The linear logic of multisets

Athanassios Tzouvaras
Department of Mathematics
University of Thessaloniki
540 06 Thessaloniki, Greece
E-mail:tzouvara@ccf.auth.gr

Abstract

We consider a cumulative hierarchy of multisets over some set of urelements, equipped with additive union $\uplus$ and a transform relation $\rhd$, and investigate the Horn fragments of Intuitionistic Linear Logic (ILL) that are interpretable in it. The operator ! is defined in an asymptotic way which causes some deviations from the linear-theoretic behavior. Soundness, completeness and partial completeness results are proved for the various fragments. Certain processes of multisets suggest rules for the multiplicatives not compatible with full ILL. One such rule added to the Horn fragment makes the system sound and complete with respect to "coherent processes".

1 Introduction.

All familiar logics except the linear one (classical, intuitionistic, modal, relevant, etc.), share a remarkable common feature: Conjunction and disjunction are idempotent operations. An explanation of this fact can be found in their semantics. Each one of these logics possesses a set-theoretic semantics, where $\wedge$ is interpreted as $\cap$ and $\vee$ as $\cup$. Thus the identities $\phi\wedge\phi=\phi\vee\phi=\phi$ are syntactic counterparts of the extensional identities $A\cap A=A\cup A=A$. It is reasonable to consider the last identities as more primitive than the first ones, since we usually identify a predicate ϕ with its extension $A_\phi=\{a:\phi(a)\}$. So the question why $\phi\vee\phi=\phi$ is reduced to the question why $A\cup A=A$.

For reasons hidden in the early history of set theory, a set came to mean a collection of *types* of objects rather than of concrete *tokens* of them. According to this view what matters with respect to elementhood is just the kind of an object x, not the concrete copies of it. Hence any series of copies $x, x, \dots$ can be suppresed to a single representative. Idempotence of $\cup$ follows then immediately: $A \cup A$ contains precisely the same types of objects as A.

Under this interpretation of formulas as extensions, a logic Λ contains exactly the syntactic rules of a calculus of extensions forming a certain kind of structure S. We express this by saying that Λ is *the logic* of S. E.g. classical logic is the logic of boolean fields of sets (i.e., boolean algebras of sets), intuitionistic logic is the logic of pseudo-boolean fields (like the structure of open sets of a topological space), modal logic is the logic of topological boolean fields (that is, boolean fields equipped with a further interior operator), and so on.

Our concern in this paper is what the effect on logic will be if we shift from ordinary sets to *multisets*, i.e. collections which account not only for types but also for tokens of objects. The demand for such collections becomes more and more urgent in applications where copies of various data, standing as resources of processes, have an existence of their own and cannot be suppressed to a single one. For instance, if data is money spent, then clearly the collections $\{\$1\}$ and $\{\$1, \$1\}$ do not coincide, and ordinary union has to be replaced by *additive union* $\{\$1\} \uplus \{\$1\} = \{\$1, \$1\}$ that captures resource preservation. Additive union is the only operation we consider here. Idempotent $\cup$ and $\cap$ can also be defined but are of minor importance. Since $\uplus$ is not idempotent, this constitutes the first basic departure from the ordinary-set paradigm. Multisets differ also from ordinary sets in that, for any given one X, the collection of submultisets of X is not closed under $\uplus$. For instance $X \uplus X \not\subseteq X$. Therefore no X can stand for a greatest multiset, and consequently no sensible notion of complement exists. In order to interpret implication we introduce a "transform relation" $x \rhd y$, which roughly means that using precisely the elements of x we can construct y. The question we address is this: What logic arises if $\uplus$ stands for conjunction and $\rhd$ stands for implication?

J.-Y. Girard has developed in [3] linear logic (LL), the multiplicative fragment of which has since stood as the major paradigm for resource sensitive logical procedures. LL possesses also a set-theoretic semantics, but $\cup$ and $\cap$ interpret only its "additive" part, which roughly coincides with classical logic. For the interpretation of the "multiplicative" part (which is the real novelty), one has to employ tensor-like products, while ! is interpreted as a topological interior. The logic of multisets is shown to be almost identical to the relevant fragments of linear logic, namely the $\{\otimes, \multimap\}$- and $\otimes, \multimap, !\}$-fragments. Some deviations, especially concerning the rules of !, shed in our view some further light on the meaning of this operator. Thus the passage from ordinary sets to multisets causes an essential simplification of semantics since, for these fragments, we can dispense with tensor products and topological closures.

The connection between the behavior of multisets and the multiplicative fragment of LL can be briefly explained as follows. Transformations of multisets, in contrast to those of ordinary sets, obey the *conservation principle*: The resources of the input and the output of the transformation are equal. Obviously this is a semantical principle. The syntactic (logical) counterpart of this principle is *non-contraction+non-weakening*, which, as is well-known, constitutes the heart of the multiplicative fragment of LL. Non-contraction can be stated as $A \not\vdash A \otimes A$, that is, nothing can be born from nothing. Non-weakening states that $(A \otimes B) \not\vdash A$, that is, nothing can perish to nothing. Thus interpreting $\otimes$ as multiset union $\uplus$ and $\multimap$ as multiset transform, $\rhd$, provides a natural model for the multiplicatives.

The paper is organized as follows: In section 2 we define the cumulative hierarchy of multisets over a set of urelements and prove some basic facts about it. In section 3 we introduce the transform relation $\rhd$, staged processes and sequents of multisets and examine the rules that these sequents satisfy. In section 4 we introduce the Horn fragment (HF) of linear logic and prove its soundness and completeness with respect to staged sequents. In section 5 a weaker kind of process and sequent is studied, the *coherent* ones. These sequents satisfy an additional rule, the cancellation rule $C_\otimes$. If CHF=HF+$C_\otimes$, then we show that CHF is sound and complete with respect to coherent sequents. In section 6 generalized multisets, processes and sequents containing the operator ! are introduced. The truth of these sequents is reduced asymptotically to the truth of ordinary sequents (staged or coherent) by means of a $\forall\exists$ definition. Here however the contraction rule for !-sequents fails. But the system !-HF is sound if we restrict ourselves to a certain subclass of !-multisets having a good normal form. Also !-HF is complete with respect to a subclass of !-staged sequents called *regular*. Analogous soundness and completeness results hold for the system

!-CHF=!-HF+$C_\otimes$ with respect to !-coherent processes.

2 Multisets.

The interest in multisets (except from marginal hints found in older books) has started to emerge rather recently (after 1960) and the literature is not very extensive. Except for a few papers that undertake to found them rigorously, like [1], the rest deal mainly with applications. Especially in the last twenty years there is a remarkable growth of applications of multisets in various areas of computer science. D. Knuth already makes considerable use of them in [6]. [1] contains a good brief survey and bibliography of main contributions to the subject up to 1989. It also contains an axiomatic foundation. However this is not really necessary in order to treat them rigorously. The framework of classical set theory ZF suffices and it is in this that we work below. Other survey articles of the multiset literature are [2] and [7].

Throughout by "set" we shall always mean an ordinary set of ZF. Capital letters $A, X, Y, \ldots$ will range over sets, while small letters $x, y, z, \ldots$ will range over multisets. Formally the notion is sufficiently captured if we take a *multiset over* X to be a mapping $x : X \to N$, where N is the set of non-negative integers.

Definition 2.1 A *multiset over a set* X is a function $x : X \to N$. The set $d(x) = \{y : x(y) \neq 0\}$ is the *domain* of x, or the set of its *types.* $x(y)$ is the *multiplicity* of y in x. We write $y \in x$ if $x(y) \neq 0$, i.e., if $y \in d(x)$. x is *finite if* $d(x)$ is finite.

We use square brackets when we write explicitly the elements of x, namely we write $x = [y_1, y_1, \ldots, y_2, y_2, \ldots]$, or $x = [y_1^{n_1}, y_2^{n_2}, \ldots]$, where n_i is the multiplicity of y_i. The empty multiset is denoted again by $\emptyset$.

(Although the elements of a multiset can be whatever, even sets, we denote them by small letters too. In fact, throughout this paper the elements of multisets will be multisets or urelements.)

M.I. Kanovich in [4] and [5] seems to have been the first to realize that for simple fragments of LL the tensor product is no more than additive union. Definition 2.2 of [4] goes as follows:

"*Taking into account the associativity and commutativity laws, we use a natural isomorphism between non-empty finite multisets of positive literals and simple (tensor) products. A multiset* $\{p_1, p_2, \ldots, p_k\}$ *is represented by the simple product* $(p_1 \otimes p_2 \otimes \cdots \otimes p_k)$*, and vice versa. For simple products* X*,* Y *representing multisets* L, M *respectively,*

(a) $(X \otimes Y)$ *represents the union of* L *and* M*;*

(b) if $L \subseteq M$ *we will say that the simple product* X *is contained in the simple product* Y*, and will write* $X \subseteq Y$*;*

(c) we write $X \cong Y$ *to indicate that* $X \subseteq Y$ *and* $Y \subseteq X$*.*"

For every set X let $M(X)$ and $FM(X)$ be the set of multisets and finite multisets, respectively, over X. The operators M, FM, like the powerset operator are monotone and send sets to sets. For every nonempty set of *urelements* A we define a hierarchy $U(A)$ of finite multisets as follows:

$$U_0(A) = A, \quad U_{n+1}(A) = U_n(A) \cup (FM(U_n(A)) \backslash \{\emptyset\}), \quad U(A) = \bigcup_{n \geq 0} U_n(A).$$

We exclude $\emptyset$ from our hierarchy because it is a "null" object for our purposes and its presence adds only unecessary complication.

The letters $a, b, c, \ldots$ range over elements of A. Clearly $U_n(A) \subseteq U_{n+1}(A)$ for each $n \geq 0$. $U(A)$ is the analog of the cumulative hierarchy of hereditarily finite sets built on the set of urelements A.

For any $X \subseteq U(A)$, $U(X)$ is defined similarly. In particular we write $U(x)$ instead of $U(d(x))$. For urelements a, we can conventionally put $d(a) = \emptyset$. Clearly, if $x \notin A$, then $x \in FM(d(x))$ and for every $Y \subset d(x)$, $x \notin FM(Y)$. Given X and $x \in U(X)$, the *rank of* x *with respect to* X, denoted $rank_X(x)$, is the least $n \in N$ such that $x \in U_n(X)$. Obviously, $rank_X(x) = 0$ iff $x \in X$. If $X = A$, we drop the subscript, i.e., $rank(x) = rank_A(x)$. Also we write $rank_y(x)$ instead of $rank_{d(y)}(x)$.

$U(A)$ is equipped with *additive union* $\uplus$ defined by

$$(x \uplus y)(z) := x(z) + y(z),$$

and *inclusion*

$$x \subseteq y := (\forall z)(x(z) \leq y(z)).$$

Clearly

$$x \subseteq y \ \& \ y \subseteq x \Rightarrow x = y.$$

Also for $x \subseteq y$, $y - x$ is defined by

$$(y - x)(z) := y(z) - x(z).$$

$x \uplus y$ is generalized to $\uplus x$, for any x, by putting

$$(\uplus x)(z) = \sum_{y \in d(x)} x(y) \cdot y(z).$$

Thus $x \uplus y = \uplus[x, y]$ and $\uplus[x] = x$.

Given x and any positive integer n, nx denotes the union of n copies of x, i.e.,

$$nx = \underbrace{x \uplus \cdots \uplus x}_{n \text{ times}} = \uplus[x^n].$$

Given x and a mapping $f : d(x) \to Y$ into another set Y, the *substitution* of elements y of x by $f(y)$ of Y which respects multiplicities, creates a new multiset denoted by $f[x]$. This is defined as follows:

Definition 2.2 Let $x \in U(A)$. Every mapping $f : X \supseteq d(x) \to Y$ is called a *substitution.* The *image of x under f,* is the multiset $f[x]$ such that:
(a) $d(f[x]) = f(d(x))$, and
(b) $(f[x])(y) = \sum\{x(z) : f(z) = y\}$.
For convenience instead of $f[x]$ we write

$$[f(y) : y \in x].$$

Definition 2.3 For every $X \subseteq U(A)$ and every $x \in U(X)$, the function $supp_X : U(X) \to FM(X)$ is defined by induction on $rank_X(x)$ as follows:
(a) $supp_X(x) = [x]$ if $x \in X$.
(b) $supp_X(x) = \uplus supp_X[x] = \uplus[supp_X(y) : y \in x]$.
$supp_X(x)$ is said to be *the support of x over X*. In particular, we write $supp(x)$ instead of $supp_A(x)$, and $supp_y(x)$ instead of $supp_{d(y)}(x)$.

In words, $supp_X(x)$ is the multiset of elements of X involved in the construction of x. The following is easy.

Lemma 2.4 *For every x, (a) $sup_x(x) = x$, (b) $supp_x([x]) = x$ and (c) $supp_{[x]}(x) = [x]$.*

3 Transforms and processes of multisets.

We fix a set of urelements A and the hierarchy $U(A)$ built on A. $rank(x)$ refers always to this hierarchy.

Lemma 3.1 *(a)* $rank_x(y) = 0$ *iff* $y \in d(x)$, *and* $rank_x(y) = 1$ *iff* $d(y) \subseteq d(x)$.

(b) For every $x \in U(y)$, $rank(x) = rank(y) + rank_y(x) - 1$.

(c) If $x \in U(y)$, *then* $rank(y) \leq rank(x)$ *unless* $x \in d(y)$.

(d) If $x \in U(y)$ *and* $y \in U(x)$, *then either* $x \in y$, *or* $y \in x$ *or* $d(x) = d(y)$.

Proof. (a) is obvious. (b) Let $x \in U(y)$ and let $rank(y) = n$ and $rank_y(x) = k$. Then

$$y \in U_n(A)\backslash U_{n-1}(A) \text{ and } x \in U_k(d(y))\backslash U_{k-1}(d(y)).$$

By the last two relations we get

$$d(y) \subseteq U_{n-1}(A) \text{ and } d(x) \subseteq U_{k-1}(d(y)),$$

whence

$$d(x) \subseteq U_{k+n-2}(A),$$

hence $x \in U_{k+n-1}(A)$. Thus $rank(x) \leq k+n-1$. From the fact that k, n are the least elements for which the above hold, we get that $rank(x) = k+n-1$.

(c) By (b), $rank(y) \leq rank(x)$, unless $rank_y(x) = 0$, i.e., by (a), $x \in d(y)$.

(d) Let $x \in U(y)$ and $y \in U(x)$. By (b), $rank(x) = rank_y(x)+rank(y)-1$, and $rank(y) = rank_x(y) + rank(x) - 1$. The last two equations yield $rank_x(y) + rank_y(x) = 2$. Then either $rank_x(y) = rank_y(x) = 1$, or $rank_x(y) = 0$ and $rank_y(x) = 2$, or $rank_y(x) = 0$ and $rank_x(y) = 2$. In the first case, by (a), $d(x) = d(y)$ and in the other cases $y \in d(x)$ and $x \in d(y)$ respectively. But these are equivalent to $y \in x$ and $x \in y$. $\square$

Lemma 3.2 *(a) If* $X \subseteq Y$ *and* $x \in U(X)$, *then* $supp_X(x) = supp_Y(x)$. *In particular, if* $d(x) \subseteq d(y)$, *then* $supp_y(x) = x$.

(b) $supp_X$ *is additive, i.e.,* $supp_X(x \uplus y) = supp_X(x) \uplus supp_X(y)$. *Consequently, for every* x *such that* $x, \uplus x \in U(X)$, $supp_X(x) = supp_X(\uplus x)$.

(c) If $y \in U(x)$ *and* $z \in U(y)$, *then* $z \in U(x)$ *and*

$$supp_x(z) = supp_x(supp_y(z)).$$

Proof. (a) Immediate from the definitions.

(b) The first claim is also immediate from the definitions. Now let $x, \biguplus x \in U(X)$ and $x = [u_1, \ldots, u_n]$. Then

$$supp_X(x) = \biguplus[supp_X(u) : u \in z] = supp_X(u_1) \uplus \cdots \uplus supp_X(u_n) =$$

$$supp_X(u_1 \uplus \cdots \uplus u_n) = supp_X(\biguplus x).$$

(c) By induction on $rank_x(z)$. Suppose it holds for $supp_x(u)$, where $u \in z$, and let $z = [u_1, \ldots, u_n]$. Then

$$supp_x(z) = supp_x(u_1) \uplus \cdots \uplus supp_x(u_n),$$

or, by the induction hypothesis,

$$supp_x(z) = supp_x(supp_y(u_1)) \uplus \cdots \uplus supp_x(supp_y(u_n)) =$$

$$supp_x([supp_y(u_1), \ldots, supp_y(u_n)]).$$

By (b), the latter is equal to

$$supp_x(\biguplus[supp_y(u_1), \ldots, supp_y(u_n)]) = supp_x(supp_y(z)). \ \Box$$

We come now to the main definition of this section.

Definition 3.3 The *transform relation* is a binary relation $\rhd$ on $U(A)$ defined as follows: $(x, y) \in \rhd$ if

(a) $y \in U(x)$, and

(b) $supp_x(y) = x$.

We write $x \rhd y$ instead of $(x, y) \in \rhd$ and say that y is a *transform of* x.

In words, $x \rhd y$ holds if y belongs to the universe built on the types of x and contains exactly the resources of x.

Comment. The transform relation is intended to capture material change subject to the conservation principle: The ultimate resources of the input and the output are equal. Chemical reactions, for example, are of this kind. For instance, the reaction

$$H_2SO_4 + 2NaOH \ \rightarrow \ Na_2SO_4 + 2H_2O$$

can be written as the multiset transform

$$[[H^2, S, O^4], [Na, O, H]^2] \rhd [[Na^2, S, O^4], [H^2, O]^2],$$

where H, S, Na, O are urelements.

Lemma 3.4 *(a) $\rhd$ is reflexive and transitive.*
(b) If $x \rhd y$ and $y \rhd x$, then either $x = y$, or $x = [y]$ or $y = [x]$.
(c) If $x \rhd y$, then $supp(x) = supp(y)$.

Proof. (a) Clearly, $x \in FM(d(x)) \subseteq U_1(d(x))$, and, by 3.2(a) $supp_x(x) = x$, hence $x \rhd x$. To check transitivity, let $x \rhd y$ and $y \rhd z$. Then
(i) $y \in U(x)$, $z \in U(y)$ and
(ii) $supp_x(y) = x$ and $supp_y(z) = y$.
By (i) and 3.2(c), $z \in U(x)$. By (ii) and 3.2(c),

$$supp_x(z) = supp_x(supp_y(z)) = supp_x(y) = x.$$

Thus $x \rhd z$.

(b) Let $x \rhd y$ and $y \rhd x$. Then $y \in U(x)$ and $x \in U(y)$. By 3.1(d), either $d(x) = d(y)$, or $x \in y$ or $y \in x$. In the first case, by 3.2(a), $supp_x(y) = y$, whereas by $x \rhd y$, $supp_x(y) = x$. Hence $x = y$. Suppose that $x \in y$, i.e., $y = [\cdots x \cdots]$. Then

$$supp_x(y) = \uplus[\cdots supp_x(x) \cdots] \subseteq x.$$

Since by the hypothesis, $supp_x(y) = x$, and for every $z \in y$, $supp_x(z) \neq \emptyset$, it follows that $y = [x]$. Similarly if $y \in x$ we find $x = [y]$.

(c) Recall that $supp(x) = supp_A(x)$. Let $x \rhd y$. By 3.2(c), $supp(y) = supp_A(y) = supp_A(supp_x(y))$. Since by the hypothesis $supp_x(y) = x$, we find

$$supp(y) = supp_A(y) = supp_A(x) = supp(x). \quad \square$$

Suppose now we are given a multiset x, representing some *initial resources*, and a transform $y \rhd z$ such that $y \subseteq x$. Putting these together we may interpret the pair $(x, (y \rhd z))$ as a *process* which transforms the part y of x into z yielding thus outcome $(x - y) \uplus z$. This can be generalized to a finite multiset $[x_1, \ldots, x_n]$ of initial resources and a finite multiset $[y_1 \rhd z_1, \ldots, y_k \rhd z_k]$ of transforms. However, since the resources of $[x_1, \ldots, x_n]$ can be represented by those of the multiset $x = x_1 \uplus \cdots \uplus x_n$, we can always consider the initial resources as consisting of a single multiset.

Lemma 3.5 *(a) Let* $y_1 \in U(x_1)$ *and* $y_2 \in U(x_2)$*. Then*

$$supp_{x_1 \uplus x_2}(y_1 \uplus y_2) = supp_{x_1}(y_1) \uplus supp_{x_2}(y_2).$$

(b) If $x_1 \rhd y_1, \ldots, x_n \rhd y_n$*, then* $(x_1 \uplus \cdots \uplus x_n) \rhd (y_1 \uplus \cdots \uplus y_n)$*.*

Proof. (a) follows immediately from 3.2(a),(b).

(b) It suffices to see it for $n = 2$. Let $x_1 \rhd y_1$ and $x_2 \rhd y_2$. Then $y_1 \in U(x_1)$, $y_2 \in U(x_2)$, $supp_{x_1}(y_1) = x_1$ and $supp_{x_2}(y_2) = x_2$. Then, clearly $(y_1 \uplus y_2) \in U(x_1 \uplus x_2)$ and, by (a),

$$supp_{x_1 \uplus x_2}(y_1 \uplus y_2) = supp_{x_1}(y_1) \uplus supp_{x_2}(y_2) = x_1 \uplus x_2.$$

Hence $(x_1 \uplus x_2) \rhd (y_1 \uplus y_2)$. $\square$

Definition 3.6 A *process* is a tuple

$$P = (x_1, \ldots, x_n, \sigma_1, \ldots, \sigma_m),$$

where $x_i \in U(A)$ and σ_j are finite multisets of pairs (y, z), with $y, z \in U(A)$, such that $y \rhd z$. Juxtaposition of multisets within a process is assumed to be equivalent to their additive union $\uplus$, so the above process is written equivalently as

$$P = (x_1 \uplus \cdots \uplus x_n, \sigma_1 \uplus \cdots \uplus \sigma_m).$$

Therefore every process can be written in the form $P = (x, \sigma)$. We write also

$$\sigma = [y_1 \rhd z_1, y_2 \rhd z_2, \ldots]$$

instead of $\sigma = [(y_1, z_1), (y_2, z_2), \ldots]$ (although, strictly speaking, $y_i \rhd z_i$ are not objects of the universe $U(A)$).

Definition 3.7 A process $P = (x, \sigma)$ is said to be *staged* if there is an enumeration of σ, $\sigma = [y_1 \rhd z_1, \ldots, y_n \rhd z_n]$, such that for every $i < n$,

$$y_{i+1} \subseteq x \uplus z_1 \uplus \cdots \uplus z_i - y_1 \uplus \cdots \uplus y_i.$$

Putting

$$P(0) = x \quad \text{and} \quad P(i+1) = P(i) \uplus z_{i+1} - y_{i+1},$$

for $i < n$, the above condition is written also

$$y_{i+1} \subseteq P(i+1).$$

The sequence $P(0), \ldots, P(n)$ is called a *stage sequence of* P. We say that P *yields* w and write

$$P \vdash w \quad \text{or} \quad x, \sigma \vdash w,$$

if for some stage sequence $P(i), i \leq n$, $P(n) = w$. In this case $P(n)$ is called *the output of* P and we denote it $out(P)$, i.e.,

$$out(P) = P(n) = w.$$

Also given $P = (x, \sigma)$ and a transform $u \rhd w$, we write

$$P \vdash (u \rhd w) \quad \text{or} \quad x, \sigma \vdash (u \rhd w),$$

if $(x \uplus u, \sigma)$ is a staged process and

$$x \uplus u, \sigma \vdash w.$$

If P is staged, the stage sequence $P(i), i \leq n$, need not be unique. However, the output $out(P)$ is independent of the particular stage sequence. To see this, let us put for a multiset σ of transforms,

$$in[\sigma] = [y : (y \rhd z) \in \sigma],$$

$$out[\sigma] = [z : (y \rhd z) \in \sigma],$$

and

$$in(\sigma) = \uplus in[\sigma], \quad \text{and} \quad out(\sigma) = \uplus out[\sigma].$$

The *input* of $P = (x, \sigma)$ is the multiset

$$in(P) = x \uplus out(\sigma).$$

Lemma 3.8 *(a) Let $P = (x, \sigma)$ be a staged process. Then for every stage sequence $P(i), i \leq n$,*

$$out(P) = in(P) - in(\sigma) = x \uplus out(\sigma) - in(\sigma).$$

(b) If $x, \sigma \vdash w$, then $supp_x(w) = x$.

Proof. (a) By the definition of $P(n)$,

$$out(P) = P(n) = x \uplus z_1 \uplus \cdots \uplus z_n - y_1 \uplus \cdots \uplus y_n = x \uplus out(\sigma) - in(\sigma).$$

(b) By (a), $x, \sigma \vdash w$ implies $w = x \uplus out(\sigma) - in(\sigma)$. Hence

$$supp_x(w) = supp_x(x) \uplus supp_x(out(\sigma)) - supp_x(in(\sigma)).$$

But for every $(y \rhd z) \in \sigma$, $supp_y(z) = y$, and by lemma 3.2(c),

$$supp_x(z) = supp_x(supp_y(z)) = supp_x(y),$$

hence $supp_x(out(\sigma)) = supp_x(in(\sigma))$. Since $supp_x(x) = x$, the first equation yields $supp_x(w) = x$. □

The expressions $P \vdash w$, $P \vdash (u \rhd w)$ are called *sequents* and are denoted by s, s_1, s_2 etc. We say that the sequent $s = (P \vdash w)$ is *true* if P is staged and yields w. Finally an expression of the form

$$\frac{s'}{s}, \quad or \quad \frac{s_1 \quad s_2}{s}$$

is said to be a *rule*. The rules $\frac{s'}{s}$, $\frac{s_1 \quad s_2}{s}$ are *true*, if the truth of s' (resp. s_1, s_2) implies the truth of s.

Theorem 3.9 *The following rules hold in $U(A)$:*

$$Ax : \frac{}{x \vdash x}, \qquad Cut_{\uplus} : \frac{x_1, \sigma_1 \vdash w \quad w, x_2, \sigma_2 \vdash u}{x_1, x_2, \sigma_1, \sigma_2 \vdash u},$$

$$L_{\uplus} : \frac{x, y, z, \sigma \vdash w}{x, y \uplus z, \sigma \vdash w}, \qquad R_{\uplus} : \frac{x_1, \sigma_1 \vdash w \quad x_2, \sigma_2 \vdash u}{x_1, x_2, \sigma_1, \sigma_2 \vdash w \uplus u},$$

$$L_{\rhd} : \frac{x_1, \sigma_1 \vdash w \quad u, x_2, \sigma_2 \vdash v \quad w \rhd u}{x_1, x_2, \sigma_1, \sigma_2, (w \rhd u) \vdash v}, \qquad R_{\rhd} : \frac{x, y, \sigma \vdash w \quad y \rhd w}{x, \sigma \vdash (y \rhd w)}.$$

(In the rules $L_{\rhd}$ and $R_{\rhd}$ the additional requirements $w \rhd u$ and $y \rhd w$ mean that the latter are true transforms.)

Proof. Ax: Here $\sigma = \emptyset$, and the process $(x, \emptyset)$ is trivially staged with $out(P) = P(0) = x$. Hence $x \vdash x$ holds.

$Cut_{\uplus}$: Suppose $x_1, \sigma_1 \vdash w$ and $w, x_2, \sigma_2 \vdash u$ are staged and hold in $U(A)$. Let

$$\sigma_1 = [y_1 \rhd z_1, \ldots, y_n \rhd z_n], \quad \sigma_2 = [s_1 \rhd t_1, \ldots, s_m \rhd t_m]$$

be appropriate enumerations of σ_1, σ_2 producing the stage sequences $P_1(i), i \leq n$, and $P_2(j), j \leq m$ be for P_1, P_2 respectively, such that $P_1(0) = x_1$, $P_1(n) = w$, $P_2(0) = x_2 \uplus w$ and $P_2(m) = u$. Consider the sequence $P(k), k \leq n + m$, defined as follows:

$P(0) = x_1 \uplus x_2$,

....

$P(n) = P_1(n) \uplus x_2 = w \uplus x_2 \uplus u = P_2(0)$,

$P(n+1) = P_2(1)$,

....

$P(n+m) = P_2(m) = u$.

Clearly, $P(k), k \leq n + m$, is a stage sequence for the process $x_1, x_2, \sigma_1, \sigma_2$ with output u.

$L_{\uplus}$: This follows immediately by the convention that juxtaposition of multisets is equivalent to their union.

$R_{\uplus}$: Similar to the verification of the cut rule.

$L_{\rhd}$: Let $P_1(i), i \leq n$, and $P_2(j), j \leq m$, be stage sequences for (x_1, σ_1) and (u, x_2, σ_2) respectively, with $P_1(0) = x_1$, $P_1(n) = w$, $P_2(0) = u \uplus x_2$, $P_2(m) = v$. Define the sequence $P(k), k \leq n + m + 1$, such that:

$P(0) = x_1 \uplus x_2$,

....

$P(n) = P_1(n) \uplus x_2 = w \uplus x_2$,

$P(n+1) = P(n) \uplus u - w = x_2 \uplus u = P_2(0)$,

$P(n+2) = P_2(1)$,

....

$P(n+m+1) = P_2(m) = v$.

Then $P(k), k \leq n + m + 1$ is a stage sequence for $(x_1, x_2, \sigma_1, \sigma_2, w \rhd u)$, with output v.

$R_{\rhd}$: Immediate by the the definition of $P \vdash (y \rhd w)$. □

4 The Horn fragment of Linear Logic

We assume the reader's familiarity with the fundamentals of Linear Logic (LL) and Intuitionistic Linear Logic (ILL) (see e.g. [3] or [8]). In particular we are dealing here with the *Horn fragment* of ILL, first studied in [4] and [5]. The language of the fragment consists of atomic formulas $p_1, p_2, \ldots$ and the connectives $\otimes$ (multiplicative conjunction) and $\multimap$ (linear implication). Following the terminology of [4] we call *simple products* formulas of the form $p_1 \otimes \cdots \otimes p_n$ and we denote them by the letters X, Y, Z, W, U possibly with subscripts. If $X_1, \ldots, X_m$ are simple products, clearly so is $X_1 \otimes \cdots \otimes X_m$. We write also $nX = \underbrace{X \otimes \cdots \otimes X}_{n}$. A *simple implication* is a formula of the form $X \multimap Y$, where X, Y are simple products. The only formulas used in the Horn fragment will be simple products and simple implications, so we can drop the adjective "simple" from now on. The letter Σ range over multisets of implications.

A *Horn sequent* of ILL is an expression of the form

$$X_1, \ldots, X_n, \Sigma \vdash Y, \quad \text{or} \quad X_1, \ldots, X_n, \Sigma \vdash (Y \multimap Z).$$

The letters S, S_1, S_2 range over Horn sequents.

Recall the following rules of the $\{\otimes, \multimap\}$-fragment of ILL (adapted for Horn sequents).

$$Ax : \frac{}{X \vdash X}, \qquad Cut : \frac{X_1, \Sigma_1 \vdash W \quad W, X_2, \Sigma_2 \vdash U}{X_1, X_2, \Sigma_1, \Sigma_2 \vdash U},$$

$$L_\otimes : \frac{X, Y, Z, \Sigma \vdash W}{X, Y \otimes Z, \Sigma \vdash W}, \qquad R_\otimes : \frac{X_1, \Sigma_1 \vdash W \quad X_2, \Sigma_2 \vdash U}{X_1, X_2, \Sigma_1, \Sigma_2 \vdash W \otimes U},$$

$$L_\multimap : \frac{X_1, \Sigma_1 \vdash W \quad U, X_2, \Sigma_2 \vdash V}{X_1, X_2, \Sigma_1, \Sigma_2, (W \multimap U) \vdash V}, \qquad R_\multimap : \frac{X, Y, \Sigma \vdash W}{X, \Sigma \vdash (Y \multimap W)}.$$

By the *Horn fragment* of ILL, or HF for short, we mean the set of Horn sequents provable by the above rules.

An *interpretation* of HF in $(U(A), \uplus, \rhd, \vdash)$, or just $(U(A), \vdash)$, is any mapping $^* : \{p_1, p_2, \ldots\} \to U(A)$ which extends to products, implications and Horn sequents as follows:

(a) If $X = p_1 \otimes \cdots \otimes p_n$, then $X^* = [p_1^*, \ldots, p_n^*]$.

(b) If $X = X_1 \otimes \cdots \otimes X_m$, then $X^* = X_1^* \uplus \cdots \uplus X_m^*$.

(c) $(X \multimap Y)^* = (X^* \rhd Y^*)$.

(d) If $\Sigma = [X_1 \multimap Y_1, \ldots, X_n \multimap Y_n]$, then $\Sigma^* = [(X_1 \multimap Y_1)^*, \ldots, (X_n \multimap Y_n)^*]$.

(e) If $S = (X, \Sigma \vdash Y)$ is a Horn sequent, then $S^* = (X^*, \Sigma^* \vdash Y^*)$.

Clearly, X^* are multisets $x \in U(A)$. However, since for an implication $X \multimap Y$ and an arbitrary *, $X^* \rhd Y^*$ need not be a *true* transform, it is necessary, for a given sequent $X, \Sigma \vdash W$, to restrict * so that all implications of Σ are mapped to true transforms. Also the interpetation of some rules require some extra restrictions on *, in order for the succedent S, S^* have a genuine process. For instance, the interpretation of the rules $L_{\multimap}$ and $R_{\multimap}$ requires that * is such that $W^* \rhd U^*$ and $Y^* \rhd W^*$ be also true transforms. Thus we give the following:

Definition 4.1 Let $S = (X, \Sigma \vdash W)$ be a sequent. An *interpretation of* S is any mapping * such that for all $(Y \multimap Z) \in \Sigma$, $(Y^* \rhd Z^*)$ are true transforms.

Given slso a rule R, an *interpretation of* R is any mapping * which turns all implications occurring in R into true transforms.

Lemma 4.2 *For every sequent $S = (X, \Sigma \vdash W)$ provable in* HF *and for every interpretation *, (X^*, Σ^*) is a staged process.*

Proof. By induction on the steps of the proof of S. It suffices to observe that whenever a rule R of HF is applied and the sequent(s) over the line are have staged processes, then so does the sequent under the line. The details are left to the reader. □

Theorem 4.3 (Soundness) *Given any set of urelements A, the structure $(U(A), \vdash)$ is a model for* HF*, i.e., for every sequent S provable in* HF*, and for any interpretation S^* of S, S^* holds in $(U(A), \vdash)$.*

Proof. Clearly, if R is a rule of HF, each interpretation R^* is one of the rules of 3.9, e.g., Cut^* is $Cut_{\uplus}$, $(L_{\otimes})^*$ is $L_{\uplus}$, $(L_{\multimap})^*$ is $L_{\rhd}$ etc., therefore, by 3.9 all these rules hold in $(U(A), \vdash)$. Now if S is a Horn sequent provable in HF, it is easy to see that S^* holds in $(U(A), \vdash)$ by an easy induction on the number of steps used in the proof of S. □

Lemma 4.4 *Let X be a product, Σ be a multiset of implications, and * be an interpretation. If (X^*, Σ^*) is a staged process, then there is a product W such that $X^*, \Sigma^* \vdash W^*$.*

Proof. By induction on $|\Sigma| = n$. For $|\Sigma| = 0$ the claim is obvious. Suppose it holds for $|\Sigma| < n$ and let $|\Sigma| = n$ and (X^*, Σ^*) be staged process, with a stage sequence $P(i), i \leq n$, produced by an enumeration of $\Sigma = [Y_1 \multimap Z_1, \ldots, Y_n \multimap Z_n]$. Then the process $(X^*, \Sigma^* - [Y_n^* \rhd Z_n^*])$ is also staged hence, by the induction hypothesis, there is a product U such that $P(n-1) = U^*$. Then $P(n) = P(n-1) \uplus Z_n^* - Y_n^* = U^* \uplus Z_n^* - Y_n^* = (U \otimes Z_n - Y_n)^*$, where $U \otimes Z_n - Y_n$ is the product whose literals are those of U plus those of Z_n minus those of Y_n. Putting $W = U \otimes Z_n - Y_n$, we are done. □

Theorem 4.5 (Completeness) *Let S be a Horn sequent such that S^* holds in $(U(A), \vdash)$ for every interpretation *. Then S is provable in* HF*.*

Proof. Let $S = (X, \Sigma \vdash W)$. By induction on the cardinality $|\Sigma| = n$ of Σ, i.e., the number of implications used in the antecedent of S.

(a) Let $|\Sigma| = 0$, i.e., $\Sigma = \emptyset$. Then $S = (X \vdash W)$ and $S^* = (X^* \vdash W^*)$ holds in $U(A)$ for every *. By definition 3.7, $X^* = W^*$ for every *. It follows that $X = W$, otherwise, clearly, we could find an interpretation * such that $X^* \neq W^*$. Hence $X \vdash W$ is provable.

(b) Suppose the claim holds for all $S = (X, \Sigma \vdash W)$ such that $|\Sigma| < n$, and let $S = (X, \Sigma \vdash W)$ be such that $|\Sigma| = n$ and S^* holds in $(U(A), \vdash)$. By definition 3.7, the process (X^*, Σ^*) is staged, i.e., there is an enumeration of Σ,

$$\Sigma = [Y_1^* \rhd Z_1^*, \ldots, Y_n^* \rhd Z_n^*]$$

and a stage sequence $P(i), i \leq n$, where

$$P(0) = X^* \textit{ and } P(i+1) = P(i) \uplus Z_i^* - Y_i^*,$$

Also $P(n) = W^*$ is the output of P. Let $P' = (X^*, \Sigma^* - [Y_n^* \rhd Z_n^*])$. Clearly P' is a staged process with stage sequence $P(i), i \leq n-1$. By lemma 4.4, there is a product U such that $P(n-1) = U^*$. Now by the induction hypothesis,

$$X, \Sigma - [Y_n \multimap Z_n] \vdash U \quad \textit{and} \quad U, (Y_n \multimap Z_n) \vdash W.$$

Using the cut rule of HF we get $X, \Sigma \vdash W$. □

5 Coherent processes.

We saw in section 3 that if $P = (x, \sigma)$ is a staged process, then

$$in(\sigma) \subseteq x \uplus out(\sigma) \text{ and } out(P) = x \uplus out(\sigma) - in(\sigma),$$

in which case we write $P \vdash out(P)$. Can these last relations be used as alternative definitions of the staged sequence and the yielding relation $\vdash$? The answer is No. However they provide *weaker* notions of process and yielding.

Definition 5.1 A process $P = (x, \sigma)$ is said to be *coherent* if

$$in(\sigma) \subseteq x \uplus out(\sigma).$$

In this case we set $out(P) = x \uplus out(\sigma) - in(\sigma)$ and say that P *weakly yields* $out(P)$. We denote this by

$$P \mid\!\sim out(P).$$

Also for a process (x, σ) *and a transform* $y \rhd w$*, we write*

$$x, \sigma \mid\!\sim (y \rhd w),$$

if $(x \uplus y, \sigma)$ *is coherent and* $x \uplus y, \sigma \mid\!\sim w$.

Expressions of the form $P \mid\!\sim w$ or $P \mid\!\sim (u \rhd w)$ are called again *sequents*.

It is easy to see that the multiset σ of transforms in a coherent process $P = (x, \sigma)$ can always be a singleton.

Lemma 5.2 *Let* $P = (x, \sigma)$ *be a coherent process, and* $\sigma = [y_1 \rhd z_1, \ldots, y_n \rhd z_n]$. *If*

$$y = y_1 \uplus \cdots \uplus y_n \text{ and } z = z_1 \uplus \cdots \uplus z_n,$$

then $y \rhd z$ *is a transform and* $x, \sigma \mid\!\sim w$ *iff* $x, (y \rhd z) \mid\!\sim w$.

Proof. That $y \rhd z$ is a transform follows from lemma 3.5(b). On the other hand, since $y = in(\sigma)$ and $z = out(\sigma)$,

$$w = x \uplus out(\sigma) - in(\sigma) = x \uplus z - y. \quad \square$$

Interpretations of HF in $(U(A), \mid\!\sim)$ are defined exactly as before, except that we now replace $\vdash$ by $\mid\!\sim$.

Theorem 5.3 (Soundness) *The rules Ax, Cut, $L_{\uplus}$, $R_{\uplus}$, $L_{\triangleright}$, $R_{\triangleright}$ hold in* $(U(A), \mid\!\sim)$.

Proof. *Ax* is obvious.

Cut: Let $x_1, \sigma_1 \mid\!\sim w$ and $w, x_2, \sigma_2 \mid\!\sim u$ hold. Then, by definition 5.1,

$$w = x_1 \uplus out(\sigma_1) - in(\sigma_1),$$

and

$$u = w \uplus x_2 \uplus out(\sigma_2) - in(\sigma_2).$$

Substituting w from the first equation in the second, we get

$$u = x_1 \uplus x_2 \uplus out(\sigma_1) \uplus out(\sigma_2) - in(\sigma_1) \uplus in(\sigma_2).$$

The last equation says precisely that $x_1, x_2, \sigma_1, \sigma_2 \mid\!\sim u$ holds true.

$L_{\uplus}$: This follows immediately by the convention that juxtaposition means union.

$R_{\uplus}$: Let $x_1, \sigma_1 \mid\!\sim w$ and $x_2, \sigma_2 \mid\!\sim u$ hold. Then

$$x_1 \uplus out(\sigma_1) - in(\sigma_1) = w$$

and

$$x_2 \uplus out(\sigma_2) - in(\sigma_2) = u.$$

Adding the corresponding sides of these equations we get

$$x_1 \uplus x_2 \uplus out(\sigma_1) \uplus out(\sigma_2) - in(\sigma_1) \uplus in(\sigma_2) = w \uplus u,$$

which means that $x_1, x_2, \sigma_1, \sigma_2 \mid\!\sim w \uplus u$ holds.

$L_{\triangleright}$: Let $x_1, \sigma_1 \mid\!\sim w$ and $u, x_2, \sigma_2 \mid\!\sim v$ hold, i.e.,

$$w = x_1 \uplus out(\sigma_1) - in(\sigma_1) \tag{1}$$

and

$$v = u \uplus x_2 \uplus out(\sigma_2) - in(\sigma_2). \tag{2}$$

Let P be the process of the sequence under the line. Then clearly,

$$in(P) = x_1 \uplus x_2 \uplus u \uplus out(\sigma_1) \uplus out(\sigma_2)$$

and

$$out(P) = in(P) - in(\sigma_1) \uplus in(\sigma_2) \uplus w.$$

Thus is suffices to prove that

$$x_1 \uplus x_2 \uplus u \uplus out(\sigma_1) \uplus out(\sigma_2) - in(\sigma_1) \uplus in(\sigma_2) \uplus w = v.$$

Now by adding the corresponding members of (1) and (2)we get

$$v \uplus w = x_1 \uplus x_2 \uplus u \uplus out(\sigma_1) \uplus out(\sigma_2) - in(\sigma_1) \uplus in(\sigma_2),$$

whence

$$v = x_1 \uplus x_2 \uplus u \uplus out(\sigma_1) \uplus out(\sigma_2) - in(\sigma_1) \uplus in(\sigma_2) \uplus w.$$

$R_{\triangleright}$: It follows by the definition of $P \mid\!\sim y \triangleright w$. □

However completeness of HF fails with respect to interpretations in $(U(A), \mid\!\sim)$.

Theorem 5.4 *There are sequents $X, \Sigma \vdash W$ unprovable in* HF *but such that $X^*, \Sigma^* \mid\!\sim W^*$ hold in $(U(A), \mid\!\sim)$.*

Proof. Consider the sequent

$$S = (X, (X \otimes U) \multimap (Y \otimes U) \vdash Y).$$

Since there is no rule of allowing the elimination of $\otimes$ from the antecedent of a sequent, S is unprovable (in fact it is unprovable in the full ILL). On the other hand its interpretation

$$S^* = (x, (x \uplus u) \triangleright (y \uplus u) \mid\!\sim y)$$

holds since the process $P = (x, (x \uplus u) \triangleright (y \uplus u))$ is coherent and $out(P) = x \uplus y \uplus u - x \uplus u = y$. □

Since every coherent process P can be of the form $(x, y \triangleright z)$, with $y \subseteq x \uplus z$, the situation is fairly simple. If, in particular, $y \triangleright x$, then P is staged. Since $y \subseteq x \uplus z$, for every u, $y(u) \leq x(u) + z(u)$. Suppose P is not staged.

Then there is a u such that $y(u) > x(u)$, and if $y(u) - x(u) = k$, then $x(u) + k \leq x(u) + z(u)$, hence $z(u) \geq k$. If u is an *atom*, then, clearly, $z(u) = k$, since the elements of z are produced by simpler elements of y through $y \rhd z$, and u has no simple constituents. Therefore all atoms of $y - (y \cap x)$ pass unchanged to z and do not affect the output w. If, however, u is not an atom, then it may be both absorbed into more complex objects, as well as be constructed by simpler elements along the same process, leading thus to circular phenomena like the one of the following example.

Example 5.1 Let $P = (x, \sigma)$ be the process with $x = [a, b, c, d, e, f]$, and σ consisting of the transforms

$$y_1 = [[a, b], c, d, e] \ \rhd \ z_1 = [[[a, b], e], [c, d]]$$

and

$$y_2 = [[c, d], a, b, f] \ \rhd \ z_2 = [[[c, d], f], [a, b]].$$

Clearly P is coherent with $out(P) = [[[a, b], e], [[c, d], f]]$. However, $t_1 = [[a, b]]$, $t_2 = [[c, d]]$ and $t_1 \not\subseteq z_1$, $t_2 \not\subseteq z_2$.

Instead of absolute atoms, we can refer to *minimal* elements with respect to a specific process P. Namely, given $P = (x, y \rhd z)$, an element u of $x \uplus z$ is *minimal* with respect to P, if there is no $v \subseteq y$ such that $v \rhd [u]$. If u is not minimal, we denote $supp(u)$ the multiset of minimal elements forming u. The following gives some information on the behavior of coherent processes.

Lemma 5.5 *Let $u \in in(P)$ such that $x(u) = 0$. Then either u is minimal and $y(u) = z(u)$, or $supp(u) \subseteq x$.*

Proof. Suppose $x(u) = 0$ and u is minimal. By the discussion above, $y(u) \leq 0 + z(u) = z(u)$, hence $y(u) = z(u)$. Suppose now u is not minimal. For simplicity, assume $u = [a, b]$, where a, b are minimal, and $z(u) = 1$. Without loss of generality we may assume that there are no other objects in z having constituents a, b, in particular $z(a) = z(b) = 0$. Then, since $[a, b]$ is not minimal, $[a, b] \subseteq y$, i.e. $y(a) = y(b) = 1$ and there is no other object in y (except a, b) having constituents a, b. In particular, $y([a, b]) = 0$. Suppose $[a, b] \not\subseteq x$. This means $x(a) = 0$ or $x(b) = 0$. Assume the former. Thus $x(a) = 0$ and $y(a) = 1$. But then, from the fact that $y(a) \leq z(a)$, we get that $z(a) = 1$, which is a contradiction. Similarly if we assume that $x(b) = 0$. □

We have seen that the logic of staged processes is the subsystem HF of LL. What is the logic of coherent processes? It is HF augmented with the following $\otimes$-*cancellation rule*:

$$C_\otimes : \frac{X \otimes Z, \Sigma \vdash Y \otimes Z}{X, \Sigma, \vdash Y}.$$

(Note that $C_\otimes$ is *not* a rule of LL). Let us put

$$\text{CHF} = \text{HF} + C_\otimes.$$

Lemma 5.6 *If* $x, \sigma, (y \rhd z) \mid\!\sim w$, *then* $x \uplus z, \sigma \mid\!\sim w \uplus y$.

Proof. By definition $x, \sigma, (y \rhd z) \mid\!\sim w$ iff

$$x \uplus z \uplus out(\sigma) - y \uplus in(\sigma) = w,$$

or, equivalently,

$$x \uplus z \uplus out(\sigma) - in(\sigma) = w \uplus y,$$

which says that $x \uplus z, \sigma \mid\!\sim w \uplus y$. □

Theorem 5.7 (Soundness and Completeness) CHF *is sound and complete with respect to* $(U(A), \mid\!\sim)$.

Proof. We have seen in theorem 5.3 that the rules of HF hold in $(U(A), \mid\!\sim)$. The interpretation of $C_\otimes$ is, clearly, the $\uplus$-cancellation rule

$$C_\uplus : \frac{x \uplus z, \sigma \mid\!\sim y \uplus z}{x, \sigma \mid\!\sim y,}$$

which is easy to verify. Therefore soundness holds.

To prove completeness, let $X, \Sigma \vdash W$ be a sequent, such that $X^*, \Sigma^* \mid\!\sim W^*$ holds for every *. We have to show that $X, \Sigma \vdash W$ is provable in the CHF. By induction on $|\Sigma|$. Suppose the claim holds for $|\Sigma| < n$ and let $|\Sigma| = n$, and $X^*, \Sigma^* \mid\!\sim W^*$. Let $(Y \multimap Z) \in \Sigma$, and let $\Sigma_1 = \Sigma - [(Y \multimap Z)]$. Then

$$X^*, \Sigma_1^*, Y^* \rhd Z^* \mid\!\sim W^*.$$

By lemma 5.6,

$$X^* \uplus Z^*, \Sigma_1^* \mid\!\sim W^* \uplus Y^*.$$

By the induction hypothesis (since $|\Sigma_1| < n$),

$$X \otimes Z, \Sigma_1 \vdash W \otimes Y.$$

The last sequent combined with $W \otimes Y, (Y -\circ Z) \vdash W \otimes Z$ and the cut rule yields

$$X \otimes Z, \Sigma_1, (Y -\circ Z) \vdash W \otimes Z,$$

or

$$X \otimes Z, \Sigma \vdash W \otimes Z.$$

Now by the last sequent and the rule $C_\otimes$, we get $X, \Sigma \vdash W$, and completeness is proved. □

6 Asymptotic behavior of processes. Storage.

Recall that given a multiset x and $n \in N$, nx denotes the union of n copies of x. We introduce now the operator ! and for every x the *formal* entity $!x$. Intuitively, $!x$ denotes the union of an *indefinite* number of copies of x. We call $!x$ a *generalized multiset*, or a !-*multiset*. In fact $!x$'s are abbreviations of "limit" objects, whose behavior is defined in terms of their standard approximations. Their meaning will become clear by definition 6.4 below. Thus $!x$'s do not extend properly the domain $U(A)$. However, for the clarity of exposition, we add these fictitious objects to those of $U(A)$, extending the latter to the universe $U^!(A)$.

Definition 6.1 $U^!(A)$ is the smallest class such that:
(a) $U(A) \subseteq U^!(A)$,
(b) $x \in U^!(A) \;\Rightarrow\; !x \in U^!(A)$, and
(c) $x, y \in U^!(A) \;\Rightarrow\; x \uplus y \in U^!(A)$.
A !-*transform* is an expression $y \rhd z$, with $y, z \in U^!(A)$, or $!(y \rhd z)$. σ ranges over multisets of !-transforms. For any σ, let $!\sigma = [!t : t \in \sigma]$. Therefore if σ is a set of !-transforms, so is $!\sigma$. A !-*process* is a pair (x, σ) with x, σ as before and a !-*sequent* an expression of the form $P \vdash w$.

The !-multisets x, and the !-transforms $y \rhd z$ are going to be approximated by ordinary multisets and ordinary transforms.

Definition 6.2 Let E be a string of !-multisets and/or !-transforms, and let $(1,\ldots,m)$ be an enumeration of all occurrences of ! inside E. Then for every m-tuple of integers $\vec{k}=(k_1,\ldots,k_m)$, $!_{\vec{k}}E$ denotes the string resulting from E, if we replace the i-th occurrence $!x$ or $!(y \rhd z))$ in E by $k_i x$ and $k_i(y \rhd z)$, respectively. $!_{\vec{k}}E$ is said to be the *$\vec{k}$-approximation of E.*

It is clear that the $\vec{k}$-approximation of E is a string of ordinary multisets and transforms with the only exception that it may contain expressions of the form $k_i(y \rhd z)$, with y,z being ordinary multisets (coming from the approximation of objects $!(y \rhd z)$), whose meaning has not yet been fixed. Now for simple multisets y,z, $n(y \rhd z)$ will be identical to the multiset of transforms $n[y \rhd z]$. In order, however, for the latter to be treated as a single transform we shall identify it with $ny \rhd nz$.

Definition 6.3 For any simple multisets and any n we set $n(y \rhd z) := (ny \rhd nz)$.

Example 6.1. (a) Let E be the process

$$!(!x_1 \uplus x_2 \uplus !x_3), ((!x_1 \uplus !x_4) \rhd !(!x_3)),$$

where x_1,x_2,x_3,x_4 are simple multisets, and let $\vec{k}=(2,1,3,0,1,2,4)$. Then $!_{\vec{k}}E$ is the string

$$2(x_1 \uplus x_2 \uplus 3x_3), ((0x_1 \uplus x_4) \rhd 2(4x_3)),$$

which, after the computations, becomes

$$2x_1 \uplus 2x_2 \uplus 6x_3, (x_4 \rhd 8x_3).$$

(b) Let $t = !((!y_1 \uplus !y_2) \rhd !(z_1 \uplus !z_2))$ and let $\vec{k}=(3,1,4,2,3)$. Then

$$!_{\vec{k}}t = 3[(y_1 \uplus 4y_2) \rhd 2(z_1 \uplus 3z_2)] = 3[(y_1 \uplus 4y_2) \rhd (2z_1 \uplus 6z_2)].$$

Before defining the truth of !-sequents, recall that although for all formulas X,Y, $X \multimap Y$ makes sense, its interpretation $X^* \rhd Y^*$ makes sense only when $(X^*,Y^*) \in \rhd$. This is why before defining $P \vdash w$ we first defined the notion $x \rhd y$. Similarly, in order to define the truth of a !-sequent $P \vdash w$, we must first define what $x \rhd y$ means when x, y are !-multisets. If, for instance, x, y, z are simple multisets, then the sequent

$$!x \uplus !y, ((!x \uplus !y) \rhd !z) \vdash !z,$$

being of the form $u, (u \rhd w) \vdash w$, should be true. Intuitively this means that given any number of copies of z, say k, we can control the resources $!x$ and the resources of the transform in to order to produce kz. That is, there must be m, n, p, q, s such that

$$mx \uplus ny, ((px \uplus qy) \rhd sz) \vdash kz. \tag{3}$$

The last is an ordinary sequent and its truth implies that $k = s$, hence

$$(\forall k)(\exists p, q)(((px \uplus qy) \rhd kz) \text{ is a true transform}).$$

If this is the case then (and only then), obviously, we can find p, q and $m = p$, $n = q$, such that (3) holds. This leads to the folowing definition.

Definition 6.4 (a) Let y, z be !-multisets. We say that $y \rhd z$ is *true* if

$$(\forall \vec{n})(\exists \vec{m})((!_{\vec{m}} y) \rhd (!_{\vec{n}} z)) \text{ is true}).$$

(b) Let $x, \sigma \vdash w$ be a !-sequent. We say that $x, \sigma \vdash w$ is *true* if
(i) every $(y \rhd z) \in \sigma$ is true, and
(ii) $(\forall \vec{n})(\exists \vec{m}, \vec{l})(!_{\vec{m}} x, !_{\vec{l}} \sigma \vdash !_{\vec{n}} w)$.
Similarly we write $P \mid\!\sim w$ if

$$(\forall \vec{n})(\exists \vec{m})(!_{\vec{m}} P \mid\!\sim !_{\vec{n}} w).$$

Also $P \vdash (u \rhd w)$ if $P, u \vdash w$. The expressions $P \vdash w$, $P \vdash (u \rhd w)$ are called !-*staged sequents* while $P \mid\!\sim w$ and $P \mid\!\sim (u \rhd w)$ are called !-*coherent sequents.* (Note that we do not allow sequents of the form $P \vdash !(u \rhd w)$ or $P \mid\!\sim !(u \rhd w)$.)

We extend now HF by adding the operator !. !-*Horn formulas* are defined in the obvious way, that is, instead of simple products we have now !-*products* defined inductively as follows:

(a) Every simple product is a !-product, and

(b) if X, Y are !-products then so are $!X$ and $X \otimes Y$.

Also if X, Y are !-products, then $X -\circ Y$ is a !-*implication*. Σ ranges over multisets of !-implications. Below the letters V, U, W range over either !- products or !-implications.

A !-*process* is a pair $P = (X, \Sigma)$. For any process $P = (X, \Sigma)$, let

$$!P = (!X, !\Sigma),$$

where $!\Sigma = [!U : U \in \Sigma]$.

Let !-HF be the system consisting of the rules of HF augmented with the following rules for !:

$$W : \frac{P \vdash W}{P, !V \vdash W} \ (weakening) \qquad C : \frac{P, !V, !V \vdash W}{P, !V \vdash W} \ (contraction)$$

$$D : \frac{P, V \vdash W}{P, !V \vdash W} \ (dereliction) \qquad S : \frac{!P \vdash W}{!P \vdash !W} \ (storage).$$

The *-interpretation of Horn formulas by multisets defined in section 5 can be extended over !-Horn formulas into $(U^!(A), \vdash)$ or $(U^!(A), \mid\!\sim)$ in the obvious way, namely $(!V)^* = !(V^*)$, V being a product or an implication. We first prove the following.

Theorem 6.5 (Weak Soundness) *All rules of* !-HF *except contraction hold in* $(U^!(A), \vdash)$*, as well as in* $(U^!(A), \mid\!\sim)$*.*

Proof. We work with $\vdash$ of definition 6.4, the case of $\mid\!\sim$ being similar. Throughout * is an arbitrary interpretation of Horn formulas into multisets.

Cut: Suppose $X_1^*, \Sigma_1^* \vdash W^*$ and $W^*, X_2^*, \Sigma_2^* \vdash U^*$ hold. Let $\vec{k}$ be a tuple assigned to the occurrences of ! in U^*. Then by the second of the above assumptions, there are $\vec{p}, \vec{q}, \vec{r}$ such that

$$!_{\vec{p}} W^*, !_{\vec{q}} X_2^*, !_{\vec{r}} \Sigma_2^* \vdash !_{\vec{k}} U^*.$$

Also by the first assumption there are $\vec{m}, \vec{n}$ such that

$$!_{\vec{m}} X_1^*, !_{\vec{n}} \Sigma_1^* \vdash !_{\vec{p}} W^*.$$

By $Cut_{\uplus}$ for simple sequents, the last two sequents imply that

$$!_{\vec{m}}X_1^*, !_{\vec{q}}X_2^*, !_{\vec{n}}\Sigma_1^*, !_{\vec{r}}\Sigma_2^* \vdash !_{\vec{k}}U^*.$$

Therefore

$$(\forall\vec{k})(\exists\vec{m}, \vec{q}, \vec{n}, \vec{n}, \vec{r})(!_{\vec{m}}X_1^*, !_{\vec{q}}X_2^*, !_{\vec{n}}\Sigma_1^*, !_{\vec{r}}\Sigma_2^* \vdash !_{\vec{k}}U^*).$$

This shows that $X_1^*, X_2^*, \Sigma_1^*, \Sigma_2^* \vdash U^*$.

$L_\otimes$, $R_\otimes$ are verified quite easily. Also $R_\multimap$ is obvious from the definition of $P \vdash y \rhd z$.

$L_\multimap$: Here besides $X_1^*, \Sigma_1^* \vdash W^*$ and $U^*, X_2^*, \Sigma_2^* \vdash V^*$, we must assume that $W^* \rhd U^*$ is a true transform, that is

$$(\forall\vec{n})(\exists\vec{m})((!_{\vec{m}}W^*) \rhd (!_{\vec{n}}U^*) \text{ is a true transform}). \tag{4}$$

Now given $\vec{k}$, there are, by the second assumption, $\vec{p}, \vec{q}, \vec{r}$ such that

$$!_{\vec{p}}U^*, !_{\vec{q}}X_2^*, !_{\vec{r}}\Sigma_2^* \vdash !_{\vec{k}}V^*. \tag{5}$$

By (4), there is an $\vec{s}$ such that

$$(!_{\vec{s}}W^*) \rhd (!_{\vec{p}}U^*) \text{ is true}. \tag{6}$$

By the first assumption and for the specific $\vec{s}$ of (6), there are $\vec{m}, \vec{n}$ such that

$$!_{\vec{m}}X_1^*, !_{\vec{n}}\Sigma_1^* \vdash !_{\vec{s}}W^*. \tag{7}$$

By (5), (6), (7) and $L_\multimap$ for simple sequents we have

$$!_{\vec{m}}X_1^*, !_{\vec{q}}X_2^*, (!_{\vec{s}}W^*) \rhd (!_{\vec{p}}U^*), !_{\vec{n}}\Sigma_1^*, !_{\vec{r}}\Sigma_2^* \vdash !_{\vec{k}}V^*.$$

Since for every $\vec{k}$ we can find $\vec{m}, \vec{q}, \vec{s}, \vec{p}, \vec{n}, \vec{r}$ such that the above holds, this means that

$$X_1^*, X_2^*, (W^* \rhd U^*), \Sigma_1^*, \Sigma_2^* \vdash V^*.$$

The rule W holds trivially if we replace the outermost occurence ! in $!V$ by $!_0$. Similarly D holds if we replace the outermost occurrence ! in $!V$ by $!_1$. Finally concerning the rule S, suppose $!P^* \vdash W^*$ holds. We have to show that given $l, \vec{m}$ there are $k, \vec{n}$ such that

$$k(!_{\vec{n}}P^*) \vdash l(!_{\vec{m}}W^*).$$

By the assumption, for the given $\vec{m}$ there are $k_1, \vec{n}_1$ such that

$$k_1(!_{\vec{n}_1}P^*) \vdash !_{\vec{m}}W^*.$$

Hence it suffices to take $\vec{n} = \vec{n}_1$ and $k = lk_1$. $\square$

This theorem seems to be able to follow also from the *Approximation Theorem* of A.Troelstra [8], pp. 46-47, but there is a critical difference in the way Troelstra defines the approximations $!_nX$ of $!X$ from that used above. Afterall, if that theorem could be applied here, we would have also soundness for contraction.

The failure of contraction is easily seen by the following.

Lemma 6.6 *(a)* $!x \not\vdash !x \uplus !x$. *(b)* $!(x \uplus !y) \not\vdash !x \uplus !y$.

Proof. (a) Let u, v be disjoint multisets and let $x = u \uplus !v$. Suppose $!x \vdash !x \uplus !x$. Then we should have

$$(\forall k, l, m, n)(\exists p, q)(p(u \uplus qv) = k(u \uplus lv) \uplus m(u \uplus nv).$$

Since u, v are disjoint, clearly, $pu = (k+m)u$ and $pqv = (kl+mn)v$, or $p = k+m$ and $pq = kl+mn$, or $(k+m)q = kl+mn$. Consequently, for all k, l, m, n, $k+m$ should divide $kl+mn$, which is absurd.

(b) Let x, y be disjoint multisets. Then $!(x \uplus !y) \not\vdash !x \uplus !y$. Indeed, otherwise we should have

$$(\forall m, n)(\exists k, l)(k(x \uplus ly) = mx \uplus ny),$$

whence $k = m$ and $kl = n$, or $ml = n$. That is for all m, n there should be an l such that $ml = n$, which is false. E.g. for $m = 2, n = 1$ there is no such l. $\square$

Given x, y we write $x \dashv\vdash y$ if $x \vdash y$ and $y \vdash x$. (Notice that if x, y are simple multisets, then $x \vdash y$ iff $x = y$). In contrast to the preceding negative result, we have the following.

Lemma 6.7 *For any* $!$*-multisets* x, y, z *the following hold:*

(a) $!!x \dashv\vdash !x$.

(b) $!(!x \uplus !y) \dashv\vdash !x \uplus !y$, *and in general*

$$!(!x_1 \uplus \cdots \uplus !x_n) \dashv\vdash !x_1 \uplus \cdots \uplus !x_n.$$

Proof. We check that (a), (b) are true according to definition 6.4.

(a) $!!x \vdash !x$: x may contain also a string of !'s, so the exact formulation of this fact amounts to the formula

$$(\forall m, \vec{n})(\exists p, q, \vec{r})(pq(!_{\vec{r}}x) = m(!_{\vec{n}}x)).$$

This is obviously true provided we take $p = 1$, $q = m$, $\vec{r} = \vec{n}$.

$!x \vdash !!x$: This is equivalent to the fact

$$(\forall m, l, \vec{n})(\exists p, \vec{r})(p(!_{\vec{r}}x) = ml(!_{\vec{n}}x)).$$

Again it suffices to take $p = ml$ and $\vec{r} = \vec{n}$.

(b) $!(!x \uplus !y) \vdash !x \uplus !y$: This is equivalent to

$$(\forall k, l, \vec{m}, \vec{n})(\exists p, q, r, \vec{s}, \vec{t})(p(q(!_{\vec{s}}\, x) \uplus r(!_{\vec{t}}\, , y)) = (k(!_{\vec{m}}x) \uplus l(!_{\vec{n}}y))).$$

Thus it suffices to take $p = 1$, $q = k$, $r = l$, $\vec{s} = \vec{m}$ and $\vec{t} = \vec{n}$.

$!x \uplus !y \vdash !(!x \uplus !y)$: This is equivalent to

$$(\forall p, q, r, \vec{s}, \vec{t})(\exists k, l, \vec{m}, \vec{n})(p(q(!_{\vec{s}}\, x) \uplus r(!_{\vec{t}}\, y)) = (k(!_{\vec{m}}x) \uplus l(!_{\vec{n}}y))).$$

It suffices to have $k\vec{m} = pq\vec{s}$ and $l\vec{n} = pr\vec{t}$ (where if $\vec{m} = (m_1, \dots, m_r)$, $k\vec{m} = (km_1, \dots, km_r)$), so we put $k = pq$, $l = pr$, $\vec{m} = \vec{s}$ and $\vec{n} = \vec{t}$. □

Corollary 6.8 !-HF$-\{C\}$ *holds in* $(U^!(A), \vdash)$.

Now we can easily see that the failure of contraction occurs when x is a *mixture* of simple multisets and !-multisets. If we restrict ourselves to !-multisets all of whose factors are !-bound, then the sequents of lemma 6.6 holds and things go smoothly. So let us give some definitions. These definitions refer both to !-multisets and to !-products.

Definition 6.9 A !-multiset x (resp. a !-product X) is said to be *normal* if it does not contain factors of the form $!!u$ and $!(!u_1 \uplus \cdots \uplus !u_n)$ (resp. $!!U$, $!(!U_1 \otimes \cdots \otimes !U_n)$.

Note that the analog of equivalence (b) of 6.7, namely the sequent

$$!X \otimes !Y \vdash !(!X \otimes !Y) \tag{8}$$

is provable in !-HF. By lemma 6.7 and (8), we get immediately that

Lemma 6.10 *(a) Every* $!$*-multiset* x *can be normalized, i.e., there is a normal* x^* *such that* $x \vdash\dashv x^*$*.*

(b) Every $!$*-product* X *can be normalized in* !-HF*, i.e., there is a normal* X^* *such that* $X \vdash X^*$ *and* $X^* \vdash X$ *are provable in* !-HF*.*

It is easy to see that we can replace every x of a sequent by its normalization without disturbing the truth of the sequent. Namely,

Definition 6.11 A !-multiset x (resp. a !-product X) is said to be *full* if its normal form is $!x_1 \uplus \cdots \uplus !x_n$, where x_i are simple multisets (resp. $!X_1 \otimes \cdots \otimes !X_n$ with X_i simple products). Equivalently, x is full if does not contain factors y not bound by !.

Lemma 6.12 *(a) For any full* x, y
(i) $!(x \uplus y) \vdash\dashv x \uplus y$*, and*
(ii) $x \uplus x \vdash\dashv x$*.*

(b) For any full X, Y*,*
(i) $!(X \otimes Y) \vdash X \otimes Y$ *and* $X \otimes Y \vdash !(X \otimes Y)$ *are provable in* !-HF*.*
(ii) $X \otimes X \vdash X$ *and* $X \vdash X \otimes X$ *are provable in* !-HF*.*

Proof. (a) (i) follows from 6.7(b). For (ii) it suffices to show that

$$!x_1 \uplus \cdots \uplus !x_n \vdash (!x_1 \uplus \cdots \uplus !x_n) \uplus (!x_1 \uplus \cdots \uplus !x_n),$$

for x_i simple multisets, which is easily verified by definition 6.4.

(b) By using rule A. □

Theorem 6.13 (Soundness of !-HF for full sequents) *For every full sequent* S *provable in* !-HF*,* S^* *is true in* $(U^!(A), \vdash)$ *for all* **.*

Proof. By corollary 6.8, all rules but contraction hold under *. By the previous lemma contraction holds also for full sequents. □

If we want to obtain some partial completeness result we must restrict even further the kind of transforms allowed. Transforms of type $y \rhd z$, where y, z are full, are not appropriate, since they yield true sequents but in general unprovable in !-HF.

Example 6.2. For instance if x, y, z, w are simple multisets, from the truth of $x, (y \rhd z) \vdash w$ we easily derive the truth of $!x, (!y \rhd !z) \vdash !w$, while from $X, (Y \multimap Z) \vdash W$ we cannot derive $!X, !Y \multimap !Z \vdash !W$. What we can derive is just $!X, !(Y \multimap Z) \vdash !W$.

Definition 6.14 A !-sequent $x, \sigma \vdash w$ (resp. $X, \Sigma \vdash W$) is said to be *regular* if x, w are full and all the elements of σ (resp. Σ) are of the form $!(y \rhd z)$ (resp. $!(Y \multimap Z)$), where y, z (resp. Y, Z) are simple multisets (resp. simple products). Equivalently, in regular sequents $\sigma = !\tau$, where τ is a multiset of simple transforms.

Theorem 6.15 (Completeness of !-HF for regular sequents) *Let $X, \Sigma \vdash W$ be a regular sequent such that $X^*, \Sigma^* \vdash W^*$ is true for every * in $(U^!(A), \vdash)$. Then $X, \Sigma \vdash W$ is provable in* !-HF.

Proof. Since we work in !-HF, we assume X, W to be normal. So it suffices to prove that the sequent is provable in the !-HF. Let for simplicity $X = !X_1 \otimes !X_2$ and $W = !W_1 \otimes !W_2$, where X_1, X_2, W_1, W_2 are simple products. Let $\Sigma = !T$, where T is a set of simple implications. Then $!X_1^* \uplus !X_2^*, !T^* \vdash !W_1^* \uplus !W_2^*$ is true. By 6.4,

$$(\forall k_1, k_2)(\exists m_1, m_2, \vec{n})(m_1 X_1^* \uplus m_2 X_2^*, !_{\vec{n}} T^* \vdash k_1 W_1^* \uplus k_2 W_2^*).$$

For $(k_1, k_2) = (1, 0), (0, 1)$, we find $m_1, m_2, \vec{n}$ and $p_1, p_2, \vec{q}$ respectively, such that

$$m_1 X_1^* \uplus m_2 X_2^*, !_{\vec{n}} T^* \vdash W_1^*,$$

and

$$p_1 X_1^* \uplus p_2 X_2^*, !_{\vec{q}} T^* \vdash W_2^*.$$

By the completeness of HF with respect to $(U(A), \vdash)$ (theorem 4.4), the last relations imply that

$$m_1 X_1 \otimes m_2 X_2, !_{\vec{n}} T \vdash W_1, \tag{9}$$

and

$$p_1 X_1 \otimes p_2 X_2, !_{\vec{q}} T \vdash W_2 \tag{10}$$

are provable in HF. So it suffices to show that from (9) we can get

$$!X_1 \otimes !X_2, !T \vdash !W_1, \tag{11}$$

and from (10) we can get

$$!X_1 \otimes !X_2, !T \vdash !W_2, \tag{12}$$

But since $!T = [!(Y_1 \multimap Z_1), \ldots, !(Y_r \multimap Z_r)]$, clearly

$$!_{\vec{n}}T = [n_1(Y_1 \multimap Z_1), \ldots, n_r(Y_r \multimap Z_r)],$$

hence (9) immediately implies (11), by weakening, contraction and storage, and (10) implies (12). □

Concerning now the logic of $(U^!(A), \mid\!\sim)$, it is easy to see that the cancellation rule $C_\otimes$ does not hold in this structure. For example, the derivation

$$\frac{!x \uplus !z \vdash !y \uplus !z}{!x \vdash !y}$$

is false in general. We can keep however this rule for !-free formulas. So let

$$! - \text{CHF} =! - \text{HF} + (C_\otimes \text{ for !-free formulas}).$$

Theorem 6.16 *(a) All rules of* !-CHF *are true in* $(U^!(A), \mid\!\sim)$*.*

(b) If S *is a regular full sequent such that* S^* *is true in* $(U^!(A), \mid\!\sim)$ *for every* **, then* S *is provable in* !-CHF*.*

Proof. (a) Precisely as theorem 6.13.

(b) Similar again to theorem 6.15. Simply we now need the rule $C_\otimes$ to infer from

$$m_1 X_1^* \uplus m_2 X_2^*, !_{\vec{n}}\Sigma^* \mid\!\sim W_i^*$$

that

$$m_1 X_1 \otimes m_2 X_2, !_{\vec{n}}\Sigma \vdash W_i$$

is provable in !-CHF. □

As an epilogue let us summarize the main results of sections 3-6. We have two main kinds of processes within multisets: The staged and the coherent ones. To these there correspond the logical systems HF (Horn fragment) and CHF (=HF $+C_\otimes$). After introducing the operator !, we have !-staged processes and !-coherent processes. To these there correspond the logical systems !-HF and !-CHF (=!-HF$+C_\otimes$ for !-free formulas) respectively. Then

1) HF is a sound and complete axiomatization of staged processes.

2) CHF is a sound and complete axiomatization of coherent processes.

3) !-HF is sound with respect to !-staged processes. And it is complete if we restrict ourselves to regular full sequents.

4) !-CHF is sound with respect to !-coherent processes. And it is complete if we restrict ourselves to regular full sequents.

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