Semantics for first-order superposition logic

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Abstract

We investigate how the sentence choice semantics (SCS) for propositional superposition logic (PLS) developed in [9] could be extended so as to successfully apply to first-order superposition logic(FOLS). There are two options for such an extension. The apparently more natural one is the formula choice semantics (FCS) based on choice functions for pairs of arbitrary formulas of the basis language. It is proved however that the universal instantiation scheme of FOL, $(\forall v)\varphi(v) \rightarrow \varphi(t)$, is false, as a scheme of tautologies, with respect to FCS. This causes the total failure of FCS as a candidate semantics. Then we turn to the other option which is a variant of SCS, since it uses again choice functions for pairs of sentences only. This semantics however presupposes that the applicability of the connective $|$ is restricted to quantifier-free sentences, and thus the class of well-formed formulas and sentences of the language is restricted too. Granted these syntactic restrictions, the usual axiomatizations of FOLS turn out to be sound and conditionally complete with respect to this second semantics, just like the corresponding systems of PLS.

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1 Introduction

In [9] we introduced and investigated various systems of propositional superposition logic (PLS). The systems of PLS extend classical propositional logic (PL). Their language is that of PL augmented with a new binary connective $|$ for the "superposition operation", while their axioms are those of PL together with a few axioms about $|$. The motivating idea was roughly this: if $\varphi|\psi$ denotes the "superposition of two states" (or, more precisely, the propositions expressing these states), as the latter is currently understood in quantum mechanics (QM), what is the *purely logical content* of the operation, that is, what can we say about the *truth* of $\varphi|\psi$ without leaving the ground of classical logic? The basic intuition is that $\varphi|\psi$ strangely expresses *both* some kind of conjunction of the properties φ and ψ (before the measurement), and simultaneously some kind of disjunction of the same properties (after the measurement, i.e., after the "collapsing" of the superposed states). This collapsing can be formalized by the help of a choice function that acts on pairs of sentences $\{\varphi, \psi\}$, turning each formula $\varphi|\psi$ into a classical one. Such functions formed the basis of a semantics for the new logic, called *sentence choice semantics* (or SCS for short), that allows $\varphi|\psi$ to present simultaneously conjunctive and disjunctive characteristics, which are manifested in the "interpolation property", i.e., the property of $\varphi|\psi$ to be strictly logically interpolated between $\varphi \wedge \psi$ and $\varphi \vee \psi$.

Although QM has been the source of motivation for introducing the logical connective of superposition, PLS is *not* a quantum logic based on orthomodular lattices (see [6] for complete information about such structures), as these logics are discussed e.g. in [7]. Nor is it in a similar vein with the content, e.g., of [1], [2], and other papers cited and discussed in [10], that belong to what can be called standard approach to quantum phenomena. As said above, the logic(s) PLS intend only to explore the logical content of the phenomenon of superposition alone. To quote from [9, p. 151]:

> "So the logic presented here is hardly the logic of superposition as this concept is currently used and understood in physics today. It is rather the logic of superposition, when the latter is understood as the 'logical extract' of the corresponding physics concept. Whether it could eventually have applications to the field of QM we don't know."

In response to questions asked by one of the referees let me add some further comments. That PLS (and its first-order extension FOLS considered in this paper) has almost no points in common with standard treatments of QM can be simply inferred from the fact that it is *not* a probabilistic theory. Probabilities have no place in this logical system (as it stands) and I cannot see how it could be revised in order to be compatible with their use. This is why the "collapsing" of the superposed sentence $\varphi|\psi$ is accomplished by means of a *choice* between φ and ψ. And up to my knowledge there is no genuine theory that relates fruitfully choice functions with probabilities. As we put it in [9, p. 151]:

> "The basic idea is that the collapse of the composite state $c_0\vec{u}_0 + c_1\vec{u}_1$ to one of the states $\vec{u}_0$, $\vec{u}_1$ can be seen, *from the point of view of pure logic,* just as a (more or less random) choice from the set of possible outcomes $\{\vec{u}_0, \vec{u}_1\}$. This is because from the point of view of pure logic probabilities are irrelevant or, which amounts to the same thing, the states $\vec{u}_0$ and $\vec{u}_1$ are considered equiprobable. In such a case the superposition of $\vec{u}_0$ and $\vec{u}_1$ is unique and the outcome of the collapse can be decided by a coin tossing or, more strictly, by a *choice function* acting on pairs of observable states, which in our case coincide with pairs of sentences of L. This of course constitutes a major deviation from the standard treatment of superposition, according to which there is not just one superposition of $\vec{u}_0$ and $\vec{u}_1$ but infinitely many, actually as many as the number of linear combinations $c_0\vec{u}_0 + c_1\vec{u}_1$, for $|c_0|^2 + |c_1|^2 = 1$."

Of course, theoretically, we could switch from $\{0, 1\}$-valuations of classical logic to $[0, 1]$-valuations of a non-classical logic. But then the interpretation of superposition would not be "within classical reasoning and commonsense", as was the aim of the original attempt. Perhaps in the future we shall attempt some non-classical interpretation through a continuous-valued logic.

In view of the above, fundamental concepts pertaining to the probabilistic character of the standard treatment of QM, such as global vs local phases, contextuality, the Kochen-Specker theorem, entanglement, etc., simply do not make any sense for our logic. However, despite of this, the systems PLS still seem to have merits. As a second referee wrote: "Even though the interpretation of superposition logics in terms of the original quantum-mechanical motivation is probably dubious, still the connective with a choice semantics is sufficiently interesting per se to justify the investigation; conceivably, the logic can find other interpretations, perhaps of some epistemic or possibilistic kind."

Let us come now to the content of the present work. A natural question, already asked in the last section of [9], is whether the logic of superposition can be extended to a quantified version, i.e., whether the systems of PLS can be extended to corresponding systems of first-order superposition logic (or FOLS for short). At a syntactic level, systems of FOLS extending corresponding systems of PLS are very easily defined. They are just extensions of classical first-order logic (FOL) by the help of the same axioms for $|$ that were used in PLS. This is because there are no new axioms for $|$ involving $\forall$ or $\exists$, as there are no plausible correlations between $|$ and the quantifiers. But at semantic level things are much more complex. First, we made sure that systems of FOLS do have semantics having characteristics quite analogous to that of SCS for PLS. Actually, an alternative semantics that we meanwhile developed for PLS in [10], the Boolean-value choice semantics (or BCS for short), turned out to be suitable also for FOLS.

However the question whether a semantics for FOLS generalizing SCS of PLS is possible, remained. One of the goals of the present paper is to show that the straightforward generalization of the semantics SCS of PLS, namely the semantics based on choice functions for all pairs of *formulas* of a first-order language L, called *formula choice semantics* (or FCS for short), does not work. Specifically, we show that the systems of FOLS fail to be true with respect to FCS (i.e., soundness fails) in an unexpected way: It is not the axioms for $|$ that fail to be tautologies of FCS but one of the fundamental axiom of FOL, namely the Universal Instantiation (UI) scheme, $\forall v\varphi(v) \rightarrow \varphi(t)$. This of course leads to the break down of FCS itself, since it cannot accommodate the most fundamental logical constant of quantified logic, the universal quantifier. This result is shown in section 3. In section 4 we show a related fact concerning non-existence of uniform choice functions.

The second major result shown in this paper is that the semantics SCS (using functions on pairs of *sentences* only rather than arbitrary formulas) *can* be applied also to FOLS, provided we shall restrict the applicability of the connective $|$ to formulas without quantifiers (unless they are classical ones), and thus restrict the class of well-formed formulas of the language $L_s = L \cup \{|\}$ of the logic of superposition. Under this restriction, FOLS is proved sound and conditionally complete with respect to SCS. This result is described in section 5.

Since the content of the present paper relies heavily on the material contained in [9], we need first to recall briefly the main notions and facts established there. This is done in the next subsection.

1.1 Overview of PLS with sentence choice semantics

This subsection overviews the main notions and facts contained in [9]. It is identical to the corresponding introductory subsection 1.1 of [10]. In general a Propositional Superposition Logic (PLS) consists, roughly, of a pair (X, K), where X is the semantic and K is the syntactic part of the logic. Actually K is a *formal system* in the usual sense of the word, and X is a set of functions that provides meaning to sentences in a way described below. PLS(X, K) will denote the propositional superposition logic with semantic part X and syntactic part K. The precise definition of PLS(X, K) will be given below.

Although the semantic part is the most intuitively appealing we start with the description of the syntactic part K. The language of all formal systems K below (or the language of PLS), L_s, is that of standard Propositional Logic (PL) $L = \{p_0, p_1, \ldots\} \cup \{\wedge, \vee, \rightarrow, \leftrightarrow, \neg\}$ augmented with the new binary connective "$|$". That is, $L_s = L \cup \{|\}$. The set of sentences of L_s, $Sen(L_s)$, is defined by induction as usual, with the additional inductive step that $\varphi|\psi$ is a sentence whenever φ and ψ are so.

Throughout the letters α, β, γ range exclusively over the set of sentences of L, $Sen(L)$, while φ, ψ, σ range over elements of $Sen(L_s)$ in general.

A formal system K consists of a set of axioms $\mathsf{Ax}(K)$ and a set of inference rules $\mathsf{IR}(K)$. The axioms of K always include the axioms of PL, while $\mathsf{IR}(K)$ includes the inference rule of PL. So let us first fix the axiomatization for PL consisting of the following axiom schemes (for the language L_s).

(P1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
(P2) $(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma))$
(P3) $(\neg\varphi \rightarrow \neg\psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \varphi)$,

together with the inference rule Modus Ponens (MP). So for every K, $\{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}\} \subset \mathsf{Ax}(K)$ and $MP \in \mathsf{IR}(K)$. In addition each K contains axioms for the new connective $|$. These are some or all of the following schemes.

(S_1) $\varphi \wedge \psi \rightarrow \varphi|\psi$
(S_2) $\varphi|\psi \rightarrow \varphi \vee \psi$
(S_3) $\varphi|\psi \rightarrow \psi|\varphi$
(S_4) $(\varphi|\psi)|\sigma \rightarrow \varphi|(\psi|\sigma)$
(S_5) $\varphi \wedge \neg\psi \rightarrow (\varphi|\psi \leftrightarrow \neg\varphi|\neg\psi)$

Provability (à la Hilbert) in K, denoted $\vdash_K \varphi$, is defined as usual. It is clear that

$$\Sigma \vdash \alpha \ \Rightarrow\ \Sigma \vdash_K \alpha,$$

where $\vdash$ denotes provability in PL. Σ is said to be *K-consistent,* if $\Sigma \nvdash_K \perp$.

Let K_0 denote the formal system described as follows.

$$\mathsf{Ax}(K_0) = \{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}\} + \{S_1, S_2, S_3\}, \qquad \mathsf{IR}(K_0) = \{MP\}.$$

Extensions of K_0 defined below will contain also the rule SV (from *salva veritate*) defined as follows.

$$(SV) \qquad \textit{from } \varphi \leftrightarrow \psi \ \textit{ infer } \varphi|\sigma \leftrightarrow \psi|\sigma,$$

$$\text{if } \varphi \leftrightarrow \psi \text{ is provable in } K_0.$$

The rule SV guarantees that if α, β are classical logically equivalent sentences, then truth is preserved if α is substituted for β in expressions containing $|$ (just as in the case with the standard connectives). Let the formal systems K_1, K_2 and K_3 be defined as follows.

$$\mathsf{Ax}(K_1) = \mathsf{Ax}(K_0), \qquad \mathsf{IR}(K_1) = \{MP, SV\},$$

$$\mathsf{Ax}(K_2) = \mathsf{Ax}(K_1) + S_4, \qquad \mathsf{IR}(K_2) = \{MP, SV\},$$

$$\mathsf{Ax}(K_3) = \mathsf{Ax}(K_2) + S_5, \qquad \mathsf{IR}(K_3) = \{MP, SV\}.$$

A consequence of SV is that if $\vdash_{K_0} (\varphi \leftrightarrow \psi)$ then, for any σ, $\vdash_{K_i} (\varphi|\sigma \leftrightarrow \psi|\sigma)$, for $i = 1, 2, 3$.

So much for the syntax of PLS. We now turn to the semantics. The axioms S_i are motivated by the intended meaning of $|$ already mentioned above, and the corresponding semantics for sentences of L_s based on choice functions. This semantics consists of pairs $\langle v, f\rangle$, where $v : Sen(L) \to \{0, 1\}$ is a usual two-valued assignment of the sentences of L, and f is a choice function for pairs of elements of $Sen(L)$, i.e., $f : [Sen(L)]^2 \to Sen(L)$ such that $f(\{\alpha, \beta\}) \in \{\alpha, \beta\}$, where for any set A, $[A]^2 = \{\{a, b\} : a, b \in A\}$. (For basic facts about choice functions the reader may consult [5].) The functions f are defined also for singletons with $f(\{\alpha\}) = \alpha$. We simplify notation by writing $f(\alpha, \beta)$ instead of $f(\{\alpha, \beta\})$, thus by convention $f(\alpha, \beta) = f(\beta, \alpha)$ and $f(\alpha, \alpha) = \alpha$. f gives rise to a function $\overline{f} : Sen(L_s) \to Sen(L)$, defined inductively as follows.

Definition 1.1 (i) $\overline{f}(\alpha) = \alpha$, for $\alpha \in Sen(L)$,
(ii) $\overline{f}(\varphi \wedge \psi) = \overline{f}(\varphi) \wedge \overline{f}(\psi)$,
(iii) $\overline{f}(\neg\varphi) = \neg\overline{f}(\varphi)$,
(iv) $\overline{f}(\varphi|\psi) = f(\overline{f}(\varphi), \overline{f}(\psi))$.

We refer to $\overline{f}$ as the *collapsing function* induced by f. Then we define the truth of φ in $\langle v, f\rangle$, denoted $\langle v, f\rangle \models_s \varphi$, as follows.

$$\langle v, f\rangle \models_s \varphi :\Leftrightarrow v(\overline{f}(\varphi)) = 1. \tag{1}$$

(In [9] we denote by M the two-valued assignments of sentences of L and write $\langle M, f\rangle$ instead of $\langle v, f\rangle$. Also we write $M \models \alpha$ instead of $M(\alpha) = 1$.)

We shall refer to the semantics defined by (1) as *sentence choice semantics,* or SCS for short. A remarkably similar notion of choice function for pairs of sentences, and its interpretation as a "conservative" binary connective, was given also independently in [4] (see Example 3.24.14, p. 479).

The reason that we used four formal systems K_0-K_3, in increasing strength, is that they correspond to four different classes of choice functions defined below.

Definition 1.2 Let $\mathcal{F}$ denote the set of all choice functions for $Sen(L)$ and let $X \subseteq \mathcal{F}$.

(i) For a set $\Sigma \subseteq Sen(L_s)$ and $X \subseteq \mathcal{F}$, Σ is said to be *X-satisfiable* if there are v and $f \in X$ such that $\langle v, f\rangle \models_s \Sigma$.

(ii) For $\Sigma \subseteq Sen(L_s)$ and $\varphi \in Sen(L_s)$, φ is an *X-logical consequence of* Σ, denoted $\Sigma \models_X \varphi$, if for every v and every $f \in X$, $\langle v, f\rangle \models_s \Sigma \Rightarrow \langle v, f\rangle \models_s \varphi$.

(iii) φ is an *X-tautology,* denoted $\models_X \varphi$, if $\emptyset \models_X \varphi$.

iv) φ and ψ are *X-logically equivalent,* denoted $\varphi \sim_X \psi$, if $\models_X (\varphi \leftrightarrow \psi)$. Also let

$$Taut(X) = \{\varphi \in Sen(L_s) : \models_X \varphi\}.$$

One of the motivating results behind the development of PLS was the following "interpolation property" of $\varphi|\psi$ with respect to $\varphi \wedge \psi$ and $\varphi \vee \psi$ (see Theorem 2.8 of [9]).

Fact 1.3 *For all $\varphi, \psi \in Sen(L_s)$,*

$$\varphi \wedge \psi \models_{\mathcal{F}} \varphi|\psi \models_{\mathcal{F}} \varphi \vee \psi,$$

while in general

$$\varphi \vee \psi \not\models_{\mathcal{F}} \varphi|\psi \not\models_{\mathcal{F}} \varphi \wedge \psi.$$

Now while the axioms of K_0 are easily seen to be $\models_{\mathcal{F}}$-tautologies, this is not the case with the axioms S_4 and S_5. They correspond to some special subclasses of $\mathcal{F}$ described below.

Definition 1.4 1) An $f \in \mathcal{F}$ is said to be *associative* if for all $\alpha, \beta, \gamma \in Sen(L)$

$$f(f(\alpha, \beta), \gamma) = f(\alpha, f(\beta, \gamma)).$$

2) An $f \in \mathcal{F}$ is said to be *regular* if for all $\alpha, \alpha', \beta \in Sen(L)$,

$$\alpha \sim \alpha' \Rightarrow f(\alpha, \beta) \sim f(\alpha', \beta)$$

where $\alpha \sim \beta$ denotes logical equivalence in PL.

Let

$$Asso = \{f \in \mathcal{F} : f \text{ is asociative}\},$$

$$Reg = \{f \in \mathcal{F} : f \text{ is regular}\},$$

We have the following simple and nice characterization of the functions in $Asso$.

Lemma 1.5 *(*[9, Corollary 2.17]) *$f \in Asso$ if and only if there is a total $<$ ordering of $Sen(L)$ such that $f = \min_<$, i.e., $f(\alpha, \beta) = \min(\alpha, \beta)$ for all $\alpha, \beta \in Sen(L)$.*

(Actually 1.5 holds for associative choice functions on an arbitrary set A, see Theorem 2.14 of [9].) Both properties of associativity and regularity are strongly desirable and would be combined. Also, in view of the above characterization of associative functions through total orderings, the following definition is natural.

Definition 1.6 A total ordering $<$ of $Sen(L)$ is *regular* if the corresponding choice function $f = \min_<$ is regular or, equivalently, if for all α, β in $Sen(L)$

$$\alpha \not\sim \beta \ \& \ \alpha < \beta \ \Rightarrow \ [\alpha] < [\beta],$$

where $[\alpha]$ is the $\sim$-equivalence class of α.

Let

$$Reg^* = Reg \cap Asso.$$

Clearly $f \in Reg^*$ iff $f = \min_<$ for a regular total ordering $<$ of $Sen(L)$.

Definition 1.7 Let $<$ be a total ordering of $Sen(L)$. $<$ is said to be $\neg$-*decreasing* if for all $\alpha, \beta \in Sen(L)$ such that $\alpha \not\sim \beta$,

$$\alpha < \beta \Leftrightarrow \neg\beta < \neg\alpha.$$

If $f \in Reg^*$, f is said to be $\neg$-*decreasing* if $f = \min_<$ for some $\neg$-decreasing $<$.

Let

$$Dec = \{f \in Reg^* : f \text{ is } \neg\text{-decreasing}\}.$$

Since $Dec \subseteq Reg^* \subseteq Reg \subseteq \mathcal{F}$, it follows that

$$Taut(\mathcal{F}) \subseteq Taut(Reg) \subseteq Taut(Reg^*) \subseteq Taut(Dec).$$

We can give now a full specification of the meaning of the notation

$$\mathrm{PLS}(X, K)$$

already introduced in the beginning of this section: given a set $X \subseteq \mathcal{F}$, and a formal system K with $\mathsf{Ax}(K) \subseteq Taut(X)$, $\mathrm{PLS}(X, K)$ is the logic with logical consequence relation $\models_X$, determined by the structures $\langle v, f\rangle$, with $f \in X$, and with provability relation $\vdash_K$. Given a logic $\mathrm{PLS}(X, K)$, the soundness and completeness theorems for it refer as usual to the connections between the relations $\models_X$ and $\vdash_K$, or between X-satisfiability and K-consistency.

At this point a word of caution is needed. As is well-known the soundness theorem (ST) and completeness theorem (CT) of a logic have two distinct formulations which are equivalent for classical logic, but need not be so in general. For the logic $\mathrm{PLS}(X, K)$ these two forms, ST1 and ST2 for Soundness and CT1 and CT2 for Completeness, are the following.

(ST1) $$\Sigma \vdash_K \varphi \;\Rightarrow\; \Sigma \models_X \varphi,$$

(ST2) $$\Sigma \text{ is } X\text{-satisfiable} \;\Rightarrow\; \Sigma \text{ is } K\text{-consistent}$$

(CT1) $$\Sigma \models_X \varphi \;\Rightarrow\; \Sigma \vdash_K \varphi,$$

(CT2) $$\Sigma \text{ is } K\text{-consistent} \;\Rightarrow\; \Sigma \text{ is } X\text{-satisfiable}.$$

ST1 and ST2 are easily shown to be equivalent for every system $\mathrm{PLS}(X, K)$. Moreover the Soundness Theorem for each one of the logics $\mathrm{PLS}(\mathcal{F}, K_0)$, $\mathrm{PLS}(Reg, K_1)$, $\mathrm{PLS}(Reg^*, K_2)$ and $\mathrm{PLS}(Dec, K_3)$ is easily established. But the equivalence of CT1 and CT2 is based on the *Deduction Theorem* (DT) which is not known to be true for every $\mathrm{PLS}(X, K)$, when K contains the inference rule SV. Recall that DT is the following implication. For all Σ, φ, ψ,

$$\Sigma \cup \{\varphi\} \vdash_K \psi \;\Rightarrow\; \Sigma \vdash_K \varphi \to \psi. \tag{2}$$

Concerning the relationship between CT1 and CT2 for $\mathrm{PLS}(X, K)$ the following holds.

Fact 1.8 CT1 $\Rightarrow$ CT2 *holds for every* PLS(X, K). *If* $\vdash_K$ *satisfies DT, then the converse holds too, i.e.,* CT1 $\Leftrightarrow$ CT2.

The system PLS$(\mathcal{F}, K_0)$, whose only inference rule is MP, satisfies CT1 $\Leftrightarrow$ CT2 as a consequence of DT. So we can just say it is "complete" instead of "CT1-complete" and "CT2-complete". The following is shown in [9, §3.1])

Theorem 1.9 PLS$(\mathcal{F}, K_0)$ *is complete.*

However in the systems over K_i, for $i > 0$, that contain the extra rule SV, the status of DT is open, so the distinction between CT1 and CT2 remains. So concerning the logics PLS(Reg, K_1), PLS(Reg^*, K_2) and PLS(Dec, K_3) it is reasonable to try to prove the weaker of the two forms of completeness, namely CT2-completeness. But even this will be proved only conditionally. Because there is still another serious impact of the lack of DT. This is that we don't know if every consistent set of sentences can be extended to a consistent and *complete* set (i.e., one that contains one of the φ and $\neg\varphi$, for every φ). Of course every consistent set Σ can be extended (e.g. by Zorn's Lemma) to a *maximal* consistent set $\Sigma' \supseteq \Sigma$. But maximality of Σ' does not guarantee completeness without DT. Because Σ' may be maximal consistent and yet there is a φ such that $\varphi \notin \Sigma'$ and $\neg\varphi \notin \Sigma'$, so $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are both inconsistent. That looks strange but we don't see how it could be proved false without DT. This property of extendibility of a consistent set to a consistent and complete one, for a formal system K, is crucial for the proof of completeness of K (with respect to a given semantics), so we isolate it as a property of K denoted $cext(K)$. It reads as follows.

($cext(K)$) *Every* K*-consistent set of sentences can be extended to a* K*-consistent and complete set.*

Then the following conditional CT2-completeness results are shown in [9, §3.2]).

Theorem 1.10 *(i)* PLS(Reg, K_1) *is* CT2*-complete if and only if* $cext(K_1)$ *is true.*

(ii) PLS(Reg^*, K_2) *is* CT2*-complete if and only if* $cext(K_2)$ *is true.*

(iii) PLS(Dec, K_3) *is* CT2*-complete if and only if* $cext(K_3)$ *is true.*

2 First-order superposition logic and their semantics

2.1 What is first-order superposition logic

First let us make precise what first-order superposition logic, or FOLS for short, is. At axiomatic level the formal systems of FOLS extend the formal system of first-order logic (FOL) exactly as the formal systems of PLS outlined in section 1.1. extend the formal system of propositional logic (PL). So we first fix an axiomatization of FOL (see e.g. [3]). If L is a first-order language with logical constants $\wedge$, $\neg$, $\forall$ and equality $=$, and variables v_i, v, u etc, let $L_s = L \cup \{|\}$, where $|$ is the new binary connective for superposition. Let $Fml(L_s)$, $Sen(L_s)$ denote the sets of *all* formulas and sentences of L_s, respectively, defined by the usual recursion, as those of L, plus the step for the connective $|$. We stress the word "all" because in some version of formalization considered below, *restrictions* to the formation of formulas of L_s, concerning the applicability of $|$, might be sensible. For example in one of the semantics of FOLS considered below formulas of the form, e.g. $\forall v \exists u(\alpha(v)|\beta(v)|\gamma(u))$ are allowed, while $(\forall v(\alpha(v)|\beta(v)))|(\exists u\gamma(u))$ are not. Thus we reserve the right to deal later only with some subsets of $Fml(L_s)$, $Sen(L_s)$.

The axioms and rules of inference of FOL are the following.

$$\mathsf{Ax}(\mathsf{FOL}) = \mathsf{Ax}(\mathsf{PL}) + \{UI, D\} + \{I_1, \ldots, I_5\}, \quad \mathsf{IR}(\mathsf{FOL}) = \{MP, GR\},$$

where GR is the generalization rule, UI (Universal Instantiation scheme) and D are the basic axioms of FOL (for the language L_s) concerning quantifiers, I_1, I_2, I_3 are the trivial axioms for $=$ (reflection, symmetry and transitivity), and finally I_4 and I_5 are the schemes of substitution of equals within terms and formulas. Specifically:

(UI) $\quad \forall v\varphi(v) \to \varphi(t)$, for every closed term t,

(D) $\quad \forall v(\varphi \to \psi(v)) \to (\varphi \to \forall v\psi(v))$, if v is not free in φ,

(I_4) $\quad (\forall v, u)(v = u \to t(v) = t(u))$,

(I_5) $\quad (\forall v, u)(v = u \to (\varphi(v) \to \varphi(u))$.

The sentences of L will be interpreted in L-structures $\mathcal{M} = \langle M, \ldots \rangle$.

Notational convention. We keep using the notational convention introduced in the previous section, that throughout the letters φ, ψ, σ will denote in general formulas of L_s, while the letters α, β, γ are reserved for formulas of L only.

The extra axioms of FOLS will be among the schemes already seen in 1.1, namely:

(S_1) $\quad \varphi \wedge \psi \to \varphi|\psi$
(S_2) $\quad \varphi|\psi \to \varphi \vee \psi$
(S_3) $\quad \varphi|\psi \to \psi|\varphi$
(S_4) $\quad (\varphi|\psi)|\sigma \to \varphi|(\psi|\sigma)$
(S_5) $\quad \varphi \wedge \neg\psi \to (\varphi|\psi \leftrightarrow \neg\varphi|\neg\psi)$

The formal systems we are going to deal with below are Λ_0, Λ_1, Λ_2 and Λ_3 defined as follows

$$\mathsf{Ax}(\Lambda_0) = \mathsf{Ax}(\mathrm{FOL}) + \{S_1, S_2, S_3\}, \quad \mathsf{IR}(\Lambda_0) = \{MP, GR\}$$

$$\mathsf{Ax}(\Lambda_1) = \mathsf{Ax}(\Lambda_0), \quad \mathsf{IR}(\Lambda_1) = \{MP, GR, SV\}$$

$$\mathsf{Ax}(\Lambda_2) = \mathsf{Ax}(\Lambda_1) + S_4, \quad \mathsf{IR}(\Lambda_2) = \{MP, GR, SV\},$$

$$\mathsf{Ax}(\Lambda_3) = \mathsf{Ax}(\Lambda_2) + S_5, \quad \mathsf{IR}(\Lambda_3) = \{MP, GR, SV\},$$

where SV is the rule Salva Veritate mentioned in section 1.1, but with Λ_0 in place of K_0. That is:

$$(SV) \qquad \textit{from } \ \varphi \leftrightarrow \psi \ \ \textit{infer } \varphi|\sigma \leftrightarrow \psi|\sigma,$$

$$\text{if } \varphi \leftrightarrow \psi \text{ is provable in } \Lambda_0.$$

Note that there are no new axioms for | in FOLS beyond those of PLS, which means that there is no natural interplay between | and quantifiers. In fact the connections one might consider between | and $\forall$, e.g. $(\forall v)(\varphi|\psi) \leftrightarrow (\forall v\varphi)|(\forall v\psi)$, or $(\exists v)(\varphi|\psi) \leftrightarrow (\exists v\varphi)|(\exists v\psi)$, either do not make sense because the formulas involved are illegitimate, or are simply false in the semantics where the formulas involved are allowed (e.g. in the semantics of [10]).

2.2 Candidate semantics for FOLS

In [10] we developed an alternative semantics (and a slightly different formalization) for PLS, based on choice function not for pairs of sentences but for pairs of elements of a Boolean algebra $\mathcal{B}$ where the classical sentences take truth values. We called this "Boolean-value choice semantics", or BCS for short. It turned out that this semantics can apply also to FOLS without extra pains, and with respect to this semantics the formal systems of FOLS satisfy some natural soundness and completeness results.

The main question addressed in this paper is whether FOLS can admit a semantics that naturally extends and generalizes the sentence choice semantics (SCS) of [9] (based on the truth definition (1) mentioned in the previous section). "Naturally" means that the semantics will continue to consist of pairs $\langle \mathcal{M}, f\rangle$, where $\mathcal{M}$ is an L-structure and f is a choice function for pairs of formulas/sentences of L, and will follow the basic reduction of truth to the Tarskian one through the relation: $\langle \mathcal{M}, f\rangle \models_s \varphi$ iff $\mathcal{M} \models \overline{f}(\varphi)$. The intricate question is about the *domain* of the choice function f. Namely, would f apply to pairs of *all* formulas of L, or only to *some* such pairs, e.g. to pairs of sentences alone? The answer to the above question is that there can be no natural extension of SCS to a "formula choice semantics", in the sense that f is allowed to apply to pairs of *arbitrary* formulas. Such a semantics fails badly for reasons independent of the connective $|$, simply as a result of incompatibility between choice of formulas with free variables and corresponding choice of formulas with substituted terms. On the other hand, it is shown that a semantics with some restrictions both to the construction of formulas, as well as to the applicability of choice functions (allowing them to apply to pairs of sentences only), can work smoothly and lead to satisfactory soundness and completeness results with respect to the axiomatization of FOLS, which is essentially the same as the one considered in [10].

First let us note that in any case, whatever the domain of f would be, the collapsing map $\overline{f}$ should satisfy conditions (i)-(iv) of Definition 1.1. So if we assume that f is defined for all pairs of quantifier-free sentences of L, then conditions (i)-(iv), in combination with the truth definition (1), suffice to define $\langle \mathcal{M}, f\rangle \models_s \varphi$ for every quantifier-free sentence of L_s. So the only missing step for the complete definition of $\langle \mathcal{M}, f\rangle \models_s \varphi$ is the definition of $\langle \mathcal{M}, f\rangle \models_s \forall v\varphi(v)$. For that we have two options, called *formula choice semantics* (FCS for short) and *sentence choice semantics* (SCS) because they are based on the use of choice functions for pairs of arbitrary formulas and for pairs of sentences alone, respectively. To distinguish them we shall use the symbols $\models_s^1$ and $\models_s^2$ for the resulting truth relations, respectively.

Option 1. Formula choice semantics (FCS) Here the set of formulas $Fml(L_s)$ of L_s is defined by the usual closure steps with respect to the connectives (including $|$) and quantifiers and every choice function f is defined on the entire $[Fml(L)]^2$. Therefore the truth definition of quantified sentences should be as follows.

$$\langle \mathcal{M}, f\rangle \models_s^1 \forall v\varphi(v) \Leftrightarrow \mathcal{M} \models \overline{f}(\forall v\varphi(v)). \tag{3}$$

It is easy to see that this definition is meaningful and effective if and only if the collapsing mapping $\overline{f}$ commutes with $\forall$, i.e., if $\overline{f}$ satisfies, in addition to conditions (i)-(iv) of Definition 1.1, the condition:

(v) $\overline{f}(\forall v\varphi) = \forall v\overline{f}(\varphi)$.[1]

[Treating $\exists$ as usual, i.e., as $\neg\forall\neg$, it follows from (v) and (iii) of 1.1 that $\overline{f}(\exists v\varphi) = \exists v\overline{f}(\varphi)$.]

Throughout this subsection we shall often refer to conditions (i)-(iv) of 1.1 together with condition (v) above as "conditions (i)-(v)" for $\overline{f}$. By (v), (3) becomes

$$\langle \mathcal{M}, f\rangle \models^1_s \forall v\varphi(v) \Leftrightarrow \mathcal{M} \models \forall v\overline{f}(\varphi(v)). \tag{4}$$

The right-hand side of (4) is an instance of Tarskian satisfaction, so it holds iff $\mathcal{M} \models \overline{f}(\varphi(v))(x)$ is true for every $x \in M$, where the elements of M are used as parameters added to L. Therefore (4) is equivalently written

$$\langle \mathcal{M}, f\rangle \models^1_s \forall v\varphi(v) \Leftrightarrow \mathcal{M} \models \overline{f}(\varphi(v))(x), \text{ for every } x \in M. \tag{5}$$

Thus (4) (or (5)) determines the truth of every sentence φ of $L_s(M)$ in $\langle \mathcal{M}, f\rangle$ with respect to $\models^1_s$. We refer to the truth relation $\models^1_s$ (for obvious reasons) as *formula choice semantics*, or FCS for short. We shall see however below that FCS fails badly not with respect to the interpretation of $|$, but because, surprisingly enough, fails to satisfy the Universal Instantiation scheme, as a consequence of the fact that f applies to pairs of formulas with free variables. So a reasonable alternative would be to restrict f to pairs of sentences alone.

Option 2. Sentence choice semantics (SCS) Assume now that the choice functions f are defined only for *sentences* of $L(M) = L \cup M$, where the latter is L augmented with the elements of M treated as parameters. We let the letters x, y, a, c range over elements of M. The question is how the collapsing $\overline{f}$ is defined in this case and for which φ of L_s. For instance, what would $\overline{f}(\forall v(\alpha(v)|\beta(v)))$ be for classical $\alpha(v)$ and $\beta(v)$? Letting $\overline{f}(\forall v(\alpha(v)|\beta(v))) = \forall v\overline{f}(\alpha(v)|\beta(v)) = \forall v f(\alpha(v), \beta(v))$ is not an option since f does not apply to pairs of open formulas. The answer is simply that for $Q \in \{\forall, \exists\}$,

$$\overline{f}(Qv(\alpha(v)|\beta(v))) \text{ are not defined.} \tag{6}$$

However this is not necessarily a dead end. It would only prompt us to define the truth of $\forall v(\alpha(v)|\beta(v))$ in $\langle \mathcal{M}, f\rangle$ not through (4), but in the Tarskian way:

$$\langle \mathcal{M}, f\rangle \models_s \forall v(\alpha(v)|\beta(v))) \Leftrightarrow \langle \mathcal{M}, f\rangle \models_s (\alpha(x)|\beta(x)),$$

[1] Otherwise, one cannot see how e.g. $\overline{f}(\forall v(\alpha|\beta))$ could be defined.

for all $x \in M$. So let us define, alternatively to (4), for every universal well-formed formula $\forall v\varphi(v)$ of L_s:

$$\langle \mathcal{M}, f\rangle \models_s^2 \forall v\varphi(v) \Leftrightarrow \langle \mathcal{M}, f\rangle \models_s^2 \varphi(x), \text{ for every } x \in M. \tag{7}$$

From (7), combined with clause (iii) of 1.1, clearly we have also that

$$\langle \mathcal{M}, f\rangle \models_s^2 \exists v\varphi(v) \Leftrightarrow \langle \mathcal{M}, f\rangle \models_s^2 \varphi(x), \text{ for some } x \in M. \tag{8}$$

(7) and (8) settle the definition with respect to $\models_s^2$ of sentences that begin with a quantifier. This also implicitly suggests that for φ that do not begin with a quantifier, the truth of φ in $\langle \mathcal{M}, f\rangle$ should be defined by means of the collapsing map $\overline{f}$ i.e.,

$$\langle \mathcal{M}, f\rangle \models_s^2 \varphi \Leftrightarrow \mathcal{M} \models \overline{f}(\varphi). \tag{9}$$

But this will immediately lead to trouble, unless we put restrictions to the formation of formulas of L_s. For consider, say, the sentence $(\forall v(\alpha|\beta))|(\exists u(\gamma|\delta))$. Then we should have

$$\langle \mathcal{M}, f\rangle \models_s^2 (\forall v(\alpha|\beta))|(\exists u(\gamma|\delta)) \Leftrightarrow \mathcal{M} \models \overline{f}[(\forall v(\alpha|\beta))|(\exists u(\gamma|\delta))] \Leftrightarrow$$

$$\mathcal{M} \models f(\overline{f}(\forall v(\alpha|\beta)), \overline{f}(\exists u(\gamma|\delta))).$$

But by (6) above, $\overline{f}(\forall v(\alpha|\beta))$ and $\overline{f}(\exists u(\gamma|\delta))$ are not defined, so the last part of the above equivalences does not make sense.

The conclusion is that if we want to employ choice functions for pairs of *sentences* only and $\models_s^2$ obeys (7) and (8), formulas like $(\forall v(\alpha|\beta))|(\exists u(\gamma|\delta))$, should not be allowed. That is, instead of the full set of formulas $Fml(L_s)$ we shall consider the restricted set of formulas $RFml(L_s)$. The latter differs from $Fml(L_s)$ in that $\varphi|\psi$ belongs to $RFml(L_s)$ iff φ and ψ are either classical or quantifier free. We shall refer to the truth relation $\models_s^2$ as *sentence choice semantics*, or SCS, just as we did with the corresponding semantics of PLS. We shall examine $\models_s^2$ in more detail in section 5.

Obviously the two semantics based on $\models_s^1$ and $\models_s^2$ are not equivalent, since they apply to different sets of sentences. However, there are sentences φ for which both truth definitions $\langle \mathcal{M}, f\rangle \models_s^1 \varphi$ and $\langle \mathcal{M}, f\rangle \models_s^2 \varphi$ make sense. But in general even for such sentences the definitions do not coincide. Actually none of them implies the other. The difference is easily detected by observing the right-hand sides of (5) and (7) for φ that do not begin with a quantifier. Namely, $\overline{f}(\varphi(v))(x)$ and $\overline{f}(\varphi(x))$ are in general inequivalent. To illustrate it, let $\varphi(v) := \alpha(v)|\beta(v)$, where $\alpha(v)$ and $\beta(v)$ are formulas of L.

Let $\mathcal{M}$ be an L-structure. To compare the two approaches, we must use an f which is meaningful in both of them, i.e., it applies to all pairs of formulas of $L(M)$. Fix such an f. Then f applies to $\{\alpha(v), \beta(v)\}$ and let $f(\alpha(v), \beta(v)) = \alpha(v)$. By (5), $\langle \mathcal{M}, f\rangle \models^1_s \forall v(\alpha(v)|\beta(v))$ iff $\mathcal{M} \models f(\alpha(v), \beta(v))(x)$, for all $x \in M$, therefore

$$\langle \mathcal{M}, f\rangle \models^1_s \forall v(\alpha(v)|\beta(v)) \Leftrightarrow \mathcal{M} \models \alpha(x), \text{ for all } x \in M. \tag{10}$$

On the other hand, by (7), $\langle \mathcal{M}, f\rangle \models^2_s \forall v(\alpha(v)|\beta(v))$ iff $\langle \mathcal{M}, f\rangle \models^2_s \alpha(x)|\beta(x)$, for all $x \in M$, hence

$$\langle \mathcal{M}, f\rangle \models^2_s \forall v(\alpha(v)|\beta(v)) \Leftrightarrow \mathcal{M} \models f(\alpha(x), \beta(x)), \text{ for all } x \in M. \tag{11}$$

The right-hand sides of (10) and (11) may be quite different, since the choices of f from the pairs $\{\alpha(x), \beta(x)\}$, for the various $x \in M$, may be non-uniform, e.g. for $x_1 \neq x_2$ we may have $f(\alpha(x_1), \beta(x_1)) = \alpha(x_1)$ and $f(\alpha(x_2), \beta(x_2)) = \beta(x_2)$. In order for the definitions (10) and (11) to be equivalent, f should be a *uniform choice function*, i.e., $f(\alpha(\vec{v}), \beta(\vec{v})) = \alpha(\vec{v})$ should imply $f(\alpha(\vec{t}), \beta(\vec{t})) = \alpha(\vec{t})$ for all pairs of formulas $\{\alpha(\vec{v}), \beta(\vec{v})\}$ and every tuple of terms $\vec{t}$ that can be substituted for $\vec{v}$. However, as we shall prove in section 4, no choice function $f : [Fml(L)]^2 \to Fml(L)$ can have this property.

3 The formula choice semantics (FCS) and the failure of universal instantiation

Since FOLS extends FOL, any proper semantics for FOLS should first of all satisfy the quantifier axioms of FOL, namely UI and D. In this section we show that unfortunately (and rather unexpectedly) FCS fails to satisfy UI.

Firstly recall that given a language L, whenever we write $\varphi(\vec{v})$, for a formula of L_s, we mean that the free variables of φ are *among* those of the tuple $\vec{v}$. Then the following can be easily verified by induction on the length of φ.

Fact 3.1 *For every choice function f for pairs of formulas and every $\varphi \in Fml(L_s)$, the free variables of $\overline{f}(\varphi)$ are included in those of φ, i.e., $FV(\overline{f}(\varphi)) \subseteq FV(\varphi)$. In particular, if φ is a sentence of L_s, then $\overline{f}(\varphi)$ is a sentence of L.*

[In general, $FV(\overline{f}(\varphi)) \subsetneq FV(\varphi)$, since, for example, we may have $\varphi(v_1, v_2) = \alpha(v_1)|\beta(v_2)$ and $f(\alpha(v_1), \beta(v_2)) = \alpha(v_1)$, so $\overline{f}(\varphi) = f(\alpha(v_1), \beta(v_2)) = \alpha(v_1)$.]

It follows from this Fact that the variables of $\overline{f}(\varphi)$ are among the variables of φ, so we may write for every $\varphi(\vec{v})$:

$$\overline{f}(\varphi(\vec{v})) = \overline{f}(\varphi)(\vec{v}). \tag{12}$$

Fact 3.2 *The scheme D is a tautology with respect to FCS.*

Proof. Take an instance of D

$$\sigma : (\forall v)(\varphi \to \psi(v)) \to (\varphi \to (\forall v)\psi(v)),$$

where φ does not contain v free, and take an arbitrary choice function satisfying conditions (i)-(v). Then clearly applying these conditions we have

$$\overline{f}(\sigma) = [(\forall v)(\overline{f}(\varphi) \to \overline{f}(\psi(v))) \to (\overline{f}(\varphi) \to (\forall v)\overline{f}(\psi(v)))].$$

Let $\overline{f}(\varphi) = \alpha$ and $\overline{f}(\psi(v)) = \beta(v)$. Then the last formula is written

$$\overline{f}(\sigma) = [(\forall v)(\alpha \to \beta(v)) \to (\alpha \to (\forall v)\beta(v))].$$

By assumption $v \notin FV(\varphi)$, and by Fact 3.1, $FV(\overline{f}(\varphi)) \subseteq FV(\varphi)$, so $v \notin FV(\alpha)$, therefore $\overline{f}(\sigma)$ is an instance of the scheme D of FOL, so it holds in every L-structure $\mathcal{M}$. Therefore $\mathcal{M} \models \overline{f}(\sigma)$, or equivalently $\langle \mathcal{M}, f\rangle \models \sigma$. $\dashv$

However the situation is quite different for the scheme *UI*. The next theorem shows that, under mild conditions for L, there is no choice function f for L with respect to which *UI* could be a scheme of tautologies.

Theorem 3.3 *Let L be a language with at least two distinct closed terms t_1, t_2, and a formula $\alpha(v)$ in one free variable such that both $\alpha(t_1) \wedge \neg\alpha(t_2)$ and $\neg\alpha(t_1) \wedge \alpha(t_2)$ are satisfiable. Then for every choice function f for $Fml(L)$ there is an L-structure $\mathcal{M}$ and a formula $\psi(v_1, v_2)$ of L_s with two free variables for which UI fails in $\langle \mathcal{M}, f\rangle$, i.e., such that $\langle \mathcal{M}, f\rangle \models^1_s (\forall v_1, v_2)\psi(v_1, v_2) \wedge \neg\psi(t_1, t_2)$.*

Proof. [Note first that the conditions required for L in the above Lemma are quite weak. E.g. any L containing three distinct constants c_1, c_2, c_3 satisfies them. For if we set $\alpha(v) = (v = c_3)$, then $\alpha(c_1) \wedge \neg\alpha(c_2)$ and $\neg\alpha(c_1) \wedge \alpha(c_2)$ are both satisfiable in L-structures.]

Now let L, t_1, t_2 and $\alpha(v)$ be as stated. Then there are structures $\mathcal{M}_1$, $\mathcal{M}_2$ such that $\mathcal{M}_1 \models \alpha(t_1) \wedge \neg\alpha(t_2)$ and $\mathcal{M}_2 \models \neg\alpha(t_1) \wedge \alpha(t_2)$. Pick a choice function f for $Fml(L)$. It suffices to show that there is a formula

$\psi(v_1,v_2)$ of L_s such that either $\langle\mathcal{M}_1,f\rangle\models^1_s(\forall v_1,v_2)\psi(v_1,v_2)\wedge\neg\psi(t_1,t_2)$, or $\langle\mathcal{M}_2,f\rangle\models^1_s(\forall v_1,v_2)\psi(v_1,v_2)\wedge\neg\psi(t_1,t_2)$. Let $\alpha(v_2)$ be the formula resulting from $\alpha(v_1)$ if we replace v_1 by the new variable v_2. We examine how f acts on the pairs of formulas $\{\alpha(v_1),\alpha(v_2)\}$ and $\{\alpha(t_1),\alpha(t_2)\}$ and consider the four possible cases.

Case 1. $f(\alpha(v_1),\alpha(v_2))=\alpha(v_1)$ and $f(\alpha(t_1),\alpha(t_2))=\alpha(t_1)$.

By assumption $\mathcal{M}_2\models\neg\alpha(t_1)\wedge\alpha(t_2)$. Arguing in the standard logic FOL, this can be written as follows:

$$\mathcal{M}_2\models(\forall v_1v_2)[v_1=t_2\wedge v_2=t_1\rightarrow\alpha(v_1)]\wedge\neg[t_2=t_2\wedge t_1=t_1\rightarrow\alpha(t_1)],$$

or

$$\mathcal{M}_2\models(\forall v_1v_2)[v_1=t_2\wedge v_2=t_1\rightarrow f(\alpha(v_1),\alpha(v_2))]\wedge$$
$$\neg[t_2=t_2\wedge t_1=t_1\rightarrow f(\alpha(t_1),\alpha(t_2))],$$

or

$$\langle\mathcal{M}_2,f\rangle\models^1_s(\forall v_1v_2)[v_1=t_2\wedge v_2=t_1\rightarrow\alpha(v_1)|\alpha(v_2)]\wedge$$
$$\neg[t_2=t_2\wedge t_1=t_1\rightarrow\alpha(t_1)|\alpha(t_2)],$$

or, since $|$ is commutative,

$$\langle\mathcal{M}_2,f\rangle\models^1_s(\forall v_1v_2)[v_1=t_2\wedge v_2=t_1\rightarrow\alpha(v_1)|\alpha(v_2)]\wedge$$
$$\neg[t_2=t_2\wedge t_1=t_1\rightarrow\alpha(t_2)|\alpha(t_1)].$$

Setting

$$\psi(v_1,v_2):=[v_1=t_2\wedge v_2=t_1\rightarrow\alpha(v_1)|\alpha(v_2)],$$

the last relation is written

$$\langle\mathcal{M}_2,f\rangle\models^1_s(\forall v_1v_2)\psi(v_1,v_2)\wedge\neg\psi(t_2,t_1),$$

thus *UI* fails in $\langle\mathcal{M}_2,f\rangle$.

Case 2. $f(\alpha(v_1),\alpha(v_2))=\alpha(v_1)$ and $f(\alpha(t_1),\alpha(t_2))=\alpha(t_2)$.

Now we use the fact that $\mathcal{M}_1\models\alpha(t_1)\wedge\neg\alpha(t_2)$. As before this is written equivalently,

$$\mathcal{M}_1\models(\forall v_1v_2)[v_1=t_1\wedge v_2=t_2\rightarrow\alpha(v_1)]\wedge\neg[t_1=t_1\wedge t_2=t_2\rightarrow\alpha(t_2)],$$

or

$$\mathcal{M}_1\models(\forall v_1v_2)[v_1=t_1\wedge v_2=t_2\rightarrow f(\alpha(v_1),\alpha(v_2))]\wedge$$

$$\neg[t_1 = t_1 \wedge t_2 = t_2 \rightarrow f(\alpha(t_1), \alpha(t_2))],$$

or

$$\langle \mathcal{M}_1, f\rangle \models^1_s (\forall v_1 v_2)[v_1 = t_1 \wedge v_2 = t_2 \rightarrow \alpha(v_1)|\alpha(v_2)] \wedge$$
$$\neg[t_1 = t_1 \wedge t_2 = t_2 \rightarrow \alpha(t_1)|\alpha(t_2)].$$

Thus putting

$$\psi(v_1, v_2) := [v_1 = t_1 \wedge v_2 = t_2 \rightarrow \alpha(v_1)|\alpha(v_2)],$$

we are done.

Case 3. $f(\alpha(v_1), \alpha(v_2)) = \alpha(v_2)$ and $f(\alpha(t_1), \alpha(t_2)) = \alpha(t_1)$.
We use again the fact that $\mathcal{M}_2 \models \neg\alpha(t_1) \wedge \alpha(t_2)$, which yields as before

$$\mathcal{M}_2 \models (\forall v_1 v_2)[v_1 = t_1 \wedge v_2 = t_2 \rightarrow \alpha(v_2)] \wedge \neg[t_1 = t_1 \wedge t_2 = t_2 \rightarrow \alpha(t_1)],$$

or

$$\langle \mathcal{M}_2, f\rangle \models^1_s (\forall v_1 v_2)[v_1 = t_1 \wedge v_2 = t_2 \rightarrow \alpha(v_1)|\alpha(v_2)] \wedge$$
$$\neg(t_1 = t_1 \wedge t_2 = t_2 \rightarrow \alpha(t_1)|\alpha(t_2)].$$

So setting

$$\psi(v_1, v_2) := [v_1 = t_1 \wedge v_2 = t_2 \rightarrow \alpha(v_1)|\alpha(v_2)]$$

we are done.

Case 4. $f(\alpha(v_1), \alpha(v_2)) = \alpha(v_2)$ and $f(\alpha(t_1), \alpha(t_2)) = \alpha(t_2)$.
We use the fact that $\mathcal{M}_1 \models \alpha(t_1) \wedge \neg\alpha(t_2)$ which translates into

$$\mathcal{M}_1 \models (\forall v_1 v_2)[v_1 = t_2 \wedge v_2 = t_1 \rightarrow \alpha(v_2)] \wedge \neg[t_2 = t_2 \wedge t_1 = t_1 \rightarrow \alpha(t_2)],$$

or

$$\langle \mathcal{M}_1, f\rangle \models^1_s (\forall v_1 v_2)[v_1 = t_2 \wedge v_2 = t_1 \rightarrow \alpha(v_1)|\alpha(v_2)] \wedge$$
$$\neg(t_2 = t_2 \wedge t_1 = t_1 \rightarrow \alpha(t_2)|\alpha(t_1)].$$

Setting

$$\psi(v_1, v_2) := [v_1 = t_2 \wedge v_2 = t_1 \rightarrow \alpha(v_1)|\alpha(v_2)]$$

we are done. This completes the proof. $\dashv$

Equivalent to *UI* (in FOL, hence also in FOLS) is the dual axiom of *Existential Generalization (EG):*

(*EG*) $\varphi(t) \rightarrow (\exists v)\varphi(v)$.

Therefore Theorem 3.3 is equivalently formulated as follows.

Corollary 3.4 *Given any language L as above, for every choice function f for $Fml(L)$ there is $\mathcal{M}$ such that EG fails in $\langle \mathcal{M}, f\rangle$, namely, there is a formula $\varphi(v_1, v_2)$ and closed terms t_1, t_2 such that $\langle \mathcal{M}, f\rangle \models_s^1 \varphi(t_1, t_2) \wedge \neg(\exists v_1, v_2)\varphi(v_1, v_2)$.*

In Theorem 3.3 the formula(s) $\psi(v_1, v_2)$ used to refute *UI* contain two free variables. We do not know if it possible to refute *UI* using a formula with a single free variable.

Also in the proof of 3.3 we used a superposed formula of the form $\alpha(v_1)|\alpha(v_2)$, which looks somewhat artificial. Can we show the failure of UI, using a superposition of the form $\alpha(\vec{v})|\beta(\vec{v})$ where $\alpha(\vec{v})$ and $\beta(\vec{v})$ are distinct formulas? The answer is yes. Specifically, by essentially the same argument we can prove the following variant of Theorem 3.3.

Theorem 3.5 *Let L be a language and assume that there exist formulas $\alpha(\vec{v})$, $\beta(\vec{v})$ and corresponding tuples of closed terms $\vec{t}$, $\vec{s}$ such that:*

(a) $\alpha(\vec{s}) = \beta(\vec{t})$,

(b) $\alpha(\vec{t}) = \beta(\vec{s})$

(c) $\alpha(\vec{t}) \wedge \neg\beta(\vec{t})$ and $\neg\alpha(\vec{t}) \wedge \beta(\vec{t})$ are satisfiable.

Then for every choice function f there is a structure $\mathcal{M}$, a formula $\psi(\vec{v})$ and closed terms $\vec{t}$ such that $\langle \mathcal{M}, f\rangle \models_s^1 (\forall\vec{v})\psi(\vec{v}) \wedge \neg\psi(\vec{t})$.

Proof. [For example if $<$ is a binary relation of L, and $\alpha(v_1, v_2) := (v_1 < v_2)$, $\beta(v_1, v_2) := (v_1 > v_2)$, $\vec{t} = \langle t_1, t_2\rangle$ and $\vec{s} = \langle t_2, t_1\rangle$, then α, β, $\vec{t}$ and $\vec{s}$ satisfy conditions (a)-(c) above.]

The argument goes exactly as in the proof of Theorem 3.3. We fix structures $\mathcal{M}_1$, $\mathcal{M}_2$ such that $\mathcal{M}_1 \models \alpha(\vec{t}) \wedge \neg\beta(\vec{t})$ and $\mathcal{M}_2 \models \neg\alpha(\vec{t}) \wedge \beta(\vec{t})$. Pick a choice function f for $Fml(L)$. It suffices to show that there is a formula $\psi(\vec{v})$ of L_s such that either $\langle \mathcal{M}_1, f\rangle \models_s^1 (\forall\vec{v})\psi(\vec{v}) \wedge \neg\psi(\vec{r})$, or $\langle \mathcal{M}_2, f\rangle \models_s^1 (\forall\vec{v})\psi(\vec{v}) \wedge \neg\psi(\vec{r})$, for $\vec{r} = \vec{t}$ or $\vec{r} = \vec{s}$. As before we examine how f acts on the pairs of formulas $\{\alpha(\vec{v}), \beta(\vec{v})\}$, and $\{\alpha(\vec{t}), \beta(\vec{t})\} = \{\alpha(\vec{s}), \beta(\vec{s})\}$, and we examine the four possible cases that arise as before. Namely:

Case 1. $f(\alpha(\vec{v}), \beta(\vec{v})) = \alpha(\vec{v})$ and $f(\alpha(\vec{t}), \beta(\vec{t})) = \alpha(\vec{t})$.

Case 2. $f(\alpha(\vec{v}), \beta(\vec{v})) = \alpha(\vec{v})$ and $f(\alpha(\vec{t}), \beta(\vec{t})) = \beta(\vec{t})$.

Case 3. $f(\alpha(\vec{v}), \beta(\vec{v})) = \beta(\vec{v})$ and $f(\alpha(\vec{t}), \beta(\vec{t})) = \alpha(\vec{t})$.

Case 4. $f(\alpha(\vec{v}), \beta(\vec{v})) = \beta(\vec{v})$ and $f(\alpha(\vec{t}), \beta(\vec{t})) = \beta(\vec{t})$.

In each of these cases we work as in the corresponding case of the proof of 3.3. Details are left to the reader. $\dashv$

Closing this section, let us remark that the failure of UI is *fatal* for any quantified logical system like Λ, in the sense that there can be no reasonable "weakening" of Λ in which $\forall$ is still in use while UI fails. For the failure of UI is quite different from the failure e.g. of the Excluded Middle (EM), which has led to a logic weaker than the classical one and yet quite interesting. The reason is that UI expresses exactly the meaning of "all", as a fundamental logical constant, while EM does not express the meaning of any logical constant.

4 The impossibility of uniform choice functions

At the end of section 2.2, comparing the truth relations $\models_s^1$ and $\models_s^2$, we said that the two notions of truth deviate even for choice functions f that are defined in both semantics, because f cannot be *uniform* when considered as a choice function in FCS. Let us make this claim precise.

Definition 4.1 A choice function $f : [Fml(L)]^2 \to Fml(L)$ is said to be *uniform* if for any two formulas $\alpha(\vec{v})$, $\beta(\vec{v})$, with $\vec{v}$ free, and any tuple $\vec{t}$ of terms substitutable for $\vec{v}$ in α, β, $f(\alpha(\vec{v}), \beta(\vec{v})) \sim \alpha(\vec{v})$ implies $f(\alpha(\vec{t}), \beta(\vec{t})) \sim \alpha(\vec{t})$, or equivalently, if the following equivalence holds:

$$[f(\alpha(\vec{v}), \beta(\vec{v}))](\vec{t}) \sim f(\alpha(\vec{t}), \beta(\vec{t})). \tag{13}$$

Note. The reason for writing $\sim$ instead of $=$ in condition (13) above is the need to cover the situation where $\alpha(\vec{v}) \sim \beta(\vec{v})$. In this case also $\alpha(\vec{t}) \sim \beta(\vec{t})$, and the choice from $\{\alpha(\vec{v}), \beta(\vec{v})\}$, as well as from $\{\alpha(\vec{t}), \beta(\vec{t})\}$, is indifferent. So if $f(\alpha(\vec{v}), \beta(\vec{v})) = \alpha(\vec{v})$, while $f(\alpha(\vec{t}), \beta(\vec{t})) = \beta(\vec{t})$, then $f(\alpha(\vec{t}), \beta(\vec{t})) \neq \alpha(\vec{t})$ while $f(\alpha(\vec{t}), \beta(\vec{t})) \sim \alpha(\vec{t})$.

Unfortunately, no choice function $f : [Fml(L)]^2 \to Fml(L)$ can be uniform, for any first-order language L, so the definition 4.1 is void.

Proposition 4.2 *For any language L there is no uniform choice function for L.*

Proof. Let L be any first-order language. Clearly we can pick a formula $\alpha(v)$ and variables v_1, v_2 such that $\alpha(v_1) \not\sim \alpha(v_2)$. Suppose f is a uniform choice function for $[Fml(L)]^2$, that is f satisfies (13). In particular this holds for the pair $\{\alpha(v_1), \alpha(v_2)\}$ and the tuple of terms $\vec{t} = \langle v_2, v_1 \rangle$. Assume without loss of generality that $f(\alpha(v_1), \alpha(v_2)) = \alpha(v_1)$. Then

$$[f(\alpha(v_1), \alpha(v_2))](\vec{t}) = \alpha(v_1)(\vec{t}) = \alpha(v_2).$$

By (13)

$$[f(\alpha(v_1), \alpha(v_2))](\vec{t}) \sim f(\alpha(v_1)(\vec{t}), \alpha(v_2)(\vec{t})) = f(\alpha(v_2), \alpha(v_1)),$$

therefore, by the above relations

$$f(\alpha(v_2), \alpha(v_1)) \sim \alpha(v_2). \tag{14}$$

But $f(\alpha(v_2), \alpha(v_1)) = f(\alpha(v_1), \alpha(v_2)) = \alpha(v_1)$ by our assumption. So $\alpha(v_1) \sim \alpha(v_2)$, a contradiction. $\dashv$

5 The sentence choice semantics (SCS) for first-order superposition logic

We come now to examine the semantics SCS for FOLS based on the truth relation $\models_s^2$ roughly described as Option 2 in section 2. As already said there, this semantics presumes that a restriction is imposed to the syntax of L_s, namely that $|$ should not apply to quantified formulas, unless they are classical. So below we shall deal with a class of formulas of L_s, called "restricted formulas/sentences". To define them we define first the class of "basic formulas/sentences".

Definition 5.1 Let L be a first-order language and let $\mathcal{M} = \langle M, \ldots \rangle$ be an L-structure.

(i) The set $BFml(L_s(M))$ of *basic formulas* of $L_s(M)$ is the smallest set of formulas X such that (a) $Fml(L(M)) \subset X$ and (b) X is closed with respect to the connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $|$ and $\neg$ (but not with respect to quantifiers). The set $BSen(L_s(M))$ of *basic sentences* of $L_s(M)$ is the subset of $BFml(L_s(M))$ of formulas without free variables.

(ii) The set $RFml(L_s(M))$ of *restricted formulas* of $L_s(M)$ is the smallest set of formulas X such that (a) $BFml(L_s(M)) \subset X$ and (b) X is closed with respect to $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$ and $\forall$, $\exists$ (but not with respect to $|$). The set $RSen(L_s(M))$ of *restricted sentences* of $L_s(M)$ is the subset of $RFml(L_s(M))$ of formulas without free variables.

We come next to choice functions. In contrast to the choice functions used in FCS, the choice functions f of SCS apply only to pairs of sentences of $L(M)$, i.e., $f : [Sen(L(M))]^2 \rightarrow Sen(L(M))$. Let us denote by $\mathcal{F}_M$ the class of all these functions. Further, SCS differs from FCS in that the collapsing function $\overline{f}$ induced from f will be defined only for *basic sentences* i.e., for elements of $BSen(L_s(M))$.

Definition 5.2 Given $f : [Sen(L(M))]^2 \to Sen(L(M))$, the function $\overline{f} : BSen(L_s(M)) \to Sen(L(M))$ is defined along the clauses of Definition 1.1 as follows:

(i) $\overline{f}(\alpha) = \alpha$, for $\alpha \in Sen(L(M))$,
(ii) $\overline{f}(\varphi \wedge \psi) = \overline{f}(\varphi) \wedge \overline{f}(\psi)$,
(iii) $\overline{f}(\neg\varphi) = \neg\overline{f}(\varphi)$,
(iv) $\overline{f}(\varphi|\psi) = f(\overline{f}(\varphi), \overline{f}(\psi))$.

It is easy to check that this definition is good, and yields $\overline{f}(\varphi)$ for every $\varphi \in BSen(L_s(M)$. Especially concerning step (iv), note that if $\varphi|\psi$ belongs to $BSen(L_s(M))$, then so do φ and ψ, hence $\overline{f}(\varphi)$ and $\overline{f}(\psi)$ are defined and are sentences of $L(M)$. Therefore $f(\overline{f}(\varphi), \overline{f}(\psi))$ is defined too.

We come to the truth definition $\langle \mathcal{M}, f\rangle \models^2_s \varphi$, where $\mathcal{M}$ is an L-structure, $f \in \mathcal{F}_M$ and $\varphi \in RSen(L_s(M))$.

Definition 5.3 $\langle \mathcal{M}, f\rangle \models^2_s \varphi$ is defined by induction on the length of φ along the following clauses. (We think of $\wedge$, $\neg$, $|$ and $\forall$ as basic connectives, the others being thought of as abbreviations.)

(i) $\langle \mathcal{M}, f\rangle \models^2_s \alpha$ iff $\mathcal{M} \models \alpha$, for $\alpha \in Sen(L(M))$.
(ii) $\langle \mathcal{M}, f\rangle \models^2_s \varphi \wedge \psi$ iff $\langle \mathcal{M}, f\rangle \models^2_s \varphi$ and $\langle \mathcal{M}, f\rangle \models^2_s \psi$.
(iii) $\langle \mathcal{M}, f\rangle \models^2_s \neg\varphi$ iff $\langle \mathcal{M}, f\rangle \not\models^2_s \varphi$.
(iv) $\langle \mathcal{M}, f\rangle \models^2_s \varphi|\psi$ iff $\mathcal{M} \models f(\overline{f}(\varphi), \overline{f}(\psi))$.
(v) $\langle \mathcal{M}, f\rangle \models^2_s (\forall v)\varphi(v)$ iff $\langle \mathcal{M}, f\rangle \models^2_s \varphi(x)$ for every $x \in M$.

It is easy to check that the above definition assigns a unique truth value to every $\varphi \in RSen(L_s(M))$. Specifically clauses (i)-(iv) attribute truth values to all basic sentences, while clause (v) is needed for quantified (non-classical) sentences.

Given a class $X \subseteq \mathcal{F}$, the X-logical consequence relation $\models^2_X$ and the notion of X-tautology, $\models^2_X \varphi$, are defined as usual. We denote by $Taut^2_X$ the set of $\models^2_X \varphi$-tautologies. We denote again by $Asso$, Reg, Reg^* and Dec the classes of associative, regular, regular and associative, and regular, associative and $\neg$-decreasing elements of $\mathcal{F}$. In particular we have

$$Dec \subset Reg^* \subset Reg \subset \mathcal{F},$$

hence

$$Taut(\mathcal{F}) \subseteq Taut(Reg) \subseteq Taut(Reg^*) \subseteq Taut(Dec).$$

Given a class $X \subseteq \mathcal{F}$ and a formal system Λ consisting of axioms (set $\models^2_X \varphi$-tautologies) and rules of inference, we shall denote by

$$\mathrm{RFOLS}(X, \Lambda)$$

the logical system having as usual semantic part X and syntactic part Λ (the prefix "R" stands for reminding that we work in a restricted class of sentences of L_s).

5.1 Soundness

The formal systems Λ_0, Λ_1, Λ_2 and Λ_3 described in section 2.1, are going to formalize the classes $\mathcal{F}$, Reg, Reg^* and Dec of choice functions. So we shall be dealing with the logics

$$\mathrm{RFOLS}(\mathcal{F},\Lambda_0),\ \mathrm{RFOLS}(Reg,\Lambda_1),\ \mathrm{RFOLS}(Reg^*,\Lambda_2),\ \mathrm{RFOLS}(Dec,\Lambda_3).$$

Since we work with restricted formulas only we must be careful with the syntax of the systems Λ_i above. Namely the following remarks are in order.

1) The formulas that can be substituted in the axiom schemes S_i above must be restricted, that is $S_i \subset RFml(L_s)$.

2) Whenever we write $\Sigma \vdash_{\Lambda_i} \varphi$, it is implicitly assumed that $\Sigma \cup \{\varphi\} \subset RFml(L_s)$

3) Next the rule SV says that if $\varphi \leftrightarrow \psi$ is provable in Λ_0, then we can derive that $\varphi|\sigma \leftrightarrow \psi|\sigma$. But since $|$ applies only to non-quantified formulas (unless they are classical), φ and ψ, hence also $\varphi \leftrightarrow \psi$, and σ must be basic formulas.

4) Given that the above conditions are satisfied, if $\varphi_1,\ldots,\varphi_n$ is a Λ_i-proof of φ from Σ, then every φ_i is restricted.

Most of the results given in this and later subsections have proofs similar to proofs of corresponding results of [10]. However the adaptations needed, especially in the proofs of completeness theorems, are rather extensive and so we give them here in full detail.

Theorem 5.4 *Let $X \subseteq \mathcal{F}$. If Λ is a system such that $\mathsf{Ax}(\Lambda) \subset Taut(X)$ and $\mathit{IR}(\Lambda) = \{MP, GR\}$, then* RFOLS$(X,\Lambda)$ *is sound. In particular* RFOLS$(\mathcal{F},\Lambda_0)$ *is sound.*

Proof. Let X, Λ be as stated and let $\Sigma \vdash_\Lambda \varphi$, for a set of sentences Σ and a sentence φ. Let $\varphi_1,\ldots,\varphi_n$, where $\varphi_n = \varphi$, be a Λ-proof of φ. As usual we show that $\Sigma \models_X \varphi_i$, for every $1 \le i \le n$, by induction on i. Given i, suppose the claim holds for all $j < i$, and let $\langle \mathcal{M}, f\rangle \models^2_s \Sigma$, for some L-structure $\mathcal{M}$ and $f \in X$. We show that $\langle \mathcal{M}, f\rangle \models^2_s \varphi_i$. If $\varphi_i \in \Sigma$ this is obvious. If $\varphi_i \in \mathsf{Ax}(\Lambda)$, then $\langle \mathcal{M}, f\rangle \models^2_s \varphi_i$, because by assumption $\mathsf{Ax}(\Lambda) \subset Taut(X)$ and $f \in X$. Next suppose φ_i is derived by the help of MP. Then there are sentences φ_j, $\varphi_k = (\varphi_j \to \varphi_i)$, for some $j,k < i$. By the induction assumption, $\langle \mathcal{M}, f\rangle \models^2_s \varphi_j$ and $\langle \mathcal{M}, f\rangle \models^2_s \varphi_k$. Therefore $\langle \mathcal{M}, f\rangle \models^2_s \varphi_i$. Finally let φ_i be derived by the help of GR, i.e., there is $j < i$ and $\varphi_j(v)$ such that $\varphi_i = (\forall v)\varphi_j(v)$. ($\Sigma$ is a set of sentences so v does not occur free in Σ.) By the induction assumption $\langle \mathcal{M}, f\rangle \models^2_s \varphi_j(x)$ for every $x \in M$. Then by the definition of $\models^2_s$, $\langle \mathcal{M}, f\rangle \models^2_s (\forall v)\varphi_j(v)$. Therefore $\langle \mathcal{M}, f\rangle \models^2_s \varphi_i$. $\dashv$

In contrast to Λ_0, the formal systems Λ_i for $i = 1, 2, 3$ contain in addition the rule SV, already mentioned in sections 1.1 and 2.1. Since however we are working in a language with syntactic restrictions we must specify it even more concretely. Recall that $\varphi|\psi$ makes sense only if φ and ψ are basic formulas, so SV takes here the form:

$$(SV) \quad \textit{For} \quad \varphi, \psi, \sigma \in BFml(L_s), \text{ if } \ \varphi \leftrightarrow \psi \text{ is provable in } \Lambda_0$$

$$\textit{infer that} \quad \varphi|\sigma \leftrightarrow \psi|\sigma.$$

Theorem 5.5 *Let $X \subseteq Reg$. If Λ is a system such that $\mathsf{Ax}(\Lambda) \subset Taut(X)$ and $\mathsf{IR}(\Lambda) = \{MP, GR, SV\}$, then* RFOLS$(X, \Lambda)$ *is sound. In particular* RFOLS(Reg, Λ_1)*,* RFOLS(Reg^*, Λ_2) *and* RFOLS(Dec, Λ_3) *are sound.*

Proof. Let $X \subseteq Reg$, $\mathsf{Ax}(\Lambda) \subset Taut(X)$ and $\mathsf{IR}(\Lambda) = \{MP, GR, SV\}$, and let $\Sigma \vdash_\Lambda \varphi$. Let $\varphi_1, \ldots, \varphi_n$, where $\varphi_n = \varphi$, be a Λ-proof of φ. We show, by induction on i, that for all $i = 1, \ldots, n$, $\Sigma \models_X \varphi_i$. Let $\langle \mathcal{M}, f\rangle \models^2_s \Sigma$, with $f \in X$. Given φ_i, the proof that $\langle \mathcal{M}, f\rangle \models^2_s \varphi_i$ (given the induction assumption) goes exactly as in the proof of Theorem 5.4, except of the case where φ_i follows from a sentence φ_j, for $j < i$, by the rule SV. It means that $\varphi_i = (\sigma|\tau \leftrightarrow \rho|\tau)$ while $\varphi_j = (\sigma \leftrightarrow \rho)$, where $\vdash_{\Lambda_0} (\sigma \leftrightarrow \rho)$. Moreover σ, ρ and τ are basic sentences. Now Λ_0 is a system satisfying the conditions of 5.4 above for $X = \mathcal{F}$, so $\models^2_{\mathcal{F}} (\sigma \leftrightarrow \rho)$. It means that for every L-structure $\mathcal{N}$ and every $g \in \mathcal{F}$, $\langle \mathcal{N}, g\rangle \models^2_s (\sigma \leftrightarrow \rho)$. Since σ, ρ and τ are basic sentences, $\overline{f}(\sigma)$, $\overline{f}(\rho)$ and $\overline{f}(\tau)$ are defined and moreover $\langle \mathcal{N}, g\rangle \models^2_s (\sigma \leftrightarrow \rho)$ is equivalent to $\mathcal{N} \models \overline{g}(\sigma) \leftrightarrow \overline{g}(\rho)$. Since this holds for every $\mathcal{N}$, $\overline{g}(\sigma) \leftrightarrow \overline{g}(\rho)$ is a classical tautology, or $\overline{g}(\sigma) \sim \overline{g}(\rho)$, for every $g \in \mathcal{F}$. In particular, $\overline{f}(\sigma) \sim \overline{f}(\rho)$. Now since $X \subseteq Reg$, $f \in X$ implies f is regular. Therefore $\overline{f}(\sigma) \sim \overline{f}(\rho)$ implies that $f(\overline{f}(\sigma), \overline{f}(\tau)) \sim f(\overline{f}(\rho), \overline{f}(\tau))$, or $\overline{f}(\sigma|\tau) \sim \overline{f}(\rho|\tau)$, therefore $\mathcal{M} \models \overline{f}(\sigma|\tau) \leftrightarrow \overline{f}(\rho|\tau)$, or $\langle \mathcal{M}, f\rangle \models^2_s (\sigma|\tau \leftrightarrow \rho|\tau)$, i.e., $\langle \mathcal{M}, f\rangle \models^2_s \varphi_i$, as required. The other claim follows from the fact that the logics in question clearly satisfy the criteria of the general statement. This completes the proof. $\dashv$

5.2 Completeness of the logic RFOLS($\mathcal{F}, \Lambda_0$)

Since the rules of Λ_0 are only MP and GR, the Deduction Theorem (DT) holds in Λ_0 so by Fact 1.8, the two forms of Completeness Theorem CT1 and CT2 are equivalent, so we can refer simply to "completeness" instead of CT1- or CT2-completeness. Further by the help of DT and standard proofs, every consistent set Σ of formulas of L_s can be extended to a complete and Henkin-complete set of formulas Σ^+ in a language L_s^+, where $L^+ \backslash L$ consists of new constants. (Recall that a set of formulas Σ is Henkin-complete, if whenever Σ contains an existential formula $\exists v \varphi(v)$, then it contains also $\varphi(c)$, for some $c \in L$, witnessing $\exists v \varphi(v)$.) Recall also that for a consistent and complete $\Sigma \subseteq Sen(L_s)$ the following hold: (a) for every $\varphi \in Sen(L_s)$, $\varphi \in \Sigma$ iff $\neg\varphi \notin \Sigma$, (b) $\varphi \wedge \psi \in \Sigma$ iff $\varphi \in \Sigma$ and $\psi \in \Sigma$, (c) if $\Sigma \vdash_K \varphi$, then $\varphi \in \Sigma$.

Before coming to the logics introduced in the previous subsection, we shall give a general criterion of satisfiability for a consistent, complete and Henkin-complete set Σ of sentences of L_s. Given such a set Σ, if we set $\Sigma_1 = \Sigma \cap Sen(L)$ (the subset of Σ that contains the classical sentences of Σ) then obviously Σ_1 is a consistent, complete and Henkin-complete set of sentences of L. By the Completeness Theorem of FOL, there exists an L-structure $\mathcal{M}$ such that, for every $\alpha \in Sen(L)$, $\alpha \in \Sigma_1$ iff $\mathcal{M} \models \alpha$. We have the following criterion of satisfiability.

Lemma 5.6 *Let $X \subseteq \mathcal{F}$ and $\Lambda \subset Taut(X)$. Let also Σ be a Λ-consistent, complete and Henkin-complete set of sentences of L_s and let $\Sigma_1 = \Sigma \cap Sen(L)$ and $\mathcal{M}$ such that*

$$\alpha \in \Sigma_1 \;\Leftrightarrow\; \mathcal{M} \models \alpha. \tag{15}$$

Then given $f \in X$, $\langle \mathcal{M}, f \rangle \models_s^2 \Sigma$ if and only if for every $\varphi \in BSen(L_s)$ (the set of basic sentences of L_s),

$$\varphi \in \Sigma \;\Rightarrow \overline{f}(\varphi) \in \Sigma. \tag{16}$$

(Actually (16) is equivalent to

$$\varphi \in \Sigma \;\Leftrightarrow \overline{f}(\varphi) \in \Sigma,$$

but the other direction follows from (16), the consistency and completeness of Σ and the fact that $\overline{f}(\neg\varphi) = \neg\overline{f}(\varphi)$.)

Proof. Pick an $f \in X$ and suppose $\langle \mathcal{M}, f \rangle \models_s^2 \Sigma$. Then by the completeness of Σ and the definition of $\models_s^2$, for every $\varphi \in BSen(L_s)$,

$$\varphi \in \Sigma \;\Leftrightarrow\; \langle \mathcal{M}, f \rangle \models_s^2 \varphi \Leftrightarrow \mathcal{M} \models \overline{f}(\varphi).$$

Now by (15), $\mathcal{M} \models \overline{f}(\varphi) \Rightarrow \overline{f}(\varphi) \in \Sigma_1 \subset \Sigma$. Therefore $\varphi \in BSen(L_s) \cap \Sigma \Rightarrow \overline{f}(\varphi) \in \Sigma$. Thus (16) holds.

Conversely, suppose (16) is true. We have to show that $\langle \mathcal{M}, f \rangle \models^2_s \Sigma$. Pick some $\varphi \in \Sigma$. Assume first that φ is basic, i.e., $\varphi \in BSen(L_s)$. Then $\overline{f}(\varphi)$ is defined. By (16) $\overline{f}(\varphi) \in \Sigma$, therefore $\overline{f}(\varphi) \in \Sigma_1$ since $\overline{f}(\varphi)$ is classical. So by (15) $\mathcal{M} \models \overline{f}(\varphi)$. This means that $\langle \mathcal{M}, f \rangle \models^2_s \varphi$, as required.

So it remains to show that $\langle \mathcal{M}, f \rangle \models^2_s \varphi$, for $\varphi \in RSen(L_s) \backslash BSen(L_s)$. In this case φ is a Boolean and quantifier combination of basic formulas. So we can prove $\langle \mathcal{M}, f \rangle \models^2_s \varphi$ by induction on the length of its construction from basic formulas. The steps of the induction for the Boolean connectives are trivial due to the completeness of Σ. Thus it suffices to prove $\langle \mathcal{M}, f \rangle \models^2_s \exists v \varphi(v)$, whenever $\exists v \varphi(v) \in \Sigma$, assuming that this is true for φ. But if $\exists v \varphi(v) \in \Sigma$, then by Henkin-completeness of Σ, $\varphi(c) \in \Sigma$ for some $c \in L$. By the induction hypothesis, $\langle \mathcal{M}, f \rangle \models^2_s \varphi(c)$. Therefore $\langle \mathcal{M}, f \rangle \models^2_s \exists v \varphi(v)$. This completes the proof. ⊣

We come to the completeness of RFOLS$(\mathcal{F}, \Lambda_0)$. The essential step of the proof is the Lemma. The pattern of proof is quite similar to that of the proof of Lemma 3.8 of [9].

Lemma 5.7 *Let $\Sigma(\vec{v}) \subset RFml(L_s)$ be a Λ_0-consistent, complete and Henkin-complete set of restricted formulas of L_s. Then $\Sigma(\vec{v})$ is $\mathcal{F}$-satisfiable.*

Proof. Let $\Sigma(\vec{v})$ be a Λ_0-consistent, complete and Henkin-complete set of formulas of L_s. Let us set $\Sigma_1(\vec{v}) = \Sigma(\vec{v}) \cap Fml(L)$. Clearly $\Sigma_1(\vec{v})$ is a consistent, complete and Henkin-complete set of classical L-formulas. By the completeness theorem of FOL, there is an L-structure $\mathcal{M} = \langle M, \ldots \rangle$ and a tuple $\vec{a} \in M$ such that

$$\alpha \in \Sigma_1(\vec{a}) \Leftrightarrow \mathcal{M} \models \alpha. \tag{17}$$

Let $L(\vec{a})$ be L augmented with the parameters $\vec{a} \in M$. Without loss of generality and because of Henkin-completeness, we may assume that $L(\vec{a}) = L$. Because otherwise we may take $\Sigma_1^*(\vec{a}) = \{\alpha(\vec{a}) : \mathcal{M} \models \alpha(\vec{a})\}$ instead of $\Sigma_1(\vec{a})$. Since for every element a_i of the sequence $\vec{a}$, $\exists v(v = a_i)$ belongs to $\Sigma_1^*(\vec{a})$, and a_i is (essentially) the unique parameter witnessing $\exists v(v = a_i)$, it follows that $a_i \in L$. Further, if we start with $\Sigma_1^*(\vec{a})$, we can easily extend it to a consistent complete and Henkin-complete set of sentences $\Sigma^*(\vec{a})$ of $L_s(\vec{a})$ such that $\Sigma(\vec{a}) \subseteq \Sigma^*(\vec{a})$.

Now $\Sigma_1(\vec{a}) \subset \Sigma(\vec{a})$. So, applying the criterion of Lemma 5.6, in order to show that $\Sigma(\vec{a})$ is satisfiable, it suffices to construct a choice function

$g : [Sen(L)]^2 \to Sen(L)$ such that for every $\varphi \in BSen(L)$,

$$\varphi \in \Sigma(\vec{a}) \Rightarrow \overline{g}(\varphi) \in \Sigma(\vec{a}). \tag{18}$$

To do that we examine for any $\varphi, \psi \in BSen(L_s)$, the possible subsets of $\Sigma(\vec{a})$ whose elements are $\varphi|\psi$, φ, ψ or their negations. These are the following:

(a1) $\{\varphi|\psi, \varphi, \psi\} \subset \Sigma(\vec{a})$
(a2) $\{\varphi|\psi, \varphi, \neg\psi\} \subset \Sigma(\vec{a})$
(a3) $\{\varphi|\psi, \neg\varphi, \psi\} \subset \Sigma(\vec{a})$
(a4) $\{\neg(\varphi|\psi), \neg\varphi, \neg\psi\} \subset \Sigma(\vec{a})$
(a5) $\{\neg(\varphi|\psi), \varphi, \neg\psi\} \subset \Sigma(\vec{a})$
(a6) $\{\neg(\varphi|\psi), \neg\varphi, \psi\} \subset \Sigma(\vec{a})$

The remaining cases,
(a7) $\{\varphi|\psi, \neg\varphi, \neg\psi\} \subset \Sigma(\vec{a})$
(a8) $\{\neg(\varphi|\psi), \varphi, \psi\} \subset \Sigma(\vec{a})$
are impossible because they contradict Λ_0-consistency and completeness of $\Sigma(\vec{a})$. Indeed, in case (a7) we have $\neg\varphi \wedge \neg\psi \in \Sigma(\vec{a})$. Also $\varphi|\psi \in \Sigma(\vec{a})$, so by S_2 and completeness, $\varphi \vee \psi \in \Sigma(\vec{a})$, a contradiction. In case (a8) $\varphi \wedge \psi \in \Sigma(\vec{a})$. Also $\neg(\varphi|\psi) \in \Sigma(\vec{a})$, so by S_1 and completeness $\neg(\varphi \wedge \psi) \in \Sigma(\vec{a})$, a contradiction.

Given a pair $\{\alpha, \beta\}$ of sentences of L, we say that "$\{\alpha, \beta\}$ satisfies (ai)" if for $\varphi = \alpha$ and $\psi = \beta$, the corresponding case (ai) above, for $1 \leq i \leq 6$, holds. We define a choice function g for L as follows:

$$g(\alpha, \beta) = \begin{cases} (i)\ \alpha, \text{ if } \{\alpha, \beta\} \text{ satisfies (a2) or (a6)} \\ (ii)\ \beta, \text{ if } \{\alpha, \beta\} \text{ satisfies (a3) or (a5)} \\ (iii) \text{ any of the } \alpha,\ \beta, \text{ if } \{\alpha, \beta\} \text{ satisfies (a1) or (a4).} \end{cases} \tag{19}$$

Claim. $\overline{g}$ satisfies the implication (18).

Proof of the Claim. We prove (16) by induction on the length of φ. For $\varphi = \alpha \in Sen(L)$, $\overline{g}(\alpha) = \alpha$, so (16) holds trivially. Similarly the induction steps for $\wedge$ and $\neg$ follow immediately from the fact that $\overline{g}$ commutes with these connectives and the completeness of $\Sigma(\vec{a})$. So the only nontrivial step of the induction is that for $\varphi|\psi$. It suffices to assume

$$\varphi \in \Sigma(\vec{a}) \ \Rightarrow \overline{g}(\varphi) \in \Sigma(\vec{a}), \tag{20}$$

$$\psi \in \Sigma(\vec{a}) \ \Rightarrow \overline{g}(\psi) \in \Sigma(\vec{a}), \tag{21}$$

and prove

$$\varphi|\psi \in \Sigma(\vec{a}) \ \Rightarrow \overline{g}(\varphi|\psi) \in \Sigma(\vec{a}). \tag{22}$$

Assume $\varphi|\psi \in \Sigma(\vec{a})$. Then the only possible combinations of φ, ψ and their negations that can belong to $\Sigma(\vec{a})$ are those of cases (a1), (a2) and (a3) above. To prove (22) it suffices to check that $\overline{g}(\varphi|\psi) \in \Sigma(\vec{a})$ in each of these cases. Note that $\overline{g}(\varphi|\psi) = g(\overline{g}(\varphi), \overline{g}(\psi)) = g(\alpha, \beta)$, where $\overline{g}(\varphi) = \alpha$ and $\overline{g}(\psi) = \beta$ are sentences of L, so (19) applies.

Case (a1): Then $\varphi \in \Sigma(\vec{a})$ and $\psi \in \Sigma(\vec{a})$. By (20) and (21), $\overline{g}(\varphi) \in \Sigma(\vec{a})$ and $\overline{g}(\varphi) \in \Sigma(\vec{a})$. By definition (19), $\overline{g}(\varphi|\psi) = g(\overline{g}(\varphi), \overline{g}(\psi))$ can be either $\overline{g}(\varphi)$ or $\overline{g}(\psi)$. So in either case $\overline{g}(\varphi|\psi) \in \Sigma(\vec{a})$.

Case (a2): Then $\varphi \in \Sigma(\vec{a})$ and $\neg\psi \in \Sigma(\vec{a})$. By (20) and (21), $\overline{g}(\varphi) \in \Sigma(\vec{a})$, $\overline{g}(\psi) \notin \Sigma(\vec{a})$. Also by (19), $\overline{g}(\varphi|\psi) = g(\overline{g}(\varphi), \overline{g}(\psi)) = \overline{g}(\varphi)$, thus $\overline{g}(\varphi|\psi) \in \Sigma(\vec{a})$.

Case (a3): Then $\neg\varphi \in \Sigma(\vec{a})$, $\psi \in \Sigma(\vec{a})$. By (20) and (21), $\overline{g}(\varphi) \notin \Sigma(\vec{a})$, $\overline{g}(\psi) \in \Sigma(\vec{a})$. By (19), $\overline{g}(\varphi|\psi) = g(\overline{g}(\varphi), \overline{g}(\psi)) = \overline{g}(\psi)$, thus $\overline{g}(\varphi|\psi) \in \Sigma(\vec{a})$. This completes the proof of the Claim.

It follows that condition (18) is true, so by Lemma 5.6, since $\mathcal{M} \models \Sigma_1(\vec{a})$ where $\Sigma_1(\vec{a}) = \Sigma(\vec{a}) \cap Sen(L)$, $\langle \mathcal{M}, g \rangle \models_s^2 \Sigma(\vec{a})$, therefore $\Sigma(\vec{a})$ is $\mathcal{F}$-satisfiable. ⊣

Theorem 5.8 (Completeness of RFOLS($\mathcal{F}, \Lambda_0$)) *Let $\Sigma(\vec{v})$ be a consistent set of restricted formulas of L_s. Then $\Sigma(\vec{v})$ is $\mathcal{F}$-satisfiable, i.e., there are $\mathcal{M}$, $f : [Sen(L)]^2 \to Sen(L)$ and $\vec{a} \in M$ such that $\langle \mathcal{M}, f \rangle \models_s^2 \Sigma(\vec{a})$.*

Proof. Let $\Sigma(\vec{v})$ be a Λ_0-consistent set of formulas. Extend $\Sigma(\vec{v})$ to a Λ_0-consistent, complete and Henkin-complete set of formulas of $L_s^+ \supseteq L_s$ such that $\Sigma(\vec{v}) \subseteq \Sigma^+(\vec{v})$. By Lemma 5.7, $\Sigma^+(\vec{v})$ is $\mathcal{F}$-satisfiable. Therefore so is $\Sigma(\vec{v})$. ⊣

5.3 Conditional completeness of the remaining systems

Coming to completeness, as in the case of PLS, the presence of SV makes the status of Deduction Theorem (DT) open. In turn the absence of DT has two consequences: (a) we don't know if CT1 and CT2 are equivalent (we only know that CT1 implies CT2) and (b) we don't know if a consistent set of formulas can be extended to a consistent and complete set (and a fortiori if it can be extended to a consistent, complete and Henkin-complete set). So, concerning the completeness of the systems based on Λ_i, for $i = 1, 2, 3$, (a) we shall be confined to the weaker form CT2 only, and (b) we shall appeal to an *extendibility principle* for the formal systems Λ_i, already used in [10].

$(cHext(\Lambda))$ *Every Λ-consistent set of formulas of L_s can be extended*

to a Λ-consistent, complete and Henkin-complete set.

We can see $cHext(\Lambda)$ as the conjunction of $cext(\Lambda)$ and $Hext(\Lambda)$, where $cext(\Lambda)$ says that every Λ-consistent set can be extended to a complete Λ-consistent set, and $Hext(\Lambda)$ says that every Λ-consistent set can be extended to a Henkin-complete Λ-consistent set.

The following Lemma will be essential for the completeness of the aforementioned logics, proved in the next section.

Lemma 5.9 *If $\Sigma \subset Sen(L_s)$ is closed with respect to $\vdash_{\Lambda_i}$, for some $i = 1, 2, 3$, and α, α' are formulas of L such that $\alpha \sim \alpha'$, then for every β, $(\alpha|\beta \leftrightarrow \alpha'|\beta) \in \Sigma$.*

Proof. Let $\alpha \sim \alpha'$. Then $\vdash_{\mathrm{FOL}} \alpha \leftrightarrow \alpha'$, hence also $\vdash_{\Lambda_0} \alpha \leftrightarrow \alpha'$. By SV it follows that for every β, $\vdash_{\Lambda_i} \alpha|\beta \leftrightarrow \alpha'|\beta$. Therefore $(\alpha|\beta \leftrightarrow \alpha'|\beta) \in \Sigma$ since Σ is $\vdash_{\Lambda_i}$-closed. $\dashv$

The following theorem is the analogue of Theorem 3.16 of [9], as well as part of Theorem 4.9 of [10].

Theorem 5.10 (Conditional CT2-completeness of RFOLS(Reg, Λ_1) *The logic* RFOLS(Reg, Λ_1) *is* CT2*-complete if and only if $cHext(\Lambda_1)$ is true.*

Proof. We prove first the easy direction. Assume $cHext(\Lambda_1)$ is false. Then either $cext(\Lambda_1)$ fails or $Hext(\Lambda_1)$ fails.

Assume first that $cext(\Lambda_1)$ fails. It follows that there is a maximal Λ_1-consistent set of formulas $\Sigma(\vec{v})$ not extendible to a Λ_1-consistent and complete set. It means that there is a formula $\varphi(\vec{v})$ such that both $\Sigma(\vec{v}) \cup \{\varphi(\vec{v})\}$ and $\Sigma(\vec{v}) \cup \{\neg\varphi(\vec{v})\}$ are Λ_1-inconsistent and hence unsatisfiable. Then clearly $\Sigma(\vec{v})$ cannot be satisfiable in any structure $\mathcal{M}$, for then $\mathcal{M}$ would also satisfy $\varphi(\vec{v})$ or $\neg\varphi(\vec{v})$. Thus CT2-completeness fails.

Next assume that $Hext(\Lambda_1)$ fails. It follows that there is a maximal Λ_1-consistent set of formulas $\Sigma(\vec{v})$ not extendible to a Λ_1-consistent and Henkin-complete set. It means that $\Sigma(\vec{v})$ contains a formula $\exists u\varphi(u, \vec{v})$ such that for any new constant c, $\Sigma(\vec{v}) \cup \{\varphi(c, \vec{v})\}$ is Λ_1-inconsistent. But then $\Sigma(\vec{v})$

cannot be satisfiable. For if $\mathcal{M}$ satisfies $\Sigma(\vec{v})$, then in particular $\exists u\varphi(u,\vec{v})$ is satisfied in $\mathcal{M}$, so also $\varphi(c,\vec{v})$ is satisfied in $\mathcal{M}$ for some $c \in M$. Therefore $\Sigma(\vec{v}) \cup \{\varphi(c,\vec{v})\}$ is satisfiable, contrary to the fact that $\Sigma(\vec{v}) \cup \{\varphi(c,\vec{v})\}$ is inconsistent. Thus indeed the Λ_1-consistent set $\Sigma(\vec{v})$ is not satisfiable, so CT2-completeness fails.

We come to the main direction of the equivalence assuming $cHext(\Lambda_1)$ is true. Then given a Λ_1-consistent set $\Sigma(\vec{v}) \subset RFml(L_s)$ of restricted formulas, we may assume without loss of generality that it is also complete and Henkin-complete. We have to find $\mathcal{M}$ and $g \in Reg$ such that $\langle \mathcal{M}, g\rangle \models^2_s \Sigma(\vec{a})$ for some $\vec{a} \in M$. It turns out that the main argument of Lemma 5.7, concerning the definition of the choice function g, works also here, with the necessary adjustments. Namely it suffices to find a choice function $g \in Reg$ such that $\langle \mathcal{M}, g\rangle \models^2_s \Sigma(\vec{a})$, where $\mathcal{M}$ and $\vec{a} \in M$ are the model and parameters such that for every $\alpha \in L(\vec{v})$,

$$\alpha \in \Sigma_1(\vec{a}) \Leftrightarrow \mathcal{M} \models \alpha,$$

where $\Sigma_1(\vec{a}) = \Sigma(\vec{a}) \cap Sen(L)$. The definition of g follows exactly the pattern of definition of g in the proof of Lemma 5.7, except that we need now to take care so that g be regular. Recall that g is regular if for all α, α', β,

$$\alpha' \sim \alpha \;\Rightarrow\; g(\alpha', \beta) \sim g(\alpha, \beta).$$

In (19) g is defined by three clauses: (i) (a2) or (a6), (ii) (a3) or (a5), (iii) (a1) or (a4).

Claim. The regularity constraint is satisfied whenever g is defined by clauses (i) and (ii) above.

Proof of Claim. Pick α, α', β such that $\alpha \sim \alpha'$. We prove the Claim for the case that $g(\alpha,\beta)$ is defined according to clause (i)-(a2). All other cases are verified similarly. That $g(\alpha,\beta)$ is defined by case (i)-(a2) of (19) means that $\alpha|\beta \in \Sigma(\vec{a})$, $\alpha \in \Sigma(\vec{a})$, $\neg\beta \in \Sigma(\vec{a})$ and $g(\alpha,\beta) = \alpha$. It suffices to see that necessarily $g(\alpha',\beta) = \alpha' \sim g(\alpha,\beta)$.

Since $\Sigma(\vec{a})$ is complete, it is closed with respect to $\vdash_{\Lambda_1}$, so by Lemma 5.9, $\alpha \sim \alpha'$ implies that $(\alpha|\beta \leftrightarrow \alpha'|\beta) \in \Sigma(\vec{a})$. Also by assumption, $\alpha|\beta \in \Sigma(\vec{a})$, hence $\alpha'|\beta \in \Sigma(\vec{a})$. Moreover $\alpha' \in \Sigma(\vec{a})$, since $\alpha \in \Sigma(\vec{a})$, and $\neg\beta \in \Sigma(\vec{a})$. Therefore case (i)-(a2) occurs too for $\alpha'|\beta$, α' and β. So, by (19), $g(\alpha',\beta) = \alpha'$, therefore $g(\alpha',\beta) \sim g(\alpha,\beta)$. This proves the Claim.

It follows from the Claim that if we define g according to (19), regularity is guaranteed unless $g(\alpha,\beta)$ is given by clause (iii), that is, unless (a1) or (a4) is the case. In such a case either both α, β belong to Σ, or both $\neg\alpha$, $\neg\beta$ belong to Σ, and (19) allows $g(\alpha, \beta)$ to be *any* of the elements α, β. So at this point we must intervene by a new condition that will guarantee regularity. This is done as follows.

Pick from each $\sim$-equivalence class $[\alpha]$, a representative $\xi_\alpha \in [\alpha]$. Recall that, by completeness, the set $\Sigma_1 = \Sigma \cap Sen(L)$ as well as its complement $\Sigma_2 = Sen(L) - \Sigma_1$ are saturated with respect to $\sim$, that is, for every α, either $[\alpha] \subset \Sigma_1$ or $[\alpha] \subset \Sigma_2$. Let $D_1 = \{\xi_\alpha : \alpha \in \Sigma_1\}$, $D_2 = \{\xi_\alpha : \alpha \in \Sigma_2\}$. Let $[D_i]^2$ be the set of pairs of elements of D_i, for $i = 1, 2$, and pick an arbitrary choice function $g_0 : [D_1]^2 \cup [D_2]^2 \to D_1 \cup D_2$. Then it suffices to define g by slightly revising definition (19) as follows:

$$g(\alpha, \beta) = \begin{cases} (i)\ \alpha, \text{ if } \{\alpha, \beta\}, \text{ satisfies (a2) or (a6)} \\ (ii)\ \beta, \text{ if } \{\alpha, \beta\} \text{ satisfies (a3) or (a5)} \\ (iii)\ \sim g_0(\xi_\alpha, \xi_\beta), \text{ if } \{\alpha, \beta\} \text{ satisfies (a1) or (a4).} \end{cases} \tag{23}$$

(The third clause is just a shorthand for: $g(\alpha, \beta) = \alpha$ if $g_0(\xi_\alpha, \xi_\beta) = \xi_\alpha$, and $g(\alpha, \beta) = \beta$ if $g_0(\xi_\alpha, \xi_\beta) = \xi_\beta$.) In view of the Claim and the specific definition of g by (23), it follows immediately that if $\alpha \sim \alpha'$ then for every β, $g(\alpha, \beta) \sim g(\alpha', \beta)$. So g is regular. Further, exactly as in Lemma 5.7 it follows that $\langle M, g\rangle \models^2_s \Sigma(\vec{a})$. This completes the proof. $\dashv$

The next two theorems are cited without proofs. They are analogues of Theorems 3.18 and 3.19 of [9], and their proofs follow the patterns of the latter with adaptations similar to the ones we used in the proofs of Theorems 5.8 and 5.10 above.

Theorem 5.11 (Conditional CT2-completeness for RFOLS(Reg^*, Λ_2)) *The logic* RFOLS(Reg^*, Λ_2) *is* CT2-*complete if and only if cHext*(Λ_2) *is true.*

Theorem 5.12 (Conditional CT2-completeness for RFOLS(Dec, Λ_3)) *The logic* RFOLS(Dec, Λ_3) *is* CT2-*complete if and only if cHext*(Λ_3) *is true.*

We shall close this section and the paper by answering a question raised in [9] (section 5 concerning future work), namely whether the extension of PLS to FOLS might help us to pass from superposition of sentences to *superposition of objects.* Such a notion may sound a little bit strange, but is closely related to "disjunctive objects" (more precisely "disjunctive multisets"), which have already been used in [8] to provide semantics for the Horn fragment of the multiplicative intuitionistic linear logic (ILL) augmented with additive disjunction. The following simple every-day example motivates sufficiently the introduction of the concept. Restaurant menus refer to entities of the form "steak or fish" (upon choice), for main dish, and "dessert or season fruit" (upon choice and season), for exit.[2] One can think of the term "steak or fish" as representing a new kind of *theoretical* entity, an object generated by the superposition of steak and fish. Of course a specific customer who dines in the restaurant does not eat "steak or fish". They eat either steak or fish, which are the *actualizations*, i.e., the possible collapses, of the superposed object. It is true that *existence* of such objects seems dubious. They look unstable and temporary, since they always collapse to their actualizations, and also elusive since they can be handled not in themselves, but only through their actualizations. However, more or less, the same is true of all theoretical entities: they are supposed to stand out there elusive in themselves for our minds, like platonic ideas, accessible only through their concrete physical realizations. Notice in particular that in the case of superposed menu items, the phrase "upon choice" that accompanies them explicitly indicates that our access to their physical realizations is obtained only by the help of a choice function.

One way to obtain (formal representation of) superposition of objects would be through the logic of superposition (namely FOLS), if in the latter one could prove that for any two objects (constants) a and b, there exists a *unique* object c satisfying the formula (in one free variable) $(v = a)|(v = b)$, i.e., if the sentence $(\forall v, u)(\exists! w)((w = v)|(w = u))$ would be a tautology. If that would be the case, we could write $c = a \uparrow b$ for the unique object c satisfying the formula $(v = a)|(v = b)$, and say that c is the *superposition of a and b*.

If the semantics FCS would not have broken down, it is easy to see that it would satisfy the above requirement, i.e., for every $\mathcal{M}$, for every choice function f for pairs of formulas and any $a, b \in M$ we would have $\langle \mathcal{M}, f\rangle \models_s (\exists! v)((v = a)|(v = b))$. This is because $\langle \mathcal{M}, f\rangle \models_s (\exists! v)((v = a)|(v = b))$ holds iff $\mathcal{M} \models (\exists! v) f(v = a, v = b)$ and the latter is obviously true no matter whether $f(v = a, v = b) = (v = a)$ or $f(v = a, v = b) = (v = b)$.

However working with the semantics SCS for FOLS described in this section we have the following situation.

Proposition 5.13 *Let L be a first-order language, $\mathcal{M} = \langle M, \ldots\rangle$ be an L-structure and $a \neq b \in M$. Then:*

[2] In popular presentations of linear logic the first kind of disjunction is construed as "multiplicative" or deterministic, while the latter is construed as "additive" or non-deterministic.

(i) There are choice functions $f \in \mathcal{F}$, such that $\langle \mathcal{M}, f\rangle \models^2_s (\exists! v)((v = a)|(v = b))$.

(ii) However there is no $f \in Reg$ such that $\langle \mathcal{M}, f\rangle \models^2_s (\exists! v)((v = a)|(v = b))$.

Proof. (i) Given $a \neq b$, clearly the only values for v that might satisfy $(v = a)|(v = b)$ are a or b. Pick f such that $f(a = a, a = b) = (a = a)$ and $f(b = b, a = b) = (a = b)$. Then clearly $\langle \mathcal{M}, f\rangle \models^2_s (a = a)|(a = b)$, while $\langle \mathcal{M}, f\rangle \not\models^2_s (b = a)|(b = b)$. Thus the only element of M that satisfies $(v = a)|(v = b)$ in $\langle \mathcal{M}, f\rangle$ is a. Similarly, if we consider f' such that $f'(a = a, a = b) = (a = b)$ and $f'(b = b, a = b) = (b = b)$, the only element of M that satisfies $(v = a)|(v = b)$ in $\langle \mathcal{M}, f'\rangle$ is b.

(ii) In contrast to (i), if $f \in Reg$ then, since $(a = a) \sim (b = b)$ we should have $f(a = a, a = b) \sim f(b = b, a = b)$. Therefore either

$$f(a = a, a = b) = (a = a), \text{ and } f(b = b, a = b) = (b = b),$$

or

$$f(a = a, a = b) = f(b = b, a = b) = (a = b).$$

In the first case both a, b satisfy $(v = a)|(v = b)$ in $\langle \mathcal{M}, f\rangle$, while in the second case none of the a, b, and hence no element of M, satisfies $(v = a)|(v = b)$. In either case $\langle \mathcal{M}, f\rangle \not\models^2_s (\exists! v)((v = a)|(v = b))$. ⊣

The preceding result shows that the attempt to represent superposition of objects through FOLS and its semantics SCS fails. We can only show that for any two objects a, b there is an object c satisfying the property $(v = a)|(v = b)$, but it is not unique. Nevertheless, superposition of objects can be introduced by an alternative way, namely through *mathematical* rather than logical means. Specifically, given a first order theory T in a language L, T can be extended to a theory $T^|$ in the language $L \cup \{|\}$, where $|$ is a new binary operation (on the objects of T). $T^|$ (with underlying logic the usual FOL) consists of the axioms of T plus some plausible axioms for $|$, analogous to the axioms S_i of section 1.1, expressing idempotence, symmetry and associativity of $|$, and possibly some further properties for the objects $a|b$.

Let us note by the way that the notation $x|y$ was first used in [8, §3]. The operation $x|y$ was defined there (for multisets and finite sets of multisets) so that idempotence, symmetry and associativity hold, and also so that the object $x|y$ be *distinct* from both x and y, i.e., $x|y \notin \{x, y\}$, unless $x = y$. In contrast, if $x|y$ is going to represent an entity that always collapses to either x or y, then necessarily $x|y \in \{x, y\}$, i.e., $|$ must behave as a choice function. Thus one can have at least two different implementations of the operation $x|y$ on objects: a "projective" one, such that $x|y \in \{x, y\}$, and a "non-projective" one, such that $x|y \notin \{x, y\}$.

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