Localizing the axioms

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Abstract

We examine what happens if we replace ZFC with a localistic/relativistic system, LZFC, whose central new axiom, denoted by *Loc*(ZFC), says that every set belongs to a transitive model of ZFC. LZFC consists of *Loc*(ZFC) plus some elementary axioms forming Basic Set Theory (BST). Some theoretical reasons for this shift of view are given. All Π_2 consequences of ZFC are provable in LZFC. LZFC strongly extends Kripke-Platek (KP) set theory minus Δ_0-Collection and minus $\in$-induction scheme. ZFC+"there is an inaccessible cardinal" proves the consistency of LZFC. In LZFC we focus on models rather than cardinals, a transitive model being considered as the analogue of an inaccessible cardinal. Pushing this analogy further we define α-Mahlo models and Π^1_1-indescribable models, the latter being the analogues of weakly compact cardinals. Also localization axioms of the form *Loc*(ZFC$+\phi$) are considered and their global consequences are examined. Finally we introduce the concept of standard compact cardinal (in ZFC) and some standard compactness results are proved.

Keywords. Localization axiom, Local ZFC, Mahlo model, standard compact cardinal.

1 Introduction

The purpose of this paper is to look at ZFC from a certain localistic/relativistic point of view. In current set theory we believe that there is an objective reality of sets, the "real world" V, the main properties of which are captured by the axioms of ZFC. In other words, the ZFC axioms are supposed to hold *in* V. This is the absolutistic point of view. An opposite view, that may be called localistic/relativistic, would consist in claiming that the ZFC axioms, especially the problematic axiom of Powerset (and perhaps Replacement), should refer not to V itself but only to several *local* models, which are counterparts of the reference frames of physics. Conceivably there are more than one ways to formalize this general idea of local truth and local models. The formal account presented in this paper is just one among them. Its main points are roughly the following: (1) All local models of ZFC (or extensions of it) that we consider are standard transitive sets. (2) There is an abundance of them across the universe.

The motivation for such a shift of view comes from the well-known relativity, first pointed out by Skolem [10], that occurs in all first-order axiomatizations of set theory. Some fundamental notions, especially cardinality and powerset, raise such unsurmountable difficulties when treated as absolute entities, that, until one comes up with a revolutionary new idea about what the powerset of an infinite set actually contains - which possibly (though not necessarily) might settle also the problem of counting its members - one would better let aside the idea that $\mathcal{P}(\omega)$ exists in V and instead be content with the idea that $\mathcal{P}(\omega)$ is a set with respect to transitive set-universes *only,* i.e., in the local/relative form $\mathcal{P}^M(\omega)=\mathcal{P}(\omega)\cap M$, where $(M,\in)$ is a transitive model of ZFC.[1] $\mathcal{P}(\omega)$ itself makes sense only as a proper class. In compensation one may assume that transitive models of ZFC exist *everywhere in* V, specifically that every set x belongs to some transitive model y. Such a view on the one hand does not have any negative impact on the study of various kinds of infinite cardinals. For example it by no means invalidates the theory of large cardinals, except of course that these are now treated as relativized entities living only in models. And on the other hand it spurs the interest in transitive models themselves, as objects of study *per se* rather than just a means. Large cardinals in particular constitute a source of ideas

[1]Throughout the term "transitive model" is used instead of the more cumbersome "standard transitive (set) model", i.e., a transitive set x equipped with the standard membership relation $\in$, so that $(x,\in)\models$ ZFC. We could just say "standard model", would transitivity not be independent from standardness. A transitive set on the other hand is implicitly thought as being structured by $\in$. In view of the Mostowski's isomorphism theorem however, a standard model of ZFC is essentially identical to a standard transitive one.

and techniques some of which can be transferred to models in order to build analogous classifications among them.

A theoretical justification of the above viewpoint is summarized in the following argument: Although we may believe that V is indeed an objective, absolute reality, it does not necessarily follow that all properties and facts concerning objects of V should be absolute too. *Some* properties may be subject always and by their nature to local constraints, so that any absolutistic judgment about them would simply not make sense. A helpful and convincing analogy comes from the universe of physical objects. According to the established paradigm of Relativity Theory, this universe is also an objective, absolute reality of things,[2] but fundamental physical magnitudes like mass, length, time, velocity, etc, are inherently relative quantities, heavily depending on the observer's reference frame. If fundamental attributes of physical objects such as mass and size are relative, why should the *type* (or *degree*) *of infinity* of an infinite set be absolute? Of course there are differences: In the case of physical universe there are experiments and measurements supporting the view of Relativity Theory, while for the universe of abstract sets one can only make assumptions. Also one tends to accept much more easily that almost all physical properties (color, shape, smell, etc) are subject to relativization, than that this is also the case with abstract properties, like number and structure, which are commonly supposed to reflect deeper and more permanent characteristics of beings. And in fact, finite cardinalities $0, 1, 2, \ldots$ do not seem to relativize in any reasonable way. But the various *infinite cardinalities* is a different matter. Among all mathematical objects these should be the most naturally expected to be inherently relative. A strong indication is the ease by which the cardinality of an infinite set can change by means of forcing constructions.

So much for the viability of the localistic/relativistic approach to set theory. The purpose of the paper is to set out a particular *implementation* of this approach through an axiomatic system and examine its logical strength and its set theoretic consequences. The paper is organized as follows:

In section 2 we define the system LZFC (from "local ZFC") whose main axiom is:

$$(Loc(\mathrm{ZFC})) \qquad \forall x \exists y (x \in y \wedge Tr(y) \wedge (y, \in) \models \mathrm{ZFC}).$$

[2] Or, at least, it *can* be. Obviously no final decision can be reached on such a metaphysical issue.

The other axioms, forming the system BST (of Basic Set Theory), are elementary assumptions like Pair, Union, etc, needed only to formulate Loc(ZFC). LZFC proves all Π_2 consequences of ZFC. Also LZFC proves the equivalence of the Found* of $\in$-induction and the scheme Found$_{On}$ of induction over the ordinals. However none of them seems to be derivable in LZFC. Consequently, transfinite induction along On is not available in LZFC. LZFC does not prove Π_2-Reflection, since LZFC $+ \Pi_2$-Reflection $\vdash Con$(LZFC). Σ_1-Collection is equivalent to Δ_0-Collection over LZFC, but it is open whether the latter proves Δ_0-Collection. The class L of constructible sets is definable (though one cannot prove in LZFC that L is an inner model of LZFC). Also standard facts and constructions, like Completeness theorem, Löwenheim-Skolem theorem, generic extensions, Mostowski collapse etc, are available in LZFC. LZFC is a strong extension of KP (Kripke-Platek set theory) minus Δ_0-Collection and minus the scheme Found* of $\in$-induction. Concerning consistency, LZFC is a subtheory of ZFC+ "there is a proper class of inaccessible cardinals". Also ZFC + "there is an inaccessible cardinal" proves the consistency of ZFC+LZFC, while ZFC + "there is a natural model of ZFC" proves the consistency of LZFC.

In section 3 we discuss infinite (uncountable) cardinals and powersets (of infinite sets) in LZFC. In view of the absence of transfinite induction, no general statement about cardinals ω_α and powersets $\mathcal{P}^\alpha(\omega)$ can be derived. Yet certain implications concerning existence and absoluteness of concrete classes like ω_1, $\mathcal{P}(\omega)$ and $H(\omega_1)$ (and more generally ω_n, $\mathcal{P}^n(\omega)$ and $H(\omega_n)$, for $n \in \omega$) can be established. For example it is proved that $H(\omega_1) \in M$ implies $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)$ and $\omega_1^M = \omega_1$; $\mathcal{P}(\omega) \in M$ implies $\omega_1^M = \omega_1$ and $H(\omega_1)^M = H(\omega)$, etc. We discuss also an ambiguity concerning the meaning of the symbols ω_α, for $\alpha > 0$, and how it can be raised.

In section 4 we define α-Mahlo models as analogues of α-Mahlo cardinals. This is a pretty natural notion: A model M of ZFC is Mahlo if the set of models of ZFC that belong to M is a stationary subset of M. Stationary, as well as closed unbounded subsets of M, are restricted to *definable* subsets of M. Definability guarantees that the property of α-Mahloness is absolute for transitive models of ZFC. It is shown in ZFC that if κ is α-Mahlo, then V_κ is an α-Mahlo model.

In section 5 we define in LZFC Π^1_1-indescribable models, as analogues of Π^1_1-indescribable (i.e., weakly compact) cardinals. Concerning the existence of such models, we show (in ZFC) that if κ is weakly compact then V_κ is Π^1_1-indescribable. Moreover, if M is Π^1_1-indescribable, then it is α-Mahlo for every $\alpha \in M$.

In section 6 we consider localization axioms of extensions of ZFC, i.e., of the form $Loc(\mathrm{ZFC} + \phi)$, or $\{Loc(\mathrm{ZFC} + \phi) : \phi \in \Gamma\}$, for some set of sentences Γ, and examine their consistency (when added to LZFC) and their impact on V. For instance it is shown that for every Π_1 or Σ_1 sentence ϕ, $Loc(\mathrm{ZFC} + \phi) + Loc(\mathrm{ZFC} + \neg\phi)$ is inconsistent. Further, $Loc(\mathrm{ZFC} + V = L)$ implies $V = L$. Also it is shown that if LZFC + $Loc(\mathrm{ZFC} + \mathrm{CH})$ + $Loc(\mathrm{ZFC} + \neg\mathrm{CH})$ is consistent, then Powerset is false, while the consistency of LZFC + $Loc(\mathrm{ZFC} + \mathrm{CH}) + Loc(\mathrm{ZFC} + \neg\mathrm{CH})$ follows from the consistency of ZFC+"there is a natural model of ZFC". Finally we show that for any definable set c and definable ordinals α, β, the theory LZFC+$Loc(\mathrm{ZFC}+|c| = \omega_\alpha) + Loc(\mathrm{ZFC} + |c| = \omega_\beta) +$ "c exists" is inconsistent.

In section 7 we consider (in ZFC) a question that arises as a result of dealing exclusively with transitive models. We can dub it "standard compactness" problem, since it is like ordinary compactness except that the models allowed are (standard) transitive ones only. Given a set Σ of sentences of a finitary language extending the language of set theory, such that $|\Sigma| = \kappa$ and every subset of Σ of cardinality $< \kappa$ has a transitive model, does Σ have a transitive model? If the answer is yes we call κ *standard compact*. We show (in ZFC): (a) ω is not standard compact, (b) every weakly compact cardinal is standard compact, and (c) if $\lambda > \omega$ is strongly compact, then every $\kappa \geq \lambda$ such that $\kappa^{<\kappa} = \kappa$ is standard compact.

2 A localized variant of ZFC.

V is the universe of sets. The membership relation between entities of V is denoted by $\in$. Let $\mathcal{L} = \{\epsilon\}$ be the language of set theory. Since ϵ is going to be interpreted only by $\in$ we shall identify ϵ with $\in$ and write for simplicity $\mathcal{L} = \{\in\}$.

Π_n, Σ_n denote the usual classes of formulas in the Lévy hierarchy (with $\Pi_0 = \Sigma_0$ being the class of bounded formulas). If S is a set theory, Σ_n^{S} and Π_n^{S} are the classes of formulas provably equivalent in S to a Σ_n and Π_n formula, respectively. Also Δ_n^{S} is the class of properties ϕ which are provably equivalent in S both to a Π_n and a Σ_n formula, i.e., there is a Σ_n formula ϕ_1 and a Π_n formula ϕ_2 such that $\mathrm{S} \vdash \phi \leftrightarrow \phi_1 \leftrightarrow \phi_2$.

Lower case letters a, b, x, y, u, v denote sets. Upper case letters A, B, M, N, X, Y denote either sets or (proper) classes, depending on the context. For example throughout the letters M, N always denote transitive *sets* which are models of ZFC.

If ϕ is a formula of $\mathcal{L}$ and u is a set, ϕ^u denotes the bounded formula resulting from ϕ if we replace each unbounded quantifier $\forall x$, $\exists x$ of ϕ with $\forall x \in u$, $\exists x \in u$, respectively. As usual writing ϕ we mean that $(V, \in) \models \phi$. So ϕ^u is equivalent to $(u, \in) \models \phi$.

The following localistic substitute of ZFC will be the main axiom of our system LZFC defined below:

$$(Loc(\mathrm{ZFC})) \qquad \forall x \exists y (x \in y \wedge Tr(y) \wedge (y, \in) \models \mathrm{ZFC}),$$

where $Tr(y)$ denotes the formula "y is transitive" and $(y, \in) \models$ ZFC abbreviates the formula $\forall \phi \in \mathrm{ZFC}((y, \in) \models \phi)$. *Loc*(ZFC) says that the class of transitive models of ZFC is an *unbounded* (or *cofinal*) subclass of V with respect to $\in$, and hence with respect to $\subseteq$ (because of transitivity).

However, the relation "$(y, \in) \models \phi$", as well as the set ZFC, as a set of formulas, cannot be defined without some elementary notions and facts from a body of absolute set theoretic truths that we call Basic Set Theory and denote by BST. This is similar to Elementary Set Theory, EST, of [4, p. 39], except that BST contains in addition Cartesian Product, while the axioms of Foundation and Choice are not included because they can be deduced from *Loc*(ZFC) (see below). So we take BST to consist of the following axioms:

(Emptyset) $\exists x (x = \emptyset)$,

(Ext) $\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \to x = y]$

(Pair) $\forall x \forall y \exists z (z = \{x, y\})$

(Union) $\forall x \exists y (y = \bigcup x)$

(Cartesian Product) $\forall x \forall y \exists z (z = x \times y)$. [The predicates, "pair", "function" etc, are Δ_0 and can be defined as in [1, p. 14].]

(Infinity) $\exists x [\emptyset \in x \wedge \forall y \in x \exists z \in x (z = y \cup \{y\})]$

(Δ_0-Separation) $\forall \bar{z} \forall a \exists b \forall y (y \in b \leftrightarrow y \in a \wedge \phi(y, \bar{z}))$,

for every Δ_0 formula ϕ not containing b free.

Lemma 2.1 *In* BST*: (i) ω exists and the axioms of Peano arithmetic* (PA) *can be proven to hold in ω endowed with the usual operations. Thus* PA $\subseteq$ BST*. (ii) The set of formulas $Fml(\mathcal{L})$ is definable, V_ω exists and the relation "$(x, \in) \models \phi(\bar{a})$" is definable.*

Proof. (i) By Infinity, let a be an inductive set. We can define ω (using Δ_0-Separation) as the set of ordinals $x \in a$ such that for every $y \leq x$ and

$y \neq 0$, y is a successor ordinal. We can see that this set is the least inductive set (details are left to the reader). The minimality of ω as inductive set amounts to the fact that ω satisfies complete induction. The operations $', +, \cdot$ on it are defined as usual and the axioms of PA are shown in BST to be true with respect to ω.

(ii) By Cartesian Product, for every set a and $n \in \omega$, $a^n = \{(x_0, \ldots, x_{n-1}) : x_i \in a\}$ is a set. Formulas of $\mathcal{L}$ are defined inductively as triples of integers, e.g. $\ulcorner v_i = v_j \urcorner = (0, i, j)$, $\ulcorner v_i \in v_j \urcorner = (1, i, j)$, etc, as in [3, p. 90]. The set $Fml(\mathcal{L})$ of formulas of $\mathcal{L}$ is a recursive (hence Δ_1^{ZFC} definable subset of ω). So is also ZFC $\subset Fml(\mathcal{L})$.

The set a^n can be identified also with the set of functions f such that $dom(f) = n$ and $rng(f) \subseteq a$. Using this identification we can define V_ω as in [3, p.81] by a Δ_1^{ZFC} definition. Finally the relation "$(x, \in) \models \phi$" is also Δ_1^{ZFC} definable by the help of V_ω (see [3, p. 91] for details). $\dashv$

Remark 2.2 Without the axiom Cartesian Product of BST, to prove that cartesian products of sets are sets one would need something like Δ_0-Collection (or Δ_0-Replacement) (see [1, prop. 3.2]). This is a rather strong axiom, while existence of cartesian products is quite elementary. We do not know if LZFC proves Δ_0-Collection (see Propositions 2.15 and 2.16 below).

Having fixed the definitions of $Fml(\mathcal{L})$, ZFC and $(x, \in) \models \phi$, we can now consider the axiom *Loc*(ZFC) given above and set

$$\mathrm{LZFC} = \mathrm{BST} + Loc(\mathrm{ZFC}).$$

For simplicity henceforth we shall write $x \models \phi$ instead of $(x, \in) \models \phi$. Sometimes we drop also the predicate $Tr(x)$ if implicitly understood, so *Loc*(ZFC) is usually written $\forall x \exists y (x \in y \ \wedge \ y \models \mathrm{ZFC})$.

First let us note, as already mentioned above, that the axioms of Choice and Foundation are deduced from *Loc*(ZFC).

Lemma 2.3 *Loc*(ZFC) *implies the axioms of* Choice *and* Foundation.

Proof. Let $x \neq \emptyset$ be a set such that for every $y \in x$, $y \neq \emptyset$. By *Loc*(ZFC), there is a transitive model M of ZFC such that $x \in M$. Then in M x has a choice function and also has a $\in$-least member. $\dashv$

Given a tuple of sets $\bar{x} = (x_1, \ldots, x_n)$ let $\bar{x} \in y$ abbreviate the formula $x_1 \in y \wedge \cdots \wedge x_n \in y$.

Lemma 2.4 *(i)* LZFC $\vdash \forall \bar{x}\ \exists y(\bar{x} \in y \wedge Tr(y) \wedge y \models$ ZFC$)$.

(ii) Let Π_2(ZFC) *be the set of* Π_2 *consequences of* ZFC. *Then* Π_2(ZFC) $\subseteq$ LZFC.

Proof. (i) Let $\bar{x} = (x_1, \ldots, x_n)$. Given any $a_1, \ldots, a_n$, $\{a_1, \ldots, a_n\}$ exists in BST. So by *Loc*(ZFC) there is a transitive model b such that $\{a_1, \ldots, a_n\} \in b$. Then $\{a_1, \ldots, a_n\} \subset b$ and $b \models$ ZFC.

(ii) Let $\phi \in \Pi_2$(ZFC). ϕ has the form $\forall \bar{x}\ \exists \bar{y} \psi(\bar{x}, \bar{y})$, where $\bar{x}$ is an n-tuple of variables, $\bar{y}$ is an m-tuple of variables and ψ is bounded. Let us work in LZFC. Pick any n-tuple of sets $\bar{a}$. It suffices to show that there is a $\bar{y}$ such that $\psi(\bar{a}, \bar{y})$. By (i) above there is a transitive model $b \models$ ZFC such that $\bar{a} \in b$. Since ϕ is a consequence of ZFC, $b \models \phi$, or $b \models \exists \bar{y} \psi(\bar{a}, \bar{y})$. Hence $\exists \bar{y} \psi(\bar{a}, \bar{y})$ since ψ is Δ_0. Thus LZFC $\vdash \phi$. $\dashv$

Remark 2.5 In contrast to ZFC, LZFC should not in general allow axioms with unbounded quantifiers, since its truths are "local", and so the variables must range in some set-model. However certain Π_2 statements expressing elementary, indisputable facts (like e.g. $\forall x, y \exists z(z = \{x, y\})$), cannot but be accepted, despite the occurrence of two alternating unbounded quantifiers. This is the case with the axioms of BST. All of them are Π_2 sentences, as one can easily check by inspecting the formulations given above.

Remark 2.6 The axioms of BST are necessary only to make possible the strict *formulation* of *Loc*(ZFC). Otherwise, that is, if we assume that *Loc*(ZFC) *is* sensible, by assuming for example that the notions "formula" and "$x \models \phi$" are primitive, then we can easily prove lemma 2.4 by working in *Loc*(ZFC)+Pair+Emptyset rather than LZFC. Since all axioms of BST are Π_2 consequences of ZFC, it follows from 2.4 (ii) that from *Loc*(ZFC)+Pair+Emptyset we can recover the rest of the axioms of BST.

Remark 2.7 Throughout we are going to make heavy use of the well-known fact that every Δ_1^{ZFC} (and hence every Σ_0^{ZFC}) formula is absolute for transitive models of ZFC. However a word of caution is needed here. Δ_1^{ZFC} formulas are absolute between transitive models of ZFC and the universe, *when we work in* ZFC (and this is done most of the time), i.e., when V is supposed to satisfy ZFC. If $V \not\models$ ZFC absoluteness of Δ_1^{ZFC} formulas is no longer guaranteed. For instance let ϕ be Σ_1 and ψ be Π_1 and ZFC $\vdash \phi \leftrightarrow \psi$. Then $M \models \phi \leftrightarrow \psi$ for any model of ZFC. But if $V \not\models$ ZFC we cannot infer that $V \models \phi \leftrightarrow \psi$, so we cannot infer absoluteness of ϕ and ψ. In our case V satisfies LZFC rather than ZFC, so this observation is in order. However, if S is a set theory such that $V \models \mathrm{S}$ and $\mathrm{S} \vdash \phi \leftrightarrow \psi$ whenever $\mathrm{ZFC} \vdash \phi \leftrightarrow \psi$, for ϕ, ψ as above, then ϕ, ψ are still absolute between V and the models of ZFC. The next lemma says that this is the case for S = LZFC.

Lemma 2.8 *If $\phi \in \Sigma_1$ and $\psi \in \Pi_1$ and* $\mathrm{ZFC} \vdash \phi \leftrightarrow \psi$*, then* $\mathrm{LZFC} \vdash \phi \leftrightarrow \psi$*. Consequently for any transitive $M \models$* ZFC*, any Δ_1^{ZFC} formula $\phi(\bar{x})$ and any $\bar{a} \in M$, $\phi(\bar{a}) \leftrightarrow M \models \phi(\bar{a})$.*

Proof. Let $\phi \in \Sigma_1$ and $\psi \in \Pi_1$ and $\mathrm{ZFC} \vdash \phi \leftrightarrow \psi$. Then $\phi \leftrightarrow \psi$ belongs to $\Pi_2(\mathrm{ZFC})$, so the claim follows from lemma 2.4 (ii). $\dashv$

For brevity we express the fact established in lemma 2.8 by saying that every Δ_1^{ZFC} formula of $\mathcal{L}$ is also Δ_1^{LZFC}.

It follows from lemma 2.8 that every set defined by a Δ_1^{ZFC} formula inside any transitive model M of ZFC with parameters in M is the same as when defined in LZFC. We often express this by saying that this set *exists in* LZFC, in the sense that its definition in LZFC does not provide a proper class. In particular this is the case with sets defined inductively by some positive inductive operator Γ_ϕ in any transitive model, for some Σ_1 formula ϕ.

Remark 2.9 Let us remark at this point, for later use, that the sentence

$$Loc(\mathrm{ZFC}) = \forall x \exists y (x \in y \wedge y \models \mathrm{ZFC})$$

is itself Π_2^{ZFC}, since "ϕ is a formula", "$\phi \in \mathrm{ZFC}$" and "$x \models \phi$" are Δ_1^{ZFC}. Moreover, by lemma 2.8, $Loc(\mathrm{ZFC})$ is also Π_2^{LZFC}.

Ordinals are defined in LZFC as usual (transitive sets linearly ordered, and hence well-ordered, by $\in$). Lower case Greek letters $\alpha, \beta, \ldots$ denote ordinals. We often write $\alpha < \beta$ instead of $\alpha \in \beta$. We denote the class of all ordinals by On. $(On, \in)$ is well-ordered, but we must be careful with the meaning of this assertion. $(On, \in)$ is well-ordered means that every *subset* of On has a least element, as a consequence of Foundation. Things however may be different for *subclasses* of On. If $X = \{\alpha : \phi(\alpha)\}$ is a subclass of On, then there is no way to ensure that X has a least element. The usual argument that amounts to pick an $\alpha \in X$ and then take the trace $\alpha \cap X$ of

X on α does not work in LZFC since, in absence of full separation, $\alpha \cap X$ need not be a set. It works only for Δ_0-classes (i.e., classes defined by Δ_0-formulas). So let us denote by Found_{On} the scheme "every subclass of On has a least element". Namely:

$$(\mathrm{Found}_{On})\ \ \exists\alpha \in On\ \phi(\alpha) \rightarrow \exists\alpha \in On[\phi(\alpha) \wedge \forall\beta < \alpha\neg\phi(\beta)].$$

Found_{On} is apparently a weak form of the full $\in$-induction scheme Found^* which says that "every class has an $\in$-least element":

$$(\mathrm{Found}^*)\ \ \exists x\phi(x) \rightarrow \exists x[\phi(x) \wedge \forall y \in x\neg\phi(y)]$$

However we shall see below (Lemma 2.11) that Found_{On} and Found^* are in fact equivalent over LZFC.

A remarkable situation where Found_{On} is involved is the following. Let us call sets x, y *equinumerous* and write $x \sim y$ if there is a bijection $f : x \rightarrow y$. Also let us write $x \precsim y$ if there is an injection $f : x \rightarrow y$, and $x \precnsim y$ if there is an injection $f : x \rightarrow y$, but $x \not\sim y$. Given any x, let $Ord(x) = \{\alpha \in On : x \sim \alpha\}$. By Choice, for every x, $Ord(x) \neq \emptyset$. However the formula $x \sim \alpha$ is Σ_1, hence, since Σ_1-Separation is not available in LZFC, we cannot ensure that $Ord(x)$ has a least element. The least element of $Ord(x)$, if it existed, would be the (absolute) cardinality of x, what we usually denote $|x|$. It follows that in LZFC alone, without Found_{On} (or Found^* according to the previous lemma), absolute cardinalities of sets cannot be defined.[3] This is rather in accordance with the spirit of LZFC, whose primary motivation was to challenge the existence of absolute infinite cardinalities and powersets. So further discussion on this issue is provided in section 3.

Given a model M, let $Def(M)$ denote the collection of its first-order definable subsets, i.e.,

$$Def(M) = \{X \subseteq M : (\exists\phi(x, \bar{y}) \in \omega)(\exists\bar{b} \in M)[M \models \forall x(x \in X \leftrightarrow \phi(x, \bar{b}))]\}.$$

The definition is absolute so $Def(M)$ exists in V. Further, if $\mathcal{X}$ is a subset of $\mathcal{P}(M)$, then $Def(M, \mathcal{X})$ denotes the collection of subsets of M second-order definable in $(M, \mathcal{X})$.

The Ramified Analytical hierarchy over M is the collection $RA(M) = \bigcup_{\alpha\in On} RA_\alpha(M)$, where

[3]Consequently the notation $|x|$ will not be used when x is a set of LZFC. Sometimes this notation is employed without actual reference to existent cardinalities as sets. For example, the notation $|x| = |y|$ is another way to say $x \sim y$, while $|x| < |y|$ means just $x \precnsim y$.

$RA_0(M) = Def(M)$,
$RA_{\alpha+1}(M) = Def(M, RA_\alpha(M))$,
$RA_\alpha(M) = \bigcup_{\beta<\alpha} RA_\beta(M)$.

Lemma 2.10 *The following facts are provable in* LZFC *and are absolute with respect to transitive models of* ZFC*:*

(i) $\exists x(x = V_\omega)$ *(the set of hereditarily finite sets exists).*

(ii) $\forall x \exists y(TC(x) = y)$ *(every set has a transitive closure).*

(iii) $\forall x \exists \alpha \in On(rank(x) = \alpha)$*, where* $rank(x) = \sup\{rank(y) + 1 : y \in x\}$*.*

(iv) $\forall \alpha \in On\ \exists x(x = L_\alpha)$*, where* L_α *is the* α*-th level of the ordinary constructible hierarchy.*

(v) Δ_1*-Separation.*

(vi) For every model M*,* $(M, Def(M))$ *as well as* $(M, RA(M))$ *exist and are models of the theories of classes* GBC *(Gödel-Bernays) and* KM *(Kelley-Morse), respectively.*

Proof. All objects involved in the clauses (i)-(vi) above have Δ_1^{ZFC} definitions, therefore Δ_1^{LZFC} definitions by 2.8, and hence they have absolute definitions inside any transitive model of ZFC containing the appropriate parameters. For instance to show existence of L_α, take a transitive model $M \models$ ZFC such that $\alpha \in M$, and construct in M the levels L_β, $\beta \leq \alpha$, of L. $\dashv$

Lemma 2.11 Found* *and* Found$_{On}$ *are equivalent over* LZFC*.*

Proof. Since the ordering $<$ on On coincides with $\in$, obviously Found* implies Found$_{On}$. Conversely, suppose Found$_{On}$ holds and let $\exists x \phi(x)$ be true. Consider the subclass of On

$$R_\phi = \{\alpha \in On : \exists x(\phi(x)\ \wedge\ rank(x) = \alpha)\}.$$

By Lemma 2.10 (iii), every set in LZFC has a rank, hence $R_\phi \neq \emptyset$. By Found$_{On}$, R_ϕ has a least element α_0. Thus $\exists x(\phi(x) \wedge rank(x) = \alpha_0)$ is true. Pick such a x. Then $\forall y \in x \neg\phi(y)$. $\dashv$

In view of the non-derivability of Found$_{On}$ in LZFC we have the following important consequence.

Remark 2.12 The familiar transfinite induction along On is not available in LZFC, except for Δ_1 subclasses of On.

Because of 2.10 (iii), we can define (non-inductively!) for every $\alpha \in On$ the class

$$V_\alpha = \{x : rank(x) < \alpha\}.$$

V_α, $\alpha \in On$, are the *layers* of the universe, since $V_\alpha \subset V_\beta$ for $\alpha < \beta$ and $V = \bigcup_\alpha V_\alpha$. Except V_α for $\alpha \leq \omega$, V_α in general need not be sets. However it is straightforward that the relativization of V_α's to any transitive model M of ZFC generates the usual cumulative hierarchy of M.

Lemma 2.13 *Let M be a transitive model of* ZFC. *Then for every $\alpha \in On^M$, $V_\alpha^M = M_\alpha = M \cap V_\alpha$.*

Let also

$$L = \bigcup_{\alpha \in On} L_\alpha$$

be the class of constructible sets. In contrast to V_α, each L_α is a set.[4] So the picture of the universe of LZFC is roughly that of Figure 1.

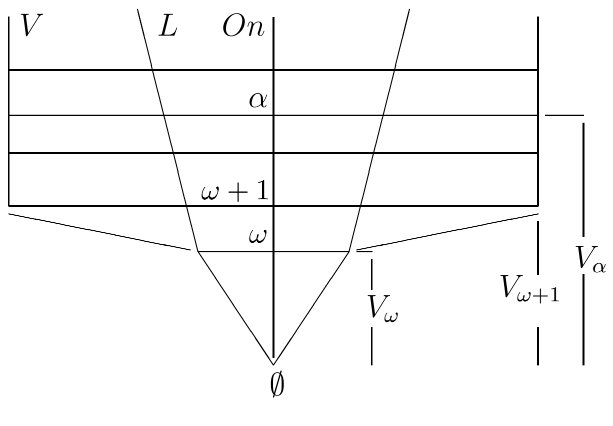

Figure 1

[4]However one should not expect LZFC to prove that L is an inner model of ZFC, since for that one would need the Replacement Axiom. After all such a requirement would not comply with the localistic spirit of LZFC, according to which only *set* models of ZFC make sense.

Remark 2.14 The picture of Figure 1 suggests that the levels V_α for $\alpha > \omega$ are all proper classes. This however need not be always true and some V_α may be sets in some cases. Of course if V_α is a proper class, so is every V_β for $\beta > \alpha$. LZFC simply does not give any information about the status of the Powerset axiom and, by so doing, is generally compatible with ZFC (see 2.23 below), although its intended interpretation points to the opposite direction. In order to refute the Powerset axiom, we need localization principles stronger than $Loc(\mathrm{ZFC})$, of the form $Loc(\mathrm{ZFC}+\phi)$ or $\{Loc(\mathrm{ZFC}+\phi) : \phi \in \Gamma\}$. See section 6.

Beside the Powerset axiom, the axiom scheme of Collection/Replacement is also questionable when referred to V. In general for a set of formulas Γ we have the scheme:

(Γ-Collection) $\forall\bar{z}\forall a\exists b[\forall x \in a\exists y\phi(x, y, \bar{z}) \to \forall x \in a\exists y \in b\ \phi(x, y, \bar{z})]$, for every formula $\phi \in \Gamma$ not containing b free.

Γ-Replacement is weaker than Γ-Collection, so we consider only the latter.

Also the following scheme of Π_2-Reflection is of interest here:

(Π_2-Reflection) $\phi \to\ \forall x\exists y[x \in y \wedge Tr(y) \wedge \phi^y]$, for every Π_2 sentence ϕ.

(Clearly, working in LZFC we may use in the above scheme Π_2^{LZFC} sentences instead of just Π_2.)

Proposition 2.15 *(i)* Δ_0-Collection *and* Σ_1-Collection *are equivalent over* LZFC*.*

(ii) $\mathrm{LZFC} + \Pi_2$-Reflection $\vdash \Sigma_1$-Collection*.*

Proof. (i) One direction is trivial. It suffices to show that Δ_0-Collection implies Σ_1-Collection over LZFC. Let $\psi(x, y) := \exists\bar{z}\phi(x, y, \bar{z})$ be a Σ_1-formula, and let $\forall x \in a\exists y\psi(x, y)$ be true in LZFC. Then $\forall x \in a\exists y\exists\bar{z}\phi(x, y, \bar{z})$. Let n be the length of the tuple $\bar{z}$. Using pairing and the Δ_0 functions $(u)_0, \ldots, (u)_n$, for an $(n+1)$-tuple u, such that $u = ((u)_0, \ldots, (u)_n)$, $\forall x \in a\exists y\exists\bar{z}\phi(x, y, \bar{z})$ is written $\forall x \in a\exists u\phi(x, (u)_0, (u)_1, \ldots, (u)_n)$. Since $\phi(x, (u)_0, (u)_1, \ldots, (u)_n)$ is (an abbreviation of) a Δ_0 formula, by Δ_0-Collection there is a b such that $\forall x \in a\exists u \in b\ \phi(x, (u)_0, (u)_1, \ldots, (u)_n)$. If $c = TC(b)$, then $\forall x \in a\exists y \in c\exists\bar{z}\phi(x, y, \bar{z})$, i.e., $\forall x \in a\exists y \in c\ \psi(x, y)$.

(ii) We work in $\mathrm{LZFC} + \Pi_2$-Reflection. Let $\phi(x, y, \bar{z})$ be a Σ_1 formula, a, $\bar{c}$ be sets and let $\forall x \in a\exists y\phi(x, y, \bar{c})$ hold true. We have to show that there is

b such that

$$\forall x \in a \exists y \in b\ \phi(x, y, \bar{c}).$$

Since ϕ is Σ_1, $\forall x \in a \exists y \phi(x, y, \bar{c})$ is a Π_2 formula. By Π_2-Reflection there is a transitive b such that $a \cup \{c_1, \ldots, c_n\} \in b$ and $(\forall x \in a \exists y \phi(x, y, \bar{c}))^b$, or $\forall x \in a \exists y \in b\ \phi(x, y, \bar{c})^b$. Since ϕ is Σ_1, ϕ^b implies ϕ, so $\forall x \in a \exists y \in b\ \phi(x, y, \bar{c})$. $\dashv$

Let TM(LZFC) denote the principle "there is a transitive model of LZFC".

Proposition 2.16 LZFC + Π_2-Reflection $\vdash$ TM(LZFC). *Consequently* LZFC + Π_2-Reflection $\vdash$ *Con*(LZFC). *Therefore if* LZFC *is consistent, then* LZFC $\nvdash$ Π_2-Reflection.

Proof. We work in LZFC + Π_2-Reflection. By Remark 2.9, the axiom *Loc*(ZFC) of LZFC is a true Π_2 sentence, hence Π_2-Reflection applies to *Loc*(ZFC). Consider the conjunction $\Phi = Loc(\text{ZFC}) \wedge \text{Pair}$. Clearly Φ is Π_2, and by assumption it holds in V, so by Π_2-Reflection there is a (nonempty) transitive set b such that $\Phi^b = (Loc(\text{ZFC}))^b \wedge \text{Pair}^b$ is the case. It suffices to show that $b \models$ LZFC. Already $b \models Loc(\text{ZFC}) \wedge \text{Pair}$, so it remains to show that b satisfies the rest of the axioms of BST. Emptyset and Extensionality are obvious in view of the transitivity of b. For Union, let $x \in b$. Then $x \in M \in b$ for some model M, so $\bigcup x \in M \in b$. For Cartesian Product, given any $x, y \in b$, $\{x, y\} \in b$ by Pair, so there is, by *Loc*(ZFC), a model $M \in b$ such that $\{x, y\} \in M$. Then $x, y \in M$, hence $x \times y \in M \in b$. Similarly for Infinity. It remains to verify Δ_0-Separation. Let $c \in b$ and $\phi(x, \bar{a})$ be Δ_0, with $\bar{a} \in b$. Let $X = \{x \in c : b \models \phi(x, \bar{a})\}$. We have to show that $X \in b$. By Pair and *Loc*(ZFC) there is a model $M \in b$ of ZFC such that $c, \bar{a} \in M$. Then clearly $X = \{x \in M : M \models x \in c \wedge \phi(x, \bar{a})\}$. Therefore $X \in M$ and hence $X \in b$. The proof that LZFC + Π_2-Reflection $\vdash TM$(LZFC) is complete. So LZFC + Π_2-Reflection $\vdash Con$(LZFC). Since by lemma 2.1 PA $\subseteq$ LZFC, Gödel's incompleteness implies that LZFC $\nvdash$ Π_2-Reflection. $\dashv$

It is open whether LZFC proves Δ_0-Collection. Also it is open whether the converse of 2.15 (ii) above is true, i.e., whether LZFC + Σ_1-Collection proves Π_2-Reflection. (If it does, then, in view of 2.15 (i) and 2.16, LZFC $\nvdash$ Δ_0-Collection).

As a byproduct of the proof of the last proposition we have the following simple fact that gives a sufficient condition in order for a set to be a model of LZFC. A transitive set $(a, \in)$ is said to be *directed* if it is upward directed as a poset, i.e., if for all $x, y \in a$ there is a $z \in a$ such that $x, y \in z$.

Corollary 2.17 *Let a be a transitive set which is the union of the transitive models of* ZFC *contained in it, that is, $a = \bigcup\{x \in a : x \models \mathrm{ZFC}\}$. If a satisfies also Pair, then $a \models \mathrm{LZFC}$. In particular, if $(a, \in)$ is a directed set of models of* ZFC*, such that $\cup a = a$, then $a \models \mathrm{LZFC}$.*

In the preceding result we can even replace models of ZFC with models of LZFC. Namely the following holds.

Lemma 2.18 *Let $(a, \in)$ be a directed set of models of* LZFC*, such that $\cup a = a$. Then $a \models \mathrm{LZFC}$.*

Proof. By directedness a satisfies *Pair.* So it suffices to show that $a \models$ *Loc*(ZFC). Let $x \in a$. Then there is $b \in a$ such that $x \in b$ and $b \models Loc(\mathrm{ZFC})$. Therefore $b \models \exists y(x \in y \wedge y \models \mathrm{ZFC})$. But then $a \models (\exists y(x \in y \wedge y \models \mathrm{ZFC}))^b$, hence $a \models \exists y(x \in y \wedge y \models \mathrm{ZFC})$, or $a \models Loc(\mathrm{ZFC})$. ⊣

Clearly if ZFC and LZFC are consistent theories, then ZFC $\not\subseteq$ LZFC and LZFC $\not\subseteq$ ZFC. Of the other set theories of the literature, close to the BST part of LZFC is Kripke-Platek set theory with infinity (KP + Infinity) (see [1], where rather the system KPU=KP+ urelements is considered). This is the system of axioms:

$$\mathrm{KP} = \{\mathrm{Empty}, \mathrm{Ext}, \mathrm{Pair}, \mathrm{Union}, \mathrm{Found}^*, \Delta_0\text{-Separation}, \Delta_0\text{-Collection}\},$$

where Found* is the scheme of $\in$-induction we already saw above to be equivalent to Found$_{On}$ (see Lemma 2.11 and before) and does not seem to follow from LZFC. It follows that

$$\mathrm{KP} + \mathrm{Infinity} - \{\Delta_0\text{-Collection}, \mathrm{Found}^*\} \subset \mathrm{LZFC}.$$

In connection with Remark 2.12, let us cite here the reasonable extensions of LZFC in which induction is valid.

Lemma 2.19

LZFC+Found$_{On}$ $\subseteq$ LZFC+Separation $\subseteq$ LZFC+Replacement $\subseteq$ LZFC+Collection.

Proof. The first inclusion follows from the discussion after Remark 2.9. The third inclusion is obvious. Concerning the inclusion LZFC + Separation $\subseteq$ LZFC + Replacement, the proof is no different from the familiar one that is used in ZFC. $\dashv$

The systems LZFC + Separation and LZFC + Replacement, apart from the fact that they restore transfinite induction, seem to be interesting in themselves extensions of LZFC.

A few further existence results for LZFC are given below.

Lemma 2.20 *(i) The Löwenheim-Skolem theorem is provable in* LZFC. *Namely, for every first-order language $\mathcal{L}$, every $\mathcal{L}$-structure $\mathcal{A} = (A, \ldots)$ and every $S \subseteq A$ such that $S \precsim \mathcal{L}$, there is an $\mathcal{L}$-structure $\mathcal{B} = (B, \ldots)$ such that $S \subseteq B$, $B \precsim \mathcal{L}$ and $\mathcal{B} \preceq \mathcal{A}$.*

(ii) The Mostowski's isomorphism theorem is provable in LZFC. *Namely if x is a set and E is a binary relation on x such that (a) E is well-founded and (b) $(x, E) \models \mathrm{Ext}$, then there is a (unique) transitive set y such that $(x, E) \cong (y, \in)$.*

(iii) The Completeness Theorem is provable in LZFC.

Proof. All three theorems, when formalized, are Π_2 sentences provable in ZFC, so the claim follows from lemma 2.4. $\dashv$

Lemma 2.21 ACA $\subset$ LZFC.

Proof. First-order Peano axioms, when transcribed into $\mathcal{L} = \{\in\}$, become Δ_0 sentences, since all quantifiers are restricted to ω. The induction axiom

$$\forall X[(0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X)]$$

becomes a Π_1 sentence, since $\forall X$ becomes $\forall x \subseteq \omega$. The arithmetic comprehension axiom is

$$\forall \bar{X}\ \exists Y \forall n(n \in Y \leftrightarrow \phi(n, \bar{X})),$$

where ϕ has no set quantifiers. In $\mathcal{L} = \{\in\}$ it becomes

$$\forall \bar{x}\ \exists y \forall z \in \omega(z \in y \leftrightarrow \psi(z, \bar{x})),$$

where ψ now is bounded, hence a Π_2 sentence provable in ZFC. Thus, in view of 2.4 (ii), both the induction axiom and the comprehension scheme of ACA are provable in LZFC, hence ACA $\subset$ LZFC. $\dashv$

Lemma 2.22 *(i) Let M be a countable transitive model of* ZFC *and let $B \in M$ be a Boolean algebra. Then it is provable in* LZFC *that there are M-generic filters $G \subseteq B$.*

(ii) For every M and generic G as above the generic extension $M[G]$ exists in LZFC.

Proof. (i) Given a countable M and the algebra $B \in M$, an M-generic filter $G \subseteq B$ is constructed by Choice as usual.

(ii) Given M, B an G as above, $M[G]$ is constructed by two inductive definitions: One that provides the set M^B of B-names over M, and another that leads from M^B and G to the G-interpretations of M^B, $I_G''M^B = M[G]$. Both definitions are inductive and absolute. So carrying them out inside any model N such that $M, B, G \in N$, is the same as carrying them out in V. ⊣

Consistency. What about the truth and consistency of LZFC? Let IC be the axiom "there exists a strongly inaccessible cardinal", IC^∞ be the axiom "there is a proper class of strongly inaccessible cardinals" and NM be the axiom "there is a natural model of ZFC" (i.e., of the form V_α). It is well known that the implications $\mathrm{IC}^\infty \rightarrow \mathrm{IC} \rightarrow \mathrm{NM}$ are strict over ZFC.

Proposition 2.23 *(i)* $\mathrm{LZFC} \subset \mathrm{ZFC} + \mathrm{IC}^\infty$.

(ii) $\mathrm{ZFC} + \mathrm{IC} \vdash Con(\mathrm{ZFC} + \mathrm{LZFC})$.

(iii) ZFC+NM $\vdash Con$(LZFC + "Every set is countable").

Proof. (i) Work in $\mathrm{ZFC} + \mathrm{IC}^\infty$. It suffices to prove that *Loc*(ZFC) holds. Then every set a belongs to some V_κ, where κ is strongly inaccessible. Since every such V_κ is a transitive model of ZFC, it follows that $\forall x \exists y (x \in y \wedge\ y \models$ ZFC$)$.

(ii) Let κ be an inaccessible in the ZFC universe. Then $V_\kappa \models$ ZFC + *Loc*(ZFC). Indeed, obviously $V_\kappa \models$ ZFC. It is well known (see [8], or [6, Ex. 12.12]) that $\{\alpha \in V_\kappa : (V_\alpha, \in) \prec (V_\kappa, \in)\}$ is closed unbounded in κ. Hence $\forall x \in V_\kappa \exists y (x \in y \wedge y \models \mathrm{ZFC})$. Thus $V_\kappa \models$ *Loc*(ZFC).

(iii) Let V_κ be a natural model of ZFC. It is well-known that κ is sufficiently large so that $H(\omega_1) \in V_\kappa$. $H(\omega_1)$ is the required model. Indeed, let $x \in H(\omega_1)$. Then $x \in V_\kappa$ and by Löwenheim-Skolem there is a countable model $N \prec V_\kappa$ such that $x \in N$. If N' is the Mostowski collapse of N, then N' is a transitive model that contains x and belongs to $H(\omega_1)$. Therefore, $H(\omega_1) \models$ *Loc*(ZFC). Moreover $H(\omega_1) \models$ "Every set is countable". ⊣

It follows from 2.23 (ii) that the consistency strength of LZFC is no greater than that of ZFC+NM. Also by 2.23 (iii), the consistency of ZFC+*Loc*(ZFC) is no greater than that of ZFC + IC. Moreover, ZFC + *Loc*(ZFC) is a good mild substitute of ZFC + IC^∞. It's worth mentioning that IC^∞ is equivalent to what in category theory is called "the axiom of universes", the origin of which goes back to Grothendieck. Roughly a "Grothendieck universe" is a transitive set closed under pairing, powerset and replacement. The axiom of universes says that every set belongs to a Grothendieck universe. It is likely that most or all of what the category theorists prove by the help of the axiom of universes, can be proved within ZFC + *Loc*(ZFC).

3 Cardinals and powersets in LZFC

Typically, we may keep talking about cardinals in LZFC, much the same way as we do in ZFC, but without expecting to prove the familiar ZFC results, due to the lack of Powerset, Replacement and also transfinite induction (Remark 2.12). The landscape of LZFC is hazy as far as absolute infinite cardinalities are concerned, and pitfalls are lurking everywhere for the visitor accustomed to ZFC.

We can define cardinals as usual. An ordinal α is a said to be a *cardinal* (in the sense of V) if it is an initial ordinal, i.e., if there is no $\beta \in On$ such that $\beta < \alpha$ and $\beta \sim \alpha$. For instance ω is a cardinal. In fact ω may be the only infinite cardinal (as it follows from Proposition 2.23 (ii)). ω_1 is the class of countable ordinals, i.e.,

$$\omega_1 = \{\alpha \in On : \alpha \precsim \omega\}.$$

In general this is a (proper) class. A class $X = \{x : \phi(x)\}$ is said to *exist,* if it is a set. So if ω_1 exists, it is a cardinal. If ω_1 is a proper class, can we infer that $\omega_1 = On$? Actually not. Because ω_1 is an initial segment of On, but in order to draw a contradiction from $On - \omega_1 \neq \emptyset$, the latter class should have a least element, which we cannot guarantee. If $\beta \in On - \omega_1$, then ω_1 would be a proper subclass of β. If M is a model of ZFC containing β, then $\omega_1 \subseteq \beta \subseteq M$, but $\omega_1^M \subsetneqq \omega_1$, i.e., $\omega_1^M \in \omega_1$, otherwise $\omega_1 = \omega_1^M$ and ω_1 would have to be a set.

If ω_1 exists, then we set $\omega_2 = \{\alpha \in On : \alpha \precsim \omega_1\}$, and similar remarks apply to this class. If ω_1 exists, then by *Loc*(ZFC) there is a model M of ZFC such that $\omega_1 \in M$. ω_1 is clearly a cardinal in M but not necessarily an absolute one with respect to M. It may be the case that $\omega_1^M < \omega_1$, and hence $\omega_1 = \omega_\alpha^M$, for some $\alpha > 1$. But even if $\omega_1^M = \omega_1$, ω_2^M (which is a set) need not be absolute, and ω_2 may be a proper class. In general, the class $X = \{\alpha \in On : \omega_\alpha \text{ exists}\}$ is defined, but we know neither whether $X = On$ nor whether $On - X$ has a least element.

Analogous comments hold about the power-class $\mathcal{P}(\omega)$ and the class

$$H(\omega_1) = \{x : TC(x) \precsim \omega\}$$

of hereditarily countable sets. If $\mathcal{P}(\omega)$ exists, then $\mathcal{P}(\omega)$ belongs to a model M and, obviously, $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)$. Since however M need not be a natural model, it is possible that $(\mathcal{P}^2(\omega))^M \neq \mathcal{P}^2(\omega)$ and, moreover, $\mathcal{P}^2(\omega)$ be a proper class. Again for the class $Y = \{\alpha : \mathcal{P}^\alpha(\omega) \text{ exists}\}$ we can say neither whether $Y = On$, nor whether $On - Y$ has a least element. Also, if $\mathcal{P}^n(\omega)$ exists for every $n \in \omega$, we cannot conclude that $\mathcal{P}^\omega(\omega)$ exists, since Replacement is missing.

Concerning $H(\omega_1)$, it is well-known that in ZFC we can code its elements by elements of $\mathcal{P}(\omega)$, constructing thus an embedding $f : H(\omega_1) \to \mathcal{P}(\omega)$. This is done by induction on the rank of the elements of $H(\omega_1)$ which goes up to ω_1. So this embedding cannot be carried out in LZFC.

The above uncertainties about absolute infinite cardinalities seem to fit to the spirit of LZFC. They prompt one to deal exclusively with models and let aside absolute uncountable infinities. However the uncertainties are settled as soon as we augment LZFC with Separation, which restores transfinite induction (see Lemma 2.19).

At this point I would like to address an ambiguity (that occurs also in the ZFC environment), concerning the meaning of the symbols ω_α. ω_α is allowed to denote alternatively (depending on the context) either an *object,* i.e., a specific ordinal, or a *property,* the property of being the α-th infinite cardinal. The ambiguity arises from the interplay of the two meanings within models of ZFC. For instance if $M \models$ ZFC, $\beta \in On \cap M$ and we write $M \models \beta = \omega_\alpha$, we refer to ω_α as a property, namely, the property "β is the α-th infinite cardinal number" (in the sense of M). The last assertion is alternatively denoted $\beta = \omega_\alpha^M$. Similarly, in the expression $M \models |x| = \omega_\alpha$, ω_α is construed as a property. Now assume that ω_α is a set. By *Loc*(ZFC) there is a model M such that $\omega_\alpha \in M$. If for some $x \in M$ we write $M \models x \sim \omega_\alpha$, then we refer to ω_α as an object which is involved in a property that is true in M. On the other hand, ω_α is still a cardinal in M, but it need not preserve also its size, i.e., we may have $\omega_\alpha = \omega_\beta^M$ for some $\beta > \alpha$. According to the usage of ω_β as a property, the latter is written equivalently $M \models \omega_\alpha = \omega_\beta$, which seems to be absurd. The absurdity is simply due to the ambiguity of the symbols $\omega_\alpha, \omega_\beta$: In the formula $M \models \omega_\alpha = \omega_\beta$, ω_α is construed as an object, while ω_β is construed as a property. The situation is no different in ZFC. Simply the (set) models we deal with there are, mostly, either *countable*, hence they do not contain real uncountable cardinals, or *natural,* in which all powersets and cardinals are absolute. The problematic situation is exactly when ω_α is uncountable, $\omega_\alpha \in M$ and $\omega_\alpha^M \neq \omega_\alpha$.

We can raise the ambiguity if we avoid using the symbols ω_α as properties and employ instead a predicate $Card(\alpha, x)$ for the property "x is the α-th infinite cardinal number". The predicate $Card(\alpha, x)$ is defined as follows. Let

$$Card(x) := x \in On \wedge \forall \beta < x(\beta \not\sim x)$$

be the property "x is a cardinal". Then the formula $Card(\alpha, x)$ is defined by the following clauses:

$$\begin{cases} Card(0, x) := [x = \omega] \\ Card(\alpha+1, x) := [Card(x) \wedge \forall y(Card(\alpha, y) \rightarrow y \precnsim x \wedge \\ \qquad\qquad \forall z(Card(z) \rightarrow z \precsim y \vee x \precsim z))] \\ Card(\alpha, x) := [(\forall \beta < \alpha \forall y(Card(\beta, y) \rightarrow y \precnsim x) \wedge \\ \qquad\qquad \forall z(Card(z) \rightarrow x \precsim z \vee \exists \gamma < \alpha \exists u(Card(\gamma, u) \wedge z \precsim u))], \\ \qquad\qquad \text{for } \alpha \text{ limit.} \end{cases}$$

Note that $Card(\alpha, x)$ is intended to be used inside models of ZFC, so the induction on α needed to verify $M \models Card(\alpha, x)$ is legitimate. Using the predicate $Card(\alpha, x)$, we write $M \models Card(\alpha, \beta)$ instead of $M \models \beta = \omega_\alpha$. If $\omega_\alpha \in M$ and ω_α happens to be the β-th cardinal of M, we express it by writing $M \models Card(\beta, \omega_\alpha)$ instead of the puzzling $M \models \omega_\alpha = \omega_\beta$. This way the ambiguity is removed.

Below we shall keep using the notation $\beta = \omega_\alpha^M$ as an abbreviation of $M \models Card(\alpha, \beta)$. Also $M \models |x| = \omega_\alpha$ will be an abbreviation of

$$M \models \exists \beta(x \sim \beta \wedge Card(\alpha, \beta)).$$

If ω_α exists and M is a model such that $\omega_\alpha \in M$, we say that ω_α is *absolute in* M if $\omega_\alpha^M = \omega_\alpha$, i.e., if $M \models Card(\alpha, \omega_\alpha)$. The following is easy to verify.

Lemma 3.1 (LZFC) *If* M, N *are models of* ZFC *such that* $M \subseteq N$, $x \in M$, $\alpha \in M$, *and* $M \models |x| = \omega_\alpha$, *then* $N \models |x| \leq \omega_\alpha$.

Proof. We just argue as usual inside the model N. $\dashv$

In general, if $n \in \omega$ and ω_n exists, we set $H(\omega_{n+1}) = \{x : TC(x) \precsim \omega_n\}$ and $\omega_{n+1} = \{\alpha \in On : \alpha \precsim \omega_n\}$.

Lemma 3.2 *In* LZFC*, for all $n \in \omega$, the following hold.*

(i) If $H(\omega_{n+1})$ exists, then so do $\mathcal{P}(\omega_n)$ and ω_{n+1}. In particular, if M is a model of ZFC *such that $H(\omega_{n+1}) \in M$, then $\mathcal{P}(\omega_n)^M = \mathcal{P}(\omega_n)$ and $\omega_{n+1}^M = \omega_{n+1}$.*

(ii) If $\mathcal{P}(\omega_n)$ exists and $\mathcal{P}(\omega_n) \in M$ then $\omega_{n+1}^M = \omega_{n+1}$ and $H(\omega_{n+1})^M = H(\omega_{n+1})$.

(iii) Suppose $\mathcal{P}^{n+1}(\omega)$ exists and $\mathcal{P}^{n+1}(\omega) \in M$. Then $\mathcal{P}(\omega_n) \in M$, hence $\omega_{n+1}^M = \omega_{n+1}$. Also $\mathcal{P}^{n+1}(V_\omega)^M = V_{\omega+n+1}^M = V_{\omega+n+1}$.

Moreover in LZFC+Separation*, the above claims are proved for every $\alpha \in On$. Namely:*

(iv) If $H(\omega_{\alpha+1}) \in M$, then $\mathcal{P}(\omega_\alpha)^M = \mathcal{P}(\omega_\alpha)$ and $\omega_{\alpha+1}^M = \omega_{\alpha+1}$.

(v) If $\mathcal{P}^\alpha(\omega) \in M$, then $\omega_\alpha^M = \omega_\alpha$ and $V_{\omega+\alpha}^M = V_{\omega+\alpha} = \mathcal{P}^\alpha(V_\omega)$.

Proof. For clarity and simplicity we show clauses (i) and (ii) for $n = 0$ and clause (iii) for $n = 1$. The inductive steps are straightforward and left to the reader.

(i) Suppose $H(\omega_1)$ is a set and M is a model such that $H(\omega_1) \in M$. Then $\mathcal{P}(\omega) \subseteq H(\omega_1) \subseteq M$, therefore $\mathcal{P}(\omega) = \mathcal{P}(\omega)^M$. Also, $\omega_1^M = \{\alpha \in On \cap M : M \models \alpha \precsim \omega\}$. Hence $\omega_1^M \subseteq \omega_1$. For the converse, let $\alpha \in \omega_1$ be an infinite ordinal. Then there is a bijection $f : \alpha \to \omega$. Clearly $f \in H(\omega_1)$, and hence $f \in M$. Since $\alpha = dom(f)$, $\alpha \in M$, therefore $\alpha \in \omega_1^M$. So $\omega_1^M = \omega_1$.

(ii) Suppose $\mathcal{P}(\omega)$ exists and let $\mathcal{P}(\omega) \in M$. We show first that $\omega_1^M = \omega_1$. As we saw above, $\omega_1^M \subseteq \omega_1$. To show the converse, pick some infinite $\alpha \in \omega_1$. It suffices to show that $\alpha \in M$ and $M \models \alpha \sim \omega$. Now there is (in V) a bijection $f : \omega \to \alpha$. Let

$$R = \{\langle m, n\rangle \in \omega \times \omega : f(m) \in f(n)\}.$$

$\omega \times \omega$ is a set and the defining property of R is Δ_0, so by Δ_0-Separation, R is a set too. Moreover R is a well-ordering of ω and $R \in \mathcal{P}(\omega \times \omega)$. Since $\mathcal{P}(\omega) \in M$, also $\mathcal{P}(\omega \times \omega) \in M$. Hence $R \in M$ and $M \models$ "(ω, R) is a well-ordering". So the order type of (ω, R) exists in M. But this order-type is α, i.e., $\alpha \in M$ and $M \models (\alpha, \in) \cong (\omega, R)$. Therefore $M \models \alpha \sim \omega$.

We come to the second claim of this clause, and let $\mathcal{P}(\omega) \in M$. We have to show that $H(\omega_1)^M = H(\omega_1)$, or $H(\omega_1) \subseteq H(\omega_1)^M$.[5] Let $x \in H(\omega_1)$, and let $f : TC(x) \to \omega$ be a bijection. Let N be a model of ZFC such that $\{\mathcal{P}(\omega), f\} \subset N$. In N we can define as usual a coding $g : H(\omega_1)^N \to \mathcal{P}(\omega)^N = \mathcal{P}(\omega)$. Now the pair $\langle x, f\rangle$ is an element of $H(\omega_1)^N$ and it is coded by $g(\langle x, f\rangle) \in \mathcal{P}(\omega)$. But since $\mathcal{P}(\omega) \in M$, $g(\langle x, f\rangle)$ is in M and from $g(\langle x, f\rangle)$ we can fully restore $\langle x, f\rangle$, i.e., $\langle x, f\rangle \in M$. Thus $x \in H(\omega_1)^M$.

(iii) We show the claim for $n = 1$. Let $\mathcal{P}^2(\omega) \in M$. Then $\mathcal{P}(\omega) \in M$, and hence $\omega_1^M = \omega_1$, by (ii). Every $\alpha \in \omega_1$ is coded by some well-ordering $R \in \mathcal{P}(\omega \times \omega)$ of ω, as we saw in (ii). Hence every $x \subseteq \omega_1$ is coded by some element of $\mathcal{P}^2(\omega \times \omega)$, or equivalently, of $\mathcal{P}^2(\omega)$. So $\mathcal{P}(\omega_1)$ is (coded by) a subset of $\mathcal{P}^2(\omega)$. This means that $\mathcal{P}(\omega_1) \in M$ and, by (ii), $\omega_2^M = \omega_2$. The other claim also follows easily.

(iv) and (v) need induction on α. Here we cannot work in any particular model of LZFC, so the induction must be carried out in V. This explains the use of Separation. $\dashv$

Concerning the converse of the claims (i)-(iii) above, some of them can be shown to be false (assuming the consistency of some basic theory). For instance it is consistent relative to ZFC +LZFC that in LZFC ω_1 exists, while $\mathcal{P}(\omega)$ is a proper class. Indeed, if ZFC +LZFC is consistent, then so is ZFC +LZFC $+\mathcal{P}(\omega) \sim \omega_2$. If K is a model of the last theory, then $H(\omega_2)^K$ is a model of LZFC + "ω_1 exists" + "$\mathcal{P}(\omega)$ does not exist".

4 Mahlo models

Transitive models of ZFC bear obvious analogies with inaccessible cardinals. Roughly a transitive $M \models$ ZFC is a "first-order counterpart" of an inaccessible cardinal, since both are transitive sets closed under the same basic closure conditions. These closure conditions are related with the two most powerful axioms of ZFC, Replacement and Powerset. First, a (strongly) inaccessible cardinal κ is closed under all functions f, in the sense that for every $\alpha \in \kappa$, $f''\alpha$ is bounded in κ. The corresponding property of a model M is that, in view of Replacement, for every $x \in M$, $f''x \in M$, provided f is *first-order definable* in M. (That is what we mean by saying that M is a first-order counterpart of an inaccessible cardinal). Second, for every cardinal $\lambda < \kappa$,

[5] The proof of this implication was provided by the referee.

$2^\lambda < \kappa$, and this obviously corresponds to the truth of Powerset in M, i.e., the fact that for every $x \in M$, $\mathcal{P}^M(x) \in M$.[6]

Consequently, a transitive M such that $M \models \mathrm{ZFC} + Loc(\mathrm{ZFC})$ is the analogue of a "quasi 1-Mahlo" cardinal in the following sense: $M \models Loc(\mathrm{ZFC})$ says that every $x \in M$ belongs to a $y \in M$ such that $y \models \mathrm{ZFC}$. That is, the set of transitive models contained in M form an unbounded (= cofinal) subclass of M under $\in$ (and $\subseteq$). This is just the property of being 1-Mahlo cardinal, except that "unbounded" should be replaced by "stationary". So M is "quasi 2-Mahlo" if $M \models \mathrm{ZFC} + Loc(\mathrm{ZFC} + Loc(\mathrm{ZFC}))$, and so on.[7] Stationarity, however, is a relative notion: It depends on what closed unbounded sets (clubs) are available. Absoluteness is obtained only if one is confined to the collection of *definable* clubs and stationary subsets of a model M. Before coming to the definition of stationary subsets of models, let us define inductively the iterated localization principles $Loc_n(\mathrm{ZFC})$, for $n \in \omega$, as follows:

$Loc_0(\mathrm{ZFC}) = Loc(\mathrm{ZFC})$,

$Loc_{n+1}(\mathrm{ZFC}) = Loc(\mathrm{ZFC} + Loc_n(\mathrm{ZFC}))$.

It is easy to check that for every $n \in \omega$, the sentence $Loc_n(\mathrm{ZFC})$ is Π_2. Moreover inductively we can see that

$$Loc_{n+1}(\mathrm{ZFC}) \ \rightarrow \ Loc_n(\mathrm{ZFC}). \tag{1}$$

Remark 4.1 Can we continue the definition of $Loc_\alpha(\mathrm{ZFC})$ for $\alpha \geq \omega$? The definition can be carried out at least along the constructive ordinals in a way analogous to that used in [5] for the definition of transfinite progressions of theories using the consistency operator: $\mathrm{T}_0 = \mathrm{T}$, $\mathrm{T}_{\alpha+1} = \mathrm{T}_\alpha + Con(\mathrm{T}_\alpha)$,

[6] Even in ZFC, the existence of a transitive model of ZFC can be thought as a weak large cardinal axiom, in view of the non-reversible implications

$$\mathrm{IC} \rightarrow \mathrm{NM} \rightarrow \mathrm{TM} \rightarrow Cns(\mathrm{ZFC}),$$

where

IC "There is an inaccessible cardinal",

NM: "There is a natural (i.e., of the form V_α) model of ZFC",

TM: "There is a transitive model of ZFC",

$Cns(\mathrm{ZFC})$: "ZFC is consistent".

[7] Note that the operator *Loc* can be applied not only to ZFC, but to any set theory S in the obvious way. Namely $Loc(\mathrm{S}) := \forall x \exists y (x \in y \wedge y \models \mathrm{S})$. In order however for the latter to make sense, S must be a *definable* set of axioms in a language $\mathcal{L}' \supseteq \mathcal{L}$. If S is defined by ϕ, then $Loc(\mathrm{S})$ is the $\mathcal{L}'$-sentence $\forall x \exists y (x \in y \wedge \forall z (\phi(z) \rightarrow y \models z))$.

$T_\alpha = \bigcup_{\beta<\alpha} T_\beta$. In ZFC one may also define Loc_α(ZFC) by using ordinals $\beta < \alpha$ as parameters. For example suppose that Loc_β(ZFC), for $\beta < \alpha$, have been defined, so that the mapping $\beta \mapsto Loc_\beta$(ZFC) is definable. Then, by Replacement, $\{Loc_\beta(\mathrm{ZFC}) : \beta < \alpha\}$ is a definable set, so, in view of footnote 7, we can set $Loc_\alpha(\mathrm{ZFC}) = Loc(\mathrm{ZFC} \cup \{Loc_\beta(\mathrm{ZFC}) : \beta < \alpha\})$. But in LZFC Replacement is not available, so $\{Loc_\beta(\mathrm{ZFC}) : \beta < \alpha\}$ need not be a set, and therefore iteration of *Loc* cannot go beyond constructive ordinals.

Recall that a cardinal κ is said to be Mahlo if the set of inaccessibles below κ is stationary in κ. Since the axioms Loc_n(ZFC) involve only the unboundedness of the class of models, just Mahloness of κ suffices in order for V_κ to satisfy Loc_n(ZFC).

Proposition 4.2 (ZFC) *Let κ be a Mahlo cardinal. Then $V_\kappa \models Loc_n(\mathrm{ZFC})$ for all $n \in \omega$.*

Proof. Let us define inductively for $n \in \omega$, that a cardinal κ is *n-unbounded* if it is inaccessible and for every $m < n$, the m-unbounded cardinals are unbounded in κ.

Claim 1. If κ is Mahlo, then κ is n-unbounded for all $n \in \omega$.

Proof. By induction on n. Trivially κ is 0- and 1-unbounded. Suppose κ is n-unbounded for $n \geq 1$. Then the $(n-1)$- unbounded cardinals are cofinal in κ. Let $\alpha < \kappa$. Let S be the set of limit points of $(n-1)$-unbounded above α. It is easy to check that S is a club. So, since κ is Mahlo, S contains an inaccessible β. This β is also a limit of $(n-1)$-unbounded cardinals, so it is an n-unbounded and lies above α. This means that the n-unbounded cardinals are cofinal in κ. Hence κ is $(n+1)$-unbounded.

Claim 2. If κ is $(n+1)$-unbounded, then $V_\kappa \models Loc_n(\mathrm{ZFC})$.

Proof. By induction on n. Let κ be 1-unbounded. Then the set $S \subset \kappa$ of inaccessibles below κ is unbounded in κ. For every $\lambda \in S$, $V_\lambda \models$ ZFC. Therefore V_κ satisfies $\forall x \exists y(x \in y \wedge y \models \mathrm{ZFC})$, i.e., $V_\kappa \models Loc_0(\mathrm{ZFC})$.

We assume that the claim holds for $n+1$ and we show it for $n+2$. Let κ be $(n+2)$-unbounded. The set $S \subset \kappa$ of $(n+1)$-unbounded cardinals is unbounded in κ. By the induction hypothesis, for every $\lambda \in S$, $V_\lambda \models Loc_n(\mathrm{ZFC})$. Therefore V_κ satisfies $\forall x \exists y(x \in y \wedge y \models \mathrm{ZFC} + Loc_n(\mathrm{ZFC}))$. The last sentence is $Loc(\mathrm{ZFC} + Loc_n(\mathrm{ZFC})) = Loc_{n+1}(\mathrm{ZFC})$.

Claims 1 and 2 yield the proof of the proposition. $\dashv$

The iterated localization principles $Loc_n(\mathrm{ZFC})$ are "weak Mahlo" principles intended to motivate the full Mahlo notion for models considered below. The latter presumes the notion of club and stationary set adapted here for that purpose. Unless otherwise stated, the definitions below are given in LZFC.

Definition 4.3 Let M be a transitive model of ZFC. A set $X \in Def(M)$ is said to be *unbounded in M,* if $(\forall x \in M)(\exists y \in X)(x \subseteq y)$. A $X \in Def(M)$ is said to be *closed,* if

$$(\forall y \in M)(y \subseteq X \wedge (y, \subseteq) \text{ is a chain } \rightarrow \cup y \in X).$$

A $X \in Def(M)$ is said to be a *club* of M if it is unbounded and closed. A $X \in Def(M)$ is said to be *stationary in* M if $X \cap Y \neq \emptyset$ for every club $Y \in Def(M)$.

For a model $M \models \mathrm{ZFC}$, a typical club of M is the set

$$\{M_\alpha : \alpha \in On \cap M\},$$

where $M_\alpha = V_\alpha^M$. For every $M \models \mathrm{ZFC}$, let

$$Club(M) = \{x \in Def(M) : x \text{ is closed unbounded in } M\},$$

$$Stat(M) = \{x \in Def(M) : x \text{ is stationary in } M\}.$$

Since $Def(M)$ is absolute, it follows that $Club(M)$ and $Stat(M)$ are absolute too. It is easy to see that for every M, $Club(M)$ is a proper subset of $Stat(M)$. For instance, if X is a club, $(y, \subseteq)$ is a chain of X and we set $Y = X - \{\cup y\}$, then $Y \in Stat(M) \backslash Club(M)$.

Given a transitive $M \models \mathrm{ZFC}$ and any unbounded $X \in Def(M)$, let $F_X^M : On^M \rightarrow On^M$ be defined as follows:

$$F_X^M(\alpha) = \mathrm{least}\{\beta : (\exists x \in X)(M_\alpha \subseteq x \subseteq M_\beta)\}.$$

Clearly $F_X^M \in Def(M)$. F_X^M is said to be the *associated function* to X with respect to M. We write simply F_X instead of F_X^M whenever M is understood. It follows from the definition that

$$(\forall \alpha \in On^M)(\exists x \in X)(M_\alpha \subseteq x \subseteq M_{F_X(\alpha)}). \tag{2}$$

Lemma 4.4 *For every M and every definable unbounded $X \subseteq M$, (a) F_X is nondecreasing, i.e., for all $\alpha < \beta \in M$, $F_X(\alpha) \le F_X(\beta)$. (b) For every $\alpha \in M$, $\alpha \le F_X(\alpha)$.*

Proof. (a) Let $\alpha < \beta$. Then $\exists x \in X(M_\beta \subseteq x \subseteq M_{F_X(\beta)})$. Since $M_\alpha \subseteq M_\beta$, we have $\exists x \in X(M_\alpha \subseteq x \subseteq M_{F_X(\beta)})$. Since $F_X(\alpha)$ is the least γ such that $\exists x \in X(M_\alpha \subseteq x \subseteq M_\gamma)$, it follows that $F_X(\alpha) \le F_X(\beta)$. (b) Just note that, by definition, $M_\alpha \subseteq M_{F_X(\alpha)}$, therefore $\alpha \le F_X(\alpha)$. $\dashv$

With the help of the function F_X one can prove the following closure properties of clubs. Since they are not going to be used in the proof of the main result of the section, Proposition 4.11, we omit the proofs.

Lemma 4.5 *(i) For any $X_1, X_2 \in Club(M)$, $X_1 \cap X_2 \in Club(M)$.*

(ii) Let $X \in Def(M)$ be a set of pairs coding a family of clubs of M. i.e., for every $x \in dom(X)$, $X_{(x)} = \{y : (x,y) \in X\}$ is a club. Then for every set $A \subseteq dom(X)$, $A \in M$, $\bigcap_{x\in A} X_{(x)} \in Club(M)$.

(iii) If $X_{(x)}, x \in M$, is an M-family of clubs of M, then $\triangle_{x\in M} X_{(x)}$ is a club (where $\triangle_{x\in M} X_{(x)}$ is the usual diagonal intersection of $X_{(x)}$). A fortiori *$\triangle_{x\in S} X_{(x)}$ is a club for every $S \in Def(M)$.*

We come to the definition of α-Mahlo models of ZFC.

Definition 4.6 (LZFC) *α-Mahlo models* of ZFC are defined inductively as follows:

(i) x is 0*-Mahlo* if x is transitive and $x \models$ ZFC.

(ii) x is $(\alpha+1)$*-Mahlo,* if x is transitive, $x \models$ ZFC and $\{y \in x : (y, \in\ )$ is an α-Mahlo model$\}$ is a stationary subset of x.

(iii) For α limit, x is α*-Mahlo* if it is β-Mahlo for all $\beta < \alpha$.

The above definition of α-Mahloness is formalized by the formula $mahlo(\alpha, x)$ defined by the following clauses (we omit only transitivity of x as implicitly understood):

$$\begin{cases} mahlo(0,x) := [x \models \text{ZFC}] \\ mahlo(\alpha+1,x) := [x \models \text{ZFC} \wedge (\forall y \in Club(x))(\exists u \in y)(mahlo(\alpha,u))] \\ mahlo(\alpha,x) := \forall\beta < \alpha\ mahlo(\beta,x), \text{ for } \alpha \text{ limit.} \end{cases} \tag{3}$$

The lack of induction on α does not prevent $mahlo(\alpha, x)$ from having a truth value for all α and x. This is because $mahlo(\alpha, x)$ is absolute, since $Club(x)$ is a Δ_1 property. Hence the induction on α needed to verify $mahlo(\alpha, x)$ can be carried out inside any model M containing x and α.

Lemma 4.7 (LZFC) *For each α, the sentence $mahlo(\alpha, x)$ is first-order and absolute for transitive models. That is, for every transitive model $M \models$ ZFC such that $\alpha, x \in M$, $mahlo(\alpha, x)$ iff $M \models mahlo(\alpha, x)$.*

Proof. By an easy induction on α, taking into account that the right-hand sides of the clauses of (3) are absolute. $\dashv$

Note that Mahloness alone (i.e., 1-Mahloness) implies the iterated localization axiom Loc_n(ZFC).

Proposition 4.8 (LZFC) *For every $n \in \omega$, if M is Mahlo then $M \models Loc_n$(ZFC).*

Proof. The proof is similar to that of proposition 4.2 so it is omitted. $\dashv$

Recall that the clubs of a cardinal κ are exactly the ranges of normal (i.e., strictly increasing and continuous) functions $f : \kappa \to \kappa$ (see e.g. [6, p. 92]). For every unbounded $X \subseteq M$ (in particular for every club), we defined above (see (2)) the associated function $F_X : On^M \to On^M$, which is nondecreasing rather than strictly increasing, and satisfies $F_X(\alpha) \geq \alpha$. Such functions can also be called normal when they are continuous.[8] Obviously every such function has fixed points above any ordinal, as usual. Using clubs X such that F_X is normal, we can relate clubs of M with clubs of κ.

Definition 4.9 Call a club $X \subseteq M$ *normal,* if the associated function F_X is normal.

Given $M \models$ ZFC, let

$$U_M = \{M_\alpha : \alpha \in M\}$$

[8] If f is simply nondecreasing, i.e., $\alpha < \beta \to f(\alpha) \leq f(\beta)$, $rng(f)$ may be bounded, which trivializes f. But if $rng(f)$ *is* unbounded, e.g. if $f(\alpha) \geq \alpha$, then strictness of monotonicity can be relaxed. This is the case with functions F_X.

be the typical club of M. For every $X \in Club(M)$, let us set

$$X^* = X \cap U_M.$$

By lemma 4.5 $X^* \in Club(M)$.

Lemma 4.10 *For every $X \in Club(M)$, X^* is a normal club.*

Proof. Since, by 4.4 (b), F_{X^*} is already nondecreasing, it suffices to show that F_{X^*} is continuous, i.e., for every limit α, $F_{X^*}(\alpha) = \sup\{F_{X^*}(\beta) : \beta < \alpha\}$. Now the elements of X^* are sets M_β. Let $X^- = \{\beta \in On^M : M_\beta \in X^*\}$. Then, by definition, for every β,

$$F_{X^*}(\beta) = \text{least}\{\gamma : (\exists x \in X^*) M_\beta \subseteq x \subseteq M_\gamma\} = \text{least}\{\gamma \in X^- : M_\beta \subseteq M_\gamma\}.$$

Therefore, for every $\beta < \alpha$, $M_\beta \subseteq M_{F_{X^*}(\beta)}$ and $M_{F_{X^*}(\beta)} \in X^*$. So

$$\bigcup_{\beta<\alpha} M_\beta = M_\alpha \subseteq \bigcup_{\beta<\alpha} M_{F_{X^*}(\beta)} = M_\gamma, \tag{4}$$

where $\sup\{F_{X^*}(\beta) : \beta < \alpha\} = \gamma$. But $\{M_{F_{X^*}(\beta)} : \beta < \alpha\} \subseteq X^*$, and the chain $\{M_{F_{X^*}(\beta)} : \beta < \alpha\}$ is in M. So, since X^* is a club, $\bigcup_{\beta<\alpha} M_{F_{X^*}(\beta)} = M_\gamma \in X^*$. Then (4) implies $F_{X^*}(\alpha) \leq \gamma$. On the other hand, by monotonicity of F_{X^*} (see 4.4 (a)), $F_{X^*}(\alpha) \geq \sup\{F_{X^*}(\beta) : \beta < \alpha\} = \gamma$. So $F_{X^*}(\alpha) = \gamma$ as required. $\dashv$

In view of lemma 4.10, a definable $Y \subseteq M$ is stationary iff it meets all normal clubs of M of the form X^* for $X \in Club(M)$. For every model $M \models$ ZFC, let $ht(M)$ (the height of M) be the supremum of the ordinals in M, that is, $ht(M) = M \cap On$.

Recall that

(i) κ is 0-Mahlo if it is strongly inaccessible.

(ii) κ is $(\alpha + 1)$-Mahlo, if the set of α-Mahlo cardinals below κ is a stationary subset of κ.

(iii) For limit α, κ is α-Mahlo if it is β-Mahlo for all $\beta < \alpha$.

Proposition 4.11 *(i) Let $M \models$ ZFC with $ht(M) = \alpha$. If $X \in Club(M)$, then $\{\beta < \alpha : M_\beta \in X^*\}$ is a club of α.*

(ii) (ZFC) *If κ is β-Mahlo, for $\beta < \kappa$, then V_κ is β-Mahlo.*

Proof. (i) Let $X \in Club(M)$ and let $X^- = \{\beta < \alpha : M_\beta \in X^*\}$. We have to show that X^- is a club of α. Let $\beta < \alpha$. It is clear that $rng(F_{X^*}) \subseteq X^-$. Since F_{X^*} is normal, it has a fixed point $\gamma > \beta$. Now $F_{X^*}(\gamma) = \gamma$ means that $\gamma \in X^-$, so X^- is unbounded. Further, let $\{\beta_\xi : \xi < \delta\}$ be an increasing sequence of X^-. Then $\{M_{\beta_\xi} : \xi < \delta\}$ is an increasing sequence of X^*. If $\beta = \sup\{\beta_\xi : \xi < \delta\}$, then $M_\beta = \bigcup_{\xi<\delta} M_{\beta_\xi}$, and $M_\beta \in X^*$, by the closedness of X^*. Therefore $\beta \in X^-$ and X^- is closed.

(ii) By induction on β. If κ is 0-Mahlo, then κ is strongly inaccessible, hence $V_\kappa \models$ ZFC, and thus V_κ is a 0-Mahlo model according to (3).

Suppose the claim holds for β and let κ be $(\beta+1)$-Mahlo. Then the set $Y = \{\lambda < \kappa : \lambda \text{ is } \beta\text{-Mahlo}\}$ is stationary in κ. Let $Y^+ = \{V_\lambda : \lambda \in Y\}$. Both Y and Y^+ are definable in V_κ. By the induction hypothesis, for every $x \in Y^+$, x is a β-Mahlo model. So it suffices to show that Y^+ is stationary in V_κ, or, in view of 4.10, that it meets all clubs X^* for $X \in Club(V_\kappa)$. Let $X \in Club(V_\kappa)$. Since $V_\kappa \cap On = \kappa$, by (i), the set $X^- = \{\alpha < \kappa : V_\alpha \in X^*\}$ is a club of κ. Therefore $Y \cap X^- \neq \emptyset$, hence $Y^+ \cap X^* \neq \emptyset$.

If β is limit then the claim follows immediately from the definitions. ⊣

5 Π^1_1-Indescribable models

The next question is whether models resembling higher large cardinals can be reasonably defined. After Mahlo the next candidate notion is that of a weakly compact model. However as is well-known weakly compact cardinals have several equivalent characterizations, through a partition property, a tree property, a compactness property, Π^1_1-indescribability, etc (see for example [6], §17). Although the most intuitively appealing characterization is the partition property, the one that seems to fit better to our context is Π^1_1-indescribability. Recall that a cardinal κ is Π^n_m-indescribable if for every $U \subseteq V_\kappa$ and every Π^n_m sentence ϕ (containing in prenex form m alternations of n-th order quantifiers starting with $\forall$), if $(V_\kappa, \in, U) \models \phi$, then there is $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \phi$. The following is standard (see [6, p. 297] for a proof).

Theorem 5.1 (Hanf-Scott) *A cardinal κ is weakly compact iff it is Π^1_1-indescribable.*

Definition 5.2 (LZFC) A transitive model $M \models \mathrm{ZFC}$ is said to be Π^1_1-*indescribable* if for every $U \in Def(M)$ and every Π^1_1 sentence ϕ, if $(M, \in, U, Def(M)) \models \phi$, then there is a transitive model $N \in M$ such that $U \cap N \in Def(N)$ and $(N, \in, U \cap N, Def(N)) \models \phi$.

In the above notation $Def(M)$, $Def(N)$ indicate the ranges for the second order quantifiers of ϕ. Π^1_1-indescribability is first-order definable and absolute for transitive models. That is, "M is Π^1_1-indescribable" iff $K \models$ "M is Π^1_1-indescribable" for any transitive model K such that $M \in K$.

That Π^1_1-indescribable models (can be consistently assumed to) exist is a consequence of the following:

Proposition 5.3 (ZFC) *If κ is weakly compact, then the model V_κ is Π^1_1-indescribable.*

Proof. This is immediate from 5.1 and lemma 5.4 below. ⊣

Lemma 5.4 (ZFC) *Let κ be a Π^1_1-indescribable cardinal. Then for every $U \in Def(V_\kappa)$, and every Π^1_1 sentence ϕ of $\mathcal{L}_2 \cup \{\mathbf{S}\}$ (where $\mathcal{L}_2$ is $\mathcal{L}$ augmented with second order variables and $\mathbf{S}(\cdot)$ is a unary predicate interpreted as U), if*

$$(V_\kappa, \in, U, Def(V_\kappa)) \models \phi,$$

then there is $\alpha < \kappa$ such that $U \cap V_\alpha$ is (first-order) definable in $(V_\alpha, \in)$ and $(V_\alpha, \in, U \cap V_\alpha, Def(V_\alpha)) \models \phi$.

Proof. Let κ be Π^1_1-indescribable. Let $U \in Def(V_\kappa)$, and let $U = \{x \in V_\kappa : V_\kappa \models \theta(x)\}$, for a first-order formula θ. Let also $\phi = \forall X \psi(X)$ be a Π^1_1 sentence, where $\psi(X)$ has no second order variables. Suppose $(V_\kappa, \in, U, Def(V_\kappa)) \models \phi$. Set $\sigma = \forall x(\mathbf{S}(x) \leftrightarrow \theta(x))$. Then clearly $(V_\kappa, \in, U, Def(V_\kappa)) \models \sigma$ and σ is first-order. So

$$(V_\kappa, \in, U, Def(V_\kappa)) \models \forall X \psi(X) \wedge \sigma,$$

or equivalently

$$(V_\kappa, \in, U) \models (\forall X)(X \in Def(V_\kappa) \rightarrow \psi(X)) \wedge \sigma. \tag{5}$$

Now it is well-known that $Def(V_\kappa)$ is Δ^1_1-definable over V_κ.[9] Therefore $(\forall X)(X \in Def(V_\kappa) \to \psi(X)) \wedge \sigma$ is Π^1_1 and hence, by Π^1_1- indescribability of κ, there is $\alpha < \kappa$ such that

$$(V_\alpha, \in, U \cap V_\alpha) \models (\forall X)(X \in Def(V_\alpha) \to \psi(X)) \ \wedge \sigma. \tag{6}$$

By the definition of σ, $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$ implies that $U \cap V_\alpha = \{x \in V_\alpha : V_\alpha \models \theta(x)\}$, that is, $U \cap V_\alpha \in Def(V_\alpha)$. Further $(V_\alpha, \in, U \cap V_\alpha) \models (\forall X)(X \in Def(V_\alpha) \to \psi(X))$ implies that

$$(V_\alpha, \in, U \cap V_\alpha, Def(V_\alpha)) \models (\forall X)\psi(X),$$

or $(V_\alpha, \in, U \cap V_\alpha, Def(V_\alpha)) \models \phi$, as required. $\dashv$

Proposition 5.5 *If M is a Π^1_1-indescribable model of* ZFC *then M is α-Mahlo for every $\alpha \in On^M$.*

Proof. By induction on α. Since M is a model of ZFC, it is 0-Mahlo. Let $\alpha = 1$. We have to show that $\{x \in M : (x, \in) \models \text{ZFC}\}$ is stationary. Let $C \in Club(M)$. There is a first-order formula $\theta(x)$ such that $x \in C \leftrightarrow M \models \theta(x)$. Let $\sigma = \forall x(\mathbf{S}(x) \leftrightarrow \theta(x))$. The fact that C is a club definable by $\theta(x)$ is expressed by writing

$$(M, \in, C, Def(M)) \models \sigma \wedge \text{“}\{x : \mathbf{S}(x)\} \text{ is a club”}.$$

The sentence $\sigma \wedge$“$\{x : \mathbf{S}(x)\}$ is a club” is first-order so, by Π^1_1-indescribability, there is $N \in M$, $N \models \text{ZFC}$, such that

$$(N, \in, C \cap N, Def(N)) \models \sigma \wedge \text{“}\{x : \mathbf{S}(x)\} \text{ is a club”}.$$

This means that $\theta(x)$ defines $C \cap N$ in N and $C \cap N$ is a club of N. So if $N_\alpha = V^N_\alpha$ for $\alpha \in N$, we can pick by induction, using Choice, sets $x_\alpha \in C \cap N$, $\alpha \in M$, such that $N_\alpha \cup (\bigcup_{\beta<\alpha} x_\beta) \subseteq x_\alpha$. If $X = \{x_\alpha : \alpha \in N\}$, then clearly $X \in M$, $X \subseteq C$ and X is a chain. Therefore $\bigcup X \in C$. But $\bigcup X = N$, so $N \in C$. It follows that the arbitrary club C of M contains a model $N \models \text{ZFC}$. Therefore $\{x \in M : (x, \in) \models \text{ZFC}\}$ is stationary in M.

[9]Namely, $X \in Def(V_\kappa) := (\exists\phi)(\forall x)(x \in X \leftrightarrow Sat(\phi, x))$, where $Sat(\phi, x)$ is the Δ^1_1 satisfaction predicate for first order formulas with parameters over V_κ.

Suppose M is $(\alpha+1)$-Mahlo. Let $C \subseteq M$ be again a club defined by $\theta(x)$ in M and let σ be as above. Then

$$(M, \in, C, Def(M)) \models \ \sigma \wedge \ \{x : \mathbf{S}(x)\} \text{ is a club } \wedge \forall X(X \text{ is a club} \rightarrow$$
$$\exists y(y \in X \wedge mahlo(\alpha, y)).$$

The last formula is Π^1_1 over $(M, \in, C, Def(M))$ and says that C is a club and that the definable set $\{x \in M : (x, \in) \text{ is } \alpha\text{-Mahlo}\}$ is a stationary set of M. By definition 5.2, there is $N \in M$, $N \models$ ZFC, such that

$$(N, \in, C \cap N, Def(N)) \models \ \sigma \wedge \ \{x : \mathbf{S}(x)\} \text{ is a club } \wedge \forall X(X \text{ is a club} \rightarrow$$
$$\exists y(y \in X \wedge mahlo(\alpha, y)).$$

This says that $C \cap N$ is a club of N defined by $\theta(x)$ in N and the set of α-Mahlo models contained in N is a stationary subset of N. It follows that N is $(\alpha+1)$-Mahlo. Moreover, by the same argument as before, we see that $N \in C$. So the arbitrary club C of M contains an $(\alpha+1)$-Mahlo model. Therefore the set $\{x \in M : (x, \in) \text{ is } (\alpha+1)\text{-Mahlo}\}$ is stationary in M, and hence M is $(\alpha+2)$-Mahlo.

Suppose α is limit and M is α-Mahlo. To show that M is $(\alpha+1)$-Mahlo the proof is essentially the same as before.

Finally, if α is limit and the claim holds for all $\beta < \alpha$, then, due to the definition of α-Mahlo, the claim holds for α. $\dashv$

Question. What other large cardinal properties (measurability, strong compactness, etc) can be adjusted to fit to models of ZFC?

6 Localizing extensions of ZFC

In section 5 we have already considered extensions of *Loc*(ZFC) of the form *Loc*(ZFC + *Loc*(ZFC)), *Loc*(ZFC + *Loc*(ZFC + *Loc*(ZFC))), etc. Here we shall consider more general extensions, namely localization principles of the form *Loc*(ZFC $+ \phi$) for various sentences ϕ independent from ZFC. In order however for *Loc*(ZFC $+ \phi$) to make sense we must first assume that ZFC $+ \phi$ not only is consistent but has a transitive model. So by analogy with the axiom TM(ZFC) ("ZFC has a transitive model"), for every such ϕ one has to accept

$$(TM(\text{ZFC}+\phi)) \qquad\qquad \exists x(Tr(x) \wedge (x, \in) \models \text{ZFC}+\phi).$$

For several natural sentences like $V = L$, CH, $V \neq L$, $\neg$CH etc, it is provable in ZFC (by usual forcing techniques, constructible sets, etc) that $TM(\text{ZFC}) \rightarrow TM(\text{ZFC} + \phi)$.[10] The same proof can be carried out (relativized) in LZFC. Actually given a transitive model $M \models \text{ZFC}$, there is, by $Loc(\text{ZFC})$, a transitive N such that $M \in N$. In N we can find a countable model M' of ZFC and then extend it by forcing, e.g. to a model $M'[G]$ of $\text{ZFC} + \neg\text{CH}$.

In the formulas occurring below as arguments in $Loc(\cdots)$ we allow the use of a constant "c". This is not a parameter, but ranges over definable classes that are proved in ZFC to be sets (like $\mathcal{P}(\omega)$, ω_1, etc). Below we refer to such classes as "terms". For the same reason ordinals occurring as parameters in formulas occurring as arguments in $Loc(\cdots)$ are definable too.

Axioms $Loc(\text{ZFC} + \phi)$, though local in essence, may have global consequences for the universe V itself. For example:

Lemma 6.1 (LZFC) *Let c be a term. Then $Loc(ZFC + V = L(c)) \rightarrow V = L(c)$.*

Proof. Assume $Loc(\text{ZFC} + V = L(c))$ and $V \neq L(c)$. Let $a \in V - L(c)$. Then there is a transitive model M of ZFC such that $\{c, a\} \in M$ and $M \models V = L(c)$. But then $a \in M = L(c)^M \subseteq L(c)$, a contradiction. $\dashv$

More generally, given a set of sentences Γ, we may extend LZFC to

$$\text{LZFC}_\Gamma = \text{LZFC} + \{Loc(\text{ZFC} + \phi) : \phi \in \Gamma\}$$

and consider its consistency and its consequences on V. The following is a simple general fact concerning the consistency of LZFC_Γ.

Proposition 6.2 *If Γ is a set of sentences such that $\{\phi, \neg\phi\} \subseteq \Gamma$ for some Σ_1^{ZFC} or Π_1^{ZFC} sentence ϕ, then LZFC_Γ is inconsistent.*

Proof. Let ϕ be a Σ_1^{ZFC} sentence (the case of Π_1^{ZFC} is the same). This means that $\phi \leftrightarrow \exists x \phi_1(x)$ holds in every model of ZFC, for some Δ_0 formula ϕ_1. ZFC_Γ contains the axioms $Loc(\text{ZFC}+\phi)$ and $Loc(\text{ZFC}+\neg\phi)$. By the first

[10]However one cannot prove in ZFC, if ZFC is consistent, the implication $TM(\text{ZFC}) \rightarrow TM(\text{ZFC} + TM(\text{ZFC}))$, otherwise $\text{ZFC} + TM(\text{ZFC}) \vdash Con(\text{ZFC} + TM(\text{ZFC}))$, contrary to Gödel's incompleteness. In particular, ZFC does not prove, if it is consistent, that there is a forcing extension $M[G]$ of a transitive model, that contains a transitive model of ZFC.

of them there is a transitive M such that $M \models \mathrm{ZFC}+\phi$. Then $M \models \exists x \phi_1(x)$, hence $M \models \phi_1(a)$ for some $a \in M$. By $Loc(\mathrm{ZFC}+\neg\phi)$ there is N such that $a \in N$ and $N \models \mathrm{ZFC}+\neg\phi$. Then $N \models \forall x \neg\phi_1(x)$. But $N \models \phi_1(a)$, since ϕ_1 is absolute. A contradiction. $\dashv$

Does any reasonable set Γ affect the status of the axioms of Powerset, Separation, Replacement, etc? (Remember that LZFC itself is compatible with ZFC). The answer is positive for Powerset. We show that if Γ contains the sentences CH and $\neg$CH, then LZFC_Γ refutes Powerset.

Given a term c and a transitive model M, let c^M denote the relativization of c with respect to M. Let us call a term c *stable* if for every transitive M, $c \subseteq M \Rightarrow c^M = c$. For instance $\mathcal{P}(\omega)$ and $H(\omega_1)$ are stable terms, while ω_1 is not.

Proposition 6.3 *Let c be a stable term. Then the theory*

$$\mathrm{LZFC}+Loc(\mathrm{ZFC}+|c| = \omega_1)+Loc(\mathrm{ZFC}+|c| \neq \omega_1)+\text{ “}c\text{ exists”}+\text{“}\mathcal{P}(\omega)\text{ exists”}$$

is inconsistent.

Proof. Suppose the above mentioned theory is consistent and let K be a model of it. In K, c and $\mathcal{P}(\omega)$ are sets. By $Loc(\mathrm{ZFC}+|c| = \omega_1)$ and Pair, we can pick a model $M \in K$ of ZFC such that $\{c, \mathcal{P}(\omega)\} \subset M$ and $M \models |c| = \omega_1$. The last relation says that in M there is a bijection $h : c^M \to \omega_1^M$. Since c is stable, $c^M = c$. Also, since $\mathcal{P}(\omega) \in M$, by Lemma 3.2 (ii), $\omega_1^M = \omega_1 \in M$. Therefore c and ω_1 are both absolute in M and $M \models c \sim \omega_1$. Hence also $K \models c \sim \omega_1$. Let $h : c \to \omega_1$ be a bijection in K. By $Loc(\mathrm{ZFC}+|c| \neq \omega_1)$ and Pair, there is a model $N \in K$ such that $\{c, h, \mathcal{P}(\omega)\} \subset N$ and $N \models |c| \neq \omega_1$. Again, by stability $c^N = c$, and by 3.2 (ii), $\omega_1^N = \omega_1$. Hence $N \models c \not\sim \omega_1$. But this contradicts the fact that N already contains a bijection $h : c \to \omega_1$. $\dashv$

Corollary 6.4 *(i) For every stable term c, the theory*

$$\mathrm{LZFC}+Loc(\mathrm{ZFC}+|c| = \omega_1)+Loc(\mathrm{ZFC}+|c| \neq \omega_1)+\text{ “}c\text{ exists”}+\text{ Powerset}$$

is inconsistent.

(ii) In particular, the theory

$$\mathrm{LZFC}+Loc(\mathrm{ZFC}+|\mathcal{P}(\omega)| = \omega_1)+Loc(\mathrm{ZFC}+|\mathcal{P}(\omega)| \neq \omega_1)+\text{Powerset},$$

or, equivalently,

$$\text{LZFC} + \mathit{Loc}(\text{ZFC} + \text{CH}) + \mathit{Loc}(\text{ZFC} + \neg\text{CH}) + \text{Powerset}$$

is inconsistent.

Proof. (i) This follows immediately from 6.3, if we replace "$\mathcal{P}(\omega)$ exists" with the stronger Powerset.

(ii) In (i) above we set $c = \mathcal{P}(\omega)$, which is stable. Then Powerset implies "$\mathcal{P}(\omega)$ exists" and the claim follows.

[It's worth noting that, for this specific term $c = \mathcal{P}(\omega)$, the claim can be alternatively proved (without appealing to 6.3) as follows: Suppose $\mathcal{P}(\omega)$ is a set. Pick a model M such that $\mathcal{P}(\omega) \in M$ and $M \models |\mathcal{P}(\omega)| = \omega_1$. Then pick a model N such that $M \in N$, hence $M \subseteq N$, and $N \models |\mathcal{P}(\omega)| \neq \omega_1$. Then either $N \models |\mathcal{P}(\omega)| < \omega_1$, or $N \models |\mathcal{P}(\omega)| > \omega_1$. The first option is obviously false. So $N \models |\mathcal{P}(\omega)| > \omega_1$. But by Lemma 3.1 (i), $M \models |\mathcal{P}(\omega)| = \omega_1$ and $M \subseteq N$ imply $N \models |\mathcal{P}(\omega)| \leq \omega_1$, a contradiction.] $\dashv$

In relation to clause (ii) of the last Corollary we point out the following (recall that NM is the assertion "there is a natural model of ZFC").

Proposition 6.5 *The theory*

$$\text{LZFC} + \mathit{Loc}(\text{ZFC} + \text{CH}) + \mathit{Loc}(\text{ZFC} + \neg\text{CH})$$

is consistent relative to ZFC + NM.

Proof. The proof is an easy strengthening of that of Proposition 2.23 (iii). Let M be a model of ZFC + NM. Then $H^M(\omega_1)$ satisfies the theory in question. Indeed, if $M_\kappa = V_\kappa^M$ is a natural model of ZFC (in the sense of M), then by the proof of 2.23 (iii), $H^M(\omega_1) \models \mathit{Loc}(\text{ZFC})$. So for every $x \in H^M(\omega_1)$, x belongs to a countable transitive model $N \in H^M(\omega_1)$. Now every such model N containing x can be generically extended to countable transitive models N_1, N_2, satisfying CH and ¬CH, respectively. Since N_1, N_2 also belong to $H^M(\omega_1)$, $H^M(\omega_1)$ satisfies both Loc(ZFC+CH) and Loc(ZFC+¬CH). $\dashv$

Finally we have a variant of 6.3 from which "$\mathcal{P}(\omega)$ exists" has been dropped.

Proposition 6.6 *Let c be a stable term and let $\alpha \neq \beta$ be two distinct definable ordinals. Then the theory*

$$\mathrm{LZFC} + Loc(\mathrm{ZFC} + |c| = \omega_\alpha) + Loc(\mathrm{ZFC} + |c| = \omega_\beta) + \text{ “}c\text{ exists”}$$

is inconsistent.

Proof. We work in the aforementioned theory and suppose $\alpha < \beta$. c is a definable set, absolute for the models they contain it, hence by $Loc(\mathrm{ZFC} + |c| = \omega_\alpha)$, there is a model M of ZFC such that $c \in M$ and $M \models |c| = \omega_\alpha$. Then, by $Loc(\mathrm{ZFC} + |c| = \omega_\beta)$, there is a model N of ZFC such that $M \in N$ and $N \models |c| = \omega_\beta$. But $M \subseteq N$ and by Lemma 3.1 $M \models |c| = \omega_\alpha$ implies $N \models |c| \leq \omega_\alpha$, i.e., $\beta \leq \alpha$, contrary to the assumption $\alpha < \beta$. $\dashv$

7 A digression: Standard compactness

For reasons explained in the introduction, one of the goals of this paper was to promote transitive models of ZFC to the status of first class citizens of the universe of sets, especially by postulating their “omnipresence”. In particular, whenever we talk about models in LZFC, we mean *transitive* models. Given that models in general is the stuff of the various notions of compactness, the confinement to transitive models induces natural refinements of corresponding compactness notions. Specifically, in ordinary compactness one infers the existence of a model for a set of sentences from the existence of models for its finite parts. A natural question arisen from this fact is the following: Can we infer the existence of a *transitive* model for a set of sentences Σ in a language $\mathcal{L}'$ extending the language $\mathcal{L}$ of set theory, from the existence of *transitive* models for certain parts of Σ? Although otherwise unrelated to the rest of the paper, this question is well-motivated by our insistence on transitive models and shall be dealt with in this section. The question we formulated above prompts the following definition.

Definition 7.1 (ZFC) A cardinal κ is said to be *standard compact* if for every set of sentences Σ of a finitary language $\mathcal{L}' \supseteq \mathcal{L}$ such that $|\Sigma| = \kappa$, if every set $A \subseteq \Sigma$ such that $|A| < \kappa$ has a transitive model, then Σ has a transitive model.

A first negative result is that the standard version of the classical compactness theorem is false.

Proposition 7.2 (ZFC or LZFC) *There is an $\mathcal{L}' \supseteq \mathcal{L}$ and a countable set Σ of sentences of $\mathcal{L}'$ such that every finite subset of Σ has a transitive model, while Σ does not. Therefore ω is not standard compact.*

Proof. Let $\mathcal{L}' = \mathcal{L} \cup \{c_n : n \in \omega\}$, and let $\Sigma = \{c_{n+1} \in c_n : n \in \omega\}$ be a set of sentences of $\mathcal{L}'$. Then clearly every finite subset of Σ has a transitive model, while Σ does not. $\dashv$

Next let us make sure that standard compact cardinals exist under the assumption of mild large cardinals. Recall that one of the equivalent definitions of a weakly compact cardinal is the following: κ is weakly compact if any set Σ of sentences of the infinitary language $\mathcal{L}_{\kappa,\kappa}$, which uses at most κ non-logical symbols and is κ-satisfiable (i.e., every $A \subseteq \Sigma$ with $|A| < \kappa$ is satisfiable), is satisfiable.

Recall also that by Mostowski's theorem 2.20 (ii), if E is a binary relation on X such that (a) E is well-founded and (b) $(X, E) \models \mathrm{Ext}$, then there is a (unique) transitive set M such that $(X, E) \cong (M, \in)$. Ext is the ordinary extensionality axiom, while well-foundedness is expressed by a sentence of $\mathcal{L}_{\omega_1,\omega_1}$ as follows:

$$\mathrm{Wf} := \neg(\exists_{n<\omega} x_n)(\bigwedge_{n<\omega}(x_{n+1} \in x_n)),$$

where $\exists_{n<\omega} x_n$ is an abbreviation of the infinite block of quantifiers $\exists x_1 \exists x_2 \cdots \exists x_n \cdots$. Every transitive set satisfies Ext and Wf. Conversely, every $\mathcal{L}$-structure (X, E) such that $(X, E) \models \mathrm{Ext} \wedge \mathrm{Wf}$ is isomorphic to a transitive model. This is a key fact by which we can prove the following:

Lemma 7.3 *Every weakly compact cardinal $\kappa > \omega$ is standard compact.*

Proof. Let $\kappa > \omega$ be a weakly compact cardinal and let Σ be a set of sentences of $\mathcal{L}' \supseteq \mathcal{L}$ such that $|\Sigma| = \kappa$. Suppose that every $A \subseteq \Sigma$ with $|A| < \kappa$ has a transitive model. Let $\Sigma' = \Sigma \cup \{\mathrm{Ext}, \mathrm{Wf}\}$. Σ' is a set of sentences of $\mathcal{L}_{\omega_1,\omega_1}$, and hence of $\mathcal{L}_{\kappa,\kappa}$. From the assumption about Σ and the remarks concerning Wf, every $A \subseteq \Sigma'$ with $|A| < \kappa$ has a (transitive) model. By weak compactness of κ, Σ' has a model (X, E). Since this satisfies Ext and Wf, it is isomorphic to a transitive model M. Thus Σ', and therefore Σ, has a transitive model. $\dashv$

Proposition 7.4 (ZFC) ω_1 *is not standard compact. Similarly for* ω_n, *for every* $n \in \omega$.

Proof. Let $\mathcal{L}' = \mathcal{L} \cup \{c\} \cup \{\dot{\alpha} : \alpha \leq \omega_1\}$, where c and $\dot{\alpha}$ are constants. We shall find a Σ that refutes standard compactness of ω_1. Let Σ be the set of the following sentences of $\mathcal{L}'$:

(1) $Ord(c)$, $Ord(\dot{\alpha})$, for all $\alpha \leq \omega_1$.

(2) $\dot{\alpha} < \dot{\beta}$, for all $\alpha < \beta \leq \omega_1$.

(3) $c > \dot{\alpha}$, for all $\alpha < \omega_1$.

(4) $c < \dot{\omega}_1$.

(5) $\forall x(x < \dot{\omega}_1 \rightarrow x$ is countable).

Clearly $|\Sigma| = \omega_1$. Let $A \subseteq \Sigma$ with $|A| < \omega_1$. Pick some V_ξ such that $\omega_1 \in V_\xi$ and let $\dot{\alpha}^{V_\xi} = \alpha$ for all $\alpha \leq \omega_1$. Then we easily see that $V_\xi \models A$ for some interpretation $c^{V_\xi} \in \omega_1$. On the other hand suppose there is a transitive structure $(K, \in)$ such that $K \models \Sigma$. Although K need not be a model of ZFC, $K \models Ord(\dot{\alpha})$ clearly entails that $\dot{\alpha}^K$ is an ordinal. In view of (2) the mapping $\alpha \mapsto \dot{\alpha}^N$ is strictly increasing. Therefore $\alpha \leq \dot{\alpha}^K$ for every $\alpha \leq \omega_1$, and hence $\omega_1 \leq \dot{\omega}_1{}^K$. By (5) every $x \in \dot{\omega}_1{}^K$ is countable, so in particular $\dot{\omega}_1{}^K = \omega_1$. In view of this and (3) and (4), we have $\alpha \leq \dot{\alpha}^K < c^K < \dot{\omega}_1{}^K = \omega_1$, hence $\alpha < c^K < \omega_1$, for all $\alpha < \omega_1$, which is a contradiction.

In the case of ω_n we just need to replace "α is countable" with the appropriate sentence defining ω_n, namely: "x is countable or of cardinality next to countable, or next to next to countable or,...., or next^n to countable". $\dashv$

The property of weak compactness (as well as that of standard compactness) contains the condition that the cardinality of non-logical symbols (or the cardinality) of Σ be $\leq \kappa$. If we drop this condition we have the property of strong compactness: κ is strongly compact if for every set Σ of sentences of $\mathcal{L}_{\kappa,\kappa}$, if Σ is κ-satisfiable, then Σ is satisfiable. An equivalent definition (see [7, p. 37]) is the following:

Definition 7.5 (ZFC) A cardinal κ is *strongly compact* if for any set X, every κ-complete filter on X can be extended to a κ-complete ultrafilter on X.

Proposition 7.6 (ZFC) *Let* $\lambda > \omega$ *be a strongly compact cardinal. Then every cardinal* $\kappa \geq \lambda$ *such that* $\kappa^{<\kappa} = \kappa$ *is standard compact.*

Proof. The proof is a variant of the proof of compactness by the use of ultraproducts (see [2, Cor. 4.1.11]). Let $\kappa \geq \lambda > \omega$, where λ is strongly compact and $\kappa^{<\kappa} = \kappa$. Let Σ be an infinite set of sentences of a language $\mathcal{L}' \supseteq \mathcal{L}$ such that $|\Sigma| = \kappa$. Suppose each $A \subseteq \Sigma$ with $|A| < \kappa$ has a transitive model. Since $\kappa^{<\kappa} = \kappa$, there is an enumeration Σ_α, $\alpha < \kappa$, of all subsets A of Σ with $|A| < |\Sigma|$. Pick and fix for each $\alpha < \kappa$ a transitive model $M_\alpha \models \Sigma_\alpha$. For every $\phi \in \Sigma$, let $\hat{\phi} = \{\beta < \kappa : \phi \in \Sigma_\beta\}$. The family $E = \{\hat{\phi} : \phi \in \Sigma\}$ is κ-complete, i.e., for every $\gamma < \kappa$ and every $\{\hat{\phi_\beta} : \beta < \gamma\} \subseteq E$, $\bigcap_{\beta<\gamma} \hat{\phi_\beta} \neq \emptyset$. This is because for every $\gamma < \kappa$ and every set $\{\phi_\beta : \beta < \gamma\}$, there is a $\delta < \kappa$ such that $\{\phi_\beta : \beta < \gamma\} = \Sigma_\delta$, so $\delta \in \bigcap_{\beta<\gamma} \hat{\phi_\beta}$. Thus the filter $\bar{E}$ on κ generated by E is κ-complete. Also $\bar{E}$ is free, otherwise some α would be in all $\hat{\phi}$, $\phi \in \Sigma$, hence $\Sigma = \Sigma_\alpha$, which is false. One can see as in [2, Cor. 4.1.11] that if D is any ultrafilter on κ extending $\bar{E}$, then $\Pi_{\alpha<\kappa} M_\alpha/D \models \Sigma$. Namely for every $\phi \in \Sigma$,

$$\{\alpha < \kappa : M_\alpha \models \phi\} \supseteq \{\alpha < \kappa : \phi \in \Sigma_\alpha\} = \hat{\phi} \in D,$$

so $\Pi_{\alpha<\kappa} M_\alpha/D \models \phi$ by the fundamental theorem of ultraproducts. It suffices to choose the ultrafilter $D \supseteq \bar{E}$ so that $\Pi_{\alpha<\kappa} M_\alpha/D$ be (isomorphic to) a transitive model. Now $\bar{E}$ is a κ-complete filter and hence λ-complete since $\lambda \leq \kappa$. But λ is strongly compact, so $\bar{E}$ can be extended to a λ-complete ultrafilter D. Since $\lambda > \omega$ and every M_α is transitive, the ultraproduct $\Pi_{\alpha<\kappa} M_\alpha/D$ is well-founded. Therefore $\Pi_{\alpha<\kappa} M_\alpha/D$ is isomorphic to a transitive $(N, \in)$. Then $(N, \in) \models \Sigma$ as required. $\dashv$

It follows from the last result that, unless strongly compact cardinals are inconsistent, it is consistent to have standard compact cardinals which are accessible, singular and even successor cardinals.

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