The logic of multisets continued: The case of disjunction

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Abstract

We continue our work [5] on the logic of multisets (or on the multiset semantics of linear logic), by interpreting further the additive disjunction $\sqcup$. To this purpose we employ a more general class of processes, called *free*, the axiomatization of which requires a new rule (not compatible with the full LL), the *cancellation rule*. Disjunctive multisets are modeled as finite sets of multisets. The $\sqcup$-Horn fragment of linear logic, with the cut rule slightly restricted, is sound with respect to this semantics. Another rule, which is a slight modification of cancellation, added to $\mathrm{HF}_\sqcup$ makes the system sound and complete.

1 Introduction

The relation between multisets and sets is roughly the relation between concrete, material objects (or "resources") on the one hand and their abstract properties on the other. Can this similarity/opposition be formulated in terms of logic and how? The logic of abstract properties is well known. Properties are true or false, they conjunct and disjunct, imply one another and are negated. Material entities on the other hand seem to be able just to exist. Yet a little more can be said. They also coexist or are incompatible, and they transform to one another respecting certain preservation rules. Thus we can talk about a "logic of material beings", or "logic of resources", meaning by that certain general rules of existence and change. In [4] an approach of the logic of resources was presented through a variant of λ-calculus applied to abstract objects which are postulated to behave like the ordinary artificial objects. Later we realized that most of the properties of abstract objects can be captured using the much simpler setup of multisets.

In [5] we identify resources with (finite) multisets and study their logic. In order to make this paper self-contained, we survey briefly below the main points of [5].

Let A be a nonempty set whose elements $a, b, c, \dots$ are thought of as urelements. A multiset over A is a function $x : A \to \mathbb{N}$ (=the set of nonnegative integers). $x(a)$ is the multiplicity of a in x. As *elements* of x we consider only those a for which $x(a) \neq 0$. x is *finite* if it has finitely many elements. Throughout $x, y, z, \dots$ range over finite multisets (over some fixed set A). A more explicit notation is $x = [a_1, a_1, \dots, a_2, a_2, \dots]$ where a_i occur with repetitions and square brackets distinguish multisets from sets. The empty multiset is denoted again by $\emptyset$.

Let $U(A)$ be the set of all finite multisets over A. We may assume $A \subset U(A)$ by identifying each $a \in A$ with the multiset $[a]$. We endow $U(A)$ with the operation of additive union $\uplus$ defined by $(x \uplus y)(z) := x(z) + y(z)$, and the "replacement relation (or operation)" $U(A) \times U(A)$ where the pair (x, y) means :"replace x by y". The replacement relation is the entire $U(A) \times U(A)$, since, at this abstract level, there is no reason to exclude any multiset from being able to replace any other one in any given procedure.

A general pattern of sequent

The entities we shall deal with in this paper fall into two categories: *Objects* of several kinds (multisets, disjunctive multisets, formulas made of atoms, conjunction and disjunction) and *transformations* of several kinds (replacements of multisets, implications). Objects and transformations are tied together and interact within a *sequent*. The general pattern of sequent used throughout is

$$O, T \vdash e, \tag{1}$$

where O is a multiset of objects, T is a multiset of transformations and e is either an object or a transformation. The meaning of (1) is: When the transformations T apply to the objects O (viewed as resources), the outcome is e. When $O = [a_1, \ldots, a_n]$ and $T = [t_1, \ldots, t_m]$, (1) is written equivalently,

$$a_1, \ldots, a_n, t_1, \ldots, t_m \vdash e.$$

Then $O, b, T \vdash e$ has the same meaning as $a_1, \ldots, a_n, b, T \vdash e$.

Sequents of multisets

In the case of multisets, (1) takes the form

$$\mu, \sigma \vdash w, \quad \text{or} \quad \mu, \sigma \vdash (y, w),$$

where $\mu = [x_1, \ldots, x_n]$ is a multiset of multisets and $\sigma = [(y_1, z_1), \ldots, (y_n, z_n)]$ is a multiset of replacements. The pair (μ, σ) is said to be a *process*. The letter P denotes processes. Given $\sigma = [(y_1, z_1), \ldots, (y_n, z_n)]$, let

$$in(\sigma) = y_1 \uplus \cdots \uplus y_n, \quad out(\sigma) = z_1 \uplus \cdots \uplus z_n.$$

The processes considered in [5] are the so-called *staged* ones: $P = (\mu, \sigma)$ is staged if there is a an enumeration $[(y_1, z_1), \ldots, (y_n, z_n)]$ of σ such that for every $i < n$,

$$y_{i+1} \subseteq ((\uplus\mu) \uplus z_1 \uplus \cdots \uplus z_i) - (y_1 \uplus \cdots \uplus y_i),$$

where $\uplus\mu = x_1 \uplus \cdots \uplus x_n$. Putting

$$P(0) = \uplus\mu \quad \text{and} \quad P(i+1) = P(i) \uplus z_{i+1} - y_{i+1},$$

for $i < n$, the above condition is written $y_{i+1} \subseteq P(i)$. The sequence $P(0), \ldots, P(n)$ is called a *stage sequence* of P. We say that P *yields* w and write $P \vdash w$ or $\mu, \sigma \vdash w$, if for some stage sequence $P(i), i \leq n$, $P(n) = w$. In this case $P(n)$ is said to be the output of P and we denote it $out(P)$, i.e., $out(P) = P(n) = w$. The notation $P \vdash w$ reads also "the sequent $P \vdash w$ holds". By definition $\mu, \sigma \vdash (y, w)$ holds iff $\mu, y, \sigma \vdash w$ does.

Sequents of formulas

The logic appropriate to axiomatize the behavior of sequents of multisets is proved to be the Horn fragment of the multiplicative intuitionistic linear logic (ILL) (see [1] for general background) without exponentials and quantifiers (i.e., the $\{\otimes, \multimap, \mathbf{1}\}$-fragment), denoted HF. The formulas of HF are either of the form (a) $p_1 \otimes \cdots \otimes p_n$, where p_i are atomic propositions, called *simple products* (or just *products*), or (b) $X \multimap Y$, where X, Y are simple products, called *implications*. The variables $X, Y, W, U \ldots$ denote simple products (although sometimes may range also over implications. See remark 1.1 below). In the case of formulas, a sequent (1) takes the form

$$\Phi, \Sigma \vdash W, \text{ or } \Phi, \Sigma \vdash (Y \multimap W),$$

where Φ is a multiset of products, Σ is a multiset of implications, and Y, W are simple products. $\mathbf{1}$ denotes the empty product.

Interpretations

Let $\mathcal{L}$ be the language of the Horn fragment. An *interpretation* of $\mathcal{L}$ in $U(A)$ consists of a (usually injective) mapping $^* : \{p_1, p_2, \ldots\} \to A$ which extends to all formulas by putting:

a) $(X \otimes Y)^* = X^* \uplus Y^*$,
b) $\Phi^* = [X_1^*, \ldots, X_n^*]$, if $\Phi = [X_1, \ldots, X_n]$,
c) $(X \multimap Y)^* = (X^*, Y^*)$,
d) $\Sigma^* = [(Y_1^*, Z_1^*), \ldots, (Y_m^*, Z_m^*)]$ if $\Sigma = [Y_1 \multimap Z_1, \ldots, Y_m \multimap Z_m]$,
e) $\mathbf{1}^* = \emptyset$ (the empty multiset).

The interpretation * extends naturally to sequents, turning sequents of formulas into sequents of multisets. We say that the sequent $\Phi, \Sigma \vdash W$, is *true under* * if (Φ^*, Σ^*) is a staged process and $\Phi^*, \Sigma^* \vdash W^*$. The sequent $\Phi, \Sigma \vdash (Y \multimap W)$ is *true under* * if $\Phi, Y, \Sigma \vdash W$ is so.

A rule $\frac{S'}{S}$ or $\frac{S_1 \quad S_2}{S}$ is *sound* with respect to *, if the truth of the sequent(s) S' (resp. S_1, S_2) implies the truth of S. Thus the pair $(U(A), \vdash)$ is a structure interpreting sequents of HF.

The rules of HF are the following:

$$Ax : \frac{}{X \vdash X}, \qquad Cut : \frac{\Phi_1, \Sigma_1 \vdash W \quad W, \Phi_2, \Sigma_2 \vdash U}{\Phi_1, \Phi_2, \Sigma_1, \Sigma_2 \vdash U},$$

$$L_\otimes : \frac{\Phi, Y, Z, \Sigma \vdash W}{\Phi, Y \otimes Z, \Sigma \vdash W}, \qquad R_\otimes : \frac{\Phi_1, \Sigma_1 \vdash W \quad \Phi_2, \Sigma_2 \vdash U}{\Phi_1, \Phi_2, \Sigma_1, \Sigma_2 \vdash W \otimes U},$$

$$L_\multimap : \frac{\Phi_1, \Sigma_1 \vdash W \quad U, \Phi_2, \Sigma_2 \vdash V}{\Phi_1, \Phi_2, \Sigma_1, \Sigma_2, (W \multimap U) \vdash V}, \qquad R_\multimap : \frac{\Phi, Y, \Sigma \vdash W}{\Phi, \Sigma \vdash (Y \multimap W)},$$

$$L_{\mathbf{1}} : \frac{\Phi, \Sigma \vdash W}{\Phi, \mathbf{1}, \Sigma \vdash W}, \qquad R_{\mathbf{1}} : \vdash \mathbf{1}.$$

Remark 1.1 i) Ax holds for any formula of HF. ii) In Cut the cut formula can be also an implication. iii) On the other hand in $L_{\multimap}$, and $R_{\multimap}$, clearly W, U must be products. Analogous remarks hold for the other rules. □

Theorem 1.2 (Soundness and completeness of HF, [5]) *i) Let S be a Horn sequent provable in* HF*. Then for every non-empty set A, and for every interpretation * of formulas of $\mathcal{L}$ in $U(A)$, S^* is true in $(U(A), \vdash)$.*

ii) Let S be a Horn sequent. If for some non-empty set A and some injective interpretation * of $\mathcal{L}$ in $U(A)$, S^* is true in $(U(A), \vdash)$, then S is provable in HF*.*

(ii) of the above theorem deviates slightly from the ordinary formulations of completeness theorems. It requires only *some* (instead of *all*) interpretation, but this must be *injective*. The precise technical reason why * must be injective in order from the truth of the multiset sequent $\Phi^*, \Sigma^* \vdash W^*$ to conclude that the Horn sequent $\Phi, \Sigma \vdash W$ is provable in HF, is explained in Lemma 3.4 of [5]. The above theorem is Theorems 3.3 and 3.5 of [5]. Injectivity of * is required also for the completeness results of this paper (e.g. 2.10).

The table below summarizes the correspondence between logical and ontological notions.

Logic	Ontology
formulas (simple products)	multisets
truth	existence
proof	process
(linear) conjunction	additive union
(linear) implication	replacement
rules of HF	corresponding rules for multisets
disjunction	?
negation	?

Table

The place of negation in the above table will remain open, since negation can hardly be conceived as an operation between objects, even if the latter are confined to multisets. So the logic of objects throughout will be *negation free*. On the other hand it is the purpose of this paper to enrich this logic with disjunction in a satisfactory way. Namely we shall find a proper formalization of *disjunctive objects* as a class of objects that extend the ordinary ones. In order to do that we shall first replace the staged processes used above with a certain weaker type of processes called *free*. In the next section we axiomatize free processes.

2 Free processes.

It is clear from the definitions of the previous section that if $P = (\mu, \sigma)$ is a staged process, then

$$in(P) \subseteq (\uplus\mu) \uplus out(\sigma) \text{ and } out(P) = (\uplus\mu) \uplus out(\sigma) - in(\sigma),$$

in which case we write $P \vdash out(P)$. Can the last relations be used as alternative definitions of the staged sequence and the yielding relation $\vdash$? The answer is negative. However they provide a strictly weaker notion of process and yielding.

Definition 2.1 A process $P = (\mu, \sigma)$ is said to be *free* if

$$in(P) \subseteq (\uplus\mu) \uplus out(\sigma).$$

In this case we set $out(P) = (\uplus\mu) \uplus out(\sigma) - in(\sigma)$ and say that P *yields* $out(P)$. We denote this by

$$P \mid\!\sim out(P).$$

Also for a process (μ, σ) *and a replacement* (y, w)*, we write* $\mu, \sigma \mid\!\sim (y, w)$*, if* (μ, y, σ) *is free and* $\mu, y, \sigma \mid\!\sim w$*.*

Remark 2.2 i) As mentioned earlier every staged process is free but not conversely. For instance the process $(x, (y, y))$, with $y \not\subseteq x$, is free (with output x) but obviously cannot be staged.

ii) Roughly staged processes are wellfounded, in the sense that some initial replacements must draw their inputs from independent resources, while the free ones may contain genuine cycles. Free processes seem to be the most general transformational procedures that can lead to an outcome. $\square$

The following simple and useful connection holds between staged and free processes.

Lemma 2.3 *Let $\mu, \sigma \mathrel{|\!\sim} w$. Then for every $u \supseteq in(\sigma)$, $\mu, u, \sigma \vdash u \uplus w$.*

Proof. Since $\mu, \sigma \mathrel{|\!\sim} w$ holds, $w = (\uplus\mu) \uplus out(\sigma) - in(\sigma)$. Let $\sigma = [(y_1, z_1), \ldots, (y_n, z_n)]$ be an arbitrary enumeration of σ. Then $u \supseteq in(\sigma) = y_1 \uplus \cdots \uplus y_n$. Let $P(0) = u \uplus (\uplus\mu)$ and $P(i+1) = P(i) \uplus z_{i+1} - y_{i+1}$. Then clearly $y_{i+1} \subseteq P(i)$, therefore $P(i)$ is a staged sequence for the process $P = (\mu, u, \sigma)$. Thus P is staged and

$$out(P) = P(n) = u \uplus (\uplus\mu) \uplus out(\sigma) - in(\sigma) = u \uplus w.$$

This proves the claim. □

Note that, having $\emptyset$ explicitly among our resources, an empty left or right side in a sequent of multisets is different from having $\emptyset$ at that side. For instance $\mathrel{|\!\sim}$ is not the same as $\emptyset \mathrel{|\!\sim} \emptyset$. The latter is obviously free and true. On the other hand, since every free P must have an output, for no free P can the sequent $P \mathrel{|\!\sim}$ hold true. Finally the interpretation of the rule $R_{\mathbf{1}}$ is the (truth of the) sequent $\mathrel{|\!\sim} \emptyset$ which does not make real sense, but we may accept as holding by convention.

Theorem 2.4 (Soundness) *The rules Ax, Cut, $L_\otimes$, $R_\otimes$, $L_\multimap$, $R_\multimap$, $L_{\mathbf{1}}$, $R_{\mathbf{1}}$ are sound in $(U(A), \mathrel{|\!\sim})$ under any interpretation.*

Proof. The proof is straightforward though tedious, and left to the patient reader. □

However completeness of HF fails with respect to $(U(A), \mathrel{|\!\sim})$. For instance it is easy to see that the sequent S:

$$X, (X \otimes U) \multimap (Y \otimes U) \vdash Y$$

is true in $(U(A), \mathrel{|\!\sim})$ under any injective interpretation, but, since there is no rule allowing the elimination of $\otimes$ from the antecedent of a sequent, S is unprovable in HF (and also in the full ILL).

In order to capture such derivations we need the following additional rule:

$$C_\otimes : \frac{\Phi, Z, \Sigma \vdash W \otimes Z}{\Phi, \Sigma, \vdash W}.$$

We call $C_\otimes$ $\otimes$-*cancellation* (or just cancellation) rule and we set

$$\mathrm{CHF} = \mathrm{HF} + C_\otimes.$$

Remark 2.5 The rule $C_\otimes$ is a bit strange, as it does not satisfy the "subformula property": Formulas occurring in the upper sequent are missing from the lower one. The only other rule for which this happens is *Cut*, and this is the reason why we want to be able to dispense with it (cut elimination). The rules however are viewed here as general laws of material transformation, and need not meet the special requirements of proof theory.

Lemma 2.6 *If $\mu, \sigma, (y, z) \mid\!\sim w$, then $\mu, z, \sigma \mid\!\sim w \uplus y$.*

Proof. By definition $\mu, \sigma, (y, z) \mid\!\sim w$ iff

$$(\uplus\mu) \uplus z \uplus out(\sigma) - y \uplus in(\sigma) = w,$$

or, equivalently,

$$(\uplus\mu) \uplus z \uplus out(\sigma) - in(\sigma) = w \uplus y,$$

which says that $\mu, z, \sigma \mid\!\sim w \uplus y$. □

Theorem 2.7 (Soundness of CHF) *If $\Phi, \Sigma \vdash W$ is provable in* CHF*, then for every interpretation* **, (Φ^*, Σ^*) is free and $\Phi^*, \Sigma^* \vdash W^*$ holds in $(U(A), \mid\!\sim)$.*

Proof. We have seen in theorem 2.4 that the rules of HF are sound in $(U(A), \mid\!\sim)$. The interpretation of $C_\otimes$ is, clearly, the $\uplus$-cancellation rule

$$C_\uplus : \frac{\mu, z, \sigma \mid\!\sim w \uplus z}{\mu, \sigma \mid\!\sim w},$$

which is easy to verify. Moreover it is easy to see that for every rule of CHF, if the sequents over the line have free interpretations, so do the sequent under the line. Therefore every derivable sequent has free interpretation and is true. □

Corollary 2.8 CHF *is consistent.*

Proof. As remarked earlier no sequent of the form $x, \sigma \mathrel{|\!\sim}$ can be true. Therefore, by soundness, no sequent of the form $X, \Sigma \vdash$ is provable in CHF. This suffices for CHF to be consistent. □

A variant of $C_\otimes$ is the following ("strange" too) rule:

$$NL_{\multimap} : \frac{\Phi, Z, \Sigma \vdash W \otimes Y}{\Phi, \Sigma, Y \multimap Z \vdash W}. \tag{2}$$

($NL_{\multimap}$ stands for "new left rule for $\multimap$", since it introduces $\multimap$ on the left in a new way.) We shall see that it is equivalent to $C_\otimes$ over HF.

Let R be a rule and T be a system of rules. The statement "the rule R is derived in the system T" means that the system T is closed under the rule R.

Lemma 2.9 *The rule* $NL_{\multimap}$ *is derived in* HF$+C_\otimes$ *and the rule* $C_\otimes$ *is derived in* HF $+ NL_{\multimap}$*. Therefore* $NL_{\multimap}$ *and* $C_\otimes$ *are equivalent over* HF*.*

Proof. i) We work in HF $+ C_\otimes$. Let the upper sequent $\Phi, Z, \Sigma \vdash W \otimes Y$ of the rule $NL_{\multimap}$ be given. Then:

$$\dfrac{\dfrac{\overline{\Phi, Z, \Sigma \vdash W \otimes Y} \qquad \dfrac{\dfrac{\dfrac{\overline{Y \vdash Y} \quad \overline{Z \vdash Z}}{Y, (Y \multimap Z) \vdash Z}L_{\multimap} \qquad \overline{W \vdash W}}{W, Y, (Y \multimap Z) \vdash W \otimes Z}R_\otimes}{W \otimes Y, (Y \multimap Z) \vdash W \otimes Z}L_\otimes}{\Phi, Z, \Sigma, (Y \multimap Z) \vdash W \otimes Z}Cut}{\Phi, \Sigma, (Y \multimap Z) \vdash W}C_\otimes.$$

ii) We work in HF $+ NL_{\multimap}$. Let the upper sequent $\Phi, Z, \Sigma \vdash W \otimes Z$ of $C_\otimes$ be given. Then:

$$\dfrac{\dfrac{\overline{Z \vdash Z}}{\vdash (Z \multimap Z)}R_{\multimap} \qquad \dfrac{\overline{\Phi, Z, \Sigma \vdash W \otimes Z}}{\Phi, \Sigma, (Z \multimap Z) \vdash W}NL_{\multimap}}{\Phi, \Sigma \vdash W}Cut. \qquad \square$$

Completeness of CHF with respect to free processes can be proved directly. However, in view of lemma 2.3, we shall reduce it to the completeness of HF with respect to staged processes.

Theorem 2.10 (Completeness of CHF) *If* $\Phi, \Sigma \vdash W$ *is a sequent such that for some injective* *, (Φ^*, Σ^*) *is free and* $\Phi^*, \Sigma^* \mid\!\sim W^*$ *holds in* $(U(A), \mid\!\sim)$, *then* $\Phi, \Sigma \vdash W$ *is provable in* CHF.

Proof. Let $\Phi, \Sigma \vdash W$ be a sequent, such that the process (Φ^*, Σ^*) is free and $\Phi^*, \Sigma^* \mid\!\sim W^*$ holds for some injective *. We shall show that $\Phi, \Sigma \vdash W$ is provable in CHF. Let $\Sigma^* = [(Y_1^*, Z_1^*), \ldots, (Y_n^*, Z_n^*)]$ and let $U = Y_1 \otimes \cdots \otimes Y_n$. Then $U^* = Y_1^* \uplus \cdots \uplus Y_n^*$ and, by lemma 2.3, $\Phi^*, U^*, \Sigma^* \vdash W^* \uplus U^*$ is a true staged process. By the completeness of HF (theorem 1.2), the sequent $\Phi, U, \Sigma \vdash W \otimes U$ is provable in HF. Applying the rule $C_\otimes$ to the last sequent, we get $\Phi, \Sigma \vdash W$. Thus the latter is provable in CHF. □

Contrary to Corollary 2.8, however, the rule $C_\otimes$ (and hence $NL_{\multimap}$) is not compatible with the full system ILL. In this system we have, besides **1**, also its dual **0**, as well as the additive constants $\top$ and $\perp$ (notice that we follow here the notation of [3] rather than the original one of [1]). As shown in [1], p. 20, for every X, $X \otimes \perp \vdash\dashv \perp$. Hence from $\perp \vdash \perp$, we get in ILL $X \otimes \perp \vdash Y \otimes \perp$ for all X, Y. Applying $C_\otimes$ to the last sequent we get $X \vdash Y$ for any X, Y, hence also $\mathbf{1} \vdash \mathbf{0}$, from which and the axioms $\vdash \mathbf{1}$ and $\mathbf{0} \vdash$, the contradiction $\vdash$ is derived.

3 Disjunctive objects and actualizations

In restaurant menus disjunctions occur either in the form "fish or stake" (upon choice), or in the form "fruit of season", which means "apple or cherry or orange or..." (depending on season). In popular presentations of linear logic the former kind of disjunction is construed as "multiplicative" (or internal or deterministic), while the latter is construed as additive (or external or non-deterministic). Given, however, that multiplicatives should not be idempotent, while the choice between "fish or fish" clearly entails "fish", this can hardly constitute a convincing example of a multiplicative disjunction applied to *objects*. In fact such examples can be found only in the special Girard's semantics (see [1]), where *negation* is available and thus multiplicative disjunction is the dual of the multiplicative conjunction. In contrast, in the negation-free realm of resources, no non-idempotent notion of disjunction is conceivable, while, on the contrary, non-idempotent conjunction is defined quite naturally.

On the other hand the "apple or cherry" pattern might be understood as a new kind of *potential* object which, although distinct from both an apple and a cherry, at some time "collapses" non-deterministically to either an apple or a cherry (but not both). Returning to our multiset framework this suggests one to extend the multiset universe with "potential multisets" $x|y$ for any two standard ones, and more generally, with objects $x_1|\cdots|x_n$ for any $x_1, \ldots, x_n \in U(A)$. The main properties of $|$ are:

i) $x \neq y \;\Rightarrow\; x|y \neq x \;\wedge\; x|y \neq y$.

ii) $x|y = y|x$ (commutativity).

iii) $(x|y)|z = x|(y|z)$ (associativity).

iv) $(x|x) = x$ (idempotence).

v) $x \uplus (y|z) = (x \uplus y)|(x \uplus z)$ ($\uplus$-distribution).

(The operation $|$ with analogous properties is used also in [4]. The operation $\odot$ of the latter paper is the analogue of $\otimes$.) A simple way to represent the objects $x_1|\cdots|x_n$ is by means of the ordinary sets $\{x_1, \ldots, x_n\}$ modulo some identifications, namely $\{x\}$ with x.

We extent $U(A)$ to $U(A)^{\{\}}$ setting $U(A)^{\{\}} = U(A) \cup P_\omega(U(A))$, where $P_\omega(U(A))$ is the set of finite subsets of $U(A)$. We call the elements of $U(A)^{\{\}}$ *disjunctive multisets* or *disjunctive objects*.

For readability we let x, y, z range over elements of $U(A)$ and α, β, γ over elements of $P_\omega(U(A))$. Define the operation $|$ on $U(A)^{\{\}}$ as follows:

$$x|y = \{x, y\}, \qquad x|\beta = \{x\} \cup \beta, \qquad \alpha|\beta = \alpha \cup \beta.$$

It follows that, practically, x is identified with $\{x\}$. The empty subset of $U(A)$ is denoted $\{\}$. This is an element of $U(A)^{\{\}}$, representing the potential object with no actualization (i.e., the "impossible") and is not to be confused with the multiset $\emptyset$.

The idea to represent disjunctive objects as subsets of the ground set $U(A)$ is essentially the same as the one used in power domains (see e.g. [2]) for representing disjunctive information but in a completely different setting.

The operation $\uplus$ of $U(A)$ can be extended to $U(A)^{\{\}}$ by setting

$$\alpha \uplus \beta = \{x \uplus y : x \in \alpha, y \in \beta\}. \tag{3}$$

Obviously $\uplus$ remains commutative and associative. Note that $\alpha \uplus \{\} = \{\}$ for every α.

Lemma 3.1 $\alpha \uplus (\beta|\gamma) = (\alpha \uplus \beta)|(\alpha \uplus \gamma)$.

Proof. $\alpha \uplus (\beta|\gamma) = \{x \uplus y : x \in \alpha, y \in \beta \cup \gamma\} =$

$$\{x \uplus y : x \in \alpha, y \in \beta\} \cup \{x \uplus y : x \in \alpha, y \in \gamma\} = (\alpha \uplus \beta) \cup (\alpha \uplus \gamma) =$$
$$(\alpha \uplus \beta)|(\alpha \uplus \gamma). \qquad \square$$

A *replacement* in $U(A)^{\{\}}$ is a pair (β, γ). Sequents in $U(A)^{\{\}}$ have the form

$$\alpha_1, \ldots, \alpha_n, (\beta_1, \gamma_1) \ldots, (\beta_m, \gamma_m) \mid\!\sim \delta,$$

where $[\alpha_1, \ldots, \alpha_n]$ is a multiset of disjunctive objects and $[(\beta_1, \gamma_1) \ldots, (\beta_m, \gamma_m)]$ is a a multiset of replacements. The left-hand side of a sequent is a *process*.

We saw above that each α is a set $\{x_1, \ldots, x_k\}$ of ordinary multisets, representing its possible actualizations. How then can we interpret a replacement (β, γ)? One can reasonably say that (β, γ) is the set of all possible actualizations of replacements arising out of all possible actualizations of α, β, that is, $(\alpha, \beta) = \{(x, y) : x \in \alpha, y \in \beta\}$. Admitting that, we turn to the crucial definition of the *truth* of the sequent say, $\alpha, (\beta, \gamma) \mid\!\sim \delta$. The idea is to reduce the truth of the latter to that of sequents of multisets arising from the various actualizations of $\alpha, \beta, \gamma, \delta$. This will inevitably involve quantification over these potential objects.

In view of the fact that (β, γ) can be identified with the set $\{(y, z) : y \in \alpha, z \in \beta\}$, one would claim that $\alpha, (\beta, \gamma) \mid\!\sim \delta$ is true if *every* actualization of α and *every* actualization of (β, γ) yield together some actualization of δ. The latter is written

$$(\forall x \in \alpha)(\forall (y, z) \in (\beta, \gamma)(\exists w \in \delta)[x, (y, z) \mid\!\sim w],$$

or, equivalently,

$$(\forall x \in \alpha)(\forall y \in \beta)(\forall z \in \gamma)(\exists w \in \delta)[x, (y, z) \mid\!\sim w]. \tag{4}$$

The generalization of (4) for the sequent

$$\alpha_1, \ldots, \alpha_n, (\beta_1, \gamma_1) \ldots, (\beta_m, \gamma_m) \mid\!\sim \delta$$

is obvious. Definition (4) is said to be *of type* $\forall, (\forall, \forall) \vdash \exists$.

One however will object that in the sequent $\alpha, (\beta, \gamma) \mathrel{|\!\sim} \delta$, the roles of β, γ are different. Namely, γ acts as an incoming resource, just like α, while β acts as an outgoing resource, just like δ. For instance under the $\forall, (\forall, \forall) \vdash \exists$ definition, the derivation

$$\frac{\alpha \mathrel{|\!\sim} \beta, \quad \gamma \mathrel{|\!\sim} \delta}{\alpha, (\beta, \gamma) \mathrel{|\!\sim} \delta}, \tag{5}$$

suggested by the rule $L_{\multimap}$, fails to be true, since the lower sequent requires $\forall$ over β, while the uppercase assumption provides only $\exists$.

Therefore in the sequent $\alpha, (\beta, \gamma) \mathrel{|\!\sim} \delta$, γ and β should be quantified like α and δ, i.e., by $\forall$ and $\exists$, respectively. Concerning the order of quantifiers, the fact that the order of objects and replacements in a process is indifferent suggests that so must be the order between the various $\forall$ and $\exists$, and this is warranted only if we collect all $\forall$'s and all $\exists$'s in separate blocks. Therefore an alternative truth definition of $\alpha, (\beta, \gamma) \mathrel{|\!\sim} \delta$ is

$$(\forall x \in \alpha)(\forall z \in \gamma)(\exists y \in \beta)(\exists w \in \delta)[x, (y, z) \mathrel{|\!\sim} w]. \tag{6}$$

Definition (6) is said to be of type $\forall, (\exists, \forall) \vdash \exists$.

The above truth definitions provide two distinct yielding relations $\mathrel{|\!\sim}_1$, $\mathrel{|\!\sim}_2$ respectively. We shall see that $\mathrel{|\!\sim}_2$ behaves better than $\mathrel{|\!\sim}_1$.

Definition 3.2 Define

$$\alpha, (\beta, \gamma) \mathrel{|\!\sim}_1 \delta \iff (4), \qquad \alpha, (\beta, \gamma) \mathrel{|\!\sim}_2 \delta \iff (6).$$

Further define

$$\alpha, (\beta, \gamma) \mathrel{|\!\sim}_i (\delta, \varepsilon) \iff \alpha, \delta, (\beta, \gamma) \mathrel{|\!\sim}_i \varepsilon.$$

The following is straightforward.

Lemma 3.3 *i) $\mathrel{|\!\sim}_1$, $\mathrel{|\!\sim}_2$, coincide with $\mathrel{|\!\sim}$ when restricted to $U(A)$.*

ii) Let P be a process. If for every replacement $(\alpha, \beta) \in P$, $\alpha \neq \emptyset$, then for every δ, $P \mathrel{|\!\sim}_1 \delta \Rightarrow P \mathrel{|\!\sim}_2 \delta$.

iii) For P without replacements, $P \mathrel{|\!\sim}_1 \delta \iff P \mathrel{|\!\sim}_2 \delta$. Moreover, $\alpha \mathrel{|\!\sim}_1 \delta \iff \alpha \mathrel{|\!\sim}_2 \delta \iff \alpha \subseteq \delta$.

Lemma 3.4 *For* $i = 1, 2$

$$\alpha_1, \ldots, \alpha_n, (\beta_1, \gamma_1), \ldots, (\beta_m, \gamma_m) \mid\!\sim_i \delta$$

iff

$$\alpha_1 \uplus \cdots \uplus \alpha_n, (\beta_1 \uplus \cdots \uplus \beta_m, \gamma_1 \uplus \cdots \uplus \gamma_m) \mid\!\sim_i \delta.$$

Proof. It suffices to verify the claim for $n = m = 2$, the general case being similar. For the relation $\mid\!\sim_2$ the claim amounts to showing the equivalence

$$(\forall x_1 \in \alpha_1)(\forall x_2 \in \alpha_2)(\forall z_1 \in \gamma_1)(\forall z_2 \in \gamma_2)(\exists y_1 \in \beta_1))(\exists y_2 \in \beta_2)$$

$$(\exists w \in \delta)[x_1, x_2, (y_1, z_1), (y_2, z_2) \mid\!\sim w].$$

$\Longleftrightarrow$

$$(\forall x \in \alpha_1 \uplus \alpha_2)(\forall z \in \gamma_1 \uplus \gamma_2)(\exists y \in \beta_1 \uplus \beta_2)$$
$$(\exists w \in \delta)[x, (y, z) \mid\!\sim w].$$

Recall that $\alpha \uplus \beta = \{x \uplus y : x \in \alpha, y \in \beta\}$. Then:

$\Rightarrow$: Let $x \in \alpha_1 \uplus \alpha_2$ and $z \in \gamma_1 \uplus \gamma_2$. Then $x = x_1 \uplus x_2$ with $x_1 \in \alpha_1, x_2 \in \alpha_2$ and $z = z_1 \uplus z_2$ with $z_1 \in \gamma_1$ and $z_2 \in \gamma_2$. By the assumption there are $y_1 \in \beta_1, y_2 \in \beta_2$ and $w \in \delta$ such that $x_1, x_2, (y_1, z_1), (y_2, z_2) \mid\!\sim w$. Then putting $y = y_1 \uplus y_2$ we have $x, (y, z) \mid\!\sim w$, with $y \in \beta_1 \uplus \beta_2$ as required.

$\Leftarrow$: Let $x_1 \in \alpha_1, x_2 \in \alpha_2$ and $z_1 \in \gamma_1, z_2 \in \gamma_2$. Then $x_1 \uplus x_2 \in \alpha_1 \uplus \alpha_2$ and $z_1 \uplus z_2 \in \gamma_1 \uplus \gamma_2$. By the assumption there are $y \in \beta_1 \uplus \beta_2$ and $w \in \delta$ such that $x_1 \uplus x_2, (y, z_1 \uplus z_2) \mid\!\sim w$. But then $y = y_1 \uplus y_2$ for some $y_1 \in \beta_1, y_2 \in \beta_2$, hence the preceding sequent is true iff the sequent $x_1, x_2, (y_1, z_1), (y_2, z_2) \mid\!\sim w$ is true, with y_1, y_2 as required. This proves the claim for $\mid\!\sim_2$. The case of $\mid\!\sim_1$ is similar. □

4 The ⊔-Horn Fragment

Let $\sqcup$ be the additive disjunction of linear logic (notation of Troelstra [3]). We extend the language $\mathcal{L} = \{\otimes, \multimap, \mathbf{1}\}$ of the Horn fragment HF to $\mathcal{L}_\sqcup = \{\otimes, \multimap, \sqcup, \mathbf{1}\}$. The $\sqcup$-products of $\mathcal{L}_\sqcup$ are formulas defined by the following recursion: $X := p; \mathbf{1}; Y \otimes Z; Y \sqcup Z$. The implications of $\mathcal{L}_\sqcup$ are formulas $X \multimap Y$ where X, Y are $\sqcup$-products. X, Y, U, W denote $\sqcup$-products. Sequents of $\mathcal{L}_\sqcup$ have the form $\Phi, \Sigma \vdash W$ (we do not allow sequents of the form $\Phi, \Sigma \vdash (Y \multimap W)$). The $\sqcup$-rules are the following:

$$L_\sqcup : \frac{\Phi, X, \Sigma \vdash W \quad \Phi, Y, \Sigma \vdash W}{\Phi, X \sqcup Y, \Sigma \vdash W}, \quad R_\sqcup : \frac{\Phi, \Sigma \vdash W}{\Phi, \Sigma \vdash W \sqcup U}.$$

An *interpretation* of $\mathcal{L}_\sqcup$ in $U(A)^{\{\}}$ is any injective mapping $^* : \{p_1, p_2, \ldots\} \to A$ which is defined as before for $\otimes$, $\multimap$, $\mathbf{1}$, plus:

$(X \sqcup Y)^* = X^* | Y^* = X^* \cup Y^*$.

Again * extends naturally to sequents in the obvious way turning them into sequents of $U(A)^{\{\}}$.

Let $\mathrm{HF}_\sqcup = \mathrm{HF} + L_\sqcup + R_\sqcup$. Call *Restricted Cut*, notation *RCut*, the cut rule in which the cut formula is *not* an implication. Let $\mathrm{HF}^r_\sqcup$ be $\mathrm{HF}_\sqcup$ with *Cut* replaced by *RCut*.

Theorem 4.1 (Soundness of $\mathrm{HF}_\sqcup$) *i) The rules of* $\mathrm{HF}^r_\sqcup$ *are sound with respect to* $\mid\!\sim_2$. *ii) The rules of* $\mathrm{HF}^r_\sqcup$ *except* $L_\multimap$ *are sound with respect to* $\mid\!\sim_1$.

Proof. In view of lemma 3.4, we may assume that in the sequents below, $|\Sigma| = 1$ and $|\Phi| = 1$, or $\Phi = \emptyset$, when Φ is simply a set of parameters, as e.g. in $L_\sqcup$.

i) *RCut*: Let $X_1, \Sigma_1 \vdash W$ and $W, X_2, \Sigma_2 \vdash U$ be the upper sequents of the derivation with W being a non-implication, hence a $\sqcup$-product. Let also $X_1^*, \Sigma_1^* \mid\!\sim W^*$ and $W^*, X_2^*, \Sigma_2^* \mid\!\sim U^*$ be true. In view of lemma 3.4 above, we may assume without loss of generality that Σ_1, Σ_2 are single element multisets, say, $\Sigma_1 = [Y_1 \multimap Z_1]$, $\Sigma_2 = [Y_2 \multimap Z_2]$ Then

$$(\forall x_1 \in X_1^*)(\forall z_1 \in Z_1^*)(\exists y_1 \in Y_1^*)(\exists w \in W^*)(x_1, (y_1, z_1) \mid\!\sim w), \tag{7}$$

and

$$(\forall w \in W^*)(\forall x_2 \in X_2^*)(\forall z_2 \in Z_2^*)(\exists y_2 \in Y_2^*)(\exists u \in U^*)(w, x_2, (y_2, z_2) \mid\!\sim u). \tag{8}$$

By (7) and (8) and basic logic, we get

$$(\forall x_1 \in X_1^*)(\forall x_2 \in X_2^*)(\forall z_1 \in Z_1^*)(\forall z_2 \in Z_2^*)(\exists y_1 \in Y_1^*)(\exists y_2 \in Y_2^*)(\exists w \in W^*)$$

$$[x_1, (y_1, z_1) \mid\!\sim w \;\&\; w, x_2, (y_2, z_2) \mid\!\sim u].$$

Now by *Cut* for standard sequents we get

$$(\forall x_1 \in X_1^*)(\forall x_2 \in X_2^*)(\forall z_1 \in Z_1^*)(\forall z_2 \in Z_2^*)(\exists y_1 \in Y_1^*)(\exists y_2 \in Y_2^*)$$

$$[x_1, x_2, (y_1, z_1)(y_2, z_2) \mathrel{|\!\!\sim} u].$$

This shows that the sequent $X_1^*, X_2^*, \Sigma_1^*, \Sigma_2^* \mathrel{|\!\!\sim}_2 U^*$ is true.

$L_\otimes$: This follows by lemma 3.4.

$R_\otimes$: Suppose $X_1^*, (Y_1^*, Z_1^*) \mathrel{|\!\!\sim}_2 W^*$ and $X_2^*, (Y_2^*, Z_2^*) \mathrel{|\!\!\sim}_2 U^*$ are true. Then

$$(\forall x_1 \in X_1^*)(\forall z_1 \in Z_1^*)(\exists y_1 \in Y_1^*)(\exists w \in W^*)[x_1, (y_1, z_1) \mathrel{|\!\!\sim} w], \tag{9}$$

and

$$(\forall x_2 \in X_2^*)(\forall z_2 \in Z_1^*)(\exists y_2 \in Y_2^*)(\exists u \in U^*)[x_2, (y_2, z_2) \mathrel{|\!\!\sim} u]. \tag{10}$$

Then (9), (10) and $R_\otimes$ for standard sequents imply

$$(\forall x_1 \in X_1^*)(\forall x_2 \in X_2^*)(\forall z_1 \in Z_1^*)(\forall z_2 \in Z_1^*)(\exists y_1 \in Y_1^*)(\exists y_2 \in Y_2^*)$$

$$(\exists v \in W^* \uplus U^*)[x_1, x_2, (y_1, z_1), (y_2, z_2) \mathrel{|\!\!\sim} v].$$

Since $W^* \uplus U^* = (W \otimes U)^*$, the above says that

$$X_1^*, X_2^*, (Y_1^*, Z_1^*), (Y_2^*, Z_2^*) \mathrel{|\!\!\sim}_2 (W \otimes U)^*$$

is true.

$L_\multimap$: Suppose $X_1^*, (Y_1^*, Z_1^*) \mathrel{|\!\!\sim}_2 W^*$ and $U^*, X_2^*, (Y_2^*, Z_2^*) \mathrel{|\!\!\sim}_2 V^*$ are true. Then

$$(\forall x_1 \in X_1^*)(\forall z_1 \in Z_1^*)(\exists y_1 \in Y_1^*)(\exists w \in W^*)[x_1, (y_1, z_1) \mathrel{|\!\!\sim} w], \tag{11}$$

and

$$(\forall u \in U^*)(\forall x_2 \in X_2^*)(\forall z_2 \in Z_1^*)(\exists y_2 \in Y_1^*)(\exists v \in V^*)[u, x_2, (y_2, z_2) \mathrel{|\!\!\sim} v]. \tag{12}$$

From (11), (12) and basic logic we get

$$(\forall x_1 \in X_1^*)(\forall x_2 \in X_2^*)(\forall z_1 \in Z_1^*)(\forall z_2 \in Z_2^*)(\forall u \in U^*)(\exists y_1 \in Y_1^*)(\exists y_2 \in Y_2^*)$$

$$(\exists w \in W^*)(\exists v \in V^*)[x_1, (y_1, z_1) \mathrel{|\!\!\sim} w \ \&\ u, x_2, (y_2, z_2) \mathrel{|\!\!\sim} v].$$

By $L_{\multimap}$ for standard sequents, this implies

$$(\forall x_1 \in X_1^*)(\forall x_2 \in X_2^*)(\forall z_1 \in Z_1^*)(\forall z_2 \in Z_2^*)(\forall u \in U^*)(\exists y_1 \in Y_1^*)(\exists y_2 \in Y_2^*)$$

$$(\exists w \in W^*)(\exists v \in V^*)[x_1, x_2, (y_1, z_1), (y_2, z_2), (w, u) \mid\!\sim v].$$

That means that

$$X_1^*, X_2^*, (Y_1^*, Z_1^*), (Y_2^*, Z_2^*), (W^*, U^*) \mid\!\sim_2 V^*$$

is true.

$R_{\multimap}$ holds by definition.

$L_{\mathbf{1}}$: Let $X^*, \Sigma^* \mid\!\sim_2 W^*$. To see that $X^*, \mathbf{1}^*, \Sigma^* \mid\!\sim_2 W^*$, i.e., $X^*, \emptyset, \Sigma^* \mid\!\sim_2 W^*$, observe that this is equivalent to $X^* \uplus \emptyset, \Sigma^* \mid\!\sim_2 W^*$. Now observe that $\alpha \uplus \emptyset = \alpha \uplus \{\emptyset\} = \{x \uplus \emptyset : x \in \alpha\} = \alpha$. So the conclusion is immediate. (Note that the empty multiset $\emptyset$ is not to be confused with the empty subset $\{\}$ of $U(A)$, which will be considered in the next section, and for which $\alpha \uplus \{\} = \{\}$.)

$L_{\sqcup}$: Suppose of $X_1^*, (Y^*, Z^*) \mid\!\sim_2 W^*$ and $X_2^*, (Y^*, Z^*) \mid\!\sim_2 W^*$ are true. Then

$$(\forall x_1 \in X_1^*)(\forall z \in Z^*)(\exists y \in Y^*)(\exists w \in W^*)[x_1, (y, z) \mid\!\sim w]$$

and

$$(\forall x_2 \in X_2^*)(\forall z \in Z^*)(\exists y \in Y^*)(\exists w \in W^*)[x_2, (y, z) \mid\!\sim w].$$

The last two relations clearly imply

$$(\forall x \in X_1^* \cup X_2^*))(\forall z \in Z^*)(\exists y \in Y^*)(\exists w \in W^*)[x, (y, z) \mid\!\sim w].$$

Since $X_1^* \cup X_2^* = (X_1 \sqcup X_2)^*$, the latter means that $(X_1 \sqcup X_2)^*, (Y^*, Z^*) \mid\!\sim_2 W^*$ is true.

$R_{\sqcup}$: Suppose $X^*, (Y^*, Z^*) \mid\!\sim_2 W^*$ is true. That is,

$$(\forall x \in X^*)(\forall z \in Z^*)(\exists y \in Y^*)(\exists w \in W^*)[x, (y, z) \mid\!\sim w].$$

Then for any formula U, and in view of the fact that $W^* \subseteq W^* \cup U^*$, the preceding relation obviously implies

$$(\forall x \in X^*)(\forall z \in Z^*)(\exists y \in Y^*)(\exists w \in W^* \cup U^*))[x, (y, z) \mid\!\sim w].$$

This means that $X^*, (Y^*, Z^*) \mid\!\sim_2 (W \sqcup U)^*$ is true.

ii) We work similarly. That $L_{\multimap}$ fails under $\mid\!\sim_1$ has been already shown (see (5)). It remains only to show that $\mid\!\sim_1$ satisfies unrestricted *Cut*, i.e., *Cut* even when the cut formula is an implication. Without loss of generality it suffices to show the implication:

$$\alpha \mid\!\sim_1 (\beta, \gamma) \ \& \ (\beta, \gamma) \mid\!\sim_1 \delta \ \Rightarrow \ \alpha \mid\!\sim_1 \delta.$$

Suppose the hypotheses of the implication hold. Now by definition 3.2, $\alpha \mid\!\sim_1 (\beta, \gamma)$ is equivalent to $\alpha, \beta \mid\!\sim_1 \gamma$, hence the hypotheses amount to

$$(\forall x \in \alpha)(\forall y \in \beta)(\exists z \in \gamma)[x, y \vdash z]$$

and

$$(\forall y \in \beta)(\forall z \in \gamma)(\exists w \in \delta)[(y, z) \vdash w].$$

From the preceding relations, given any $x \in \alpha$, and choosing *any* $y \in \beta$, there is, by the first of the above relations, a $z \in \gamma$ such that $x \uplus y = z$. By the second relation, for *these specific* y, z there is $w \in \delta$ such that $z - y = w$. The two equations imply $x = w$, hence $(\forall x \in \alpha)(\exists w \in \delta)(x = w)$, or $\alpha \mid\!\sim_1 \delta$. $\square$

Remark 4.2 Unrestricted *Cut* is not valid in $(U(A)^{\{\}}, \mid\!\sim_2)$. To see this it suffices to find $\alpha, \beta, \gamma, \delta$ in $U(A)^{\{\}}$ such that

$$\alpha \mid\!\sim_2 (\beta, \gamma) \ \& \ (\beta, \gamma) \mid\!\sim_2 \delta \text{ and } \alpha \not\mid\!\sim_2 \delta.$$

Now $\alpha \mid\!\sim_2 (\beta, \gamma) \iff \alpha, \beta \mid\!\sim_2 \gamma \iff$ (by lemma 3.3 (iii)) $\alpha \uplus \beta \subseteq \gamma$. Also we easily see that $(\beta, \gamma) \mid\!\sim_2 \delta \iff \gamma \subseteq \beta \uplus \delta$. Therefore it suffices to find $\alpha, \beta, \gamma, \delta$ such that

$$\alpha \uplus \beta \subseteq \gamma, \ \gamma \subseteq \beta \uplus \delta \text{ and } \alpha \not\subseteq \delta.$$

A fortiori it suffices to find α, β, δ such that

$$\alpha \uplus \beta \subseteq \beta \uplus \delta \text{ and } \alpha \not\subseteq \delta.$$

The following example shows this:

Example. Let $\alpha = \{[a,b],[2a,b]\}$, $\beta = \{[a,c,d],[c,d]\}$, $\delta = \{[2a,b],[b]\}$. Then $\alpha \nsubseteq \delta$, and yet

$$\alpha \uplus \beta = \{[2a,b,c,d],[3a,b,c,d],[a,b,c,d]\},$$

$$\beta \uplus \delta = \{[3a,b,c,d],[a,b,c,d],[2a,b,c,d],[b,c,d]\},$$

i.e., $\alpha \uplus \beta \subseteq \beta \uplus \delta$. □

Remark 4.3 The rule $C_\otimes$ no longer holds in $(U(A)^{\{\}}, \mid\!\sim_i)$. To see this consider a sequent of the form $\alpha \mid\!\sim_i \beta$ (with $\sigma = \emptyset$). By 3.3, this amounts to $\alpha \subseteq \beta$. Therefore if $C_\otimes$ were valid in $(U(A)^{\{\}}, \mid\!\sim_i)$, the implication $\alpha \uplus \beta \subseteq \alpha \uplus \gamma \Rightarrow \beta \subseteq \gamma$ should be valid for any α, β, γ. But the example of the previous remark shows that this is false. □

Because of the failure of $C_\otimes$, we consider instead the rule $NL_{-\circ}$ (see (2)). We have seen (lemma 2.9) that $C_\otimes$ and $NL_{-\circ}$ are equivalent over HF. But when working in $\mathrm{HF}_\sqcup$ the situation is a bit subtler. For instance, in contrast to the previous remark, the following holds.

Lemma 4.4 *$NL_{-\circ}$ holds with respect to $\mid\!\sim_2$ (but fails with respect to $\mid\!\sim_1$).*

Proof. $NL_{-\circ}$ fails in $(U(A)^{\{\}}, \mid\!\sim_1)$ for the same reasons that the rule $L_{-\circ}$ does so. Concerning its truth in $(U(A)^{\{\}}, \mid\!\sim_2)$, we have to show that for all X, Y, Z, W, Σ, and for every interpretation *, $X^*, \Sigma^*, (Y^*, Z^*) \mid\!\sim_2 W^*$ is true iff $X^* \uplus Z^*, \Sigma^* \mid\!\sim_2 W^* \uplus Y^*$ is true. Without loss of generality we can take $\Sigma = [Y_1 -\!\circ Z_1]$, so we have to show that

$$\alpha, (\beta_1, \gamma_1), (\beta, \gamma) \mid\!\sim_2 \delta \iff \alpha \uplus \gamma, (\beta_1, \gamma_1) \mid\!\sim_2 \delta \uplus \beta,$$

or equivalently,

$$(\forall x \in \alpha)(\forall z_1 \in \gamma_1)(\forall z \in \gamma)(\exists y_1 \in \beta_1)(\exists y \in \beta)(\exists w \in \delta)[x, (y_1, z_1), (y, z) \mid\!\sim w]$$

$$\iff$$

$$(\forall x \in \alpha \uplus \gamma)(\forall z_1 \in \gamma_1)(\exists y_1 \in \beta_1)(\exists w \in \delta \uplus \beta)[x, (y_1, z_1) \mid\!\sim w].$$

Now it follows from the truth of free processes that

$$x, (y_1, z_1), (y, z) \mid\!\sim w \iff x \uplus z, (y_1, z_1) \mid\!\sim w \uplus y,$$

so the left hand side of the preceding equivalence is written

$(\forall x \in \alpha)(\forall z_1 \in \gamma_1)(\forall z \in \gamma)(\exists y_1 \in \beta_1)(\exists y \in \beta)(\exists w \in \delta)[x \uplus z, (y_1, z_1) \mathrel{|\!\!\sim} w \uplus y]$,

and this is clearly equivalent to

$$(\forall x \in \alpha \uplus \gamma)(\forall z_1 \in \gamma_1)(\exists y_1 \in \beta_1)(\exists w \in \delta \uplus \beta)[x, (y_1, z_1) \mathrel{|\!\!\sim} w]. \quad \square$$

Let

$$\mathrm{NHF}^r_{\sqcup} = \mathrm{HF}^r_{\sqcup} + NL_{\multimap}.$$

Theorem 4.5 (Soundness and completeness of $\mathrm{NHF}^r_{\sqcup}$)
$\mathrm{NHF}^r_{\sqcup}$ *is sound and complete with respect to* $(U(A)^{\{\}}, \mathrel{|\!\!\sim}_2)$.

Proof. Soundness follows from theorem 4.1 and lemma 4.4.

Completeness. Let S be a sequent such that S^* is true in $(U(A)^{\{\}}, \mathrel{|\!\!\sim}_2)$ for some injective *. We shall show that S is provable in $\mathrm{NHF}^r_{\sqcup}$. For simplicity suppose the antecedent of S is of the form X, Σ, i.e., $\Phi = [X]$. (After all, if $\Phi = [X_1, \ldots, X_n]$, then Φ can be provably replaced by the $\sqcup$-product $X_1 \otimes \cdots \otimes X_n$.)

Case 1. The succedent of S is not an implication, i.e., $S = (X, \Sigma \vdash W)$, where W is a $\sqcup$-product. We show that S is provable by induction on $|\Sigma|$. Suppose first that $|\Sigma| = 0$, that is, $S = (X \vdash W)$. X, W are $\sqcup$-products, and $X \otimes (Y \sqcup Z) \dashv\vdash (X \otimes Y) \sqcup (X \otimes Z)$ are provable in $\mathrm{HF}^r_{\sqcup}$, hence we may assume that X and W have the form $X = X_1 \sqcup \cdots \sqcup X_n$ and $W = W_1 \sqcup \cdots \sqcup W_m$, where X_i, W_j are simple products. Therefore $X^* = \{X_1^*, \ldots, X_n^*\}$ and $W^* = \{W_1^*, \ldots, W_m^*\}$. By assumption $X^* \mathrel{|\!\!\sim}_2 W^*$ is true. By lemma 3.3 (iii), $X^* \subseteq W^*$. By the latter and the fact that * is injective, $\{X_1, \ldots, X_n\} \subseteq \{W_1, \ldots, W_m\}$. Thus by the rule $R_{\sqcup}$ we easily see that the sequent $X \vdash W$ is provable in $\mathrm{NHF}^r_{\sqcup}$.

Let now the claim hold for all sequents with $|\Sigma| < n$ and consider a sequent $X, \Sigma \vdash W$ with $|\Sigma| = n$, $n > 0$, such that $X^*, \Sigma^* \mathrel{|\!\!\sim}_2 W^*$ is true for some injective *. Let $\Sigma = \Sigma_1 \uplus [Y \multimap Z]$. Then $X^*, \Sigma_1^*, (Y^*, Z^*) \mathrel{|\!\!\sim}_2 W^*$. By the proof of 4.4, the latter implies $X^* \uplus Z^*, \Sigma_1^* \mathrel{|\!\!\sim}_2 W^* \uplus Y^*$, or $(X \otimes Z)^*, \Sigma_1^* \mathrel{|\!\!\sim}_2 (W \otimes Y)^*$. By the induction hypothesis, $X \otimes Z, \Sigma_1 \vdash W \otimes Y$, or $X, Z, \Sigma_1 \vdash W \otimes Y$. By the rule $NL_{\multimap}$, $X, \Sigma_1, Y \multimap Z \vdash W$, or $X, \Sigma \vdash W$.

Case 2. S is of the form $X, \Sigma \vdash U \multimap W$. Then S^* is true means that $X^*, \Sigma^* \mathrel{|\!\!\sim}_2 (U^*, W^*)$ is true, which by definition means that $X^*, U^*, \Sigma^* \mathrel{|\!\!\sim}_2 W^*$ is true. By case 1, $X, U, \Sigma \vdash W$ is provable in $\mathrm{NHF}^r_{\sqcup}$. Using $R_{\multimap}$, $X, \Sigma \vdash U \multimap W$ is provable in $\mathrm{NHF}^r_{\sqcup}$. $\square$

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