NEARNESS RELATIONS IN DISCRETELY ORDERED RINGS

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Abstract

Certain weak versions of the Shepherdson's condition for open induction in a discretely ordered ring are given, and their relationships to each other and to the underlying group structure of the ring are studied.

1 Introduction

This is a paper whose motivation comes from logic, whereas its results are rather algebraic. This seems to be fairly common when dealing with discretely ordered rings (henceforth *dors* for short), a sort of structures often interesting both to the logician and the algebraist.

The borderline between the logician's and the algebraist's territory seems to be *open induction*. This is clearly a concept of logic but we do not need to spell it out here. J.C. Shepherdson [1] found an equivalent algebraic characterization (see below). This is the only known type of dor of interest to the logician which possesses an algebraic characterization. Above them are the dors satisfying stronger (i.e., involving quantifiers) schemes of induction; below them are the dors satisfying no induction at all, but only algebraic properties (e.g. euclidean dors).

The motivation behind this paper came from the question about the conditions under which a discretely ordered group (dog for short) can be expanded to a dor, perhaps with extra properties, such as euclidean, or satisfying open induction. (Note that the analogous expandability problem for dors satisfying something stronger than open induction, i.e., moderately strong fragments of PA, has been solved long ago: A countable dog expands to a dor of this kind iff it is recursively saturated). The question is addressed and partially answered in [2]. In the course of this work certain natural weakenings of the above mentioned criterion of Shepherdson appeared to be of independent interest. The interest comes from the fact that these weak conditions are interwoven with the underlying non archimedean group structure and an assigned field of reals.

Below we fix some definitions. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are the familiar structures. A *discretely ordered ring* (*dor* for short) $R = (R, +, \times, <, 0, 1)$ is a commutative ring with an ordering $<$ such that 1 is the immediate successor of 0 and $+, \times$ are both strictly monotonic with respect to $<$, i.e., $x > y \Rightarrow x + z > y + z$ and $x > y \;\&\; z > 0 \;\Rightarrow x \times z > y \times z$.

Obviously we may assume that that for every dor R, $\mathbb{Z}$ is a subring of R. We refer to the elements of $\mathbb{Z}$ as *standard* and to the elements of $R \setminus \mathbb{Z}$ as *nonstandard*.

The dor R is a said to be a $\mathbb{Z}$-*dor* if for all $x \in R$ and for all $n > 1$ there exist exists $y \in R$ and $k \in \mathbb{N}$ such that $x = ny + k$ and $k < n$. (That is, if euclidean division by standard divisors exists in R.)

R is said to be *euclidean* if

$$(\forall x, y)(\exists z)(\exists u < y)(x = y \times z + u). \tag{1}$$

Clearly every euclidean dor is a $\mathbb{Z}$-dor.

For every dor R, let $R^+ = \{x \in R : x \geq 0\}$ be the positive part of R. Obviously R^+ is to R what $\mathbb{N}$ is to $\mathbb{Z}$. Note that R is a $\mathbb{Z}$-dor iff R^+ satisfies all axioms of Peano arithmetic restricted to the language of addition (the so-called Presburger arithmetic). $R^+ \models \mathrm{OI}$ means that R^+ satisfies open induction, i.e. induction for formulas of the language of arithmetic without quantifiers. Instead of giving the precise definition, we just state the equivalent algebraic condition of J.C. Shepherdson. For any dor R let R^{fr} be the field of fractions over R and R^{rc} be the real closure of R. Shepherdson's result says that $R^+ \models \mathrm{OI}$ iff the following condition holds:

$$\text{(S)} \qquad (\forall x \in R^{rc})(\exists a \in R)(|x - a| < 1). \tag{2}$$

Besides this highly nontrivial result the following easy characterization of euclidean dors is also useful: R is euclidean iff

$$\text{(F)} \qquad (\forall x \in R^{fr})(\exists a \in R)(|x-a|<1). \tag{3}$$

Conditions S and F above say that every element of the fields R^{rc}, R^{fr} respectively has an integral part in R. Therefore clearly S$\Rightarrow$ F (over the rest axioms for dors). Note that elements of R^{fr} will be denoted $\dfrac{x}{y}$ rather than x/y, because the latter notation is to be used below with another meaning.

Discretely ordered groups (dogs) are defined in the obvious way, and every dor is a dog for $+$. Moreover R is a a $\mathbb{Z}$-dor iff it is a $\mathbb{Z}$-dog. A natural problem is to find conditions for a $\mathbb{Z}$-dog (under $+$) to be a reduct of a dor (resp. satisfying S or F). A partial solution is given in [2].

Given a dor or dog H, it is often convenient to work in the *divisible closure* of H, $div(H)$.

2 Nearness relations in non-archimedean ordered structures

Given a dog $G = (G, +, <, 0, 1)$ (actually weaker structures suffice also) consider the following equivalence relations in G^+:

$x \equiv y := (\exists n \in \mathbb{N})(x < y \;\&\; y < nx) \vee (y < x \;\&\; x < ny)$.

$x \approx y := x \equiv y \;\&\; |x-y| \not\equiv x$.

$x \sim y := x - y \in \mathbb{Z}$.

Let $\mu(x)$, $\delta(x)$ and $[x]$ denote the equivalence classes of x under $\equiv$, $\approx$ and $\sim$ respectively. Clearly for nonstandrad x, $[x] \subseteq \delta(x) \subseteq \mu(x)$. However for standard n, $\mu(n) = [n] = \mathbb{N}$, but $\delta(n) = \{n\}$, since $m \approx n$ means $|m-n| \not\equiv m$, hence $|m-n| = 0$. All these equivalence classes are convex subsets of G^+ and inherit the ordering of H. If $x \equiv y$, we say that x, y are *of the same magnitude* and $\mu(x)$ is the *magnitude class* of x. If $x \approx y$ we say that x, y are *near to each other* and $\delta(x)$ is the *neighborhood* of x. $x \ll y$ is another notation for $\mu(x) < \mu(y)$.

Most facts and notions concerning $\equiv$ and $\approx$ can be found in Harnik [3]. Let

$$M_1(G) = G^+/\equiv = \{\mu(x) : x \in G^+\},$$

and

$$M_2(G) = G^+/\approx = \{\delta(x) : x \in G^+, x \gg 1\} \vee \{0\}$$

be the sets of equivalence classes of G^+ with respect to $\equiv$ and $\approx$ respectively.

For any pair of elements $x, y > 0$ such that $x \equiv y$, x/y is the real number

$$x/y = \sup\{p \in \mathbb{Q}^+ : py < x\}.$$

If $x \ll y$ we set $x/y = 0$. For any $a \in G^+$, let

$$F(a) = \{\pm x/a : x \in G^+, \mu(x) \leq \mu(a)\},$$

and

$$F(G) = \{\pm x/y : x, y \in G^+, \mu(x) \leq \mu(y)\}.$$

For a dor R, $F(R)$ has the obvious meaning.

Note. Throughout the paper the notation x/y is always used with the above meaning. Ordinary fractions on the other hand are written $\dfrac{x}{y}$.

The letters μ, ν, λ range over magnitude classes. Denoting $\mu(0)$ by 0 and $\mu(1)$ by 1, $M_1(G)$ is always a totally ordered set with first and second element $0, 1$ respectively, and $\mu : G^+ \to M_1(G)$ is a surjective mapping with the following properties:

Lemma 2.1 *i)* $\mu(x) = 0$ *iff* $x = 0$, *and* $\mu(n) = 1$ *iff* $n \in \mathbb{N}$.
ii) $\mu(px) = \mu(x)$, *for all* $p \in \mathbb{Q}^+$.
iii) $\mu(|px + qy|) \leq \max\{\mu(x), \mu(y)\}$.
If G *has also a multiplication* $\times$,
(iv) $\mu(x_1) = \mu(x_2)$ *and* $\mu(y_1) = \mu(y_2) \ \Rightarrow\ \mu(x_1 \times y_1) = \mu(x_2 \times y_2)$.
(v) $\mu(x) < \mu(y)$ *&* $z > 0 \ \Rightarrow\ \mu(x \times z) < \mu(y \times z)$.

By (iii) above, $\mu + \nu = \max\{\mu, \nu\}$, hence $+$ is trivial on $M_1(G)$ and we ignore it. If G is a ring R, then, by (iv) and (v), $\times$ induces a multiplication $*$ on $M_1(R)\backslash\{0\}$. For simplicity we keep writing $M_1(R)$ rather than $M_1(R)\backslash\{0\}$. $(M_1(R), *, <, 1)$ is an infinite commutative ordered monoid with unit 1. Below we spell out the precise definition.

Definition 2.2 *A totally ordered commutative monoid (henceforth* ordered monoid *or just* monoid*) is a structure* $A = (A, *, <, 1)$ *such that:*

i) $(A, *, 1)$ *is a commutative monoid with identity* 1*,*

ii) $(A, <, 1)$ *is a (totally) ordered set with least element* 1*, and*

iii) for all $x, y, z \in A$*,* $x < y \;\Rightarrow\; x * z < y * z$*.*

The monoid A *above is said to be* euclidean *if for all* $x, y \in A$*,*

$x < y \;\Rightarrow\; (\exists z)(y = x * z)$*.*

A *is said to be* radically closed *if for every* $x \in A$ *and* $n \in \mathbb{N}$*,*

$n > 0 \;\Rightarrow\; (\exists y)(y^n = x)$*.*

Lemma 2.3 (Harnik [3]) *For any* $x, y, z \in G^+$ *such that* $x \equiv z$*, the following hold:*

(i) $x/y = 1$ *iff* $x \approx y$*.*

(ii) $x \approx x'$ *and* $y \approx y' \;\Rightarrow\; x/y = x'/y'$*.*

(iii) If $x \not\approx y$*, then* $x < y \iff x/z < y/z$*.*

(iv) $(x/y) \cdot (y/z) = x/z$*.*

(v) $(px)/y = p(x/y)$ *for every* $p \in \mathbb{Q}^+$*.*

(vi) $(x + y)/z = x/z + y/z$*.*

If we are in a ring with multiplication $\times$*, then moreover*

(vii) $(x/y) \cdot (z/w) = (x \times z)/(y \times w)$*.*

Lemma 2.4 *i) If* R *is a dor, then* $F(R)$ *is a field.*

ii) If R *is euclidean, then for every nonstandard* $a > 0$ *,* $F(a) = F(R)$*.*

Proof. i) is immediate by (vii) of the previous lemma. ii) It suffices to show that $F(a) = F(b)$ for all b nonstandard, or that given $x \equiv a$ and b there is $y \equiv b$ such that $x/a = y/b$. By the property, we can divide $b \times x$ by a and find y and $z < a$ such that $b \times x = a \times y + z$. Then $b \times x \approx ay$, and by the preceding lemma, $b \times x/a \times y = 1$ or $x/a = y/b$. □

By (i)-(iii) of 2.1, $\mu : G^+ \to M_1(G)$ is a *valuation* of the group G in the sense (essentially) of [4]. Namely it is the *natural* valuation, assigning to each element its magnitude class. The following definition was originally found in [3], but it can be quite general in terms of valuations (see [4] for the notion of v-*independent* vectors, where v is a valuation).

Definition 2.5 The elements $x_1, \dots, x_n \in G^+$ are said to be *strongly independent* (s.i. for short) if they are μ-independent, i.e., if for all $p_1, \dots, p_n \in \mathbb{Q}$, such that $(p_1, \dots, p_n) \neq (0, \dots, 0)$,

$$\mu(|p_1x_1 + \cdots + p_nx_n| = \max\{\mu(x_i) : i \leq n \ \&\ p_i \neq 0\}.$$

Since by 2.1 (iii), we always have

$$\mu(|p_1x_1 + \cdots + p_nx_n| \leq \max\{\mu(x_i) : i \leq n\},$$

it follows that $x_1, \dots, x_n$ are *not* s.i. iff there are $p_1, \dots, p_n$, not all zero, such that

$$\mu(|p_1x_1 + \cdots + p_nx_n| < \max\{\mu(x_i) : i \leq n \ \&\ p_i \neq 0\}.$$

Clearly if $x_1, \dots, x_n$ are s.i., then they are linearly independent.

Lemma 2.6 (Harnik [3]) *i) Let $x_i, \dots, x_n$ be given. If $\mu(x_i) \neq \mu(x_j)$ for all $i \neq j$, then $x_1, \dots, x_n$ are s.i.*

ii) Let $\mu(x) = \mu(x_1) = \cdots = \mu(x_n)$. Then $x_1, \dots, x_n$ are s.i. iff the reals $x_1/x, \dots, x_n/x$ are linearly independent over the rationals.

Hence for $x, x_1, \dots, x_n \in \mu$, $x_1, \dots, x_n$ is a maximal s.i. subset of μ iff $x_1/x, \dots, x_n/x$ is a basis of $F(x)$.

We turn now to $M_2(G)$. Let the letters $\delta, \varepsilon, \vartheta$ range over neighborhoods. Recall that we consider neighborhoods of nonstandard elements only, plus the zero element.

Note that neighborhoods are contained in magnitude classes. For every $\mu \in M_1(G)$ let $M_2(\mu) = \{\varepsilon : \varepsilon \subset \mu\}$. We write $\mu(\varepsilon)$ for the magnitude class of ε, $\delta \equiv \varepsilon$, if $\mu(\delta) = \mu(\varepsilon)$ and $\delta \ll \varepsilon$ if $\mu(\delta) < \mu(\varepsilon)$. Also, by 2.3 (ii), for every $\delta \equiv \varepsilon$, and all $x, x' \in \delta$, $y, y' \in \varepsilon$, $x/y = x'/y'$, so it makes sense to write δ/ε for this common real number.

Here are some elementary properties of the $\approx$-classes. The verification is left to the reader.

Lemma 2.7 *i) $x \approx x'$ and $y \approx y'$ $\Rightarrow$ $(x + y) \approx (x' + y')$. Hence we may set $\delta(x) + \delta(y) = \delta(x + y)$. In particular, $\delta \ll \varepsilon$ $\Rightarrow$ $\delta + \varepsilon = \varepsilon$.*

ii) For every δ and every $n > 0$, $n|x$ for some $x \in \delta$, so it makes sense to write $p\delta$ for every positive rational p, and $(p + q)\delta = p\delta + q\delta$. Hence for all

$\varepsilon_1, \ldots, \varepsilon_n$ and all non-negative rationals $p_1, \ldots, p_n$, $p_1\varepsilon + \cdots + p_n\varepsilon_n$ is defined by setting,

$$p_1\delta(x_1) + \cdots + p_n\delta(x_n) = \delta(p_1x_1 + \cdots + p_nx_n).$$

Moreover:

$$p_1\varepsilon_1 + \cdots + p_n\varepsilon_n = \sum\{p_k\varepsilon_k : \mu(\varepsilon_k) \text{ is greatest}\}.$$

iii) If $\varepsilon \equiv \delta \equiv \eta$ and $\varepsilon/\delta = \eta/\delta$, then $\varepsilon = \eta$.
iv) $\varepsilon/\delta < 1 \iff \varepsilon < \delta$.
If G is a ring with multiplication $\times$, moreover the following hold:
v) $x \approx x'$ and $y \approx y' \Rightarrow x \times y \approx x' \times y'$, hence we may set $\delta(x) \odot \delta(y) = \delta(x \times y)$.
vi) $\delta < \varepsilon$ & $\eta > 0 \Rightarrow \delta \odot \eta < \varepsilon \odot \eta$.

Thus $(M_2(G), +)$ is a semigroup (though not a strictly ordered one since $\delta \ll \varepsilon$ implies $\delta + \varepsilon = \varepsilon$, hence in general, $\delta < \varepsilon \Rightarrow \delta + \eta \leq \varepsilon + \eta$), and also a (quasi)-vector space over $\mathbb{Q}$. If in addition we start with a ring $R = (R, +, \times)$, with some extra work employing bases (see [2]) we can show that $(M_2(R), +, \odot)$, where $\odot$ is the multiplication on $M_2(R)$ induced by $\times$, is a semiring.

The semiring $(M_2(R), +, \odot)$ is said to be *euclidean* if $(\forall\varepsilon \ll \delta)(\exists\eta)(\varepsilon\odot\eta = \delta)$ and *radically closed* if $(\forall\varepsilon \neq 0)(\forall n > 0)(\exists\eta)(\eta^n = \varepsilon)$. (Using the fact that $\varepsilon \ll \eta \Rightarrow \varepsilon + \eta = \eta$, we can easily see that this definition of the euclidean property is equivalent to the standard one for this particular kind of structure).

3 Strong bases

Recall from the introduction that for any dor (or dog) R, $div(R)$ is the divisible closure of R which is a linear $\mathbb{Q}$-space, having thus linear bases from R. Working now with strong independence rather than simple one, it is natural to ask whether $div(R)$ continues to have a basis consisted of strongly independent elements. If R is countable, the answer is affirmative, and such bases are indeed a very effective tool in the study of non-archimedean groups.

Definition 3.1 *Let $(H, +)$ be a divisible ordered structure. A linear basis B of H over $\mathbb{Q}$ is said to be* strong *if B is a s.i. set.*

Lemma 3.2 *Let $(H,+,<)$ be a divisible ordered structure, $a \in H^+$ and $\{e_1,\ldots,e_k\}$ be a s.i. set of elements of H^+. Then we can always find $e_{k+1} \in H^+$ such that $\{e_1,\ldots,e_k,e_{k+1}\}$ is s.i. and $a \in \langle e_1,\ldots,e_k,e_{k+1}\rangle$.*

Proof. Let $X = \{e_1,\ldots,e_k\}$. If $X \cup \{a\}$ is s.i. we just set $e_{k+1} = a$. Otherwise there is a $X_0 \subseteq X$, say $X_0 = \{e_1,\ldots,e_{k_0}\}$ and non-zero rationals p, p_i such that

$$c = |pa + p_1e_1 + \cdots + p_{k_0}e_{k_0}| \ll e_m, a, \text{ for all } i \leq k_0. \tag{4}$$

If $c = 0$, then $a \in \langle e_1,\ldots,e_{k_0}\rangle$, hence we are done.

Suppose $c \neq 0$. Let $\mu(X_0) = \max\{\mu(e_i) : e_i \in X_0\}$. By the fact that X_0 is s.i. and $p_i \neq 0$, it follows that $\mu(|p_1e_1 + \cdots + p_{k_1}e_{k_1}|) = \mu(X_0)$. Also if $\mu(a) < \mu(X_0)$ or $\mu(a) > \mu(X_0)$, then either $\mu(c) = \mu(X_0)$ or $\mu(c) = \mu(a)$, contrary to (4). Therefore $\mu(a) = \mu(X_0)$, and $\mu(c) < \mu(a) = \mu(X_0)$. Clearly $a \in \langle X \cup \{c\}\rangle$. Thus if $X \cup \{c\}$ is s.i., then it suffices to set $e_{k+1} = c$.

Suppose $X \cup \{c\}$ is not s.i. Then there is a subset $X_1 \subseteq X$, say $X_1 = \{e_1,\ldots,e_{k_1}\}$ and non-zero q, qi such that

$$c_1 = |qc + q_1e_1 + \cdots + q_{k_1}e_{k_1}| \ll e_m, c, \text{ for all } i \leq k_1. \tag{5}$$

If $c_1 = 0$, then $c \in \langle X_1\rangle$, hence $a \in \langle X\rangle$. Otherwise let $\mu(X_1) = \max\{\mu(e_i) : e_i \in X_1\}$. Arguing as before we see that $\mu(c) = \mu(X_1)$ and $\mu(c_1) < \mu(c)$ and $c \in \langle X \cup \{c_1\}\rangle$. Hence, since already $a \in \langle X \cup \{c\}\rangle$, we get $a \in \langle X \cup \{c_1\}\rangle$. If $X \cup \{c_1\}$ is s.i., it suffices to set $e_{k+1} = c_1$.

If $X \cup \{c_1\}$ is not s.i., the process continues and we find $c_2, c_3, \ldots$ such that $X \cup \{c_i\}$ is not s.i., $a \in< X \cup \{c_i\} >$ and $\mu(c_{i+1}) < \mu(c_i)$. We claim that the process will terminate at some i, i.e., $X \cup \{c_i\}$ will be s.i. Indeed, let $\{\mu_1,\ldots,\mu_s\}$ be the classes of the elements of X. Since $\mu(c_i) \neq \mu(c_j)$, there will be a c_t such that $\mu(c_t) \notin \{\mu_1,\ldots,\mu_s\}$. Then, clearly $X \cup \{c_t\}$ is s.i. and $a \in \langle X \cup \{c_t\}\rangle$. Setting $e_{k+1} = c_t$ we are done. This completes the proof. $\square$

Theorem 3.3 *i) Every countable divisible structure $(H,+)$ has a strong basis.*

ii) For every countable dog G, $div(G)$ has a strong basis B such that $B \subseteq G^+$.

Proof. i) Let $a_1, a_2, \ldots$ be an enumeration of H^+. It suffices to construct inductively a sequence (probably finite) $e_1, e_2, \ldots$ of s.i. elements of H^+, such that for each n there is a k such that $\{a_1, \dots, a_n\} \subseteq \langle e_1, \dots, e_k \rangle$. Suppose that for a given $n \in \mathbb{N}$ we have found $e_1, \dots, e_k$ such that $\{a_1, \dots, a_n\} \subseteq \langle e_1, \dots, e_k \rangle$. Then use 3.2 to extend $\{e_1, \dots, e_k\}$ to a s.i. set $\{e_1, \dots, e_k, e_{k+1}\}$ such that $a_{n+1} \in \langle e_1, \dots, e_{k+1} \rangle$.

ii) By i) let $B' = \{e'_1, e'_2, \dots\}$ be a strong basis of $div(G)$ such that $B' \subset div(G)^+$. If $e'_i = \dfrac{e_i}{n_i}$, where $e_i \in G^+$ and $n_i > 0$, it is easy to see that $B = \{e_1, e_2, \dots\}$ is also a strong basis of $div(G)$. □

Given a strong basis B of H, let for every $\mu \in M_1(H)$

$$B_\mu = B \cap \mu \text{ and } B{\restriction}\mu = \{e \in B : \mu(e) \leq \mu\}.$$

Proposition 3.4 *i) If B is a strong basis, then every element of μ is a linear combination of elements of $B{\restriction}\mu$.*

ii) If B is a strong basis, then for every μ, B_μ is a maximal s.i. subset of μ.

Proof. i) Let $x \in \mu$ and $x = p_1 e_1 + \cdots + p_m e_m$. It suffices to show that $e_i \leq \mu$. Assume the contrary and let, say, $e_1, \dots, e_s > \mu$. Then $|x - p_{s+1}e_{s+1} - \cdots - p_m e_m| = |p_1 e_1 + \cdots + p_s e_s|$. But the left hand side of this equation is an element $\leq \mu$, hence $\not\equiv e_j$, for all $j = 1, \dots, s$, which contradicts the fact that $e_1, \dots, e_s$ are s.i.

ii) Clearly B_μ is s.i. and assume it is not maximal. Then there is $b \in \mu \backslash B$ such that $B_\mu \cup \{b\}$ is s.i. But since $B{\restriction}\mu$ produces the elements of μ, b is written $b = \sum_i p_i e_i + u$, for $e_i \in B_\mu$, and $\mu(u) < \mu$, which contradicts the strong independence of $B_\mu \cup \{b\}$. □

4 Integral approximations of elements in fields over dors

For a dor R it is important to know the proximity of its elements to those of the fields R^{fr} and R^{rc}. The prototype of such nearness conditions is Shepherdson's algebraic characterization of open induction S. S can equivalently be stated as follows:

S$_\sim$: $(\forall x \in R^{rc})(\exists a \in R)(x \sim a)$.

Replacing $\sim$ by $\approx$ and $\equiv$ we get the weaker versions:

S$_\approx$: $(\forall x \in R^{rc})(x \not\equiv 1 \Rightarrow (\exists a \in R)(x \approx a))$.
S$_\equiv$: $(\forall x \in R^{rc})(\exists a \in R)(x \equiv a)$.
Similarly the condition F for the euclidean property is equivalent to:
F$_\sim$: $(\forall x \in R^{fr})(\exists a \in R)(x \sim a)$,
which leads to the weakenings:
F$_\approx$: $(\forall x \in R^{fr})(x \not\equiv 1 \Rightarrow (\exists a \in R)(x \approx a))$.
F$_\equiv$: $(\forall x \in R^{fr})(\exists a \in R)(x \equiv a)$.

Let us consider also nearness of roots of elements of R to elements of R:
R$_\sim$: $(\forall a \in R)(\forall n > 0)(\exists b \in R)(b^n \sim a)$.
R$_\approx$: $(\forall a \in R)(\forall n > 0)(\exists b \in R)(b^n \approx a)$.
R$_\equiv$: $(\forall a \in R)(\forall n > 0)(\exists b \in R)(b^n \equiv a)$.

Finally consider the principles:
U: $(\forall a \gg 1)(F(a) = F(R))$.
RC: $F(R)$ is real closed.

Note that properties U and RC refer essentially to the underlying ordering and $+$, so they make sense also for groups.

The following implications are obvious:

$$\begin{tikzcd}
\text{euclidean} \arrow[r, leftrightarrow] & \mathrm{F}_\sim \arrow[r] & \mathrm{F}_\approx \arrow[r] & \mathrm{F}_\equiv \\
\mathrm{OI} \arrow[u] \arrow[r, leftrightarrow] & \mathrm{S}_\sim \arrow[u] \arrow[r] \arrow[d] & \mathrm{S}_\approx \arrow[u] \arrow[r] \arrow[d] & \mathrm{S}_\equiv \arrow[u] \arrow[d] \\
 & \mathrm{R}_\sim \arrow[r] & \mathrm{R}_\approx \arrow[r] & \mathrm{R}_\equiv
\end{tikzcd}$$

Proposition 4.1 *Let $R = (R, +, \times, <)$ be a dor, and let $*, \odot$ be the multiplications induced by $\times$ on $M_1(R), M_2(R)$ respectively. Then*
i) F$_\sim$ $\iff$ *R is euclidean and $S_\sim$* $\iff$ $R \models$ OI.
ii) F$_\equiv$ $\iff$ *$(M_1(R), *, <, 1)$ is a euclidean monoid.*
iii) R$_\equiv$ $\iff$ *$(M_1(R), *, <, 1)$ is a radically closed monoid.*
iv) F$_\approx$ $\iff$ F$_\equiv$ + U $\iff$ *$(M_2(R), +, \odot)$ is a euclidean semiring.*
v) R$_\approx$ $\iff$ *$(M_2(R), +, \odot)$ is a radically closed semiring.*
vi) R$_\equiv$ + U + RC $\Rightarrow$ R$_\approx$.
vii) F$_\equiv$ + R$_\equiv$ $\iff$ S$_\equiv$.

Proof. (i) This is immediate from the definitions and Shepherdson's result.

ii) Let $F_\equiv$ hold and $\mu < \nu \in M_1(R)$. Choosing $a \in \mu$ and $b \in \nu$, we have from $F_\equiv$ that there is $c \in R$ such that $\frac{b}{a} \equiv c$. Then $b \equiv ac$, hence $\mu * \mu(c) = \nu$. For the converse, let $\frac{b}{a} \in R^{fr}$. If $b \ll a$ then $\frac{b}{a} \equiv 0$. If $b \equiv a$, then $\frac{b}{a} \equiv n$ for some $n > 0$. If $b \gg a$, then by the assumption there is λ such that $\mu(a) * \lambda = \mu(b)$, hence $\frac{b}{a} \equiv c$ for every $c \in \lambda$.

iii) Similar to (ii).

iv) The equivalence $F_\approx \iff (M_2(R), +, \odot)$ is a euclidean semiring, follows immediately from the definition. We prove the other one. Suppose $F_\approx$ holds. Then $F_\equiv$ holds. Concerning U it suffices to show that given $a \equiv b$ and nonstandard c, there is $x \equiv c$ such that $a/b = x/c$. Since $a \equiv b$ and c is nonstandard, clearly $b \ll ac$. Then $\frac{ac}{b} \gg 1$ and by $F_\approx$, there is $x \in R$ such $\frac{ac}{b} \approx x$, or $bx \approx ac$. By 2.3, $bx/ac = 1$, or $x/c = a/b$.

Conversely suppose $F_\equiv$ and , U hold true and let $\frac{b}{a} \gg 1$, in R^{fr}. Then $a \ll b$ and by (ii) there is λ such that $\mu(a) * \lambda = \mu(b)$. Pick some $u \in \lambda$. Then $au \equiv b$ and by U, there is $c \equiv u$ such that $b/au = c/u$, or $b/au = ac/au$, or $ac/b = 1$ and equivalently $\frac{b}{a} \approx c$. Thus $F_\approx$ holds.

v) This is immediate from the definition of radically closed semiring.

vi) Let $a \in R$. By $R_\equiv$ there is $d \in R$ such that $d^n \equiv a$. By RC, $(a/d^n)^{1/n} \in F(R)$, hence, by U, there is b such that $(a/d^n)^{1/n} = b/d$. Then $a/d^n = b^n/d^n$, or $a \approx b^n$. So $R_\approx$ holds true.

vii) One direction is obvious. We prove $F_\equiv$+$R_\equiv \Rightarrow S_\equiv$. Let $F_\equiv$ and $R_\equiv$ hold. Then by (ii), (iii) above $M_1(R)$ is euclidean and root closed. We have to show that for every $r \in R^{rc}$, $r > 0$, there is $a \in R$ such that $a \equiv r$. If $r \equiv 1$ the claim is obvious. It suffices to show the claim for $r \gg 1$.

Let $r \gg 1$ and let $f(r) = a_n r^n + \cdots + a_1 r + a_0 = 0$ for some $f(x) \in R[x]$. If for all $i \neq j$, $0 \leq i, j \leq n$, $|a_i r^i| \not\equiv |a_j r^j|$, then clearly $|f(r)| \equiv \max\{|a_i r^i| : i \leq n\}$, contrary to the fact that $f(r) = 0$. Therefore there are $i \neq j$ such that $|a_i r^i| \equiv |a_j r^j|$. Hence if $j < i$, $|a_i r^{i-j}| \equiv |a_j|$. Since $M_1(R)$ is euclidean, there is a $b \in R$ such that $|a_j| \equiv |a_i| b$, whence $|a_j| \equiv |a_i| b \equiv |a_i| r^{i-j}$, or $b \equiv r^{i-j}$. Also by the fact that $M_1(R)$ is radically closed, there is a such that $a^{i-j} \equiv b$, from which we get $a^{i-j} \equiv r^{i-j}$, and finally $a \equiv r$. $\square$

Given a nonstandard $a \in R^+$, a polynomial $f(x) \in F(a)[x]$ has the form

$$f(x) = (a_n/a)x^n + (a_{n-1}/a)x^{n-1} + \cdots + (a_1/a)x + a_0/a,$$

where for all $i \leq n$, either $|a_i| \equiv a$ or $|a_i| \ll a$, i.e., $a_i/a = 0$. For such an f, let $f^* \in R[x]$ be the polynomial

$$f^*(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Lemma 4.2 *i) For every f as above, if $f^*(r) = 0$, then either $r = 0$, or $r > 0$ and $\frac{1}{l} < r < m$, or $r < 0$ and $-m < r < -\frac{1}{l}$ for some standard positive integers l, m.*

ii) For every $f \in F(a)[x]$ and every $b \in R$ such that $|b| \equiv a$,

$$f(b/a) = 0 \iff |f^*(\frac{b}{a})| \ll a.$$

Proof. i) Let $f^*(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with all $a_i \neq 0$, and $|a_i| \equiv |a_j|$. Suppose $f^*(r) = 0$, $r > 0$ and assume on the contrary that $r > n$ for all $n \in \mathbb{N}$. Thus $i < j \Rightarrow r^i \ll r^j$, and since $a_i \equiv a_j$, it follows that if $|f^*(r)| \equiv |a_n r^n|$ which is a contradiction. Similarly if $r < \frac{1}{l}$ for all standard l, then $\frac{1}{r} > l$ for all l, and $\frac{1}{r}$ is a root of the polynomial $a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$, whose coefficients are also of the same magnitude, so the same argument as before we reach a contradiction. The case for $r < 0$ is similar.

ii) $f(b/a) = 0 \iff$

$$(a_n/a)(b/a)^n + (a_{n-1}/a)(b/a)^{n-1} + \cdots + (a_1/a)(b/a) + a_0/a = 0 \iff$$
$$(a_n b^n + a_{n-1}b^{n-1}a + \cdots a_1 b a^{n-1} + a_0 a^n)/a^{n+1} = 0 \iff$$
$$|a_n b^n + a_{n-1}b^{n-1}a + \cdots a_1 b a^{n-1} + a_0 a^n| \ll a^{n+1} \iff$$
$$\frac{|a_n b^n + a_{n-1}b^{n-1}a + \cdots a_1 b a^{n-1} + a_0 a^n|}{a^n} \ll a \iff |f^*(\frac{b}{a})| \ll a.$$

□

Harnik [3], p. 425, says that if G is recursively saturated, then it can be shown that the field $F(G)$ is real closed. This means that for every dor R such that $R \models \mathrm{I}\Sigma_0$, $F(R)$ is real closed. In fact the following stronger implication holds.

Lemma 4.3 $\mathrm{S}_{\approx} \Rightarrow \ \mathrm{RC}$. *A fortiori, if* $R \models \mathrm{OI}$ *then* R *satisfies* RC, *i.e.,* $F(R)$ *is real closed.*

Proof. Note that $\mathrm{S}_{\approx} \Rightarrow \mathrm{F}_{\approx} \Rightarrow \mathrm{U}$ (by 4.1 (vi)), hence $F(R) = F(a)$ for all $a \gg 1$. We show that $F(a)$ is real closed.

(a) Let $x/a > 0$. We have to show that there is y/a such that $x/a = (y/a)^2$. Clearly $(xa)^{1/2} \in R^{rc}$ and $(xa)^{1/2} \gg 1$, hence by the assumption there is $y \in R$ such that $(xa)^{1/2} \approx y$. We easily see that y is as required.

(b) Let $f(x) \in F(a)[x]$ be of odd degree. $f(x)$ is of the form

$$f(x) = (a_n/a)x^n + \cdots + (a_1/a)x + a_0/a.$$

Consider $f^*(x)$. This is of odd degree hence it has a root $r \in R^{rc}$. For simplicity assume $r > 0$. By 4.2 (i), $\frac{1}{l} < r < m$ for standard l, m. Let $r_1 < r < r_2$ be the roots of f^* immediately before and after r. Take $t \in R$, $t \gg 1$, such that $t(r-r_1), t(r_2-r) > 1$. By the size of r, $tr \equiv t$. Since $tr \gg 1$, by $\mathrm{S}_{\approx}$, there is $s \in R$ such that $tr \approx s$, say, $tr = s + \theta$, $\theta \in R^{rc}$, $\theta \ll s \equiv t$. From these we get $tr_1 < tr - 1 = s + \theta - 1 < tr < s + \theta + 1 = tr + 1 < tr_2$, or

$$r_1 < \frac{s+\theta-1}{t} < r < \frac{s+\theta+1}{t} < r_2.$$

Since $f^*(r) = 0$, either

$$f^*(\frac{s+\theta-1}{t}) < 0 < f^*(\frac{s+\theta-1}{t}),$$

or

$$f^*(\frac{s+\theta-1}{t}) > 0 > f^*(\frac{s+\theta+1}{t}).$$

In both cases

$$|f^*(\frac{s+\theta-1}{t})| < |f^*(\frac{s+\theta+1}{t}) - f^*(\frac{s+\theta-1}{t})|.$$

Now it is easy to see that the right-hand side of the above inequality is $\ll a$, hence $|f^*(\frac{s+\theta-1}{t})| \ll a$, thus from 4.2, $f((s+\theta-1)/t) = 0$. Since $\theta \ll t$, $(s+\theta-1)/t = s/t$ and finally $f(s/t) = 0$, that is, f has a root in $F(R)$. $\square$

Lemma 4.4 $\mathrm{S}_{\equiv} + \mathrm{U} + \mathrm{RC} \Rightarrow \mathrm{S}_{\approx}$.

Proof. Suppose R is as stated and that there is an $r \gg 1$ in R^{rc} such that for every $a \in R$, $a \not\approx r$. We shall reach a contradiction.

First, by $S_{\equiv}$ there is an $a \in R^+$ such that $a \equiv r$. Let $\mu = \mu_R(a)$ and let $B_\mu = \{e_i : i \in I\}$ be a maximal set of strongly independent elements of μ. We claim that $B_\mu \cup \{r\}$ is strongly independent. Assume the contrary. Then there are rationals $p, p_1, \ldots, p_k \neq 0$ and $e_{i_1}, \ldots, e_{i_k} \in B\mu$ such that $|pr + p_1 e_{i_1} + \cdots + p_k e_{i_k}| \ll r$, hence $pr \approx |p_1 e_{i_1} + \cdots + p_k e_{i_k}|$, from which (using the fact that R is a $\mathbb{Z}$-dor) it follows that $r \approx b$ for some $b \in R$, which contradicts our assumption. Now the fact that B_μ is s.i. in μ and that $B_\mu \cup \{r\}$ is also s.i. is equivalent (see 2.6 (ii)) to the fact that $\{e_i/a : i \in I\}$ is a basis of the vector space $F(R)$, while $\{e_i/a : i \in I\} \cup \{r/a\}$ is linearly independent in $F(R(r))$, therefore $r/a \notin F(R)$.

Now since $r \in R^{rc}$, there is a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ of $R[x]$ such that $f(r) = 0$. As we have shown in 4.1, (vi), there are $i \neq j$ such that $|a_i r^i| \equiv |a_j r^j|$. Let us gather together all the monomials of $f(r)$ of maximum magnitude and let us assume without loss of generality that they form the sum $g(r) = a_k r^k + a_{k-1} r^{k-1} + \cdots + a_1 r + a_0$. Then $|a_m r^m| \equiv |a_0|$, for all $m = 1, \ldots, k$, and clearly $|g(r)| \ll a_0$ (otherwise $|f(r)| \equiv |g(r)| \equiv |a_0|$, contradicting $f(r) = 0$). Equivalently we have

$$r^k + \frac{a_{k-1}}{a_k} r^{k-1} + \cdots + \frac{a_1}{a_k} r + \frac{a_0}{a_k} \ll \frac{a_0}{a_k}. \tag{6}$$

Now $S_{\equiv}$+U implies $F_{\equiv}$+U which also implies $F_{\approx}$, hence for every $i \leq k-1$ there is a $b_i \in R$ such that $\dfrac{a_i}{a_k} \approx b_i$ (and $b_i \equiv r^{k-i} \equiv a^{k-i}$). Replacing in (6) each $\dfrac{a_i}{a_k}$ by b_i, an easy computation yields

$$r^k + b_{k-1} r^{k-1} + \cdots + b_1 r + b_0 \ll b_0 \text{ and } b_0 \equiv r^k \equiv a^k. \tag{7}$$

From (7) we get

$$(r/a)^k + (b_{k-1}/a)(r/a)^{k-1} + \cdots + (b_1/a^{k-1})(r/a) + b_0/a^k = 0,$$

or $h(r/a) = 0$, where $h(x) = x^k + (b_{k-1}/a) x^{k-1} + \cdots + (b_1/a^{k-1}) x + b_0/a^k$. But $h(x) \in F(R)[x]$, and, by (c) of the hypotheses, $F(R)$ is real closed, hence $r/a \in F(R)$, which contradicts our previous conclusion. This proves the claim. □

Corollary 4.5 *a) The following are equivalent (over the axioms of dor's):*
i) $\mathrm{S}_{\approx}$
ii) $\mathrm{S}_{\equiv} + \mathrm{U} + \mathrm{RC}$.
iii) $\mathrm{F}_{\equiv} + \mathrm{R}_{\equiv} + \mathrm{U} + \mathrm{RC}$.

Proof. Immediate from lemmas 4.1, 4.3, 4.4. □

The main implications of 4.1 and 4.5 are summarized in the following diagram:

$$\begin{tikzcd}
 & \mathrm{F}_{\equiv} + \mathrm{U} \arrow[d, leftrightarrow] & & \\
 & \mathrm{F}_{\approx} \arrow[r] & \mathrm{F}_{\equiv} & \\
\mathrm{S}_{\equiv} + \mathrm{U} + \mathrm{RC} \arrow[r, leftrightarrow] & \mathrm{S}_{\approx} \arrow[u] \arrow[d] \arrow[r] & \mathrm{S}_{\equiv} \arrow[u] \arrow[d] \arrow[r, leftrightarrow] & \mathrm{F}_{\equiv} + \mathrm{R}_{\equiv} \\
 & \mathrm{R}_{\approx} \arrow[r] & \mathrm{R}_{\equiv} & \\
 & \mathrm{R}_{\equiv} + \mathrm{U} + \mathrm{RC} \arrow[u] & &
\end{tikzcd}$$

We shall show below using counterexamples that *no* single arrow in the above diagram can be reversed. First a lemma.

Lemma 4.6 *Let R be a dor and let $x, y \in R$ such that $x \equiv y$. Then*
i) x/y is irrational iff x, y are s.i.
ii) x/y is non-algebraic if for every $n > 0$, the set $\{x^i y^{n-i} : 0 \leq i \leq n\}$ is s.i.

Proof. i) By definition x, y are not s.i. iff there is $p \in \mathbb{Q}$ such that $py + x \ll y$, or, in view of lemma 2.3, iff there is p such that $(py + x)/y = 0$, or iff there is p such that $x/y + p = 0$, hence iff $x/y \in \mathbb{Q}$.
ii) x/y is algebraic iff there is a polynomial f such that $f(x/y) = 0$, or $a_n(x/y)^n + \cdots + a_1(x/y) + a_0 = 0$ for some $a_i \in \mathbb{Q}$, or $a_n x^n + a_{n-1}x^{n-1}y + \cdots + a_1 x y^{n-1} + a_0 y^n \ll y^n$, or $\{x^i y^{n-i} : 0 \leq i \leq n\}$ is s.i. □

Proposition 4.7 *i)* $\mathrm{F}_{\equiv} \not\Rightarrow \mathrm{R}_{\equiv}$, *ii)* $\mathrm{R}_{\equiv} \not\Rightarrow \mathrm{F}_{\equiv}$, *iii)* $\mathrm{F}_{\equiv} \not\Rightarrow \mathrm{S}_{\equiv}$, *iv)* $\mathrm{R}_{\equiv} \not\Rightarrow \mathrm{S}_{\equiv}$, *v)* $\mathrm{F}_{\equiv} \not\Rightarrow \mathrm{U}$, *vi)* $\mathrm{F}_{\equiv} \not\Rightarrow \mathrm{F}_{\approx}$, *vii)* $\mathrm{S}_{\equiv} \not\Rightarrow \mathrm{S}_{\approx}$, *viii)* $\mathrm{R}_{\equiv} \not\Rightarrow \mathrm{R}_{\approx}$, *ix)* $\mathrm{F}_{\approx} \not\Rightarrow \mathrm{S}_{\approx}$, *x)* $\mathrm{R}_{\approx} \not\Rightarrow \mathrm{R}_{\equiv} + \mathrm{U} + \mathrm{RC}$.

Proof. i) Let R be a dor such that $(M_1(R), *, <, 1)$ is of the form $\{1, \mu, \mu^2, \ldots\}$, where $1 < \mu < \mu^2 < \cdots$. There is an abundance of such rings, e.g. that generated by $\mathbb{Z} \cup \{x\}$, i.e. $R = \mathbb{Z}[x]$, where x is construed as an infinite (nonstandard) *divisible* element (i.e., $n|x$ for all standard $n > 0$) and $1 \ll x \ll x^2 \ll \cdots$. Every such monoid is obviously euclidean. However it cannot be radically closed since e.g. there is no ν such that $\nu^2 = \mu$. More generally it is shown in [5] that if $(M_1(R), *, <, 1)$ is radically closed, then its order must be dense. It follows by proposition 4.1 (ii) and (iii), that in such an R, $\mathrm{F}_\equiv$ holds, while $\mathrm{R}_\equiv$ is false. Therefore $\mathrm{F}_\equiv \not\Rightarrow \mathrm{R}_\equiv$.

ii) $\mathrm{R}_\equiv \not\Rightarrow \mathrm{F}_\equiv$: In [5] we gave a simple example of a radically closed monoid which is not euclidean. Here we have to find an R such that $(M_1(R), *, <, 1)$ is of this kind. Then $\mathrm{R}_\equiv \not\Rightarrow \mathrm{F}_\equiv$ will follow again by 4.1 (ii) and (iii). Let R be the ring generated by $\mathbb{Z} \cup \{x^{1/n}, y^{1/n} : n > 0\}$, where x, y are infinite divisible numbers such that for all $n \in \mathbb{N}$, $x^n \ll y$. Every $f(x, y) \in R$ consists of monomials $x^a y^b$, with $a, b \in Q^+$. Since $x^n \ll y$, it follows easily that $x^a y^b \ll x^c y^d$ iff $b < d$, or $b = d$ and $a < c$. For every $f \in R$, let $deg_y(f)$ be the greatest exponent of y in f, and let $deg_x(f, y)$ be the greatest exponent of x occurring in the monomials with degree $deg_y(f)$. Then $f \ll g$ iff $deg_y(f) < def_y(g)$, or $deg_y(f) = def_y(g)$ and $deg_x(f, y) < def_x(g, y)$. Therefore $f \equiv g$ iff $deg_y(f) = def_y(g)$ and $deg_x(f, y) = def_x(g, y)$. Thus each class $\mu(f)$ can be identified to a monomial $x^a y^b$. For any $x^a y^b$ and any $n > 0$, the n-th root of $x^a y^b$ is $x^{a/n} y^{b/n}$, hence $M_1(R)$ is radically closed. On the other hand, there is no monomial $x^a y^b$ such that $(x^a y^b)x \equiv y$. Indeed the latter requires $a + 1 = 0$ and $b = 1$, which is impossible since $a, b \in \mathbb{Q}^+$. Thus $M_1(R)$ is not euclidean and the claim is proved.

iii) and iv) Since by 4.1 (vii), $\mathrm{S}_\equiv \Leftrightarrow \mathrm{F}_\equiv + \mathrm{R}_\equiv$, it follows immediately from (i) and (ii) above that $\mathrm{F}_\equiv \not\Rightarrow \mathrm{S}_\equiv$ and $\mathrm{R}_\equiv \not\Rightarrow \mathrm{S}_\equiv$.

v) Let R be a dor generated by $\mathbb{Z} \cup \{x, y\}$, where x, y are infinite divisible elements such that $x \equiv y$ and x, y are strongly independent. Then, by the lemma 4.6 i), x/y is irrational and it is easy to see that $F(x) = F(y) = \mathbb{Q}[x/y]$. (Concerning the existence of x, y such that x/y is irrational, if H is a recursively saturated dog, then $F(H)$ contains all recursive positive reals. See [3] for more details.) Since $\mu(x) = \mu(y) = \mu$, clearly $M_1(R) = \{1, \mu, \mu^2, \ldots\}$, therefore $M_1(R)$ is a euclidean monoid, i.e. R satisfies $\mathrm{F}_\equiv$. On the other hand, $x^2, y^2 \in \mu^2$ and $x^2/y^2 = (x/y)^2$, hence $F(x^2) = \mathbb{Q}[(x/y)^2]$. Taking e.g. x, y so that $x/y = \sqrt{2}$, the fields $\mathbb{Q}[x/y]$ and $\mathbb{Q}[(x/y)^2]$ are distinct, hence U is false in R. This proves $\mathrm{F}_\equiv \not\Rightarrow \mathrm{U}$.

vi) Since by 4.1 (iv), $F_{\approx} \Leftrightarrow F_{\equiv} + U$, $F_{\equiv} \not\Rightarrow F_{\approx}$ follows immediately from (v).

vii) $S_{\equiv} \not\Rightarrow S_{\approx}$: In view of the fact that $S_{\approx} \Leftrightarrow S_{\equiv} + U + RC$ and $S_{\equiv} \Leftrightarrow F_{\equiv} + R_{\equiv}$, it suffices to show that $F_{\equiv} + R_{\equiv} \not\Rightarrow RC$ (or $F_{\equiv} + R_{\equiv} \not\Rightarrow U$), i.e., to find R such that $(M_1(R), *, <, 1)$ is euclidean and radically closed but the field $F(a)$ for some a is not real closed.

Let R be the dor generated by $\mathbb{Z} \cup \{x^{1/n}, y^{1/m} : m, n > 0\}$, where x, y are as in the example of (v) above, i.e., $x \equiv y$ and x, y are s.i. It is easy to see that if $\mu = \mu(x) = \mu(y)$, $M_1(R) = \{\mu^{m/n} : m \geq 0, n > 0\}$, with the obvious ordering and multiplication, hence $M_1(R)$ is both euclidean and radically closed. Then $F(x) = \mathbb{Q}[x/y]$ and this field is clearly not real closed. (In fact neither U holds in R, since $F(x^{1/2}) = \mathbb{Q}[(x/y)^{1/2}] \neq F(x)$.)

viii) $R_{\equiv} \not\Rightarrow R_{\approx}$: It suffices to show that if in the dor R of (vii) above we take x/y to be non-algebraic, then R does not satisfy $R_{\approx}$. (Since there are recursive non-algebraic reals, the existence of such x/y follows from the analogous remark of clause (v).) Indeed in this case, for every n, $(x/y)^{1/n}$ is also non-algebraic, so by lemma 4.6 (ii), for any k and any pairs of rationals $(a_i, b_i), i \leq k$, such that $a_i + b_i$=fixed,

$$\{x^{a_1}y^{b_1}, \ldots, x^{a_k}y^{b_k}\} \text{ is s.i.} \tag{8}$$

We claim that there is no $u \in R$ such that $u^2 \approx (x + y)$.

Assume the contrary and let u be such that $u^2 \approx x + y$. Since $x + y \in \mu$, u must belong to $\mu^{1/2}$. A (strong) basis of $div(R)$ is formed by the elements $x^{k/n}y^{l/n}$, hence u is written as a finite sum $u = \sum_i p_i x^{a_i} y^{b_i} + w$, where $p_i \in \mathbb{Q}$, $a_i + b_i = 1/2$ and $w < \mu^{1/2}$. Therefore $(\sum_i p_i x^{a_i} y^{b_i} + w)^2 \approx x + y$, or

$$[\sum_i (p_i x^{a_i} y^{b_i})^2 + \sum_{ij} 2p_i p_j x^{a_i + a_j} y^{b_i + b_j} + z] \approx x + y,$$

where $z = w^2 + \sum_i 2p_i x^{a_i} y^{b_i} w < \mu$. But then

$$|\sum_i (p_i x^{a_i} y^{b_i})^2 + \sum_{ij} 2p_i p_j x^{a_i + a_j} y^{b_i + b_j} - x - y| = z \ll x + y,$$

which means that the set $\{x, y, x^{2a_i} y^{2b_i}, x^{a_i + a_j} y^{b_i + b_j} : i, j\}$ is not s.i. Since $2a_i + 2b_i = a_i + a_j + b_i + b_j = 1$, this contradicts (8) and the claim is proved.

ix) Consider the ring $R = \mathbb{Z}[x]$ of clause (i). This is euclidean, hence it satisfies $\mathrm{F}_{\approx}$. Moreover R satisfies U and $F(R) = \mathbb{Q}$. Hence RC fails for R. In view of the fact that $\mathrm{S}_{\approx} \Leftrightarrow \mathrm{S}_{\equiv} + \mathrm{U} + \mathrm{RC}$, it follows that $\mathrm{F}_{\approx} \not\Rightarrow \mathrm{S}_{\approx}$

x) It suffices to show that $\mathrm{R}_{\approx} \not\Rightarrow \mathrm{RC}$. Let R be the dor generated by $\mathbb{Z} \cup \{x^{1/n} : n > 0\}$. It is easy to see that $M_2(R)$ is radically closed. Therefore $\mathrm{R}_{\approx}$ holds for R. Now $F(R) = \mathbb{Q}$, hence RC fails. □

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