A combinatorial result related to the consistency of New Foundations

Athanassios Tzouvaras

Department of Mathematics
Aristotle University of Thessaloniki
541 24 Thessaloniki, Greece.
e-mail:tzouvara@math.auth.gr

Abstract

We prove a combinatorial result for models of the 4-fragment of the Simple Theory of Types (TST), TST_4. The result says that if $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ is a standard transitive and rich model of TST_4, then $\mathcal{A}$ satisfies the $\langle 0, 0, n\rangle$-property, for all $n \geq 2$. This property has arisen in the context of the consistency problem of the theory New Foundations (NF). The result is a weak form of the combinatorial condition (existence of ω-extendible coherent triples) that was shown in [5] to be equivalent to the consistency of NF. Such weak versions were introduced in [6] in order to relax the intractability of the original condition. The result strengthens one of the main theorems of [5, Theorem 3.6], which is equivalent just to the $\langle 0, 0, 2\rangle$-property.

1 Introduction

For more than 70 years the consistency of the set theory NF (New Foundations)[1] continues to be an open problem. After the work of Grishin [2], [3],

[1]This paper is not actually dealing with NF. It only deals with some combinatorial properties of models of the related Simple Theory of Types (TST), so the reader is not required to be familiar with NF. Nevertheless, for completeness of the presentation I have included most relevant definitions and facts in section 2. For further background material and proofs of the facts, the interested reader can consult [1].

who showed that (a) the whole system NF is equivalent to its fragment NF_4, and (b) the fragment NF_3 is consistent, it became clear that the consistency of NF (that is, the consistency of NF_4 relative to ZFC) can be reduced to a hard combinatorial problem. The specific combinatorics involved is rather peculiar and known techniques, such as Ramsey type theorems and partition calculus, do not seem to be helpful. Specifically, it concerns finite partitions, and the corresponding finite Boolean algebras, over three layers of infinite sets, each of which is (or approximates) the collection of all subsets of the previous one, the layers being roughly of the form A, $\mathcal{P}(A)$ and $\mathcal{P}^2(A)$, for A infinite. Given finite partitions u, v, w of A, $\mathcal{P}(A)$ and $\mathcal{P}^2(A)$, respectively, the triple $\langle u, v, w\rangle$ is said to be *coherent,* if the corresponding Boolean algebras $Bool(u)$, $Bool(v)$, $Bool(w)$ are isomorphic in a very strong way: There are bijections $f : Bool(u) \to Bool(v)$ and $g : Bool(v) \to Bool(w)$, such that (a) f, g are $\subseteq$-isomorphisms (i.e., usual Boolean isomorphisms), (b) f, g preserve the reduced cardinality of the sets (defined below), i.e., $\|f(X)\| = \|X\|$ and $\|g(Y)\| = \|Y\|$, and (c) f, g are $\in$-isomorphisms, i.e., $X \in Y \Leftrightarrow f(X) \in g(Y)$, for all $X \in Bool(u)$ and $Y \in Bool(v)$. The latter condition means that $Bool(u)$ distributes over the sets of $Bool(v)$, exactly as $Bool(v)$ distributes over the corresponding sets of $Bool(w)$. It is not hard to prove existence of coherent triples. The difficulty begins when we want to *extend* existing coherent triples to finer ones by adding arbitrary new sets, or simply to complete a single given partition into a coherent triple.

In order to explain briefly the motivation and give the perspective for the result of the present paper, let me say that this is part of ongoing work initiated with [5] and aiming to prove the consistency of NF by forcing. The importance of the extendibility property for coherent triples lies in the fact that if extendible triples exist over a model of TST_4, then they can be used as *forcing conditions* in order to obtain a model of the fragment NF_4 of NF, and hence a model of NF itself, since it is known that the latter is equivalent to NF_4. This is because a model of NF_4 is (generated by) a model $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ of TST_4 (the fragment of TST, the Simple theory of Types, consisting of four levels), plus a "type-shifting automorphism" for $\mathcal{A}$, i.e., a pair of bijections $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ which preserve both $\subseteq$ and $\in$. Our plan is, starting with a countable model M of ZFC and an "appropriate" model $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ of TST_4, to generically add a type-shifting automorphism $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ for $\mathcal{A}$ in $M[G]$, forcing with a set of extendible coherent triples. If this can be done, then $\mathcal{A}$ together with the pair $\langle f_1, f_2\rangle$, gives rise to a model of NF_4 in $M[G]$, and hence to a model of NF (see the Remarks 1.1 below).

In [5] we showed that the consistency of NF is equivalent to a certain strong extendibility condition (called ω-extendibility) concerning coherent triples. At the same time, this particular formulation revealed the extraordinary complexity and hardness of the problem.

In view of the intractability of the existence of ω-extendible pairs, we considered in [6] considerably weaker properties, called "augmentability properties". Of those again the simplest ones are the so-called $\langle n, 0, 0\rangle$-, $\langle 0, n, 0\rangle$- and $\langle 0, 0, n\rangle$-properties, for $n \geq 2$ (we define them below). The first two of them are relatively easy to prove, for all $n \geq 2$, and were established in [6]. On the other hand, the property $\langle 0, 0, n\rangle$ is tougher. The proof of $\langle 0, 0, 2\rangle$-property is actually the main result of [5] (lemma 3.5 and theorem 3.6) (under an equivalent formulation in terms of 1-extendibility). Also in [6, lemma 17], it was shown that the $\langle 0, 0, n\rangle$-property holds for n-partitions containing a single infinite set. So what remained open was the $\langle 0, 0, n\rangle$-property for $n > 2$ and for n-partitions containing at least two infinite sets. This is stated and discussed in [6] as the main open question. The aim of the present paper is to settle this problem. We prove that the $\langle 0, 0, n\rangle$-property indeed holds true in every rich model of TST_4, for all n-partitions and for every $n \geq 2$.

Remarks 1.1 Before closing this introduction, and in view of the above mentioned plan to extend generically a model M of ZFC containing a model $\mathcal{A}$ of TST_4 to a model $M[G]$ so that $\mathcal{A}$ becomes in $M[G]$ essentially a model of NF, the following remarks are in order:

(1) It is not known exactly what the *appropriate kind* of model $\mathcal{A}$ of TST_4 is that one could hopefully try to turn to a model of NF. We know that $\mathcal{A}$ should not satisfy the axiom of choice (AC), in particular $\mathcal{A}$ should not be a full model $\mathcal{A} = \langle A, \mathcal{P}(A), \mathcal{P}^2(A), \mathcal{P}^3(A)\rangle$ in the sense of M. This is because if $\mathcal{A} \models \mathrm{AC}$ in M, then also $\mathcal{A} \models \mathrm{AC}$ in the generic extension $M[G]$, so if we assume that $M[G]$ contains a type-shifting automorphism $\langle f_1, f_2\rangle$ for $\mathcal{A}$, then $\langle \mathcal{A}, f_1, f_2\rangle$ essentially satisfies NF+AC, which is a contradiction, since it is well-known that $\mathrm{NF} \vdash \neg\mathrm{AC}$. This means that the initial model $\mathcal{A}$ should be in some sense "symmetric".

(2) On the other hand a property that is needed for elementary constructions inside $\mathcal{A}$, is the "splitting property" (SP): "Every infinite set splits into two infinite subsets". We call a model $\mathcal{A}$ of TST_4 satisfying SP *rich.* This

property is also naturally defined for Boolean algebras of sets. Equivalently, a model $\mathcal{A} = \langle A_0, A_1, A_2, A_3 \rangle$ is rich if all the algebras A_i, for $i > 0$, are rich. The models of TST_4 involved in the proof of the result of this paper are rich. As far as we know there is no indication that SP cannot hold for models of NF.

2 Preliminaries

Throughout our metatheory will be ZFC. $\in$ will denote the membership relation of the ground world. The next subsection 2.1 contains definitions and facts concerning the theories TST and NF. The reader who does not want to bother with them may skip it and proceed to 2.2, with the proviso that throughout the rest of the paper, he will replace the term "rich model of TST_4" with "full model of TST_4", i.e., with a structure of the form $\mathcal{A} = \langle A, \mathcal{P}(A), \mathcal{P}^2(A), \mathcal{P}^3(A) \rangle$.

2.1 Standard material

The language L_{TST} of the Theory of Simple Types (TST) has a binary predicate symbol ε and typed variables x^i_j, for all $i, j \in \mathbb{N}$. The superscript i indicates the type. As usual, instead of subscripts we may use different letters, y^j, z^k etc. The atomic formulas of L_{TST} are $x^i \varepsilon y^{i+1}$ and $x^i = y^i$. The other formulas are built from the atomic formulas using connectives and quantifiers as usual. The axioms of TST are the following schemes of comprehension and extensionality:

(Co) $(\exists x^{i+1})(\forall y^i)(y^i \varepsilon x^{i+1} \Leftrightarrow \phi(y^i))$, for every $\phi(y^i) \in L_{\mathrm{TST}}$ possibly with extra free variables.

(Ex) $(\forall x^i)(x^i \varepsilon y^{i+1} \Leftrightarrow x^i \varepsilon z^{i+1}) \ \Rightarrow\ y^{i+1} = z^{i+1}$.

A model of TST is a sequence $\mathcal{A} = \langle A_0, A_1, \ldots, R \rangle$, where each A_i interprets the variables of type i, and $R \subseteq \bigcup_i (A_i \times A_{i+1})$ is a binary relation that interprets ε. The model $\mathcal{A}$ is *standard transitive* (s.t.) if for every $i \geq 0$, $A_{i+1} \subseteq \mathcal{P}(A_i)$ and $R = \in \restriction \bigcup_i (A_i \times A_{i+1})$. Without serious loss of generality (see [5]) we may restrict ourselves to standard transitive models. In that case we drop R and write simply $\mathcal{A} = \langle A_0, A_1, \ldots \rangle$. If for every $i \geq 0$ $A_{i+1} = \mathcal{P}(A_i)$, i.e., $\mathcal{A} = \langle A, \mathcal{P}(A), \mathcal{P}^2(A) \ldots \rangle$, for some infinite set A, the model $\mathcal{A}$ is said to be *full*, sometimes denoted $\langle\langle A\rangle\rangle$. Henceforth every model of TST will be standard transitive.

For $n > 0$, a formula ϕ of L_{TST} is an *n-formula*, if every variable of ϕ is of type $< n$. Let TST_n be the subtheory of TST whose axioms are those of TST restricted to n-formulas. A (standard transitive) model of TST_n is an n-sequence $\mathcal{A} = \langle A_0, A_1, \ldots, A_{n-1}\rangle$ such that $A_{i+1} \subseteq \mathcal{P}(A_i)$. In particular below we shall be confined to TST_4 and its models $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$.[2] In particular a full model of TST_4 has the form $\mathcal{A} = \langle A, \mathcal{P}(A), \mathcal{P}^2(A), \mathcal{P}^3(A)\rangle$ (which in [6] is called also a *staircase*).

If $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ is a model of TST_4, the sets A_i, $0 \le i \le 3$, are the *levels* of the model. The elements of the bottom level A_0 are treated as "atoms" or urelements, i.e., as having no set structure. The elements of the other levels are ordinary sets and for $i \ge 0$, A_{i+1} is a Boolean subalgebra of $\mathcal{P}(A_i)$.

Given a model $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ of TST_4, a *type-shifting automorphism* for $\mathcal{A}$ is a triple of bijections

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

which preserve $\in$, i.e., for all $a \in A_0$, $x \in A_1$, $y \in A_2$,

$$a \in x \Leftrightarrow f_0(a) \in f_1(x), \text{ and } x \in y \Leftrightarrow f_1(x) \in f_2(y).$$

This is equivalent to say that there is a pair of bijections

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

which preserve both $\subseteq$ and $\in$, i.e.,

$$x_1 \subseteq x_2 \Leftrightarrow f_1(x_1) \subseteq f_1(x_2),\ y_1 \subseteq y_2 \Leftrightarrow f_2(y_1) \subseteq f_2(y_2),\ x \in y \Leftrightarrow f_1(x) \in f_2(y).$$

More generally, if $\mathcal{A} = \langle A_0, A_1, \ldots\rangle$ is a model of TST, a type shifting automorphism is an $\in$-preserving sequence of bijections

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots$$

The language L_{NF} of NF consists of the predicate ε and untyped variables $x, y, \ldots$. The atomic formulas of L_{NF} are $x\varepsilon y$ and $x = y$. A formula of L_{NF}

[2]This is because the theory NF, which is closely connected with TST, is equivalent to its subtheory NF_4.

is called *stratified* if it results from a formula of L_{TST} if we erase all type superscripts from its variables. The axioms of NF are stratified comprehension and extensionality:

(StCo) $(\exists x)(\forall y)(y\varepsilon x \Leftrightarrow \phi(y))$, for every stratified $\phi(y) \in L_{\mathrm{NF}}$, possibly with extra free variables.

(Ex) $(\forall x)(x\varepsilon y \Leftrightarrow x\varepsilon z) \Rightarrow\ y = z$.

A model of NF is of the form $\langle K, E\rangle$, where $E \subseteq K^2$ is a binary relation on K interpreting ε.

Basic Fact. There is a model of NF iff there is a model of TST with a type shifting-automorphism. Specifically (cf. [4]):

(1) If $\mathcal{A} = \langle A_0, A_1, \ldots\rangle$ is a s.t. model of TST with a type- shifting automorphism $\langle f_0, f_1, \ldots\rangle$ and we define the relation E on A_0 by

$$aEb \Leftrightarrow a \in f_0(b),$$

then the structure $K_{\mathcal{A}} = \langle A_0, E\rangle$ is a model of NF.

(2) If $\langle K, E\rangle$ is a model of NF, let $A_n = K \times \{n\}$, for every $n \in \omega$. Let also $R \subseteq \bigcup_n (A_n \times A_{n+1})$ be the relation defined by

$$\langle a, n\rangle R\langle b, n+1\rangle \Leftrightarrow aEb.$$

If we set $\mathcal{A}_K = \langle A_0, A_1 \ldots\rangle$, then $\mathcal{A}_K$ is a model of TST and the bijections $f_n : A_n \to A_{n+1}$, defined by $f(\langle a, n\rangle) = \langle a, n+1\rangle$ form a type-shifting automorphism for $\mathcal{A}_K$. [$\mathcal{A}_K$ is in general a nonstandard model, but using the level collapsing of [5], we can turn it into an (almost isomorphic) standard transitive model of TST.]

We often conflate the structures K and $\mathcal{A}_K$. Also, given $\langle K, E\rangle$, we refer to the TST structure $\mathcal{A}_K$ as the model of TST *underlying* K.

2.2 Material needed for the present result

Definition 2.1 A Boolean algebra of sets B is said to be *rich* if for every infinite set $X \in B$ there are infinite sets $X_1, X_2 \in B$ such that $X_1 \cup X_2 \in X$ and $X_1 \cap X_2 = \emptyset$. A model $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ is said to be *rich* if every A_i, $i > 0$, is a rich Boolean algebra.

Clearly every full model, or every model of TST_4 satisfying the axiom of choice, is rich but not the other way around. The property of richness will be essential for the proof of the main result of this paper.

Given $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$, we shall be mainly concerned with finite partitions of the sets A_i, $i \leq 2$ the sets of which belong to the next level A_{i+1}. Specifically we are interested in a certain strong similarity relation between such partitions, with respect (a) to the Boolean structure and (b) to the cardinality of the corresponding elements. Since the actual cardinalities of A_i may be different (for example when $\mathcal{A}$ is full), we shall employ the notion of *reduced cardinality,* that is, we shall distinguish only between sets which are finite with cardinalities $m \neq n$, and between finite and infinite ones. All infinite sets have the same reduced cardinality. We denote the reduced cardinality of a set X by $\|X\|$. If X is finite, then we set $\|X\| = |X| = n$. If X is infinite we set $\|X\| = \infty$. For every positive integer n we shall use throughout the notation

$$[n] = \{1, \dots, n\}.$$

Definition 2.2 Let A be a set and let $n \geq 2$. An *n-partition* of A, is an n-tuple $u = \langle x_1, \dots, x_n\rangle$ of subsets of A, such that (a) $x_i \notin \{\emptyset, A\}$, for all $i \in [n]$, (b) $x_i \cap x_j = \emptyset$ for $i \neq j$ and (c) $\bigcup_{i=1}^{n} x_i = A$. n is the *length* of u.[3]

It is important to stress that partitions are treated here as *ordered tuples* rather than just sets. If $u = \langle x_1, \dots, x_n\rangle$ is an n-partition of A it seems more appropriate to denote the i-th element of the partition u by x_i^u, rather than just x_i. In this notation

$$u = \langle x_1^u, \dots, x_n^u\rangle.$$

However, if there is no danger of confusion, we can keep writing x_i instead of x_i^u.

Definition 2.3 Let $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ be a model of TST_4. For every $i \leq 2$, $Part_{\mathcal{A}}(A_i)$ denotes the set of all finite partitions u of A_i such that $x_i^u \in A_{i+1}$ for every $i \leq n$ where n is the length of u. If there is no danger of confusion we write $Part(A_i)$ instead of $Part_{\mathcal{A}}(A_i)$.

Sometimes, given a model $\mathcal{A}$ as above, we refer just to "partitions of A_i", while we always mean elements of $Part_{\mathcal{A}}(A_i)$.

[3] In section 3 below we shall need to refer also to the trivial partition $\{\emptyset, A\}$ of a set A. This partition will be denoted simply by $\emptyset$.

Definition 2.4 Let A, B be infinite sets, and let $u = \langle x_1, \ldots, x_n \rangle$, $v = \langle y_1, \ldots, y_m \rangle$ be finite partitions of A, B respectively. We say that u and v are *similar* and write $u \sim v$, if (a) $m = n$, and (b) $\|x_i\| = \|y_i\|$ for every $i \in [n]$.

Every finite partition u on a set A generates a finite non-trivial Boolean algebra, denoted by $Bool(u)$, whose atoms are the elements of u. Boolean algebras, in contrast to partitions, are treated as sets, rather than tuples. Conversely, let $\mathcal{B}$ be a non-trivial finite Boolean algebra on A. $\mathcal{B}$ has a set of atoms, denoted by $Atom(\mathcal{B})$. If $|Atom(\mathcal{B})| = n$, then for every 1-1 enumeration $u : [n] \to Atom(\mathcal{B})$, u is an n-partition of A and $Bool(u) = \mathcal{B}$.

Definition 2.5 Two non-trivial finite Boolean algebras $\mathcal{B}_1, \mathcal{B}_2$ on the sets A_1, A_2 respectively are said to be *similar,* notation $\mathcal{B}_1 \sim \mathcal{B}_2$, if there are similar partitions $u \sim v$ such that $\mathcal{B}_1 = Bool(u)$ and $\mathcal{B}_2 = Bool(v)$. That is, $Bool(u) \sim Bool(v) \iff u \sim v$.

If $\mathcal{B} = Bool(u)$ and $u = \langle x_1, \ldots, x_n \rangle$, then each $X \in \mathcal{B}$ is uniquely written as the union of some elements of u, namely there is a unique $I \subseteq [n]$ such that $X = \bigcup_{i \in I} x_i$. Throughout we shall use the convenient notation X_I^u to denote this set $\bigcup_{i \in I} x_i$, i.e., let

$$X_I^u = \bigcup_{i \in I} x_i = \bigcup_{i \in I} x_i^u.$$

In this notation the letter X (as well as x) plays the role of a bound variable, so we could replace it, say, by Y and write Y_I^u instead of X_I^u. This will be done below when we refer to distinct partitions u, v and the corresponding algebras $Bool(u)$, $Bool(v)$.

Note that for any $i \in [n]$, $X_{\{i\}}^u = x_i^u$, i.e., the notation X_I^u consistently extends the notation x_i^u adopted above. Also $X_{[n]}^u = A$ and $X_\emptyset^u = \emptyset$. Thus for every n-partition u

$$Bool(u) = \{X_I^u : I \in \mathcal{P}([n])\}. \tag{1}$$

Lemma 2.6 *Let $\mathcal{B}_1, \mathcal{B}_2$ be finite non-trivial Boolean algebras. $\mathcal{B}_1 \sim \mathcal{B}_2$ iff there is a Boolean isomorphism $f : \mathcal{B}_1 \to \mathcal{B}_2$ such that $\|f(X)\| = \|X\|$ for every $X \in \mathcal{B}_1$. This isomorphism has the form $f(X_I^u) = Y_I^v$, where $\mathcal{B}_1 = Bool(u)$, $\mathcal{B}_2 = Bool(v)$.*

Proof. Let $\mathcal{B}_1 \sim \mathcal{B}_2$. Then $\mathcal{B}_1 = Bool(u)$, $\mathcal{B}_2 = (v)$ and $u \sim v$ for some partitions u, v. Let $u = \langle x_1, \ldots, x_n \rangle$, $v = \langle y_1, \ldots, y_n \rangle$. Then $\|x_i\| = \|y_i\|$ for all $i \in [n]$. Define $f : \mathcal{B}_1 \to \mathcal{B}_2$ by setting $f(X_I^u) = Y_I^v$ for all $I \in \mathcal{P}([n])$. This in particular yields $f(x_i^u) = f(X_{\{i\}}^u) = Y_{\{i\}}^v = y_i^v$, for every $i \in [n]$. Clearly for all $I, J \in \mathcal{P}([n])$,

$$X_I^u \subseteq X_J^u \Leftrightarrow I \subseteq J \Leftrightarrow Y_I^v \subseteq Y_J^v,$$

thus f preserves $\subseteq$ and hence it is a Boolean isomorphism. Also, $\|x_i\| = \|y_i\|$ immediately implies $\|X_I^u\| = \|Y_I^v\|$ for every $I \in \mathcal{P}([n])$, therefore $\|f(X)\| = \|X\|$ for every $X \in \mathcal{B}_1$.

Conversely, let $f : \mathcal{B}_1 \to \mathcal{B}_2$ be an isomorphism such that $\|f(X)\| = \|X\|$ for every $X \in B_1$. Let $Atom(\mathcal{B}_1)$ be the set of atoms of $\mathcal{B}_1$ and let $u = \langle x_1, \ldots, x_n \rangle$ be an enumeration of $Atom(\mathcal{B}_1)$. Since f sends atoms to atoms, if $y_i = f(x_i)$, then $v = \langle y_1, \ldots, y_n \rangle$ is an enumeration of $Atom(\mathcal{B}_2)$. By assumption $\|y_i\| = \|f(x_i)\| = \|x_i\|$, i.e., $u \sim v$. Thus $\mathcal{B}_2 = Bool(v)$ and $\mathcal{B}_1 \sim \mathcal{B}_2$. Moreover, $f(X_I^u) = Y_I^v$. ⊣

When $u \sim v$, we refer to the isomorphism $f : Bool(u) \to Bool(v)$ such that $f(X_I^u) = Y_I^v$ as the *canonical* isomorphism.

Before going on let us fix the following notational conventions that facilitate reading.

Notational conventions: Given a model $\mathcal{A} = \langle A_0, A_1, A_2, A_3 \rangle$ of TST_4, the letters

X, x, x_1, etc, range over elements of A_1 (hence subsets of A_0),

Y, y, y_1, etc, range over elements of A_2 (hence subsets of A_1),

Z, z, z_1, etc, range over elements of A_3 (hence subsets of A_2).

a, b, c, etc, range over elements of A_0.

The letters u, u_1, u', etc, range over finite partitions of A_0, i.e., elements of $Part_{\mathcal{A}}(A_0)$. Since its elements belong to A_1, u has the form $u = \langle x_1, \ldots, x_n \rangle$.

The letters v, v_1, v', etc, range over elements $Part_{\mathcal{A}}(A_1)$.. Since its elements belong to A_2, v has the form $v = \langle y_1, \ldots, y_n \rangle$.

The letters w, w_1, w', etc, range over elements of $Part_{\mathcal{A}}(A_2)$. Since its elements belong to A_3, w has the form $w = \langle z_1, \ldots, z_n \rangle$.

In particular, X_I^u denote sets of the Boolean algebra $Bool(u)$, Y_I^v denote sets of the Boolean algebra $Bool(v)$, Z_I^w denote sets of the Boolean algebra $Bool(w)$.

Definition 2.7 Let $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ be a model of TST_4, and let u, v, w be n-partitions of A_0, A_1, A_2, respectively, for $n \geq 2$. The triple $\langle u, v, w\rangle$ is said to be *coherent,* notation $coh(u, v, w)$, if:

(a) $u \sim v \sim w$ (hence $Bool(u) \sim Bool(v) \sim Bool(w)$), and

(b) The canonical isomorphisms $f : X^u_I \mapsto Y^v_I$ and $g : Y^v_I \mapsto Z^w_I$ between $Bool(u), Bool(v)$ and $Bool(v), Bool(w)$ are also $\in$-preserving, i.e.,

$$(\forall I, J \subseteq [n])(X^u_I \in Y^v_J \iff Y^v_I \in Z^w_J). \tag{2}$$

Lemma 2.8 *Given n-partitions $u = \langle x_1, \ldots, x_n\rangle$, $v = \langle y_1, \ldots, y_n\rangle$, $w = \langle z_1, \ldots, z_n\rangle$, such that $u \sim v \sim w$, $\langle u, v, w\rangle$ is coherent iff:*

$$(\forall I \subseteq [n])(\forall i \in [n])(X^u_I \in y^v_i \iff Y^v_I \in z^w_i). \tag{3}$$

Proof. It suffices to show that condition (2) is equivalent to (3). Clearly (2) implies (3). For the converse, suppose (3) holds and let $X^u_I \in Y^v_J$. Since $Y^v_J = \bigcup_{i\in J} y^v_i$ and y^v_i are disjoint, there is a unique $i_0 \in J$ such that $X^u_I \in y^v_{i_0}$. By (3), $Y^v_I \in z^w_{i_0}$. Hence $Y^v_I \in Z^w_J$, since $Z^w_J = \bigcup_{i\in J} z^w_i$. Thus $X^u_I \in Y^v_J \Rightarrow Y^v_I \in Z^w_J$. The other direction is shown similarly. Therefore (3) implies (2). $\dashv$

Example. Given a model $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$, let $u = \langle x_1, x_2\rangle$ be a 2-partition of A_0 with $x_1 = \{a\}$, and $x_2 = A_0 - \{a\}$, for some $a \in A_0$. Let $v = \langle y_1, y_2\rangle$ be a 2-partition of A_1 with $y_1 = \{x\}$, and $y_2 = A_1 - \{x\}$, for some $x \in A_1$. Let $w = \langle z_1, z_2\rangle$ be a 2-partition of A_2 with $z_1 = \{y\}$, and $z_2 = A_2 - \{y\}$, for some $y \in A_2$.

Clearly $u \sim v \sim w$. In order for $\langle u, v, w\rangle$ to be coherent, the following must be the case:

(a) $\emptyset \in y_i \Leftrightarrow \emptyset \in z_i$. (In particular $x = \emptyset \Leftrightarrow y = \emptyset$.)

(b) $A_0 \in y_i \Leftrightarrow A_1 \in z_i$. (In particular $x = A_0 \Leftrightarrow y = A_1$.)

(c) $x_1 \in y_i \Leftrightarrow y_1 \in z_i$. (In particular $x = \{a\} \Leftrightarrow y = \{x\}$.)

(c) $x_2 \in y_i \Leftrightarrow y_2 \in z_i$. (In particular $x = A_0 - \{a\} \Leftrightarrow y = A_1 - \{x\}$.)

3 Extendibility

Let us fix a model $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ of TST_4. The main combinatorial problem about coherent triples over $\mathcal{A}$ is their *extendibility.* That is, given a coherent triple $\langle u, v, w\rangle$ over A_0, A_1, A_2, and a set $x \in A_1$, or $y \in A_2$, or

$z \in A_3$, to find a coherent triple $\langle u', v', w' \rangle$, such that u', v', w' extend (i.e., refine) u, v, w, respectively, and accommodate also x, y or z.

First some terminology and notation. Given two finite partitions u_1, u_2, say of A_0, we say that u_2 *refines* u_1, and write $u_1 \sqsubseteq u_2$, if every set of u_2 is a subset of a set of u_1. If u_1 is an n-partition, u_2 is an m-partition and $u_1 \sqsubseteq u_2$, then clearly $n \leq m$. However here we are interested not in refinements of isolated partitions, but in refinements of triples $\langle u, v, w \rangle$. Despite of this, for the needs of the present paper, one might just use the "extension relation" $\sqsubseteq$ defined by:

$$\langle u, v, w \rangle \sqsubseteq \langle u', v', w' \rangle := u \sqsubseteq u' \wedge v \sqsubseteq v' \wedge w \sqsubseteq w'.$$

Nevertheless, in view of the discussion in section 1, according to which extendible coherent triples are intended to be used as forcing conditions, a more refined notion of extension is needed, which we cite here for reasons of precision, as well as for future reference.

Let $u_1 \sqsubseteq u_2$, where u_1 is an n partition and u_2 is an m partition, for $n < m$. Then each element $X_I^{u_1}$ of $Bool(u_1)$ coincides with an element $X_J^{u_2}$ of the larger algebra $Bool(u_2)$. So there is a unique injection

$$e_{u_2}^{u_1} : \mathcal{P}([n]) \to \mathcal{P}([m])$$

that defines the sets of $Bool(u_1)$ in terms of the atoms of $Bool(u_2)$. Namely, for every $I \in \mathcal{P}([n])$,

$$X_I^{u_1} = X_{e_{u_2}^{u_1}(I)}^{u_2}.$$

We call $e_{u_2}^{u_1}$ the *extension mapping* between u_1 and u_2. We drop the indices from e when there is no danger of confusion. The following properties of extension mappings are easy to check.

Lemma 3.1 *Let $u_1 \sqsubseteq u_2$, where u_1 is an n-partition and u_2 is an m-partition. Then the extension mapping $e = e_{u_2}^{u_1}$ has the following properties:*

(a) If for every $i \in [n]$ we write $e(i) = e(\{i\})$, then the sets $\{e(i) : i \in [n]\}$ form a partition of $[m]$.

(b) For every $I, J \in \mathcal{P}([n])$, $I \subseteq J \Leftrightarrow e(I) \subseteq e(J)$. In particular, for every $I \in \mathcal{P}([n])$, $e(I) = \bigcup\{e(i) : i \in I\}$.

(c) If $u \sqsubseteq u_1 \sqsubseteq u_2$, then $e_{u_2}^{u} = e_{u_1}^{u} \circ e_{u_2}^{u_1}$.

In view of (b) of the preceding lemma, it suffices to define the extension mapping e on the singleton elements of $\mathcal{P}([n])$, or equivalently, "identifying"

$\{i\} \in \mathcal{P}([n])$ with $i \in [n]$, it suffices to define $e : [n] \to \mathcal{P}([m])$ so that $x_i^{u_1} = X_{e(i)}^{u_2}$, and then extend it to the whole $\mathcal{P}([n])$ by setting $e(I) = \bigcup\{e(i) : i \in I\}$.

Given $u_1 \sqsubseteq u_2$ and $v_1 \sqsubseteq v_2$ such that $u_1 \sim v_1$ and $u_2 \sim v_2$, under what conditions is $e_{u_2}^{u_1} = e_{v_2}^{v_1}$? Recall from Lemma 2.6 of the preceding section that $u \sim v$ iff the Boolean algebras $Bool(u)$, $Bool(v)$ are isomorphic via the canonical isomorphism $f : Bool(u) \to Bool(v)$ such that $f(X_I^u) = Y_I^v$.

Lemma 3.2 *Let u_1, v_1, u_2, v_2 be partitions such that $u_1 \sim v_1$, $u_2 \sim v_2$ $u_1 \sqsubseteq u_2$, and $v_1 \sqsubseteq v_2$. Then $e_{u_2}^{u_1} = e_{v_2}^{v_1}$ iff the canonical isomorphism between $Bool(u_2)$ and $Bool(v_2)$ extends the canonical isomorphism between $Bool(u_1)$ and $Bool(v_1)$.*

Proof. Let $f : Bool(u_2) \to Bool(v_2)$ be the canonical isomorphism, i.e., $f(X_I^{u_2}) = Y_I^{v_2}$. Let $e_1 = e_{u_2}^{u_1}$ and $e_2 = e_{v_2}^{v_1}$. Then f extends the canonical isomorphism between $Bool(u_1)$ and $Bool(v_1)$ iff for each $X_I^{u_1} \in Bool(u_1)$, $f(X_I^{u_1}) = Y_I^{v_1}$, or equivalently $f(X_{e_1(I)}^{u_2}) = Y_{e_2(I)}^{v_2}$. But $f(X_{e_1(I)}^{u_2}) = Y_{e_1(I)}^{v_2}$, hence $Y_{e_1(I)}^{v_2} = Y_{e_2(I)}^{v_2}$, for every I. This holds iff $e_1(I) = e_2(I)$ for every I, i.e., iff $e_1 = e_2$. ⊣

The right notion of extension for triples of (similar) partitions is given in the following definition (as is customary in forcing, we write $p \leq q$ for "p extends q" rather than $q \leq p$):

Definition 3.3 Let $\langle u_1, v_1, w_1\rangle$ and $\langle u_2, v_2, w_2\rangle$ be triples such that $u_1 \sim v_1 \sim w_1$, $u_2 \sim v_2 \sim w_2$, $u_1 \sqsubseteq u_2$, $v_1 \sqsubseteq v_2$ and $w_1 \sqsubseteq w_2$. We say that $\langle u_2, v_2, w_2\rangle$ *extends* $\langle u_1, v_1, w_1\rangle$ and write $\langle u_2, v_2, w_2\rangle \leq \langle u_1, v_1, w_1\rangle$ if the canonical isomorphisms $f_2 : Bool(u_2) \to Bool(v_2)$ and $g_2 : Bool(v_2) \to Bool(w_2)$ extend the corresponding canonical isomorphisms $f_1 : Bool(u_1) \to Bool(v_1)$ and $g_1 : Bool(v_1) \to Bool(w_1)$, i.e., if $f_1 = f_2{\restriction}Bool(u_1)$ and $g_1 = g_2{\restriction}Bool(v_1)$.

Lemma 3.4 *Let $\langle u_1, v_1, w_1\rangle$ and $\langle u_2, v_2, w_2\rangle$ be triples such that $u_1 \sim v_1 \sim w_1$, $u_2 \sim v_2 \sim w_2$, $u_1 \sqsubseteq u_2$, $v_1 \sqsubseteq v_2$ and $w_1 \sqsubseteq w_2$. Then:*

(i) $\langle u_2, v_2, w_2\rangle \leq \langle u_1, v_1, w_1\rangle$ iff $e_{u_2}^{u_1} = e_{v_2}^{v_1} = e_{w_2}^{w_1}$.

(ii) If $\langle u_2, v_2, w_2\rangle \leq \langle u_1, v_1, w_1\rangle$ and $coh(u_2, v_2, w_2)$, then $coh(u_1, v_1, w_1)$.

Proof. (i) Let $\langle u_2, v_2, w_2\rangle \leq \langle u_1, v_1, w_1\rangle$. By definition 3.3, the canonical isomorphisms between $Bool(u_2)$, $Bool(v_2)$ and $Bool(v_2)$, $Bool(w_2)$ extend the canonical isomorphisms between $Bool(u_1)$, $Bool(v_1)$ and $Bool(v_1), Bool(w_1)$, respectively. But, by Lemma 3.2, this holds iff $e^{u_1}_{u_2} = e^{v_1}_{v_2}$ and $e^{v_1}_{v_2} = e^{w_1}_{w_2}$.

(ii) Let f_1, g_1 be the canonical isomorphisms between $Bool(u_1)$, $Bool(v_1)$ and $Bool(v_1)$, $Bool(w_1)$, respectively, and let f_2, g_2 be the canonical isomorphisms between $Bool(u_2)$, $Bool(v_2)$ and $Bool(v_2)$, $Bool(w_2)$, respectively. By $coh(u_2, v_2, w_2)$, the pair f_2, g_2 is $\in$-preserving. By $\langle u_2, v_2, w_2\rangle \leq \langle u_1, v_1, w_1\rangle$, f_1, g_1 are the restrictions of f_2, g_2 to $Bool(u_1)$ and $Bool(v_1)$, respectively, hence the pair f_1, g_1 is also $\in$-preserving. Therefore $coh(u_1, v_1, w_1)$. $\dashv$

Trivially, as follows from Lemma 3.4, $\langle u_2, v_2, w_2\rangle \leq \langle u_1, v_1, w_1\rangle$ implies $\langle u_1, v_1, w_1\rangle \sqsubseteq \langle u_2, v_2, w_2\rangle$. Although in this paper the role of $\leq$ is not crucial (since we actually seek coherent extensions of the triple $\langle\emptyset, \emptyset, \emptyset\rangle$ of trivial partitions), our definitions of this section concerning extendibility are given with respect to $\leq$ rather than $\sqsubseteq$.

Let a coherent triple $\langle u, v, w\rangle$ over $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ be given. A natural extendibility requirement for $\langle u, v, w\rangle$ is the following: Given a set $x \in A_1$, we wish to find a coherent triple $\langle u', v', w'\rangle$ such that $\langle u', v', w'\rangle \leq \langle u, v, w\rangle$ and $x \in Bool(u')$. In such a case we say that the triple $\langle u', v', w'\rangle$ *accommodates* x. Analogously, given $\langle u, v, w\rangle$ and $y \in A_2$, we wish to find a coherent $\langle u', v', w'\rangle$ such that $\langle u', v', w'\rangle \leq \langle u, v, w\rangle$ and $y \in Bool(v')$, thus accommodating y. And finally, given $\langle u, v, w\rangle$ and $z \in A_3$, we wish to find a coherent $\langle u', v', w'\rangle$ such that $\langle u', v', w'\rangle \leq \langle u, v, w\rangle$ and $z \in Bool(w')$, thus accommodating z. It follows that the extendibility requirement for the triple $\langle u, v, w\rangle$ splits into three particular cases (in [5] we refer to them as A_1- A_2 and A_3-extendibility respectively), which makes the formulation of the property a little bit cumbersome. In order to treat them all in a unified and concise way, let t range over $A_1 \cup A_2 \cup A_3$, and let $\langle u, v, w\rangle$ be a triple. Then the extendibility condition amounts to the existence of a coherent triple $\langle u', v', w'\rangle$ such that $\langle u', v', w'\rangle \leq \langle u, v, w\rangle$ and $t \in Bool(u') \cup Bool(v') \cup Bool(w')$.

Definition 3.5 Let $\langle u, v, w\rangle$ be a coherent triple over $\mathcal{A}$. We say that $\langle u, v, w\rangle$ is *1-extendible,* or just *extendible,* if for every $t \in A_1 \cup A_2 \cup A_3$, there is a coherent triple $\langle u', v', w'\rangle$ such that $\langle u', v', w'\rangle \leq \langle u, v, w\rangle$ and $t \in Bool(u') \cup Bool(v') \cup Bool(w')$. Inductively:

$\langle u, v, w\rangle$ is $(n+1)$-*extendible,* if for every $t \in A_1 \cup A_2 \cup A_3$, there is a triple $\langle u', v', w'\rangle$ such that $\langle u', v', w'\rangle \leq \langle u, v, w\rangle$, $\langle u', v', w'\rangle$ is n-extendible, and $t \in Bool(u') \cup Bool(v') \cup Bool(w')$.

$\langle u, v, w\rangle$ is said to be *ω-extendible* if it is n extendible for all $n \geq 1$.

When dealing with extensions of coherent triples, one can start with the triple of trivial partitions, denoted for simplicity, $\langle\emptyset, \emptyset, \emptyset\rangle$ (or equivalently the trivial Boolean algebras $\{\emptyset, A_0\}$, $\{\emptyset, A_1\}$, $\{\emptyset, A_2\}$). It was proved in [5]) that (a) NF is consistent iff for every n, there is a model of TST $\langle A_0, A_1, A_2, A_3\rangle$ in which $\langle\emptyset, \emptyset, \emptyset\rangle$ is n-extendible. (b) In any rich model of TST which is, roughly, an elementary submodel of a full model, $\langle\emptyset, \emptyset, \emptyset\rangle$ is 1-extendible (Th. 3.6).

The property of n-extendibility, for $n > 1$, is actually very hard to prove even for the trivial triple $\langle\emptyset, \emptyset, \emptyset\rangle$, mainly because it involves iterated extendibility. For that reason we considered in [6] some weaker extendibility properties. A natural such weakening is "augmentability", defined below.

Definition 3.6 Let $\langle u, v, w\rangle$ be a coherent triple over A_0, A_1, A_2, and let $n_1, n_2, n_3 \geq 2$. We say that $\langle u, v, w\rangle$ is $\langle n_1, n_2, n_3\rangle$*-augmentable* if for every n_1-partition u_1 of A_0, every n_2-partition v_1 of A_1 and every n_3-partition w_1 of A_2, there is a coherent triple $\langle u', v', w'\rangle$ such that $\langle u', v', w'\rangle \leq \langle u, v, w\rangle$ and $u_1 \sqsubseteq u'$, $v_1 \sqsubseteq v'$ and $w_1 \sqsubseteq w'$.

It is easy to check that n-extendible triples, for sufficiently large n, are $\langle n_1, n_2, n_3\rangle$-augmentable (cf. [6]). Even so, however, the general $\langle n_1, n_2, n_3\rangle$-augmentability property is messy. We shall be confined only to the case where $\langle n_1, n_2, n_3\rangle$ is $\langle n, 0, 0\rangle$, $\langle 0, n, 0\rangle$, and $\langle 0, 0, n\rangle$, and the extendible triple is $\langle\emptyset, \emptyset, \emptyset\rangle$.

Definition 3.7 We say that the model $\mathcal{A}$ satisfies the *$\langle n, 0, 0\rangle$-, $\langle 0, n, 0\rangle$-, or $\langle 0, 0, n\rangle$-property,* if the triple $\langle\emptyset, \emptyset, \emptyset\rangle$ is $\langle n, 0, 0\rangle$-augmentable, $\langle 0, n, 0\rangle$-augmentable, or $\langle 0, 0, n\rangle$-augmentable, respectively. Specifically:

(a) The $\langle n, 0, 0\rangle$-property holds in $\mathcal{A}$, if for every n-partition u of A_0, there are n-partitions v, w of A_1, A_2 respectively such that $coh(u, v, w)$.

(b) The $\langle 0, n, 0\rangle$-property holds in $\mathcal{A}$, if for every n-partition v of A_1, there are n-partitions u, w of A_0, A_2 respectively such that $coh(u, v, w)$.

(c) The $\langle 0, 0, n\rangle$-property holds in $\mathcal{A}$, if for every n-partition w of A_2, there are n-partitions u, v of A_0, A_1 respectively such that $coh(u, v, w)$.

The first two of the above properties are rather easy and were proved in [6, Cor. 14] to hold in rich models. Also $\langle 0, 0, 2\rangle$-augmentability is equivalent to A_3-extendibility of $\langle\emptyset, \emptyset, \emptyset\rangle$ and that was one of the main results of [5] (Th. 3.6). So what was left of this group of tractable extendibility conditions was the $\langle 0, 0, n\rangle$-property for $n \geq 3$.

4 Simplifying partitions. Simple partitions

Let $u = \langle x_1, \ldots, x_n \rangle$ be an n-partition of an infinite set A, with $n \geq 2$. Then, for each $i \in [n]$, $\|x_i\| \in \mathbb{N}^* \cup \{\infty\}$, where $\mathbb{N}^* = \{1, 2, \ldots\}$, and for at least one i, $\|x_i\| = \infty$. Let us call the n-tuple $\langle \|x_1\|, \ldots, \|x_n\| \rangle$, the *signature* of u, in symbols $sign(u)$. Obviously, for any two n-partitions u, v, $u \sim v$ iff $sign(u) = sign(v)$. Of course the simpler the signature of a partition, the easier to handle it. Already in [6] it was observed that when one is interested in "asymptotic" results, e.g. whether the $\langle 0, 0, n \rangle$-property holds for arbitrarily large n, one can restrict one's attention to "simple" partitions, that is, partitions whose sets are either infinite or singletons. This is because every finite partition u of an infinite set has a simple refinement $u' \sqsupseteq u$ (by dismantling every finite set of u into its singletons).

Definition 4.1 A finite partition $u = \langle x_1, \ldots, x_n \rangle$ of an infinite set A is said to be *simple* if for each $i \in [n]$, $\|x_i\| = 1$ or ∞.

If u is a simple n-partition, then n_1 of its sets, with $1 \leq n_1 \leq n$, are infinite, and the rest $n_2 = n - n_1$ are singletons. Without loss of generality, whenever a simple partition is given in the form of a tuple $u = \langle x_1, \ldots, x_n \rangle$, we assume that the first n_1 of the x_i's are the infinite ones and the rest n_2 are the singletons. In such a case, the signature of u has the form

$$sign(u) = \langle \underbrace{\infty, \ldots, \infty}_{n_1}, \underbrace{1, \ldots, 1}_{n_2} \rangle,$$

so it is reasonable to simplify it by writing

$$sign(u) = \langle n_1, n_2 \rangle.$$

The simplest of all cases is when $sign(u) = \langle n, 0 \rangle$, that is, when all sets of u are infinite. Such a partition u is called *uniform.*

In [6] (proposition 4.5 and lemma 5.6) it is shown that restricting ourselves to simple partitions occasions no loss of generality. Namely it is shown that ω- extendibility and ω-augmentability of simple partitions, imply ω- extendibility and ω-augmentability in general. In our case we are interested just in the following:

Lemma 4.2 *Suppose that for all $n \geq 2$, the $\langle 0, 0, n \rangle$-property holds in the model $\mathcal{A} = \langle A_0, A_1, A_2, A_3 \rangle$, for all simple n-partitions. Then it holds for all n-partitions.*

Proof. Let w be an n-partition of A_2 of arbitrary signature. We have to find partitions u, v on A_0, A_1, respectively, such that $coh(u, v, w)$. Let $w' \sqsupseteq w$ be the refinement of w consisting of the infinite sets of w and the singletons of elements belonging to the finite sets of w. w' is a simple m-partition for some $m \geq n$. By assumption there are simple partitions u', v' of A_0, A_1, respectively such that $coh(u', v', w')$, i.e., there are $f : Bool(u') \to Bool(v')$ and $g : Bool(v') \to Bool(w')$ which are $\in$- and $\subseteq$ isomorphisms. Since $w \sqsubseteq w'$, $Bool(w) \subseteq Bool(w')$. Then the reverse image $g^{-1''}Bool(w)$ is a Boolean subalgebra of $Bool(v')$ generated by a partition $v \sqsubseteq v'$, i.e., $g^{-1''}Bool(w) = Bool(v)$, and $v \sim w$. Similarly $f^{-1''}Bool(v)$ is a Boolean subalgebra of $Bool(u')$ generated by a partition $u \sqsubseteq u'$, i.e., $f^{-1''}Bool(v) = Bool(u)$, and $u \sim v$. Finally $coh(u, v, w)$ since the restrictions of f, g to $Bool(u)$ and $Bool(v)$ respectively are $\in$-isomorphisms. $\dashv$

In view of lemma 4.2, henceforth we can deal with simple partitions only. The known partial results about the $\langle 0, 0, n\rangle$-property are the following:

• The $\langle 0, 0, 2\rangle$-property is true for all 2-partitions ([5, Lemma 3.5]).

• The $\langle 0, 0, n\rangle$-property is true for all simple partitions of signature $\langle 1, n-1\rangle$, i.e., containing a single infinite set ([6, Lemma 17]).

Thus it remains to show that the $\langle 0, 0, n\rangle$-property holds for any simple partition of signature $\langle n_1, n_2\rangle$, where $n_1 \geq 2$ and $n_2 \geq 0$. This will be proved in the next section.

5 The result

Theorem 5.1 *Let $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ be a rich model (with infinite A_0). Let w be a partition of A_2 of signature $\langle n_1, n_2\rangle$, where $n_1 \geq 2$ and $n_2 \geq 0$. Then there are partitions u, v of A_0 and A_1, respectively, such that $coh(u, v, w)$.*

A few words about the proof. First, the reader who still feels uncomfortable with general rich models of TST may think that the model $\mathcal{A}$ we are working in is full, i.e., of the form $\langle A, \mathcal{P}(A), \mathcal{P}^2(A), \mathcal{P}^3(A)\rangle$. This is an object familiar to everyone, and nothing differs in the proof after this replacement. Now the heart of the proof of $\langle 0, 0, 2\rangle$-property in [5] is the following: If we assume that the $\langle 0, 0, 2\rangle$-property is false, then we are led through a rather long and unpredictable series of logical combinations to the conclusion that for every 2-partition u of A_0 and every 2-partition v of A_1 the elements of $Bool(u)$ must be distributed over the sets of v in just two *prescribed ways.* This already sounds unnatural, and it remains to show the existence of u and v such that $Bool(u)$ is not distributed over v in any of these ways (which is rather easy). For a long time we have been attempting to prove the $\langle 0,0,n\rangle$-property along the pattern of the $\langle 0,0,2\rangle$-proof. These attempts were leading to a tremendous increase of complexity and finally to failure. The present proof emerged only when the specific line of thought used in $n=2$ was abandoned. The idea is the following: Given w we need to find u, v such that the elements of $Bool(u)$ distribute over the sets of v exactly as the elements of $Bool(v)$ distribute over the sets of w. Schematically, given w we have to find u, v such that $\frac{v}{u} = \frac{w}{v}$. This is actually possible because there are only finitely many distribution patterns, while there is a vast variety of partitions of an infinite set.

The rest of this section will be devoted to the proof of 5.1. Fix a rich model $\mathcal{A} = \langle A_0, A_1, A_2, A_3\rangle$ of TST$_4$. For every $\langle n_1, n_2\rangle$, with $n_1 \geq 2$ and $n_2 \geq 0$, let

$$\mathcal{U}_{\langle n_1,n_2\rangle} = \{u : u \in Part_{\mathcal{A}}(A_0) \ \wedge \ sign(u) = \langle n_1, n_2\rangle\},$$

$$\mathcal{V}_{\langle n_1,n_2\rangle} = \{v : v \in Part_{\mathcal{A}}(A_1) \ \wedge \ sign(v) = \langle n_1, n_2\rangle\},$$

$$\mathcal{W}_{\langle n_1,n_2\rangle} = \{w : w \in Part_{\mathcal{A}}(A_2) \ \wedge \ sign(w) = \langle n_1, n_2\rangle\}.$$

Clearly for all $u \in \mathcal{U}_{\langle n_1,n_2\rangle}$, $v \in \mathcal{V}_{\langle n_1,n_2\rangle}$, and $w \in \mathcal{W}_{\langle n_1,n_2\rangle}$, $u \sim v \sim w$. So, in view of lemma 2.8, what we have to prove is

$$(\forall w \in \mathcal{W}_{\langle n_1,n_2\rangle}(\exists u \in \mathcal{U}_{\langle n_1,n_2\rangle})(\exists v \in \mathcal{V}_{\langle n_1,n_2\rangle})(\forall I \in \mathcal{P}([n]))(\forall i \in [n])$$
$$(X_I^u \in y_i^v \Leftrightarrow Y_I^v \in z_i^w). \tag{4}$$

Fix $n_1 \geq 2$, $n_2 \geq 0$, $n = n_1 + n_2$, and a $w_0 \in \mathcal{W}_{\langle n_1,n_2\rangle}$. That is,

$$w_0 = \langle z_1, \ldots, z_{n_1}, z_{n_1+1}, \ldots, z_n\rangle,$$

where all z_i for $i \in [n_1]$ are infinite, while all z_i for $i \in [n]-[n_1]$ are singletons. In view of (4), it suffices to prove

$$(\exists u \in \mathcal{U}_{\langle n_1,n_2\rangle})(\exists v \in \mathcal{V}_{\langle n_1,n_2\rangle})(\forall I \in \mathcal{P}([n]))(\forall i \in [n])$$
$$(X_I^u \in y_i^v \Leftrightarrow Y_I^v \in z_i^{w_0}). \tag{5}$$

Now, with a minor adjustment, we may assume that in (5) I ranges over sets $\neq \emptyset$ and $[n]$ only. This is because the sets $X^u_\emptyset = \emptyset$ and $X^u_{[n]} = A_0$ are constant for every u, and when w_0 is given the location of $\emptyset$ and A_1 among the sets of w_0 is also given. So we can confine ourselves to those $v \in \mathcal{V}_{\langle n_1,n_2\rangle}$ for which

$$\forall i(\emptyset \in y^v_i \Leftrightarrow \emptyset \in z^{w_0}_i) \wedge \forall i(A_0 \in y^v_i \Leftrightarrow A_1 \in z^{w_0}_i). \tag{6}$$

For example, if $\emptyset \in z^{w_0}_1$ and $A_1 \in z^{w_0}_3$, we need only consider those $v \sim u$ for which $\emptyset \in y^v_1$ and $A_0 \in y^v_3$. Given w_0 and v we express the fact that they satisfy (6) by saying that v and w_0 satisfy the same *initial conditions.* Thus, henceforth, we shall deal with the set

$$\mathcal{V}^*_{\langle n_1,n_2\rangle} = \{v \in \mathcal{V}_{\langle n_1,n_2\rangle} : v \text{ satisfies } (6)\},$$

instead of $\mathcal{V}_{\langle n_1,n_2\rangle}$, and with

$$\mathcal{P}^*([n]) = \mathcal{P}([n]) - \{\emptyset, [n]\},$$

instead of $\mathcal{P}([n])$. In view of $\mathcal{V}^*_{\langle n_1,n_2\rangle}$ and $\mathcal{P}^*([n])$, (5) is written:

$$(\exists u \in \mathcal{U}_{\langle n_1,n_2\rangle})(\exists v \in \mathcal{V}^*_{\langle n_1,n_2\rangle})(\forall I \in \mathcal{P}^*([n]))(\forall i \in [n])$$
$$(X^u_I \in y^v_i \Leftrightarrow Y^v_I \in z^{w_0}_i). \tag{7}$$

Given $u \in \mathcal{U}_{\langle n_1,n_2\rangle}$ and $v \in \mathcal{V}_{\langle n_1,n_2\rangle}$, the *distribution function* of $Bool(u)$ over $Bool(v)$ is the function $D^v_u : \mathcal{P}^*([n]) \to [n]$ defined by

$$D^v_u(I) = i \Leftrightarrow X^u_I \in y^v_i,$$

for all $I \in \mathcal{P}^*([n])$. Similarly, given $v \in \mathcal{U}_{\langle n_1,n_2\rangle}$ and $w \in \mathcal{W}_{\langle n_1,n_2\rangle}$, the *distribution function* of $Bool(v)$ over $Bool(w)$ is the function $D^w_v : \mathcal{P}^*([n]) \to [n]$ defined by

$$D^w_v(I) = i \Leftrightarrow Y^v_I \in z^w_i,$$

for all $I \in \mathcal{P}^*([n])$. Note that for a given signature $\langle n_1, n_2\rangle$, the distribution functions of $Bool(u)$ over $Bool(v)$, or of $Bool(v)$ over $Bool(w)$, are finitely many, namely n^{2^n-2}. Using distribution functions and initial conditions, the sentence

$$(\forall I \in \mathcal{P}^*([n]))(\forall i \in [n])(X^u_I \in y^v_i \Leftrightarrow Y^v_I \in z^{w_0}_i)$$

becomes

$$(\forall I \in \mathcal{P}^*([n]))(D^v_u(I) = D^{w_0}_v(I)),$$

i.e.,

$$D_u^v = D_v^{w_0}.$$

So finally, (7) is equivalent to

$$(\exists u \in \mathcal{U}_{\langle n_1,n_2\rangle})(\exists v \in \mathcal{V}^*_{\langle n_1,n_2\rangle})(D_u^v = D_v^{w_0}). \tag{8}$$

So in order to prove 5.1, it suffices to prove (8). To illustrate the idea we shall first prove (8) for uniform w_0 only (i.e., all of its sets are infinite), namely for w_0 such that $sign(w_0) = \langle n, 0\rangle$. Then we shall slightly modify the proof in order to work in the general case.

Lemma 5.2 *For every $n \geq 2$, there is a $v \in \mathcal{V}^*_{\langle n,0\rangle}$ with the following property:*

(†) For every function $D : \mathcal{P}^*([n]) \to [n]$, there is a $u \in \mathcal{U}_{\langle n,0\rangle}$ such that $D_u^v = D$.

Proof. Consider all distribution functions $D : \mathcal{P}^*([n]) \to [n]$ and enumerate them in the form D_l, $1 \leq l \leq n^{2^n-2}$. Using the fact that $\mathcal{A}$ is rich, we can pick for each l a partition $u_l \in \mathcal{U}_{\langle n,0\rangle}$ of A_0 so that all $Bool(u_l)$, for $1 \leq l \leq n^{2^n-2}$, are almost disjoint, i.e., $Bool(u_l) \cap Bool(u_m) = \{A_0, \emptyset\}$ for all $l \neq m$. This is clearly possible since the sets $Bool(u_l)$ are finite. In order for v to satisfy (†), it suffices that for each l, $D^v_{u_l} = D_l$, or equivalently, for each I and l, $X_I^{u_l} \in y^v_{D_l(I)}$. We set for every $i \in [n]$,

$$K_i = \{X_I^{u_l} : D_l(I) = i, 1 \leq l \leq n^{2^n-2}, I \in \mathcal{P}^*([n])\}.$$

Now $X_I^{u_l} \neq X_J^{u_m}$ for all $l \neq m$ and $I, J \in \mathcal{P}^*([n])$, because $Bool(u_l) \cap Bool(u_m) = \{A_0, \emptyset\}$. Also $X_I^{u_l} \neq X_J^{u_l}$ for any l and any $I \neq J$. Therefore the sets K_i, $i \in [n]$, are all disjoint (some of them may be empty). Hence we can obviously extend K_i to form a partition v of A_1 consisting of infinite sets, i.e., such that $K_i \subseteq y_i^v$ for all $i \in [n]$. Moreover, since all sets of v are infinite, obviously we can arrange that v satisfies the same initial conditions as w_0, i.e., that v satisfies (6). Thus $v \in \mathcal{V}^*_{\langle n,0\rangle}$. For every $I \in \mathcal{P}^*([n])$ and every $1 \leq l \leq n^{2^n-2}$, we have

$$D_l(I) = i \Leftrightarrow X_I^{u_l} \in K_i \Leftrightarrow X_I^{u_l} \in y_i^v \Leftrightarrow D^v_{u_l}(I) = i.$$

Therefore $D_l = D^v_{u_l}$, and hence v satisfies (†). This completes the proof of the lemma. $\dashv$

Corollary 5.3 *If $sign(w_0) = \langle n, 0\rangle$, then theorem 5.1 holds for w_0.*

Proof. Given $w_0 \in \mathcal{W}_{\langle n,0\rangle}$, pick by the previous lemma a $v \in \mathcal{V}^*_{\langle n,0\rangle}$ satisfying property (†). Let $D = D^{w_0}_v$. By (†) there is a $u \in \mathcal{U}_{\langle n,0\rangle}$ such that $D^v_u = D = D^{w_0}_v$. This means that

$$(\exists u \in \mathcal{U}_{\langle n,0\rangle})(\exists v \in \mathcal{V}^*_{\langle n,0\rangle})(D^v_u = D^{w_0}_v),$$

i.e., (8) is true. ⊣

The fact that the partitions u, v, w_0 are uniform is of key importance for the proof of lemma 5.2, as we do not need to care about the size of each K_i that must be included in y^v_i. Actually 5.2 fails in general for partitions containing singletons. To see that, suppose $sign(w_0) = \langle n_1, n_2\rangle$ with $n_2 \neq 0$. Let $v \in \mathcal{V}^*_{\langle n_1,n_2\rangle}$ satisfying property (†) and let $i \in [n] - [n_1]$. Then y^v_i must be a singleton. Let $D_1, D_2 : \mathcal{P}^*([n]) \to [n]$ be functions such that $D_1(I) = D_2(J) = i$, for some $I, J \in \mathcal{P}^*([n])$ such that $I \subseteq [n] - [n_1]$, while $J \subseteq [n_1]$. By (†), there are $u_1, u_2 \in \mathcal{U}_{\langle n_1,n_2\rangle}$ such that $D_1 = D^v_{u_1}$ and $D_2 = D^v_{u_2}$. Then $D^v_{u_1}(I) = D^v_{u_2}(J) = i$, and consequently, $\{X^{u_1}_I, X^{u_2}_J\} \subseteq y^v_i$. But $I \subseteq [n] - [n_1]$ implies that $X^{u_1}_I$ is finite, while $J \subseteq [n_1]$ implies that $X^{u_2}_J$ is infinite. Therefore $X^{u_1}_I \neq X^{u_2}_J$, and hence $|y^v_i| > 1$, a contradiction.

In fact the existence of singletons, or, more precisely, the fact that the functions D throw sets into singletons, is the only reason for which lemma 5.2 fails in general. So in order to cope with the general case, a simple solution is to consider distribution functions which just do *not* throw sets into singletons at all. Namely, given a signature $\langle n_1, n_2\rangle$ as above, with $n = n_1 + n_2$, let us say that $D : \mathcal{P}^*([n]) \to [n]$ is *restricted* if $rng(D) \subseteq [n_1]$. Since $n_1 \geq 2$, such functions are by no means trivial. Then we have the following variant of 5.2.

Lemma 5.4 *Let $sign(w_0) = \langle n_1, n_2\rangle$. Then there is a $v \in \mathcal{V}^*_{\langle n_1,n_2\rangle}$ with the following properties:*

(††) For every restricted function $D : \mathcal{P}^*([n]) \to [n_1]$, there is a $u \in \mathcal{U}_{\langle n_1,n_2\rangle}$ such that $D^v_u = D$.

(†††) $D^{w_0}_v$ is restricted.

Proof. The proof is similar to that of 5.4. The number of all restricted functions $D : \mathcal{P}^*([n]) \to [n_1]$ is $n_1^{2^n-2}$. So take an enumeration D_l, $1 \leq l \leq n_1^{2^n-2}$ of all of them. Using again the richness of $\mathcal{A}$, we pick for each l a partition $u_l \in \mathcal{U}_{\langle n_1,n_2\rangle}$ of A_0 so that all $Bool(u_l)$, for $1 \leq l \leq n_1^{2^n-2}$, are almost disjoint, i.e., $Bool(u_l) \cap Bool(u_m) = \{A_0, \emptyset\}$ for all $l \neq m$. Set again for $i \in [n_1]$,

$$K_i = \{X_I^{u_l} : D_l(I) = i, 1 \leq l \leq n_1^{2^n - 2}, I \in \mathcal{P}^*([n])\}.$$

v will be defined so that for the infinite sets y_i^v, $i \in [n_1]$, $K_i \subseteq y_i^v$. Next we consider the initial conditions of v. If A_1 or $\emptyset$ belongs to a singleton $z_i^{w_0}$, for $i \in [n] - [n_1]$, then A_0 or $\emptyset$ must be in the singleton y_i^v. This is the only condition posed on the singletons of v. Otherwise the rest of the singletons y_j^v, $j \in [n] - [n_1]$, are defined arbitrarily. Thus (††) is fulfilled. Finally, in order to conform with (†††), we only need to arrange that $Bool(v) - \{A_1, \emptyset\}$ does not meet any of the singletons $z_{n_1+1}^{w_0}, \ldots, z_n^{w_0}$. This is fairly easy. For instance, if it happens that $z_k^{w_0} = \{Y_I^v\}$ for some choice of the sets of v, we can slightly modify it, by moving appropriately one or more elements from Y_I^v to another set, thus getting another partition $v' \in \mathcal{V}^*_{\langle n_1, n_2 \rangle}$ satisfying (††) and so that $Y_I^{v'}$ no longer belongs to $z_k^{w_0}$. Any v defined by the above prescriptions satisfies (††) and (†††). This completes the proof of the lemma. $\dashv$

Proof of Theorem 5.1. Let $w_0 \in \mathcal{W}_{\langle n_1, n_2 \rangle}$ be given. By lemma 5.4 there is a $v \in \mathcal{V}^*_{\langle n_1, n_2 \rangle}$ satisfying (††) and (†††). Let $D = D_v^{w_0}$. By (†††) D is restricted. Hence by (††) there is $u \in \mathcal{U}_{\langle n_1, n_2 \rangle}$ such that $D = D_v^{w_0} = D_u^v$. Thus (8) holds true. This completes the proof of theorem 5.1. $\dashv$

Concluding Remarks. How does the settlement of the $\langle 0, 0, n \rangle$-property affect NF consistency? If the $\langle 0, 0, n \rangle$-property were disproved for an arbitrary rich model $\mathcal{A}$ of TST_4, that would mean that there is no NF model whose "underlying" TST model is rich. This is because if NF plus the splitting property (SP) (see Remark 1.1) is consistent, and $\langle K, E \rangle$ is a model of this theory, then the TST_4 model $\mathcal{A}_K$ resulting from K (see section 2.1) is rich and has a type-shifting automorphism $\langle f_1, f_2 \rangle$. The finite pieces of $\langle f_1, f_2 \rangle$ are ω-extendible coherent pairs, a property much stronger than the $\langle 0, 0, n \rangle$-property, a contradiction.

Now that the question was settled in the affirmative, we can only say that the forcing program described in the Introduction, is a little more likely to be successful. We say "a little" because the $\langle 0, 0, n \rangle$-property is much weaker than ω-extendibility. On the other hand, the methods employed in the proof of the former may be instructive in our attempts to tackle the full problem.

Yet the question remains: What is the *kind* of models of TST that would be likely to make the forcing program work? That has been partly answered in Remark 1.1 (1). Working with just rich models, having no other features, the program does not seem to have a chance to work, because the same method would then reasonably work also for models of TST satisfying AC, which we know is false. Models of TST with some kind of symmetry are probably needed. The exact type of symmetry is not yet known. Perhaps it will be understood if we analyze in depth the models of TST that result from models of NF.

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