Combinatorics related to NF consistency

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Key words and phrases. NF consistency, coherent pair, coherent triple, n-extendible coherent pair, n-augmentable coherent pair, simple partition.

Abstract

We elaborate on and refine certain aspects of the approach to NF consistency developed in [8], through coherent pairs and their extendibility properties. Some further results notions and problems are presented. First a quick proof of NF_3 consistency is provided. Next an alternative equivalent formalization of coherent pair, in terms of "coherent triples" of partitions is given. Extendibility is closely inspected and it is shown that instead of general partitions, only "simple partitions", i.e., partitions consisting of infinite and one-element sets, can be used throughout. Also a property weaker than n-extendibility, called "n-augmentability", is presented. Some particular n-augmentability questions are proved in the affirmative, while others, especially the appealing $(0, 0, n)$-augmentability, remain open. A partial case of this question is settled, while the source of its hardness is discussed. Finally it is briefly sketched how all these questions can be phrased as combinatorial problems of ZFC alone, without any reference to models of TST.

1 Introduction

In this paper we elaborate on the approach to NF consistency developed in [8] and try to refine, sharpen and improve some of the notions and results presented there. The paper is organized as follows: In section 2 we survey the basic definitions and results of [8], giving in addition a slightly different formalization of pieces of type-shifting automorphisms, in terms of coherent triples of partitions. In section 3 we give a short proof of the consistency of the fragment NF_3 using coherent pairs adjusted to models of TST_3. In section 4 a closer examination of the key property of n-extendibility is attempted which leads to a reduction of partitions and coherent pairs to *simple* partitions and *simple* coherent pairs. In section 5 the weaker property of n-augmentability is considered, which follows naturally from the "unfolding" of the extendibility formulation. In subsection 5.1, the special cases of $(n, 0, 0)$- and $(0, n, 0)$-augmentability are considered and proved for the trivial pair of a rich model of TST_4. In contrast $(0, 0, n)$-augmentability of the trivial pair, for $n > 2$, is still open and in subsection 5.2 we discuss certain aspects of this question and prove a partial result. This is a particularly appealing and natural question whose affirmative answer would be a nice strengthening of theorem 3.6 of [8], since the hard case of that result is equivalent to $(0, 0, 2)$-augmentability. All extendibility and augmentability questions are purely combinatorial in essence, asking how elements of finite Boolean algebras distribute over the atoms of corresponding similar Boolean algebras lying at next higher levels of a TST model. So in section 6 we describe briefly how all the preceding notions and questions can be phrased as combinatorial problems without any mention of TST models, just referring only to full models which are quite familiar objects of ZFC.

2 Survey of coherent pairs

Recall that as a consequence of the fundamental contributions [7] and [3] the following are equivalent:

(1) NF is consistent

(2) NF_4 is consistent

(3) There is a model $\mathcal{A} = (A_0, A_1, A_2, A_3)$ of TST_4 with a type-shifting automorphism

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3,$$

or just $\in$- and $\subseteq$-preserving bijections

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3.$$

(Grishin observed that f_0 can be recovered from f_1 by setting $f_0(a) = x$ if $f_1(\{a\}) = \{x\}$).

The idea then is to try to construct the automorphism $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ by forcing for a suitable model $\mathcal{A} = (A_0, A_1, A_2, A_3)$ of TST_4. *Coherent pairs* over a model $\mathcal{A} \models \mathrm{TST}_4$ were introduced in [8] as finite approximations of a type-shifting automorphism $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$. They were intended to be used as forcing conditions, a generic subset of which would yield the required automorphism. However, in order for that to work, coherent pairs should be *extendible* in the ordinary sense, That is, given a pair p and an element $t \in A_1 \cup A_2 \cup A_3$, there should be a $q \leq p$ that captures t. But extendibility is proved to be an exceptionally hard combinatorial problem.

Throughout we use only standard transitive (henceforth s.t.) models of TST. As shown in [8, lemma 1.2], one can confine oneself to such models without any serious loss of generality. E.g. for every $A \neq \emptyset$, the sequence $(A, \mathcal{P}(A), \mathcal{P}^2(A), \ldots, \in)$ is a s.t. model of TST. Such a model is called *full* and is denoted by $\langle\langle A \rangle\rangle$. If A is infinite $\langle\langle A \rangle\rangle$ is uncountable. To find a countable model we can take a countable elementary submodel of $\langle\langle A \rangle\rangle$. Such a model is standard but not transitive.

Though the intuitive meaning of coherent pairs is clear, the formal definition as given in [8] contains some inaccuracies which do not affect the proofs but might confuse the reader. Below we give a corrected and simplified version based on finite partitions and finite Boolean algebras.

Definition 2.1 Let A_1, A_2 be infinite sets such that $|A_1| = |A_2|$, and w_1, w_2 be finite partitions of A_1, A_2 respectively. We say that w_1 and w_2 are *similar* and write $w_1 \sim w_2$, if there is a bijection $p : w_1 \to w_2$ such that $|p(x)| = |x|$ for every $x \in w_1$. In that case we write $p : w_1 \sim w_2$.

Every finite partition[1] w on a set A generates a finite Boolean algebra denoted $B(w)$ whose set of atoms is w. Conversely every (nontrivial) finite Boolean algebra B on A has a set of atoms, denoted by $Atom(B)$, that constitutes a finite partition of A.

[1]All finite partitions w on a set A considered throughout will be assumed to contain only *nontrivial* sets, i.e., for every $x \in w$, $x \neq A, \emptyset$.

Definition 2.2 Two finite Boolean algebras B_1, B_2 on the sets A_1, A_2 respectively are said to be *similar,* notation $B_1 \sim B_2$, if the partitions produced by their atoms are similar.

It follows from the above definitions that for any partitions w_1, w_2 of A_1, A_2,

$$w_1 \sim w_2 \iff B(w_1) \sim B(w_2).$$

Also if $p : w_1 \sim w_2$ is a similarity, p extends to $B(w_1)$ by setting for every $X \in B(w_1)$,

$$p^*(X) = \bigcup\{p(x) : x \in w_1 \ \wedge \ x \subseteq X\}.$$

p^* is a Boolean-algebra isomorphism between $B(w_1)$ and $B(w_2)$, for which moreover $|p^*(X)| = |X|$. For simplicity we drop the star from p and write $p : B(w_1) \sim B(w_2)$ instead of $p^* : B(w_1) \sim B(w_2)$. Note that

$$p : B(w_1) \sim B(w_2) \Rightarrow p : B(w_1) \cong B(w_2),$$

but not conversely.

Definition 2.3 Let $\mathcal{A} = (A_0, A_1, A_2, A_3)$ be a model TST_4. A *coherent pair* (c.p. in brief) over $\mathcal{A}$ is a pair $p = (p_1, p_2)$ of finite 1-1 mappings with the following properties:

(a) $dom(p_1)$ is a finite Boolean subalgebra of A_1, $rng(p_1) = dom(p_2)$ is a finite Boolean subalgebra of A_2, and $rng(p_2)$ is a finite Boolean subalgebra of A_3. We set $u_1 = dom(p_1)$, $u_2 = rng(p_1) = dom(p_2)$ and $u_3 = rng(p_2)$.

(b) $p_1 : u_1 \sim u_2$ and $p_2 : u_2 \sim u_3$.

(c) p_1, p_2 are $\in$-isomorphisms, i.e., for every $x \in u_1$ and $y \in u_2$,

$$x \in y \iff p_1(x) \in p_2(y).$$

Given c.p.'s $p = (p_1, p_2)$ and $q = (q_1, q_2)$ we say that p *extends* q, and denote it by $p \leq q$, if $p_1 \supseteq q_1$ and $p_2 \supseteq q_2$.

Instead of $p = (p_1, p_2)$ we often write more suggestively

$$p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3.$$

Before going on let us fix and make explicit some notational conventions which already have been used above and will facilitate greatly the reader.

Notational conventions: Given any model $\mathcal{A} = (A_0, A_1, A_2, A_3)$ of TST_4, the letters

X, x, x_1, etc denote exclusively elements of A_1,

Y, y, y_1, etc denote exclusively elements of A_2,

Z, z, z_1, etc denote exclusively elements of A_3.

Also

u_1, u_2, u_3 (as well as v_1, v_2, v_3) are reserved for finite Boolean subalgebras of A_1, A_2, A_3, respectively, and

w_1, w_2, w_3 are reserved for partitions included in A_1, A_2, A_3, that is, for partitions of the underlying sets A_0, A_1, A_2, respectively.

An alternative formalization: Coherent triples. Since to each c.p. $p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3$ there are associated three finite Boolean algebras u_1, u_2, u_3, the domains and ranges of p_1, p_2, we might alternatively consider "coherent triples" of Boolean algebras (u_1, u_2, u_3) instead of pairs of functions. Moreover, as we saw above, speaking about finite Boolean algebras is tantamount to speaking about finite partitions. So instead of triples of Boolean algebras, we may consider just triples of partitions (w_1, w_2, w_3) of A_0, A_1, A_2, respectively. Most often partitions come up as sequences rather than sets, e.g. $w_1 = (x_1, \ldots, x_n)$, $w_2 = (y_1, \ldots, y_n)$, $w_1 = (z_1, \ldots, z_n)$. In such a case the elements of the algebras $B(w_i)$ can be indexed by means of the sets $I \subseteq [n]$ as follows (throughout $[n]$ denotes the set $\{1, \ldots, n\}$, for every $n \geq 1$): For every $I \subseteq [n]$ let

$$X_I = \bigcup \{x_i : i \in I\}, \tag{1}$$

and similarly for $Y_I \in B(w_2)$, $Z_I \in B(w_3)$. Obviously for every $X \in B(w_1)$ (resp. $Y \in B(w_2)$, $Z \in B(w_3)$) there is a unique $I \subseteq [n]$ such that $X = X_I$ (resp. $Y = Y_I$, $Z = Z_I$).[2] For a given I, we often refer to X_I, Y_I, Z_I as "corresponding" sets, with respect to the correspondences $p_1(x_i) = y_i$ and $p_2(y_i) = z_i$, since clearly $p_1(X_I) = Y_I$ and $p_2(Y_I) = Z_I$ for each $I \subseteq [n]$. The following definition can be used as an alternative equivalent to definition 2.3:

[2]The letters w_i denote, somewhat ambiguously, either a set or a sequence, depending on the context. For example in the notation "$B(w_1)$", w_1 is just a set. But the indexing of the sets $X \in B(w_1)$ by $I \subseteq [n]$ clearly depends on a particular ordering of w_1. This is why for more clarity we should write $X_I^{w_1}$ rather than X_I, where now w_1 refers to a sequence. If w_1' is a permutation of w_1, then in general $X_I^{w_1'} \neq X_I^{w_1}$. This notation is employed in the discussion of section 5.2.

Definition 2.4 Let (w_1, w_2, w_3) be a triple of partitions of A_0, A_1, A_2, respectively. (w_1, w_2, w_3) is said to be a *coherent triple,* (c.t. for short), notation $Co(w_1, w_2, w_3)$, if

(a) There are $p_1 : w_1 \sim w_2$, and $p_2 : w_2 \sim w_3$.

(b) Let $w_1 = (x_1, \ldots, x_n)$, $w_2 = (y_1, \ldots, y_n)$, $w_3 = (z_1, \ldots, z_n)$, be enumerations of w_1, w_2, w_3 such that $p_1(x_i) = y_i$, and $p_2(y_i) = z_i$. Then

$$X_I \in y_i \iff Y_I \in z_i, \tag{2}$$

for all $i \in [n]$ and all $I \subseteq [n]$.

Given triples of partitions $w = (w_1, w_2, w_3)$, $w' = (w'_1, w'_2, w'_3)$ we say that w' *extends* w and write $w' \leq w$, if $w'_1 \sqsupseteq w_1$, $w'_2 \sqsupseteq w_2$ and $w'_3 \sqsupseteq w_3$, where $w'_i \sqsupseteq w_i$ means that w'_i *refines* w_i, i.e., each element of w'_i is a subset of some element of w_i.

Remark 2.5 (i) Note that condition (c) of 2.3, that $\in$ is preserved by p_1, p_2, is equivalent to

$$X_I \in Y_J \iff Y_I \in Z_J, \tag{3}$$

for all $I, J \subseteq [n]$. However it is easy to check that condition (2) suffices for (3) to hold, that is, (2) and (3) are equivalent.

(ii) The relation $w' \leq w$ for c.t's is the analog of $p \leq q$ for c.p.'s

(iii) The relation between coherent pairs and coherent pairs is simply the following:

$$Co(w_1, w_2, w_3) \iff \text{there is a c.p.}$$

$$p = B(w_1) \xrightarrow{p_1} B(w_2) \xrightarrow{p_2} B(w_3).$$

(iv) Coherent pairs and triples over a model $\mathcal{A}$ of TST are *not* elements of $\mathcal{A}$, since they are "unstratified objects". Their relationship to $\mathcal{A}$ is that of proper classes to a model of ZFC. If one wants to treat them formally one has to extend TST to a "second-order" variant TST^c which is able to accommodate unstratified objects like coherent pairs. Models of TST$_4^c$ have the form $(\mathcal{A}, C)$, where $\mathcal{A}$ is a model of TST$_4$ and C is a certain subset $\bigcup_{i=0}^3 A_i$. For details see [8], p. 294.

Example 1. The simplest example of a c.p. is that in which u_1, u_2, u_3 are the trivial Boolean subalgebras of A_1, A_2, A_3, respectively, and p_1, p_2 are the trivial isomorphisms between them. Namely let: $u_1 = \{\emptyset, A_0\}$, $u_2 = \{\emptyset, A_1\}$,

$u_3 = \{\emptyset, A_2\}$, $p_i(\emptyset) = \emptyset$, for $i = 1, 2$, $p_1(A_0) = A_1$, and $p_2(A_1) = A_2$. We denote this pair by $o^{\mathcal{A}}$. I.e.,

$$o^{\mathcal{A}} = \{\emptyset, A_0\} \xrightarrow{o_1} \{\emptyset, A_1\} \xrightarrow{o_2} \{\emptyset, A_2\}.$$

We refer to $o^{\mathcal{A}}$ as the *trivial c.p.* of $\mathcal{A}$.

EXAMPLE 2. Let
$u_1 = (\emptyset, A_0, \{a\}, -\{a\})$, for some $a \in A_0$,
$u_2 = (\emptyset, A_1, \{x\}, -\{x\})$, for some $x \in A_1$ such that $x \neq \{a\}, -\{a\}$,
$u_3 = (\emptyset, A_2, \{y\}, -\{y\})$, for some $y \in A_2$ such that $y \neq \{x\}, -\{x\}$.
If $p_1 : u_1 \to u_2$, $p_2 : u_2 \to u_3$ are the mappings preserving the above orderings of u_i, it is easy to check that $p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3$ is a c.p.

As already said above, coherent pairs (or coherent triples) are intended to be used as forcing conditions a generic subset of which would provide the required type-shifting automorphism of the model $\mathcal{A}$ of TST_4. So the key property for (some of) them should be extendibility.

Definition 2.6 Let $p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3$ be a c.p. We say that p is *extendible* if for every $t \in A_1 \cup A_2 \cup A_3$, there is a pair $v_1 \xrightarrow{q_1} v_2 \xrightarrow{q_2} v_3$ such that $q \leq p$ and $t \in v_1 \cup v_2 \cup v_3$. When such a $q = (q_1, q_2)$ exists, we say for simplicity that q *captures* t, and denote this by $q \leq p \frown \{t\}$.

Are there extendible c.p's? We can prove that there are (theorem 2.13 below), but it is far more easy to give examples of *non-extendible* c.p's rather than extendible ones.

EXAMPLE 3. Consider the pair of Example 2 above:
$u_1 = (\emptyset, A_0, \{a\}, -\{a\})$,
$u_2 = (\emptyset, A_1, \{x\}, -\{x\})$, such that $x \neq \{a\}, -\{a\}$,
$u_3 = (\emptyset, A_2, \{y\}, -\{y\})$, such that $y \neq \{x\}, -\{x\}$, with

$$p_1 : (\emptyset, A_0, \{a\}, -\{a\}) \to (\emptyset, A_1, \{x\}, -\{x\})$$

$$p_2 : (\emptyset, A_1, \{x\}, -\{x\}) \to (\emptyset, A_2, \{y\}, -\{y\})$$

If $|x| \neq |y|$, $p = (p_1, p_2)$ is non-extendible. For if $q = (q_1, q_2)$ and $q \leq p \frown \{x\}$, then necessarily $q_1(x) = y$, hence $|q_1(x)| = |x| = |y|$, a contradiction.

The preceding example gives an idea of the hardness of the extendibility problem. Extendibility is a "chain-reaction" generating property: If $p =$

(p_1, p_2) is a given pair and, say, $y \in u_2$, in order for p to be extendible we must make sure that for any $x_1 \in y$ there exists a $y_1 \in p_2(y)$, as well as a z_1 so that p extends to a q that captures x_1, y_1, z_1; then for any $x_2 \in y_1$ we must find y_2, z_2 captured by an extension of q and so on. It follows that extendibility alone, as defined above, is by no means adequate. Even if we are able extend p to q to capture a new element t, q need not be further extendible, and the procedure will stop. What we need is a property of *iterated* extendibility up to ω iterations.

Definition 2.7 Let $p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3$ be a pair. p is said to be *1-extendible* if it is extendible. p is said to be *$(n+1)$-extendible* if for every $t \in A_1 \cup A_2 \cup A_3$ there is a pair $q = (q_1, q_2)$ such that , $q \leq p \frown \{t\}$ and q is n-extendible. p is said to be *ω-extendible* if it is n-extendible for all $n \geq 1$.

We shall see below that the trivial pair $o^{\mathcal{A}}$ is 1-extendible for any sufficiently rich $\mathcal{A}$. As to n-extendibility, for all $n \geq 1$, this is exactly the required property.

Theorem 2.8 (Main Theorem [8]) *Let M be a countable model of* ZFC *in which for every $n \in \mathbb{N}$, there is a s.t. model $\mathcal{A}$ of* TST *that contains an n-extendible c.p. Then there is a generic extension $M[G]$ of M that contains a model of* NF*. Conversely, if M contains a model of* NF*, then in M there is a s.t. model $\mathcal{A}$ of* TST *that contains an n-extendible c.p., for every $n \geq 1$.*

The most natural candidate pair to be extendible would be the trivial pair $o^{\mathcal{A}}$ of Example 1. The main theorem above can be equivalently formulated as follows:

Theorem 2.9 (Main Theorem [8]) *Let M be a countable model of* ZFC *in which for every $n \in \mathbb{N}$, there is a s.t. model $\mathcal{A}$ of* TST *such that $o^{\mathcal{A}}$ is n-extendible. Then there is a generic extension $M[G]$ of M that contains a model of* NF*. Conversely, if M contains a model of* NF*, then there is a s.t. model $\mathcal{A} \in M$ such that $o^{\mathcal{A}}$ is n-extendible, for every $n \geq 1$.*

Roughly n-extendibility works as follows: If for every n there is model $\mathcal{A}_n \in M$ of TST_4 such that $o^{\mathcal{A}_n}$ is n-extendible, then, by compactness, there is $\mathcal{A} \models \mathrm{TST}_4$ in M such that $o^{\mathcal{A}}$ is ω-extendible. If further $\mathcal{B}$ is a *saturated* elementary extension of $\mathcal{A}$ in M and we set

$$P_\omega = \{p : p \text{ is } \omega\text{-extendible over } \mathcal{B}\},$$

then $(P_\omega, \leq)$ *is* a forcing notion, and setting $f = \bigcup G$, for any generic G, f is the required type-shifting automorphism of $\mathcal{B}$. Thus $(\mathcal{B}, f)$ yields a model of NF in $M[G]$.[3]

Are there models $\mathcal{A}$ of TST having n-extendible pairs? More simply: Are there $\mathcal{A}$ such that $o^{\mathcal{A}}$ is n-extendible? All we know is that there exist $\mathcal{A}$ for which $o^{\mathcal{A}}$ is 1-extendible. In any case extendibility capabilities of $o^{\mathcal{A}}$ depend on properties of the underlying model $\mathcal{A}$. The properties of $\mathcal{A}$ mainly employed in [8] were "richness" and "regularity". Here are the definitions:

Definition 2.10 A model $\mathcal{A}$ of TST is called *regular* if for every $x \in A$,

$$x \text{ is finite} \iff \mathcal{A} \models Fin(x). \tag{4}$$

Definition 2.11 The Boolean algebra A_{i+1} is said to be *rich* if for every infinite (with respect to the ground model) $x \in A_{i+1}$, there is a $x_1 \in A_{i+1}$ such that $x_1 \subseteq x$ and both x_1 and $x - x_1$ are infinite. The structure $\mathcal{A}$ is said to be *rich* if every level A_{i+1}, for $i \geq 0$, is rich.

If $\mathcal{A}$ is regular, then the property of richness is definable in $\mathcal{A}$. Moreover the following holds:

Lemma 2.12 *Let $\langle\langle D\rangle\rangle$ be a full model of* TST *and let $\mathcal{A}$ be a standard transitive model isomorphic to an elementary submodel of $\langle\langle D\rangle\rangle$. Then $\mathcal{A}$ is regular and rich.*

Theorem 2.13 ([8]) *Let $\langle\langle D\rangle\rangle$ be a full model of* TST *(with infinite D) and let $\mathcal{A}$ be a standard transitive model isomorphic to an elementary submodel of $\langle\langle D\rangle\rangle$. Then the trivial pair $o^{\mathcal{A}}$ is extendible.*

3 A quick proof of NF_3 consistency

There are several proofs of the consistency of NF_3, due to V.N Grishin [3], M. Boffa and P. Casalegno [2] and R. Kay (see [1], p. 59). In this section, as an

[3] One may ask whether $(P_\omega, \leq)$ is a nontrivial forcing notion, that is, one producing a strict extension $M[G] \supset M$ (e.g. whether $(P_\omega, \leq)$ is separative, see [5]). The answer is that the question has no bearing on the issue of NF consistency. For if $(P_\omega, \leq)$ is trivial and $M[G] = M$, that simply means that the sought type-shifting automorphism $f = \bigcup G$ is already in M!

application of coherent pairs adapted to the fragment NF_3, we give another short and simple proof of this result. The cost to be paid for the simplicity is that the model of NF_3 exists not in the ground model M of ZFC but in a generic extension of it.

Recall that a model of NF_3 exists iff there is a model (A_0, A_1, A_2) of TST_3 together with a type-shifting automorphism $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$. Grishin in [3] observed that in order for $(A_0, A_1, A_2) \models \mathrm{TST}_3$ to be a model of NF_3, only a mapping $f_1 : A_1 \to A_2$ is needed, which will be a Boolean isomorphism. (This result was used also in the case [8] for the reduction of NF_4 consistency.)

Lemma 3.1 NF_3 *is consistent iff there is a model* (A_0, A_1, A_2) *of* TST_3 *such that there is Boolean-algebra isomorphism* $f : A_1 \cong A_2$.

Proof. The property is obviously necessary. Conversely, suppose there is a model (A_0, A_1, A_2) of TST_3 and a Boolean-algebra isomorphism $f : A_1 \to A_2$. Put $f_1 = f$ and define $f_0 : A_0 \to A_1$ by setting

$$f_0(a) = x \iff f_1(\{a\}) = \{x\}.$$

Then f_0 is a bijection because f_1 sends atoms to atoms. Moreover (f_0, f_1) is a type-shifting automorphism from (A_0, A_1) onto (A_1, A_2). Indeed, by the definition of f_0 we have that for every a, $f_1(\{a\}) = \{f_0(a)\}$. So for every $a \in A_0$ and $x \in A_1$

$$f_0(a) \in f_1(x) \iff \{f_0(a)\} \subseteq f_1(x) \iff f_1(\{a\}) \subseteq f_1(x).$$

Since f_1 preserves $\subseteq$, we have

$$f_1(\{a\}) \subseteq f_1(x) \iff \{a\} \subseteq x \iff a \in x.$$

Combining the above equivalences we get

$$a \in x \iff f_0(a) \in f_1(x).$$

This says that (f_0, f_1) is a type-shifting automorphism. $\dashv$

Now we adjust the notion of coherent pair (defined initially over models of TST_4) to models of TST_3. Necessarily it won't be a pair of finite mappings (p_1, p_2), but only a single mapping p, namely an isomorphism between finite Boolean algebras such that, in addition, $|p(x)| = |x|$ for every $x \in dom(p)$. We shall keep, however, calling it "coherent pair". The following is the adaptation of definition 2.3 to TST_3.

Definition 3.2 Let $\mathcal{A} = (A_0, A_1, A_2)$ be a countable model of TST_3. A *coherent pair over* $\mathcal{A}$ is a 1-1 mapping $u_1 \xrightarrow{p} u_2$, such that

(1) $u_1 = dom(p) \subset A_1$, $u_2 = rng(p) \subset A_2$,

(2) u_1, u_2 are finite subalgebras of the Boolean algebras A_1, A_2 respectively, and

(3) $p : u_1 \sim u_2$, and in addition $|p(x)| = |x|$ for all $x \in u_1$.

Obviously, given a model $\mathcal{A} = (A_0, A_1, A_2)$ of TST_3, coherent pairs $u_1 \xrightarrow{p} u_2$ over $\mathcal{A}$ are finite approximations of an isomorphism $A_1 \xrightarrow{f} A_2$ which is required to turn $\mathcal{A}$ into a model of NF_3. Using these pairs as forcing conditions, we can force f to occur in a generic extension $M[G]$, provided the forcing conditions are extendible. But this, in contrast to the hardness of extendibility in NF_4 case, can be easily shown to hold.

Theorem 3.3 *Let M be a countable model of* ZFC *and let $\mathcal{A} = (A_0, A_1, A_2) \in M$ be a countable rich and regular model of* TST_3. *Then there is a generic extension $M[G]$ of M containing an isomorphism $f : A_1 \to A_2$. Hence $M[G]$ contains a model of* NF_3.

Proof. Let M, $\mathcal{A} \in M$ be as in the hypothesis. Let $(P, \leq)$ be the forcing notion in M, where

$$P = \{p : p \text{ is a coherent pair over } \mathcal{A}\},$$

and $p \leq q := p \supseteq q$.

<u>Claim.</u> Each $p \in P$ is extendible, i.e., (a) for every $x \in A_1$, there is a $q \leq p$ such that $x \in dom(q)$, and (b) for every $y \in A_2$, there is a $q \leq p$ such that $y \in rng(q)$.

Proof of the Claim. (a) Let $u_1 = dom(p)$, $u_2 = rng(p)$. By assumption u_1, u_2 are finite Boolean subalgebras of A_1, A_2 and $p : u_1 \sim u_2$. p maps the atoms of u_1, onto the atoms u_2, and we can take enumerations of the sets of atoms of $w_1 = (x_1, \ldots, x_n)$, $w_2 = (y_1, \ldots, y_n)$ of the algebras u_1, u_2 so that, $u_1 = B(w_1)$, $u_2 = B(w_2)$ and $p(x_i) = y_i$. w_1, w_2 are partitions of the sets A_0, A_1 respectively and for each $i = 1, \ldots, n$, $|x_i| = |p(x_i)| = |y_i|$. Given $x \in A_1$, let $w_1 \oplus x$ be the smallest partition that refines w_1 and accommodates x. That is,

$$w_1 \oplus x = \{x_i^0, x_i^1 : i = 1, \ldots, n\},$$

where $x_i^0 = x_i \cap x$, $x_i^1 = x_i \cap (A_0 - x)$, $i = 1, \dots, n$. Let $v_1 = B(w_1 \oplus x)$. We have to find a set $y \in A_2$ and a mapping $q : v_1 = B(w_1 \oplus x) \sim v_2 = B(w_2 \oplus y)$ such that $q \le p$ and $q(x) = y$. What we need to do is just split each set y_i of w_2 into two subsets y_i^0 and y_i^1, such that $|y_i^0| = |x_i^0|$ and $|y_i^1| = |x_i^1|$. This is always possible, because $|y_i| = |x_i|$ and our model $\mathcal{A}$ is rich and regular. By regularity we do not bother about the internal and external meaning of finiteness. So if x_i^0 is finite with, say m elements, we pick a $y_i^0 \subseteq y_i$ with m elements. If both x_i^0, x_i^1 are infinite, then using richness we can split y_i into two infinite subsets y_i^0, y_i^1. Then it suffices to define q by setting $q(x_i^0) = y_i^0$ and $q(x_i^1) = y_i^1$. Clearly q is the required extension of p.

(b) This case is quite similar to the previous one. Given p and $y \in A_2$ we find as before $x \in A_1$ and $q : B(w_1 \oplus x) \sim B(w_2 \oplus y)$ such that $q \le p$ and $q(x) = y$. This completes the proof of the Claim.

In view of the Claim, $(P, \le)$ is an ordinary forcing notion on M. If G is a P-generic set and $f = \bigcup G$, then clearly $dom(f) = A_1$, $rng(f) = A_2$, so $f : A_1 \cong A_2$. $\dashv$

4 Extendibility revisited

4.1 Unfolding extendibility

We are going here to examine more thoroughly the property of n-extendibility. Recall that given c.p.'s $p = (p_1, p_2)$, $q = (q_1, q_2)$ and elements $t_1, \dots, t_k \in A_1 \cup A_2 \cup A_3$ the notation

$$q \le p \frown \{t_1, \dots, t_k\}$$

abbreviates the fact that q extends p and captures $t_1, \dots, t_k$, that is, $q \le p$ and $\{t_1, \dots, t_k\} \subseteq dom(q_1) \cup dom(q_2) \cup rng(q_2)$. Let us set

$$\theta_0(p) := \text{“}p \text{ is a coherent pair”},$$

$$\theta_n(p) := \text{“}p \text{ is } n\text{-extendible”}.$$

The definition of n-extendibility given in 2.7 above is inductive. Namely:

$$\theta_{n+1}(p) := (\forall t)(\exists q)(q \le p \frown \{t\} \ \wedge \ \theta_n(q)). \tag{5}$$

and

$$\theta_\omega(p) := \bigwedge_n \theta_n(p).$$

If we "unfold" θ_n into a plain formula, we shall get inductively the formulas

$$\theta_n(p) = (\forall t_1)(\exists q_1)(\forall t_2)(\exists q_2)\cdots(\forall t_n)(\exists q_n)[q_1 \leq p \frown \{t_1\} \wedge \bigwedge_{i=2}^{n} q_i \leq q_{i-1} \frown \{t_i\}], \tag{6}$$

where q_i range over coherent pairs and t_i range over elements of $A_1 \cup A_2 \cup A_3$.[4]

We see that $\theta_n(p)$ is of high logical complexity, a sort of "Π_n-formula", with n alternating quantifiers. On can easily verify by induction on n that properties θ_n, $n \geq 1$, become stronger and stronger as n grows:

$$(\forall p)[\theta_{n+1}(p) \Rightarrow \theta_n(p)]. \tag{7}$$

It is open whether this hierarchy of θ_n is proper or collapses at a certain level. The following is only a partial answer to the question.

Proposition 4.1 *Let M be a countable model of* ZFC *and let $\mathcal{A} \in M$ be model of* TST_4 *elementarily embeddable to a full model. Then*

$$(\forall p)[\theta_1(p) \Rightarrow \theta_2(p)] \tag{8}$$

is false with respect to $\mathcal{A}$.

Proof. Suppose $\mathcal{A}$ is as in the hypothesis and let (8) holds true.

Claim.

$$(\forall p)[\theta_n(p) \Rightarrow \theta_{n+1}(p)]. \tag{9}$$

Proof of the Claim. We prove (9) by induction on n. For $n = 1$, (9) is (8). Suppose (9) holds for n. Then, by (5),

$$\theta_{n+1}(p) = (\forall t)(\exists q)(q \leq p \frown \{t\} \wedge \theta_n(q)),$$

and

$$\theta_{n+2}(p) = (\forall t)(\exists q)(q \leq p \frown \{t\} \wedge \theta_{n+1}(q)).$$

Since by the induction hypothesis $\theta_n(q) \Rightarrow \theta_{n+1}(q)$ is true, it follows from the preceding equations that so is $\theta_{n+1}(p) \Rightarrow \theta_{n+2}(p)$.

[4] As we have noticed in remark 2.5 (iv), coherent pairs are not elements of models of TST, but of an extended theory TST^c. For the same reason θ_n (as well as the formulas χ_n defined later) are not formulas of L_{TST} but of L_{TST^c}. So it doesn't make sense to write $\mathcal{A} \models \theta_n(p)$. To relax the reader from the technicalities of using L_{TST^c}, we shall say instead that "$\theta_n(p)$ holds with respect to $\mathcal{A}$".

Now by theorem 2.13, $\theta_1(o^{\mathcal{A}})$ holds true, so by (8) and (9), $\theta_n(o^{\mathcal{A}})$ holds true for all $n \geq 1$, therefore $\theta_\omega(o^{\mathcal{A}})$. This means (see the proof of [8, 2.8] for details) that if $\mathcal{B}$ is a saturated elementary extension of $\mathcal{A}$, and $P_\omega = \{p : \theta_\omega(p)$ holds with respect to $\mathcal{B}\}$, then forcing with $(P_\omega, \leq)$ yields a generic type-shifting automorphism f for $\mathcal{B}$, $f \in M[G]$. We have $\mathcal{B} \equiv \mathcal{A} \equiv \langle\langle D\rangle\rangle$ for some full model $\langle\langle D\rangle\rangle$ and $\langle\langle D\rangle\rangle \models$ AC, where AC is the choice axiom adjusted to the language of TST. So $\mathcal{B} \models$ AC. Now $\mathcal{B}$ is an ambiguous model, and if $\mathcal{B}^*$ is the induced model of NF, the clearly $\mathcal{B}^* \models$ AC. But this contradicts Specker's result [6] that NF $\vdash \neg$AC. $\dashv$

Before elaborating further on properties θ_n we shall first deal with a simplification of all objects considered so far (c.p.'s, Boolean algebras, c.t. partitions, etc). All these objects are defined in terms of finite partitions. The simplification concerns the kind of partitions involved. The simplest kind of finite partitions are those whose sets are either infinite or singletons. We shall call such partitions "simple". And we shall see that all extendibility questions about general c.p.'s can be reduced, without any loss of generality, to questions about "simple c.p." only, that is, c.p.'s whose domains are (essentially) simple partitions.

4.2 Simple partitions and simple extendibility

Recall that all partitions w_i, $i = 1, 2, 3$, considered in this paper are nontrivial, in the sense that for every $x \in w_i$, $x \neq A_{i-1}, \emptyset$. Each of the underlying sets A_i is (countably) infinite, so every w_i contains at least one infinite set and possibly several finite ones. To simplify things let us consider partitions whose sets are only either infinite or singletons.

Definition 4.2 A finite partition w (resp. a finite Boolean algebra u) of an infinite set A is said to be *simple* if each $x \in w$ (resp. each atom of the algebra) is either a singleton or an infinite set. A c.p. $p = w_1 \xrightarrow{p_1} w_2 \xrightarrow{p_2} w_3$, is said to be *simple* if each u_i is a simple Boolean algebra. Similarly a c.t. $w = (w_1, w_2, w_3)$ is *simple* if all partitions w_i are simple.

Recall that for partitions w, w' of a set A, $w \sqsubseteq w'$ denotes the fact that w' is a refinement of w'. Also we have already defined in the proof of theorem 3.3, for every finite partition $w = \{x_1, \ldots, x_n\}$ of a set A and every $x \subseteq A$, the smallest refinement

$$w \oplus x = \{x \cap x_i : x_i \in w\} \cup \{(A - x) \cap x_i : x_i \in w\}$$

of w that accommodates w.

Definition 4.3 (i) Let w be a partition of A. The *simple refinement* of w, denoted $sr(w)$, is the partition resulting from w if we replace each finite $x \in A$ with the sets $\{a\}$, $a \in x$.

Clearly, if w is finite, then $sr(w)$ is the $\sqsubseteq$-least simple partition that refines w.

Let $p = u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3$ be a simple c.p. and let $w_i = Atom(u_i)$, the set of atoms of the algebra u_i. Let t be a new element of $A_1 \cup A_2 \cup A_3$, say $t = x \in A_1$. In order for p to be extendible on x, it is necessary and sufficient that it be extendible on each of the elements of $w \oplus x$, and further on each of the elements of $sr(w \oplus x)$. So given a pair $p = w_1 \xrightarrow{p_1} w_2 \xrightarrow{p_2} w_3$, let

$$S(p) = \{t : (\exists t_1 \in w_i)(t \subset t_1) \wedge (|t| = 1 \vee |t| = \infty)\}.$$

In words, $S(p)$ is the set of all singletons or infinite sets which are proper subsets of elements of some w_i.

Definition 4.4 A simple c.p. $p = (p_1, p_2)$ is said to be *n-simply extendible* if $\theta^s_n(p)$ holds, where

$$\theta^s_0(p) := \text{“}p \text{ is a simple c.p.”},$$

and

$$\theta^s_{n+1}(p) := (\forall t \in S(p))(\exists q)(q \leq p \frown \{t\} \wedge \theta^s_n(q)). \tag{10}$$

p is *ω-simply extendible* if $\theta^s_\omega(p)$ holds, where $\theta^s_\omega(p) := \bigwedge_n \theta^s_n(p)$.

Proposition 4.5 *Let $(\mathcal{A}, C)$ be a (recursively) saturated model of* TST^c.
(i) If $\theta^s_\omega(p)$, then $(\forall t)(\exists q)(q \leq p \frown \{t\} \wedge \theta^s_\omega(q))$.
(ii) $(\forall p)[\theta_\omega(p) \Leftrightarrow \theta^s_\omega(p)]$ with respect to $\mathcal{A}$. In particular $\theta_\omega(o^{\mathcal{A}}) \Leftrightarrow \theta^s_\omega(o^{\mathcal{A}})$.

Proof. (i) Suppose that $\theta^s_\omega(p)$ is the case and let $t \in A_i$ be a new element. w_i is a partition of A_{i-1} so let $sr(w_i \oplus t) = \{t_1, \ldots, t_k\}$ be the elements of the simple refinement of w_i augmented with t. We have to show that there is q such that $q \leq p \frown \{t\}$ and $\theta^s_\omega(q)$. The proof is similar to that of Claim 1 of theorem [8, 2.8], so we omit it.

(ii) Trivially $(\forall p)[\theta_\omega(p) \Rightarrow \theta^s_\omega(p)]$, with respect to any $\mathcal{A}$.

For the converse, suppose p is a simple pair such that $\theta^s_\omega(p)$ holds. We show by induction on n that $\theta^s_\omega(p) \Rightarrow \theta_n(p)$.

Let $n = 1$, and let $t \in A_i$ be a new element. We have to show that there is a $q \leq p \frown \{t\}$. Let $sr(w_i \oplus t) = \{t_1, \ldots, t_k\}$, where t_i are either singletons or infinite sets contained in the atoms of w_i. By assumption p is k-simply augmented, so there are simple c.p.'s $q_1, \ldots, q_k$ such that $q_1 \leq p \frown \{t_1\}$ and $q_i \leq q_{i-1} \frown \{t_i\}$ for $2 \leq i \leq k$. Therefore $q \leq p \frown \{t_1, \ldots, t_k\}$, and hence $q \leq p \frown \{t\}$. Therefore $\theta^s_\omega(p) \Rightarrow \theta_1(p)$.

Suppose now that the claim is true for n. Let t a new element and let again $sr(w_i \oplus t) = \{t_1, \ldots, t_k\}$. Then as in (i) above we can show that there is a q such that $q \leq p \frown \{t\}$ and $\theta^s_\omega(q)$. By the induction hypothesis, $\theta^s_\omega(q) \Rightarrow \theta_n(q)$ is true. Therefore $(\forall t)(\exists q)(\leq p \frown \{t\} \wedge \theta_n(q))$. But that means that $\theta_{n+1}(p)$ holds. So $\theta^s_\omega(p) \Rightarrow \theta_{n+1}(p)$. This completes the proof. $\dashv$

It follows from 4.5 (ii) that with respect to saturated models of TST^c, ω-simple extendibility is no weaker than the full ω-extendibility.

5 Augmentability

Let us return to the unfolded formulation (6) of θ_n:

$$\theta_n(p) = (\forall t_1)(\exists q_1)(\forall t_2)(\exists q_2)\cdots(\forall t_n)(\exists q_n)[q_1 \leq p \frown \{t_1\} \wedge \bigwedge_{i=2}^{n} q_i \leq q_{i-1} \frown \{t_i\}].$$

Since for every ϕ, $(\exists x)(\forall y)\phi \Rightarrow (\forall y)(\exists x)\phi$ is logically valid, the preceding formula logically implies

$$(\forall t_1)(\forall t_2)\cdots\forall t_n)(\exists q_1)(\exists q_2)\cdots(\exists q_n)[q_1 \leq p \frown \{t_1\} \wedge \bigwedge_{i=2}^{n} q_i \leq q_{i-1} \frown \{t_i\}]. \tag{11}$$

Moreover, obviously

$$(\exists q_1)(\exists q_2)\cdots(\exists q_n)[q_1 \leq p \frown \{t_1\} \wedge \bigwedge_{i=2}^{n} q_i \leq q_{i-1} \frown \{t_i\}] \Rightarrow$$
$$(\exists q)[q \leq p \frown \{t_1, \ldots, t_n\}]. \tag{12}$$

From (11) and (12) we get

$$(\forall t_1)(\forall t_2)\cdots(\forall t_n)(\exists q)[q \le p \frown \{t_1,\ldots,t_n\}].$$

This formula is a natural weakening of $\theta_n(p)$. We denote it by $\chi_n(p)$, and call the property it expresses *n-augmentability.* That is, we set for every $n \ge 1$,

$$\chi_n(p) := (\forall t_1)(\forall t_2)\cdots(\forall t_n)(\exists q)[q \le p \frown \{t_1,\ldots,t_n\}], \tag{13}$$

and

$$\chi_\omega(p) := \bigwedge_n \chi_n(p).$$

It follows from (6), (11) and (12) that for all $n \ge 2$

$$\theta_n(p) \Rightarrow \chi_n(p), \tag{14}$$

while

$$\theta_1(p) \Leftrightarrow \chi_1(p). \tag{15}$$

Definition 5.1 Let $p = (p_1, p_2)$ be a c.p. over $\mathcal{A}$. We call p *n-augmentable* if $\chi_n(p)$ holds. p is *ω-augmentable* if it is n-augmentable for all $n \ge 1$.

$\chi_n(p)$ expresses also an extendibility property of p but of a different kind: It says that *any* n new elements can be adjoined to p to give an extension $q \le p$, but with no claim as to the extendibility capabilities of q. (Observe how much lower is the logical complexity of χ_n compared to that of θ_n.)

Lemma 5.2 *If for each n there is $\mathcal{A}_n \models \mathrm{TST}$ containing an n-augmentable pair, then there is a $\mathcal{B} \models \mathrm{TST}$ containing an ω-augmentable pair.*

Proof. The proof is again similar to that of 2.8 in [8], for showing the existence of ω-extendible pairs. Namely, we consider the theory $T = \mathrm{TST}^c + \{\chi_n(b) : n \in \mathbb{N}\}$, where c is a new constant. Then the result follows by compactness. See [8, Th. 2.8] for details. $\dashv$

Since the elements $t_1, \ldots, t_n$ of formula (13) above can be distributed arbitrarily among A_1, A_2, A_3, so that n_1 of them belong to A_1, n_2 to A_2 and n_3 to A_3, where $n_1 + n_2 + n_3 = n$, instead of "n-augmentable" we shall use the more suggestive term "(n_1, n_2, n_3) -augmentable". Moreover, instead of arbitrary subsets $w_1 \subseteq A_1$, $w_2 \subseteq A_2$, $w_3 \subseteq A_3$ with $|w_i| = n_i$, $i = 1, 2, 3$, we can take w_i to be *partitions* of the corresponding underlying sets. And, finally, we can take w_i to be just *simple* partitions.

Definition 5.3 Let $\mathcal{A}$ be a model of TST_4, p be a simple pair over $\mathcal{A}$ and $n_1, n_2, n_3 \in \mathbb{N}$. p is said to be (n_1, n_2, n_3)*-simply augmentable* if for any simple partitions $w_1 \subseteq A_1$, $w_2 \subseteq A_2$, $w_3 \subseteq A_3$ with $|w_i| = n_i$, there is a simple pair q over $\mathcal{A}$ such that $q \leq p \frown w_1 \cup w_2 \cup w_3$. p is said to be ω*-simply augmentable* if it is (n_1, n_2, n_3)-simply augmentable for all $n_1, n_2, n_3 \geq 1$.

Let

$$\chi^s_{(n_1,n_2,n_3)}(p) := \text{ “}p \text{ is } (n_1, n_2, n_3)\text{-simply augmentable”},$$

and

$$\chi^s_\omega(p) := \bigwedge_{n_1,n_2,n_3 \geq 1} \chi^s_{(n_1,n_2,n_3)}(p).$$

Let us write $(n_1, n_2, n_3) \leq (l_1, l_2, l_3)$ if $n_i \leq l_i$ for each $i = 1, 2, 3$.

Lemma 5.4 *For any p, (n_1, n_2, n_3), (l_1, l_2, l_3),*

$$(n_1, n_2, n_3) \leq (l_1, l_2, l_3) \ \wedge \ \chi^s_{(l_1,l_2,l_3)}(p) \ \Rightarrow \ \chi^s_{(n_1,n_2,n_3)}(p). \tag{16}$$

Proof. Indeed, given p and $(n_1, n_2, n_3) \leq (l_1, l_2, l_3)$ such that $\chi^s_{(l_1,l_2,l_3)}(p)$, let $w_i \subseteq A_i$ be simple partitions such that $|w_i| = n_i$. Since each w_i contains at least an infinite set, and $n_i \leq l_i$, we can refine w_i to a simple partition w'_i such that $|w'_i| = l_i$. This is done either by adding new singletons that we subtract from an infinite set, or by splitting an infinite set into two infinite subsets (here we need the property of richness for A). Now each w'_i, $i = 1, 2, 3$, forms a simple partition with $|w'_i| = l_i \geq n_i$, so by $\chi^s_{(l_1,l_2,l_3)}(p)$, there is a c.p. $q \leq p \frown B(w'_1) \cup B(w'_2) \cup B(w'_3)$. Since $B(w'_i) \supseteq B(w_i)$, we have $q \leq p \frown B(w_1) \cup B(w_2) \cup B(w_3)$, and we are done. ⊣

Lemma 5.5 *If for every (n_1, n_2, n_3) there is $\mathcal{A}$ and p over $\mathcal{A}$ such that $\chi^s_{(n_1,n_2,n_3)}(p)$, then there is $\mathcal{B}$ and q over $\mathcal{B}$ such that $\chi^s_\omega(q)$.*

Proof. (Sketch, details in [8, 2.8].) By compactness again. Consider the theory

$$T = \mathrm{TST}^c + \{\chi^s_{(n_1,n_2,n_3)}(b) : n_1, n_2, n_3 \geq 1\},$$

where b is a new constant. To show that T is finitely satisfiable, take a finite subset $\Sigma = \{\chi^s_{(k_i,l_i,m_i)}(c) : i = 1, \ldots, n\}$ of T. Let $k = \max\{k_i : i = 1, \ldots, n\}$, $l = \max\{l_i : i = 1, \ldots, n\}$, $m = \max\{m_i : i = 1, \ldots, n\}$. By assumption there are $\mathcal{A}$ and p such that $\chi^s_{(k,l,m)}(p)$ holds with respect to $\mathcal{A}$. Also $(k_i, l_i, m_i) \leq (k, l, m)$ for all $i = 1, \ldots, n$. So by (16), all $\chi^s_{(k_i,l_i,m_i)}(p)$, $i = 1, \ldots, n$, hold with respect to $\mathcal{A}$. Equivalently, $T = \mathrm{TST}^c + \Sigma$ is satisfied in an expansion $(\mathcal{A}, C)$ of $\mathcal{A}$. ⊣

Lemma 5.6 *For every simple c.p. p, $\chi_\omega(p) \Leftrightarrow \chi^s_\omega(p)$ with respect to any model $\mathcal{A}$ of* TST*. In particular $\chi_\omega(o^{\mathcal{A}}) \Leftrightarrow \chi^s_\omega(o^{\mathcal{A}})$.*

Proof. Trivially, $\chi_\omega(p) \Rightarrow \chi^s_\omega(p)$. For the converse, suppose p is a simple pair and $\chi^s_\omega(p)$ holds with respect to $\mathcal{A}$. Let $w_i \subseteq A_i$, $i = 1,2,3$, be any finite partitions. It suffices to show that there is $q \leq p \frown \bigcup_i w_i$. Note that $q \leq p \frown \bigcup_i w_i$ iff $q \leq p \frown \bigcup_i B(w_i)$. Let $w'_i = sr(w_i)$. By $\chi^s_\omega(p)$ there is q such that $q \leq p \frown \bigcup_i B(w'_i)$, and hence $q \leq p \frown \bigcup_i B(w_i)$. This shows that $\chi_{(n_1,n_2,n_3)}(p)$ for all (n_1, n_2, n_3). Therefore $\chi_\omega(p)$. $\dashv$

It follows from lemma 5.6 that in order to prove the existence of ω-augmentable pairs, it suffices to restrict the search to n-simply augmentable pairs for each n.

Despite the above reductions, the general problem of (n_1, n_2, n_3)-simple augmentability for $o^{\mathcal{A}}$ is still very hard to tackle. Tractable cases of this seem to be the special subcases of $(n, 0, 0)$-, $(0, n, 0)$- and $(0, 0, n)$-simple augmentability.

5.1 $(n, 0, 0)$- and $(0, n, 0)$- simple augmentability

We always refer to an underlying model $\mathcal{A} = (A_0, A_1, A_2, A_3)$ of TST over which coherent pairs and triples are considered. A triple of partitions (w_1, w_2, w_3), $w_i \subseteq A_i$, is said to be *similar* if $w_1 \sim w_2 \sim w_3$. The *size* of a triple (w_1, w_2, w_3) is n, if $|w_1| = |w_2| = |w_3| = n$.

Given a simple partition w, let $inf(w)$ and $sin(w)$ denote the sets of infinite sets and of singletons of w, respectively. A partition w such that $|inf(w)| = m$ and $|sin(w)| = l$ is called an *(m, l)-partition.* (m, l) is called the *index* of w and we write $ind(w) = (m, l)$. In that case $m + l = n$ is the size of w. Clearly, if $w_1 \sim w_2 \sim w_3$, then all w_i are of the same index and size.

Lemma 5.7 *Let $\mathcal{A}$ be a rich model and let w_1, w_2 be simple partitions of A_0, A_1 respectively such that $p_1 : w_1 \sim w_2$. Then there is partition w_3 of A_2 such that $Co(w_1, w_2, w_3)$.*

Proof. Since $w_1 \sim w_2$, $ind(w_1) = ind(w_2) = (m, l)$. Let $inf(w_1) = (x_1, \ldots, x_m)$, $sin(w_1) = (x_{m+1}, \ldots, x_n)$, $inf(w_2) = (y_1, \ldots, y_m)$, $sin(w_2) = (y_{m+1}, \ldots, y_n)$ such that $p_1(x_i) = y_i$ for all $i = 1, \ldots, n$. p_1 extends to the sets $X \in B(w_1)$ as we have seen in section 2. We have to find a partition $w_3 = (z_1, \ldots, z_n)$ of A_2, such that $z_1, \ldots, z_m$ are infinite while $z_{m+1}, \ldots, z_n$ are singletons, and for all $X \in B(w_1)$, and all $1 \leq i \leq n$,

$$X \in y_i \iff p_1(X) \in z_i.$$

We first define z_i for $m+1 \leq i \leq k$. Take such an i. If there is a $X \in B(w_1)$ such that $X \in y_i$, then clearly $y_i = \{X\}$, so let us put $z_i = \{p_1(X)\}$. If on the contrary $y_i \cap B(w_1) = \emptyset$, then we choose an arbitrary singleton z_i such that $z_i \cap B(w_2) = \emptyset$. This way we have defined z_i, for all $m+1 \leq i \leq n$.

Next we define the infinite sets z_i, for $1 \leq i \leq m$. Let $K = A_2 - \bigcup\{z_i : m+1 \leq i \leq n\}$. For each $i = 1, \ldots, m$, let $E_i = y_i \cap B(w_1)$, and let $D_i = \{p_1(X) : X \in y_i \cap B(w_1)\}$. Each D_i is a finite subset of the infinite set K. Using the richness of $\mathcal{A}$ we can find a partition of K into m infinite subsets $z_1, \ldots z_m$ such that $D_i \subset z_i$. This completes the definition of z_i. Their choice clearly guarantees the truth of the equivalences $X \in y_i \iff p_1(X) \in z_i$. Thus w_3 is as required. $\dashv$

Corollary 5.8 *Let $\mathcal{A}$ be a rich model.*

(i) For every simple partition w_1, there are w_2, w_3 such that $Co(w_1, w_2, w_3)$.

(ii) For every simple partition w_2, there are w_1, w_3 such that $Co(w_1, w_2, w_3)$.

Proof. (i) Given w_1, pick an arbitrary w_2 such that $w_1 \sim w_2$. Then use lemma 5.7 to find w_3 such that $Co(w_1, w_2, w_3)$.

(ii) Given w_2, pick an arbitrary w_1 such that $w_1 \sim w_2$. Then use again lemma 5.7 to find w_3 such that $Co(w_1, w_2, w_3)$. $\dashv$

The above immediately implies the following.

Corollary 5.9 *Let $\mathcal{A}$ be rich. Then for all n, $\chi^s_{(n,0,0)}(o^{\mathcal{A}})$ and $\chi^s_{(0,n,0)}(o^{\mathcal{A}})$ hold true with respect to $\mathcal{A}$.*

Note that Corollary 5.8 is a strengthening of A_1- and A_2-extendibility of theorem 2.13 (theorem 3.6. of [8]).

It is of some interest to observe that, in contrast to lemma 5.7, we have the following impossibility result:

Lemma 5.10 *Let $\mathcal{A}$ be a rich model. Then:*

(i) There are partitions w_2, w_3 and $p_2 : w_2 \sim w_3$ such that $Co(w_1, w_2, w_3)$ for no partition w_1.

(ii) There are partitions w_1, w_3 such that $Co(w_1, w_2, w_3)$ for no partition w_2.

Proof. (i) We shall use the simplest kind of partitions, namely binary ones. Pick an infinite and coinfinite set $y_0 \in A_2$ which is "consistent", that is, $(\forall x)(x \in y_0 \Rightarrow -x \notin y_0)$, and consider the partition $w_2 = (y_0, -y_0)$. Next take an infinite and coinfinite z_0 such that $\{y_0, -y_0\} \subseteq z_0$, and let $p_2(y_0) = z_0$ and $p_2(-y_0) = -z_0$. Clearly $p_2 : w_2 \sim w_3$. We claim that there is no w_1 such that $Co(w_1, w_2, w_3)$. Suppose not and let $Co(w_1, w_2, w_3)$ for some $w_1 = (x, -x)$ and $p_1(x) = y_0$, $p_1(-x) = -y_0$. Then we must have $x \in y_0 \Leftrightarrow y_0 \in z_0$ and $-x \in y_0 \Leftrightarrow -y_0 \in z_0$. Since $\{y_0, -y_0\} \subseteq z_0$, we must have $\{x, -x\} \subseteq y_0$, which contradicts the consistency of y_0.

(ii) We again use binary partitions. Pick an infinite and coinfinite x_0 and set $w_1 = (x_0, -x_0)$. Next set $z_0 = \{y \in A_2 : x_0 \notin y\}$. Since z_0 is definable, it belongs to A_3 and is infinite and coinfinite. Put $w_3 = (z_0, -z_0)$. We claim that there is no $w_2 = (y, -y)$ such that $Co(w_1, w_2, w_3)$ under the obvious correspondences. Suppose not and let $w_2 = (y, -y)$ be one such. Then it should be $x_0 \in y \Leftrightarrow y \in z_0$. But $y \in z_0 \Leftrightarrow x_0 \notin y$, and hence $x_0 \in y \Leftrightarrow x_0 \notin y$, a contradiction. $\dashv$

Since in lemma 5.10 we use binary partitions, we think of this as an indication that the proof of A_3-extendibility of $o^{\mathcal{A}}$ in lemma 3.5 of [8], specifically case 3 of the proof, cannot be simplified significantly. Yet we guess that A_3-extendibility can be strengthened to hold for an arbitrary number of elements instead of a single one. Equivalently, we guess that $(0, 0, 2)$-augmentability can be strengthened to $(0, 0, n)$-one, for all $n \geq 2$. For the time being this is still open. In the next section we offer a partial result and some discussion concerning this problem.

5.2 Remarks on $(0, 0, n)$-augmentability

Recall that in order to prove 1-extendibility of $o^{\mathcal{A}}$, one has to consider an arbitrary $x \in A_1$ (resp. $y \in A_2$, and $z \in A_3$) and try to find corresponding elements y, z (resp. x, z, and x, y) so that the triple of binary partitions $(x, -x)$, $(y, -y)$, $(z, -z)$ is coherent. But this obviously coincides with proving $(2,0,0)$-augmentability (resp. $(0,2,0)$ and $(0,0,2)$-augmentability. Therefore $(2,0,0)$-, $(0,2,0)$- and $(0,0,2)$- augmentability have already been settled by lemma 3.6 of [8], where the above properties are called A_1-, A_2- and A_3-extendibility, respectively. Thus, in view of Corollary 5.9 above, the only open and seemingly tractable problem of this type is $(0,0,n)$-simple augmentability for $n \geq 3$. As was the case with A_3-extendibility compared with A_1- and A_2-extendibility, this problem is expected to be much harder than $(n,0,0)$- and $(0,n,0)$-augmentability, settled in the previous section.[5]

Obviously the question of $(0,0,n)$-simple-augmentability of $o^{\mathcal{A}}$, for $n \geq 3$, amounts to the following:

QUESTION. Let $\mathcal{A}$ be a sufficiently rich $\mathcal{A}$ (e.g. $\mathcal{A}$ is isomorphic to an elementary submodel of a full model). Let w_3 be any simple partition of A_2 with $|w_3| \geq 3$. Do there exist similar simple partitions w_1, and w_2 of A_0, A_1 respectively such that $Co(w_1, w_2, w_3)$?

Below we offer some remarks with respect to this QUESTION. We work over a fixed sufficiently rich model $\mathcal{A}$ of TST_4. Let us fix a simple enumerated partition $w_3 = (z_1, \ldots, z_n)$ of A_2 with $|w_3| = n \geq 3$ and $ind(w_3) = (m, n-m)$. In this enumeration we assume that the first m sets $z_1, \ldots, z_m$ are the infinite ones while the next $n-m$ elements $z_{m+1}, \ldots, z_n$ are the singletons. We want to show that there exist enumerated partitions $w_1 = (x_1, \ldots, x_n)$, $w_2 = (y_1, \ldots, y_n)$, with corresponding elements $x_i \mapsto y_i \mapsto z_i$, such that $Co(w_1, w_2, w_3)$.[6]

Recall from section 2, that given an enumerated partition $w_1 = (x_1, \ldots, x_n)$, we denote by X_I the set $\bigcup\{x_i : i \in I\}$, and similarly for Y_I, Z_I. Now if w_1 varies, to avoid ambiguity, we should write $x_i^{w_1}$ and $X_I^{w_1}$

[5]In general, the construction "from left to right" is the easy one, while the construction "from right to left" is the hard one. The reason of this asymmetry is simply the strong asymmetry of the relation $x \in y$: In every reasonably rich structure (like a rich model of TST), given x, one can find y such that $x \in y$ possessing almost any prescribed properties, e.g. with y being finite, or cofinite, or infinite and coinfinite. In contrast, given y, the prescribed choices for x such that $x \in y$ are drastically restricted by the very extension of y. If e.g. y is a set of singletons or a set of cofinite sets, obviously no other choice is possible.

[6]Note that it suffices to prove the statement not for each particular n, but for every *sufficiently large* n, i.e., for w_3 with $|w_3| \geq n_0$, where n_0 is any given number. For if that was the case for such w_3, that would hold also for w_3' with all smaller cardinalities. Indeed, given w_3' such that $|w_3'| < n_0$, just extend arbitrarily w_3' to a finer partition w_3 such that $|w_3| \geq n_0, |w_3'|$. Then any coherent pair that captures w_3, captures also w_3'.

rather than x_i and X_I, respectively. ($x_i^{w_1}$ of course means "the i-set of the sequence w_1".) In view of the relation (2) of definition 2.4, the fact that there exists w_1, w_2 such that $Co(w_1, w_2, w_3)$ has the following formulation:

$$(\exists w_1)(\exists w_2)(w_1 \sim w_2 \sim w_3 \wedge$$

$$(\forall I \subseteq [n])(\forall i \in [n])[X_I^{w_1} \in y_i^{w_2} \Leftrightarrow Y_I^{w_2} \in z_i^{w_3}]. \tag{17}$$

We have fixed only the following partial result.

Lemma 5.11 *Suppose $ind(w_3) = (1, n-1)$, i.e., the partition w_3 contains a unique infinite set, the rest being singletons. Then (17) is true, and hence the* QUESTION *above is answered in the affirmative.*

Proof. We argue by contradiction. Let the negation of (17)

$$(\forall w_1)(\forall w_2)(w_1 \sim w_2 \sim w_3 \ \Rightarrow$$

$$(\exists I \subseteq [n])(\exists i \in [n])[X_I^{w_1} \in y_i^{w_2} \Leftrightarrow Y_I^{w_2} \notin z_i^{w_3}]. \tag{18}$$

be true. Let z_0 be the unique infinite set of w_3. Then z_0 is cofinite. Fix a partition w_1 of A_0 such that $w_1 \sim w_3$ whose unique infinite set is x_0 and let

$$\mathcal{Y} = \{w_2 : w_2 \sim w_3 \ \wedge \ \forall y(inf(w_2) = \{y\} \Rightarrow B(w_1) \subseteq y)\}.$$

Since $B(w_1)$ is finite, clearly $\mathcal{Y}$ is infinite. By (18),

$$(\forall w_2 \in \mathcal{Y})(\exists I \subseteq [n])(\exists i)[X_I^{w_1} \in y_i^{w_2} \Leftrightarrow Y_I^{w_2} \notin z_i^{w_3}]].$$

But for every $w_2 \in \mathcal{Y}$, if y_0 is the element corresponding to x_0 and z_0, $B(w_1) \subseteq y_0$. Therefore (18) implies

$$(\forall w_2 \in \mathcal{Y})(\exists I)(Y_I^{w_2} \notin z_0).$$

Equivalently,

$$(\forall w_2 \in \mathcal{Y})(\exists Y \in B(w_2))(Y \notin z_0),$$

or

$$(\forall w_2 \in \mathcal{Y})(B(w_2) \cap -z_0 \neq \emptyset).$$

The latter easily implies that $-z_0$ must be infinite, which is a contradiction since $-z_0$ is finite. ⊣

Towards proving in the affirmative the general statement of the QUESTION, we have tried to generalize the method used in the proof of $(0,0,2)$-augmentability of $o^{\mathcal{A}}$, in lemma 3.5 of [8]. The nontrivial case of the proof was the one where $ind(w_3)=(2,0)$, i.e., w_3 was a binary partition $(z,-z)$, consisting of two infinite sets. The proof was by contradiction again. Namely we assumed that (18) is true for *all* $w_1=(x,-x)$ and *all* $w_2=(y,-y)$. But then (18) should hold also for the permutations of the partitions w_1, w_2, that is, the partitions $(-x,x)$ $(-y,y)$. The exploitation of this fact led eventually to a contradiction (actually considering only the permutations of w_1 suffices).

Does this idea work in the case of an arbitrary simple partition w_3, with $|w_3|\geq 3$? For simplicity one may consider the case where $ind(w_3)=(n,0)$ (n infinite sets, no singletons).[7] Even so, the situation is extremely complicated. The main obstacle in transferring the above roughly sketched proof of $(0,0,2)$-augmentability to the case of $(0,0,n)$-augmentability for $n\geq 3$, is the tremendous increase of complexity produced by the relation (18) even for the case $n=3$. Namely, given a $(n,0)$-partition $w_3=(z_1,\ldots,z_n)$, we fix temporarily $(n,0)$-partitions $w_1=(x_1,\ldots,x_n)$ and $w_2=(y_1,\ldots,y_n)$ that are supposed to satisfy (18). Let π, σ denote permutations of w_1,w_2, or equivalently $\pi,\sigma\in S_n$, since we just set $\pi(x_i)=x_{\pi(i)}$ and similarly for y_i. Let also

$$\pi(w_1)=(\pi(x_1),\ldots,\pi(x_n))=(x_{\pi(1)},\ldots,x_{\pi(n)}),$$

and similarly for $\pi(w_2)$. Then (18) yields: For all w_1,w_2 such that $w_1\sim w_2\sim w_3$

$$(\forall\pi,\sigma\in S_n)(\exists I\subseteq[n])(\exists i\in[n])[X_I^{\pi(w_1)}\in y_i^{\sigma(w_2)}\Leftrightarrow Y_I^{\sigma(w_2)}\notin z_i^{w_3}]. \tag{19}$$

(19) is a combinatorial statement involving three types of entities: (a) elements of the set $[n]$, having cardinality n, (b) elements of the set $\mathcal{P}([n])$ (or, actually, of $\mathcal{P}^*([n])=\mathcal{P}([n])-\{\emptyset,[n]\}$), having cardinality 2^n, and elements of the set S_n, having cardinality $n!$. For $n=2$, we have $|[2]|=|\mathcal{P}^*([2])|=|S_2|=2$. In that case (especially if we take $\sigma=id$) (19) reduces to a Boolean combination of no more than 8 concrete equivalences of the form $X_I^{\pi(w_1)}\in y_i^{\sigma(w_2)}\Leftrightarrow Y_I^{\sigma(w_2)}\notin z_i^{w_3}$. As a consequence these formulas can be controlled and manipulated so that eventually a contradiction can emerge. But if we make a step ahead and take $n=3$, we have $|[3]|=3$, $|\mathcal{P}^*([3])|=6$ and $|S_3|=6$. In such a case (19) reduces to a conjunction of $|S_3\times S_3|=36$

[7] We may reasonably assume that if this case is settled, then the general case of index (m,l) can also be settled by easy adjustments.

clauses each of which consists of a disjunction of $|\mathcal{P}^*([3])\times[3]| = 18$ concrete equivalences of the form $X_I^{\pi(w_1)} \in y_i^{\sigma(w_2)} \Leftrightarrow Y_I^{\sigma(w_2)} \notin z_i^{w_3}$. If we attempt to turn this formula into a disjunction of conjunctions in order to exploit all chances for reaching a contradiction, we shall face the monstrous number of 18^{36} disjuncts! (All the previous discussion concerns a fixed particular pair of partitions w_1, w_2.) That makes any attempt to reduce (19) to a set of specific consequences, unattainable and infeasible. Perhaps an approach through the *structural* consequences of (19) could prove successful.

6 Extendibility with no reference to TST

The discussion of the extendibility and augmentability properties of c.p.'s always takes place in ZFC but relatively to models of TST. In order to focus on the combinatorial character of these issues alone one can relax the role of particular models of TST by restricting oneself to the most natural and common of these, namely the *full* models

$$(A_0, \mathcal{P}(A_0), \mathcal{P}^2(A_0), \mathcal{P}^3(A_0))$$

of TST_4, which are just sequences of consecutive powersets. Given an infinite set A_0 (preferably countable for simplicity) let $A_1 = \mathcal{P}(A_0)$, $A_2 = \mathcal{P}(A_1)$ and $A_3 = \mathcal{P}(A_2)$. We call such a sequence (A_0, A_1, A_2, A_3), a *4-staircase*.

In order to be able to talk about similarity of partitions in the sets A_1, A_2, A_3, we consider all infinite cardinalities as identical, denoted ∞. We denote this *reduced cardinality* of a set X by $\|X\|$ and write $\|X\| = |X| = n$ if X is finite with n elements, and $\|X\| = \infty$ if X is infinite.

Replacing the ordinary notion of equipollence of sets $|X| = |Y|$ by reduced equipollence $\|X\| = \|Y\|$, coherent pairs can be formulated for any staircase $\mathcal{A} = (A_0, A_1, A_2, A_3)$. A pair of functions $p = (p_1, p_2)$ is a *coherent pair* if the definition given above for coherence holds for p with the relation $\|X\| = \|Y\|$ in place of $|X| = |Y|$. Namely

Definition 6.1 Let A_1, A_2 be infinite sets and w_1, w_2 be finite partitions of A_1, A_2 respectively. We say that w_1 and w_2 are *similar* and write $w_1 \sim w_2$, if there is a bijection $p : w_1 \to w_2$ such that $\|p(x)\| = \|x\|$ for every $x \in w_1$. In that case we write $p : w_1 \sim w_2$. If B_1, B_2 are finite Boolean algebras, then $B_1 \sim B_2$ if $Atom(B_1) \sim Atom(B_2)$.

Definition 6.2 Let $\mathcal{A} = (A_0, A_1, A_2, A_3)$ be a 4-staircase. A *coherent pair* over $\mathcal{A}$ is a pair $p = (p_1, p_2)$ of finite 1-1 mappings with the following properties:

(1) $dom(p_1)$ is a finite Boolean subalgebra of A_1, $rng(p_1) = dom(p_2)$ is a finite Boolean subalgebra of A_2, and $rng(p_2)$ is a finite Boolean subalgebra of A_3. We set $u_1 = dom(p_1)$, $u_2 = rng(p_1) = dom(p_2)$ and $u_3 = rng(p_2)$.

(2) $p_1 : u_1 \sim u_2$ and $p_2 : u_2 \sim u_3$.

(3) p_1, p_2 are $\in$-isomorphisms: For every $x \in u_1$ and $y \in u_2$,

$$x \in y \iff p_1(x) \in p_2(y).$$

We believe that the study of extendibility properties of coherent pairs might constitute a serious research project towards the solution of NF consistency. But independently of that, extendibility questions are genuine combinatorial problems interesting in themselves. Of course dealing with staircases instead of general models of TST is a restriction rather than a generalization concerning the results one may obtain (in the sense that if some ϕ holds with respect to all staircases, it doesn't follow that it holds for all models of TST). However, people working or just being interested in ordinary set theoretic combinatorics (e.g. partition calculus) can get interested in problems concerning coherent pairs more easily through the framework of staircases rather than through TST models. The purpose of using the term "4-staircase" instead of "full model of TST_4" is simply to disconnect the issue from the milieu of TST, its language, models etc, that might bother a combinatorist.

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