$ in each of the cases $A \cup B_i \subset \Sigma$, for $1 \leq i \leq 5$.

1) $A \cup B_1 \subset \Sigma$: We have $h(\alpha, \beta) = \alpha$, $h(\alpha, \gamma) = \alpha$. Then

$$h(\beta, h(\alpha, \gamma)) = h(\beta, \alpha) = \alpha = h(\alpha, \gamma) = h(\gamma, h(\alpha, \beta)),$$

so the Claim holds.

2) $A \cup B_2 \subset \Sigma$: Same as before.

3) $A \cup B_3 \subset \Sigma$: We have $h(\alpha, \beta) = \alpha$, $h(\alpha, \gamma) = \gamma$. Thus $h(\beta, h(\alpha, \gamma) = h(\beta, \gamma)$, so $h(\beta, h(\alpha, \gamma)$ is also undefined. We se that only $h(\gamma, h(\alpha, \beta)) = \gamma$ is defined, so the Claim holds vacuously.

4) $A \cup B_4 \subset \Sigma$: We have $h(\alpha, \beta) = \beta$, $h(\alpha, \gamma) = \alpha$. Thus $h(\gamma, h(\alpha, \beta)) = g(\gamma, \beta)$ is undefined, and the Claim holds vacuously as before.

5) $A \cup B_5 \subset \Sigma$: We have $h(\alpha, \beta) = \beta$, $h(\alpha, \gamma) = \gamma$. Thus $h(\beta, h(\alpha, \gamma)) = g(\beta, \gamma)$ is undefined, and we are done again. This completes the proof of Claim 1.

Claim 2. Let

$$S = \{\langle \alpha, \beta \rangle : \{\alpha, \beta\} \in dom(h) \wedge h(\alpha, \beta) = \alpha\},$$

and let $<_1$ be the transitive closure of S. Then $<_1$ is a regular partial ordering on $Sen(L)$.

Proof of Claim 2. By Claim 1, h is an associative partial choice function, so as in the proof of Theorem 2.14 we can see that the transitive closure $<_1$ of S is a partial ordering. Also that $<_1$ is a regular ordering follows from the Claim of Theorem 3.16. This completes the proof of Claim 2.

Now clearly the partial ordering $<_1$ of Claim 2 extends to a regular total ordering $<$ of $Sen(L)$. Then it suffices to define g by setting $g = \min_<$. Since for every $\alpha, \beta \in Sen(L)$, if the pair $\langle \alpha, \beta \rangle$ satisfies some of the cases (a2), (a3), (a5), (a6), $\alpha < \beta$ if and only if $h(\alpha, \beta) = \alpha$, clearly g extends h. Moreover

$$
g(\alpha, \beta) = \begin{cases}
(i) \ \alpha, \ \text{if } \langle \alpha, \beta \rangle \text{ satisfies (a2) or (a6)}, \\
(ii) \ \beta, \ \text{if } \langle \alpha, \beta \rangle \text{ satisfies (a3) or (a5)}, \\
(iii) \ \min_<(\alpha, \beta), \ \text{if } \langle \alpha, \beta \rangle \text{ satisfies (a1) or (a4)}.
\end{cases} \tag{36}
$$

Thus it follows as in Lemma 3.8 that $\langle M, g \rangle \models_s \Sigma$, that is, Σ is Reg^*-satisfiable. This completes the proof of the theorem. $\dashv$

Finally we come to the conditional completeness of $\mathrm{PLS}(Dec, K_3)$.

Theorem 3.19 (Conditional CT2-completeness for $\mathrm{PLS}(Dec, K_3)$) *The logic* $\mathrm{PLS}(Dec, K_3)$ *is CT2-complete if and only if* $cext(K_3)$ *is true.*

Proof. Again one direction of the equivalence is shown exactly as the corresponding direction of Theorem 3.16. We come to the other direction assuming $cext(K_3)$ is true. Fix a K_3-consistent set Σ. By $cext(K_3)$ we may assume that Σ is also complete. Let $M \models \Sigma_1$, where $\Sigma_1 = \Sigma \cap Sen(L)$. We show that there exists a choice function g such that $g = \min_<$, where $<$ is a $\neg$-decreasing regular total ordering of $Sen(L)$, and $\langle M, g \rangle \models \Sigma$. g is essentially defined as in the previous theorem plus an extra adjustment that guarantees $\neg$-decreasingness. Namely, let h be the function defined exactly as in the proof of Theorem 3.18.

Claim. h is $\neg$-decreasing, i.e., whenever $h(\alpha, \beta)$ and $h(\neg\alpha, \neg\beta)$ are defined, then

$$
h(\alpha, \beta) = \alpha \ \Leftrightarrow \ h(\neg\alpha, \neg\beta) = \neg\beta. \tag{37}
$$

Proof of Claim. We must check that whenever $\{\alpha, \beta\}$ and $\{\neg\alpha, \neg\beta\}$ satisfy some of the cases (a2), (a3), (a5) and (a6), then (37) holds true. Thus we must examine the combinations of α, β, $\alpha|\beta$, $\neg\alpha|\neg\beta$ and their

negations that can belong to Σ. There is a total of 16 possible combinations of these sentences. Of them the combinations

$$U_1 = \{\alpha|\beta, \neg(\neg\alpha|\neg\beta), \alpha, \beta\}$$
$$U_2 = \{\neg(\alpha|\beta), \neg\alpha|\neg\beta, \neg\alpha, \neg\beta\}$$

do not allow definition of h since in these cases either both α, β or both $\neg\alpha$, $\neg\beta$ belong to Σ. Next we have 10 combinations that contradict K_3-consistency and completeness of Σ. These are:

$$F_1 = \{\alpha|\beta, \neg\alpha|\neg\beta, \alpha, \beta\}$$
$$F_2 = \{\alpha|\beta, \neg\alpha|\neg\beta, \neg\alpha, \neg\beta\}$$
$$F_3 = \{\alpha|\beta, \neg(\neg\alpha|\neg\beta), \neg\alpha, \neg\beta\}$$
$$F_4 = \{\alpha|\beta, \neg(\neg\alpha|\neg\beta), \neg\alpha, \beta\}$$
$$F_5 = \{\alpha|\beta, \neg(\neg\alpha|\neg\beta), \alpha, \neg\beta\}$$
$$F_6 = \{\neg(\alpha|\beta), \neg\alpha|\neg\beta, \alpha, \beta\}$$
$$F_7 = \{\neg(\alpha|\beta), \neg\alpha|\neg\beta, \neg\alpha, \beta\}$$
$$F_8 = \{\neg(\alpha|\beta), \neg\alpha|\neg\beta, \alpha, \neg\beta\}$$
$$F_9 = \{\neg(\alpha|\beta), \neg(\neg\alpha|\neg\beta), \alpha, \beta\}$$
$$F_{10} = \{\neg(\alpha|\beta), \neg(\neg\alpha|\neg\beta), \neg\alpha, \neg\beta\}.$$

Notice that of the preceding sets, F_4, F_5, F_7 and F_8 yield a contradiction because of the axiom S_5. For instance consider $F_4 = \{\alpha|\beta, \neg(\neg\alpha|\neg\beta), \neg\alpha, \beta\}$. It contains $\neg\alpha, \beta$, thus it proves $\neg\alpha \wedge \beta$. By S_5, F_4 proves $(\alpha|\beta \leftrightarrow \neg\alpha|\neg\beta)$. F_4 also contains $\alpha|\beta$, thus it proves $\neg\alpha|\neg\beta$. But it besides contains $\neg(\neg\alpha|\neg\beta)$, so $F_4 \vdash_{K_3} \bot$. Thus the only combinations that can be contained in Σ are the following:

$$C_1 = \{\alpha|\beta, \neg\alpha|\neg\beta, \alpha, \neg\beta\} \subset \Sigma$$
$$C_2 = \{\alpha|\beta, \neg\alpha|\neg\beta, \neg\alpha, \beta\} \subset \Sigma$$
$$C_3 = \{\neg(\alpha|\beta), \neg(\neg\alpha|\neg\beta), \alpha, \neg\beta\} \subset \Sigma$$
$$C_4 = \{\neg(\alpha|\beta), \neg(\neg\alpha|\neg\beta), \neg\alpha, \beta\} \subset \Sigma.$$

It is easy to verify that in each of the cases $C_i \subset \Sigma$, for $1 \leq i \leq 4$, (37) is true in view of the definition (35) of h. For example in case $C_4 \subset \Sigma$, necessarily $h(\alpha, \beta) = \alpha$, while $h(\neg\alpha, \neg\beta) = \neg\beta$. This completes the proof of Claim 1.

As in the proof of 3.18, let

$$S = \{\langle \alpha, \beta \rangle : \{\alpha, \beta\} \in dom(h) \wedge h(\alpha, \beta) = \alpha\},$$

and let $<_1$ be the transitive closure of S. As shown in 3.18, $<_1$ is a regular partial ordering. Moreover here, in view of the Claim, $<_1$ is $\neg$-decreasing.

So by a standard application of Zorn's Lemma, $<_1$ extends to a regular $\neg$-decreasing total ordering $<$ of $Sen(L)$. If we set $g = \min_<$, then g satisfies (36) of the previous theorem and thus $\langle M, g \rangle \models_s \Sigma$. Therefore Σ is Dec-satisfiable. $\dashv$

The following is open.

Question 3.20 *If $cext(K_i)$ are true for $i = 1, 2, 3$, do the logics $\mathrm{PLS}(Reg, K_1)$, $\mathrm{PLS}(Reg^*, K_2)$ and $\mathrm{PLS}(Dec, K_3)$ satisfy the form* CT1 *of Completeness Theorem?*

3.3 Some closing remarks on axiomatization

Before closing this section on axiomatization of superposition logics, let us notice that all axioms S_1-S_5 introduced above are true also for the connectives $\wedge$ and $\vee$. That is, none of the S_i can be used to discriminate $|$ from $\wedge$ and $\vee$. This looks somewhat strange, since we showed semantically that the converse of S_1 and S_2 are not tautologies. However this cannot be formulated in the straightforward way, namely as the *schemes $\varphi|\psi \not\rightarrow \varphi \wedge \psi$* and $\varphi \vee \psi \not\rightarrow \varphi|\psi$ (the latter are false, e.g. for $\varphi = \psi$). It means that the axiomatic systems K_i, for $i = 0, 1, 2, 3$ introduced above are *interpretable* in the standard propositional logic PL, through the obvious interpretations $I_\wedge$ and $I_\vee$ that interpret $|$ as $\wedge$ or $\vee$, respectively. These are defined inductively in the obvious way for standard connectives, while for $|$ we have $(\varphi|\psi)^{I_\wedge} = \varphi \wedge \psi$ and $(\varphi|\psi)^{I_\vee} = \varphi \vee \psi$. Then clearly for any φ and for I being some of these interpretations,

$$\vdash_{K_i} \varphi \ \Rightarrow \ \vdash \varphi^I,$$

that is, for every Dec-tautology φ (to consider the strongest system Dec of choice functions), φ^I is a classical tautology. However both of the aforementioned interpretations are not "faithful", which means that the converse of the above implication is not true. For example for classical sentences α, β, $(\alpha|\beta)^{I_\wedge} = \alpha \wedge \beta$, hence $(\alpha|\beta \rightarrow \alpha)^{I_\wedge} = \alpha \wedge \beta \rightarrow \alpha$. Then $\alpha \wedge \beta \rightarrow \alpha$ is a classical tautology while $\alpha|\beta \rightarrow \alpha$ is not a K_i-tautology.

The question is if there exist any further axioms, appropriate for some finer class $X \subset Dec$, which can distinguish $|$ from $\wedge$ and/or $\vee$. The answer is yes. For example a further condition that can be imposed to $\neg$-decreasing orderings is one that concerns the position of the special classes $\bot$ and $\top$ in this ordering. For example we may require that our decreasing orderings $<$ satisfy the condition $\top < \bot$. Let $Dec_{\top < \bot}$ denote the class of these total

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orderings of $Sen(L)$. It is rather straightforward that the additional axiom needed to characterize $Dec_{\top<\bot}$ is

$\quad (S_6) \;\; \bot|\top$.

Thus $\bot|\top$ is a $Dec_\bot$-tautology while $(\bot|\top)^{I_\wedge} = \bot \wedge \top$ is not a classical tautology, which means that the logic corresponding to $Dec_{\top<\bot}$ is not interpretable in PL through $I_\wedge$. We might also require that $\top$ is the least element of $<$. If this class of orderings is denoted by $Dec_\top$, the additional scheme needed to characterize $Dec_\top$ is

$\quad (S_6') \;\; \alpha|\top$.

Again $Dec_{\top<\bot}$ is not interpretable in PL through $I_\wedge$. Similarly if $Dec_{\bot<\top}$, $Dec_\bot$ denote the classes of decreasing orderings $<$ such that $\bot < \top$ and $\bot < \alpha$, for every α, respectively, then the needed corresponding axioms are

$\quad (S_7) \;\; \neg(\bot|\top)$

and

$\quad (S_7') \;\; \neg(\alpha|\bot)$,

respectively. These logics are not interpretable in PL through $I_\vee$.

4 Future work

Our future work will focus on two goals. The first goal is to develop some alternative semantics for superposition logics. We have already found a second semantics based again on choice functions, but this time the choice applies not to pairs of sentences but to pairs of elements of a Boolean algebra $\mathcal{B}$, in which the standard sentences of PL take truth values. We can refer to this as "Boolean-value choice semantics" (BCS) to distinguish it from the "sentence choice semantics" (SCS) used in the present paper.

The second goal is to extend PLS to *first-order superposition logic* (abbreviated FOLS). Such an extension might help us to pass from superposition of sentences/formulas to *superposition of objects*. Given two objects (constants) a and b, let us consider the formula (in one free variable) $(x = a)|(x = b)$. If our logic can prove that there exists a *unique* object c satisfying this formula, then we can set $c = a \uparrow b$ and say that c is the *superposition of a and b*. Thus in order for such an operation to be defined for all objects x, y the sentence $(\forall x, y)(\exists! z)((z = x)|(z = y))$ must be a tautology. So the question is whether there is a suitable formalization of FOLS in which this sentence can be stated and proved.

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