An axiomatization of “very” within systems of set theory

Athanassios Tzouvaras

Dept. of Mathematics, Univ. of Thessaloniki
540 06 Thessaloniki, Greece
E-mail:tzouvara@auth.gr

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Abstract

A structural (as opposed to Zadeh's quantitative) approach to fuzziness is given, based on the operator "very", which is added to the language of set theory together with some elementary axioms about it. Due to the axiom of foundation and to a lifting axiom, the operator is proved trivial on the cumulative hierarchy of ZF. So we have to drop either foundation or lifting. Since fuzziness concerns complemented predicates rather than sets, a class theory is needed for the very operator. And of them the Kelley-Morse (KM) theory is more appropriate for reasons of class existence. Several definable realizations of the very-operator are presented in KM^-. In the last section we consider the operator "very" without the lifting axiom on classes of urelements. To each structurally fuzzy set X a traditional quantitative fuzzy set $\overline{X}$ is assigned – its quantitative representation. This way we are able partly to recover ordinary fuzzy sets from the structurally fuzzy ones.

Keywords. Fuzzy set, very-operator, axiomatic set theory, non-well-founded set.

1 Introduction

Current set theories, like ZF, GB etc., are all supposed to be theories about *crisp* collections. The reason is that their membership relation $\in$ (i.e., the underlying logic) is two-valued. Now let us imagine a set theorist wanting to build an axiomatic theory T capable to capture not only crisp but also *fuzzy* sets, without leaving the ground of old classical logic. In what should T differ from ZF? Our set theorist would have to examine the axioms of ZF and decide which of them should be retained, which should be dropped and what the new candidate axioms could be (if any). But if we inspect the axioms of ZF one by one, we can hardly find one which cannot be true of fuzzy sets. In fact the axioms of ZF are *neutral* with respect to crispness/fuzziness debate. And the reason is quite simple: These axioms have historically emerged in a discourse of ideas which had nothing to do with the above mentioned dichotomy. So our set theorist should necessarily keep the axioms of ZF as they are. He intends to keep also the underlying logic, since classical logic is a safe and familiar ground. So he is looking for a way to allow more than two membership degrees without changing neither the set axioms nor the logic. How could that be achieved?

In the popular fuzzy set theory of L. Zadeh, the goal of multiple membership degrees is obtained through the well-known *quantitative* method of assigning real numbers to elementhood. But in this way we only simulate fuzzy sets within the domain of standard ones. We do not build them from scratch as primitive objects. Can there be alternative *qualitative*, i.e. *structural* ways to represent fuzziness? In [11] we attempted to do that by the help of nonstandard natural numbers. That was also a simulation of fuzziness, but with emphasis on its structure rather than the current applications.

There is still another option: We can allow more than two membership degrees if we imitate the way we treat fuzziness in everyday reasoning - and this is what we examine in this paper. I believe that the logic of everyday reasoning is, in its essence, classical (although I know that many people would not agree with that), since it is vastly based on yes-no dichotomies. Yet the dichotomies may become *subtler and subtler* through the use of words like "very", "very much", "a little", "more than", "less than" etc. For instance, people are divided not just to tall and non-tall, but also to very tall and not very tall, to very very tall and not very very tall etc, and relations of the form "taller than" between them are established. Thus in the framework of natural language we obtain several membership degrees by means of *comparatives* for fuzzy predicates on the one hand, like "more elegant than", "taller than", "less expensive than", and *intensity operators* on the other, like "very", "very very", "a little", "much" etc. In this setting a predicate is crisp, exactly if it does not admit comparatives and intensifiers. For example it is due to the crispness of the predicate "pregnant", that the phrases "Mary is more pregnant than Alice", or "Mary is very pregnant" sound absurd.

Both comparatives and intensity operators are *structural* notions. The former are essentially order relations, while the latter are operators in the usual mathematical sense of the term. For the purpose of this paper we shall ignore the comparatives and focus only to "very". This operator alone yields an infinite sequence of membership degrees. For example, for the predicate "hot", the operator gives the sequence of predicates "very hot", "very very hot", etc. To formalize it let the variables x and X range over *fuzzy* (in general) sets and classes, respectively, and let $\in$ be the *fuzzy membership* between them. Let also v be an extra operator of our language, intended to mean "very", and sending every class (=predicate) to a class $v(X) \subseteq X$. Using v, the above predicates are formally written: $X \supseteq v(X) \supseteq v^2(X)$, etc. In this setting a collection X is *crisp* if $v(X) = X$.

One may still wonder what it means for a collection X to be fuzzy (i.e., non-crisp), since for any x, either $x \in X$ or $x \notin X$. The answer is that, by our assumption, $\in$ no more expresses definite belonging, since we are living in a universe of fuzzy collections. Definite belonging is now expressed rather by $\in_1$, where $x \in_1 X$ means $x \in v(X)$. So for X to be non-crisp, it means that there are x such that $x \notin_1 X$ and $x \notin_1 -X$, which holds precisely when $v(X) \neq X$, or $v(-X) \neq -X$. Later we shall postulate (see F4 below) that $v(X) \neq X$ iff $v(-X) \neq -X$.

To sum up: While the axioms of ZF are currently supposed to refer to crisp sets, and fuzzy sets are only simulated within the latter, here we assume the contrary, i.e., that the axioms refer primarily to fuzzy sets, and that the crisp sets form only a special subuniverse of ZF.

One of course has to offer some axioms about v. In order to do this in a precise way let us first extend the language L of set theory to $L_v = L \cup \{v\}$ by adding a new unary operation symbol v. The formulas of L_v are constructed in the obvious way. Next we require the axiom schemes of ZF (namely separation and replacement) to be true for formulas of L_v. Finally we add the axioms:

(F0) $(\forall x)(\exists y)(v(x) = y)$ (v is a total operator).
(F1) $v(x) \subseteq x$.
(F2) $x \subseteq y \ \Rightarrow \ v(x) \subseteq v(y)$.
(F3) $v(x) = x \Rightarrow v(\{x\}) = \{x\}$ (Lifting axiom).

Let ZF_v be the theory consisting of the axioms of ZF for the language L_v together with F0-F3.

Remarks 1.1 1) Intuitively $v(x)$ is the *definite* or *good* part of x. This "goodness" may be either absolute, or relative to each particular x. In the first case v must be idempotent, i.e., $v = v^2$, since the (absolutely) good part of a good part is the good part itself. If $v(x)$ is good only relative to x, then in general $v^2 \neq v$. In mathematics most "goodness" operators are absolute, acting as interior operators. They catch all the intended "good elements" out of a certain set just in one step.

2) Let us come to motivate the above axioms. F0 and F1 are obvious. If $v(X)$ is absolute, then F2 is also obvious, since absolute "goodness" operators are monotonic. F2 follows also from F5 which is discussed below. However, although F5 is desirable and has a good motivation, some of the realizations of "very" discussed in section 4 do not satisfy it. Another source of justification could be the "standard" quantitative fuzzy sets due to Zadeh. There "very" is used rather in fuzzy logic as a determination of true and false. If the truth value of "p is true" is $a \in [0,1]$, then the truth value of "p is very true" is usually taken to be a^2 (see e.g. [13]). But this can be easily transferred to fuzzy sets. Recall that in Zadeh's approach, if V is the universe of crisp sets, a fuzzy class is a mapping $f : V \to [0,1]$, and $f(x)$ is the membership degree of x. Passing to f the preceding idea of squaring, $v(f) : V \to [0,1]$ should be the mapping for which,

$$v(f)(x) = (f(x))^2.$$

Since $f \subseteq g$ if $(\forall x)(f(x) \leq g(x))$, and for every $a \in [0,1]$, $a^2 \leq a$, we have for every f, $v(f) \subseteq f$. Moreover, if $f \subseteq g$, then $f(x) \leq g(x)$ for every x, hence $f(x)^2 \leq g(x)^2$, for every x, therefore $v(f) \subseteq v(g)$.

3) Due to the foundation axiom, the universe of ZF_v, just like that of ZF, consists of levels V_α, $\alpha \in On$. So we need a "lifting axiom" which will relate fuzziness/crispness of sets of a certain rank α to that of sets of rank $\alpha + 1$. F3 is this axiom. It says that if x is crisp, then so is $\{x\}$. Such an axiom may be strongly disputable since it mixes types: $v(x) = x$ is a property of x with respect to its elements, while $v(\{x\}) = \{x\}$ is a property of $\{x\}$ with respect to its single element x. On the other hand F3 aims exclusively at handling *cumulative* sets which form the universe of ZF and are not met outside mathematical practice. So the author is conscious of its counter-application character. We use it as an attempt to see the possible formalizations of "very" in the ZF universe.

Seen this way F3 is a modest assumption about the action of v on finite sets. Note that if we accept that $v(\{x\}) = \{x\}$ for every x, then this, combined with F1, F2 above, trivializes v, i.e., v is the identity. Indeed, suppose $v(\{x\}) = \{x\}$ for every x. Let y be any set. For every $z \in y$, $\{z\} \subseteq y$ and, by F2, $v(\{z\}) \subseteq v(y)$. Hence $\{z\} \subseteq v(y)$, or $z \in v(y)$. Thus

$y \subseteq v(y)$. By F1, $v(y) = y$ for every y. Despite its modesty however, we shall see in the next section that F3, when combined with the foundation axiom, trivializes the very operator.

The "very" operator as treated in this paper is similar to some basic notion of the theory of "rough sets" created by Z. Pawlak in [10]. Briefly, on a universe A one assumes an indiscernibility relation R. For any subset X of A the lower approximation of X is the set $L(X)$ consisting of all x such that $R(x) \subseteq X$, while the upper approximation of X is the set $U(X) = \bigcup\{R(x) : R(x) \cap X \neq \emptyset\}$. $L(X)$ behaves like $v(X)$ and is idempotent if R is an equivalence relation. However, as explained in [10], the theory of rough sets and the theory of fuzzy sets have distinct, rather complementary objectives, as they focus on the granularity and the graduality of knowledge respectively.

There have been also older attempts to axiomatize fuzzy sets in the style of ZF, which however differ significantly from the present one. For the interested reader we mention [3], [9], [7] as well as [12], where the operator "very" has been considered. [7] deals with an axiomatization of functions that was given by von Neumann in 1925. The author shows that the notions of set, fuzzy set and multiset can simultaneously be handled in this setting. [3] and [9] on the other hand employ a ternary membership relation $\varepsilon(x, y, z)$ (instead of the usual $\in$) with the intended meaning: x belongs to y with degree z.

After finishing the first draft of this paper, I learned that P. Hájek used a very-true operator "vt" in his paper [4]. However this is a paper about fuzzy logic rather than set theory, so "vt" applies to formulas and the intended meaning of $\mathrm{vt}\phi$ is "ϕ is very true". He introduces for vt the following axioms (which augment those of his system BL):

(1) $\mathrm{vt}\phi \to \phi$,

(2) $\mathrm{vt}(\phi \to \psi) \to (\mathrm{vt}\phi \to \mathrm{vt}\psi)$,

(3) $\mathrm{vt}(\phi \vee \psi) \to (\mathrm{vt}\phi \vee \mathrm{vt}\psi)$.

We just note that (1) is the analogue of F1 and (2) is roughly the analogue of F5 introduced later, since (2) implies $\mathrm{vt}(\phi \wedge \psi) \leftrightarrow (\mathrm{vt}\phi \wedge \mathrm{vt}\psi)$. From this point on however the paper has no more common elements with the present one.

The view discussed above about the need of a set theory in which fuzzy sets would be primitive objects rather than just simulations in a universe of crisp ones, seems to be shared also by P. Hájek and Z. Haniková in [5], where, in order to obtain that, they switch from classical to (some system of) fuzzy logic and build in it a set theory as close to ZF as possible. For example [5] presents such a theory FST inside Hájek's basic fuzzy logic BL and constructs a (non-trivial) interpretation of this theory (and its logic) to ZFC with classical logic.

2 Failure of "very" in ZF_v

Strangely enough, the only operator v satisfying F0-F3 in ZF_v is the identity. Let v be a mapping satisfying F0-F3 and let $Cr = \{x : v(x) = x\}$ be the class of crisp sets with respect to v.

Proposition 2.1 (ZF_v)
i) $x \subseteq Cr \;\Rightarrow\; x \in Cr$.
ii) $x \subseteq Cr \;\Rightarrow \mathcal{P}(x) \subseteq Cr \;\Rightarrow\; \mathcal{P}(x) \in Cr$.

Proof. i) Let $x \subseteq Cr$. For any $y \in x$, $y \in Cr$, hence, by F3, $v(\{y\}) = \{y\}$. Now $y \in x$ means $\{y\} \subseteq x$ and, by F2, $v(\{y\}) \subseteq v(x)$, hence $\{y\} \subseteq v(x)$, or $y \in v(x)$. Therefore $x \subseteq v(x)$. By F1, $v(x) = x$, hence $x \in Cr$.

ii) Let $x \subseteq Cr$. Take $y \in \mathcal{P}(x)$. Then $y \subseteq x \subseteq Cr$. By (i) above, $y \in Cr$, hence $\mathcal{P}(x) \subseteq Cr$. The other implication follows from (i) above. follows from (ii). ⊣

Corollary 2.2 *The only mapping v satisfying* F0-F3 *in* ZF_v *is the identity, i.e.,* $Cr = V$.

Proof. Let V be the universe of ZF_v. Then exactly as in ZF, we can see that $V = \cup_{\alpha \in On} V_\alpha$, where V_α are defined as usual. It suffices to show that $V_\alpha \subseteq Cr$. By induction on α.

a) $V_0 = \emptyset \subseteq Cr$.

b) Let α be limit, and let $V_\beta \subseteq Cr$ for all $\beta < \alpha$. Then $V_\alpha = \cup_{\beta<\alpha} V_\beta \subseteq Cr$.

c) Let $V_\alpha \subseteq Cr$. Then by (ii) of 2.1, $V_{\alpha+1} = \mathcal{P}(V_\alpha) \subseteq Cr$. ⊣

3 Dropping foundation

The proof of 2.2 above fails either if $V \neq \cup_{\alpha\in On} V_\alpha$, i.e., if we drop the foundation axiom, or if we drop the lifting axiom F3. In this section we shall do the first; in the last section we shall do the second. So in this and the next section we shall work either in ZF_v^- (=ZF_v minus foundation) or is some analogous theory of classes. The natural basic candidate class theories are GB (Gödel-Bernays) and the stronger one KM (Kelley-Morse). We choose to work in KM_v because exactly of its strength. Namely, as we shall see later on, the iterates $v^\alpha(X)$, for $\alpha \in On$, can be shown to exist in KM_v, though not in GB_v.

Recall that KM is the theory of classes with strong comprehension (i.e., for every formula $\phi(x)$ of L, with any kind of quantifiers, there is a class X such that $x \in X \iff \phi(x)$. This is in contrast to what happens with GB, where the above holds only for "normal" formulas, i.e., those with set quantifiers only). Let KM^- be KM minus foundation.

Let $L = \{\in\}$ be the language of KM with variables $X, Y, \ldots$ for (fuzzy) classes. We think of $\in$ as *fuzzy* membership. As usual, X is a *set* if $X \in Y$ for some Y, and we use lowercase letters $x, y, \ldots$ to range over sets. Let again $L_v = L \cup \{v\}$. We accept again the axioms F0-F3 of the preceding section, applied now to classes, together with some further principles. Namely, we postulate the following:

(F0) $(\forall X)(\exists Y)(v(X) = Y)$.
(F1) $v(X) \subseteq X$.
(F2) $X \subseteq Y \;\Rightarrow\; v(X) \subseteq v(Y)$.
(F3) $v(x) = x \Rightarrow v(\{x\}) = \{x\}$.
(F4) $v(X) = X \iff v(-X) = -X$.
(F5) $v(X \cap Y) = v(X) \cap v(Y)$.
(F5*) $v(\bigcap_n X_n) = \bigcap_n v(X_n)$, for every sequence of classes (X_n), $n \in \omega$.

Finally let KM_v^- be the theory consisted of the axioms of KM^- for the formulas of L_v, together with axioms F0-F5 (or F0-F5*, depending on the occasion). Throughout the rest of the paper, we work in KM_v^- or in $\mathrm{KM}_v^- - \{\mathrm{F5}, \mathrm{F5}^*\}$. Whenever the presence of F5 or F5* is necessary, this will be stated explicitly.

In the next section we give (non-trivial) examples of *definable* operations in KM$^-$ satisfying all or some of the principles F0-F5 above. Obviously this constitutes a proof of the of the non-triviality of KM$_v^-$ (i.e., relative to the consistency of KM$^-$ plus $(\exists X)(v(X) \neq X)$).

Remarks 3.1 1) Note that F0, combined with F1, implies the corresponding principle for sets, i.e., the principle $(\forall x)(\exists y)(v(x) = y)$. This is because if X is a set, the $v(X)$ is also a set, since $v(X) \subseteq X$ and in KM$_v$ every subclass of a set is a set.

2) F4 says that X is crisp if and only its complement is such. And indeed the fuzziness or the crispness of a predicate P refers simultaneously to *both* parts of the pair P, $\neg P$, not just to one of them. This is in agreement with the quantitative definition of "very" considered in section 1. Recall that the complement $-f$ of the mapping f is defined by $(-f)(x) = 1 - f(x)$. Recall also that $v(f)$ was defined there by $v(f)(x) = f(x)^2$. Since $f = g$ iff $f(x) = g(x)$ for all x, it follows that $v(f) = f$ iff $v(f)(x) = f(x)$, or $f(x)^2 = f(x)$, and this is possible only if $f(x) = 1$ or 0 for every x, i.e., iff f is crisp. But then so is $-f$.

3) Note that since $v(\emptyset) = \emptyset$, by F4, also $v(V) = V$. This agrees with the quantitative approach, where V is crisp, being identical to the constant function $V(x) = 1$.

4) F5 and F5* are desirable but not always satisfied by the concrete realizations of "very". One could argue that the "good" elements of $X \cap Y$ are no different from the good elements of X which are in common with the good elements of Y. For example the very hot-and-humid days are exactly those which are simultaneously very hot and very humid. This seems to be true when the good part $v(X)$ is absolute, but not when $v(X)$ is relative to X. Some of the realizations of "very" given in the next section do not satisfy F5. On the other hand, definitely we cannot have $v(X) \cup v(Y) = v(X \cup Y)$. Indeed $X \cup Y$ may very well be crisp while each of the X, Y are not. For instance for any fuzzy X, we have $V = X \cup -X$ and $v(V) = V = v(X \cup -X)$. Since $v(X) \subset X$, clearly $V \supset v(X) \cup v(-X)$.

Does the quantitative approach support F5? The answer depends on how $f \cap g$ is defined in this approach. (The non-unique definitions of the basic set theoretic operations is one of the main drawbacks of the quantitative fuzzy set theory.) The commonest definition of $f \cap g$ is by $(f \cap g)(x) = \min\{f(x), g(x)\}$ (and $(f \cup g)(x) = \max\{f(x), g(x)\}$). According to this, it is easy to see that $v(f) \cap v(g) = v(f \cap g)$, as well as $v(f) \cup v(g) = v(f \cup g)$. However this entails in general that $f \cap -f \neq \emptyset$ (and $f \cup -f \neq V$). Another definition is $(f \cap g)(x) = \max\{0, f(x)+g(x)-1\}$ (and $(f \cup g)(x) = \min\{1, f(x)+g(x)\}$, respectively). This definition respects $f \cap -f = \emptyset$ and $f \cup -f = V$ but $v(f) \cap v(g) = v(f \cap g)$ is no longer true.

F1 and F5 and the fact that $v(V) = V$ make v a quasi interior operator on the boolean algebra of all classes (that is an interior operator without the property $v^2 = v$). The crisp classes are the "open" classes of this weak topology. However as follows from F4, the open classes are also closed, hence the open classes coincide to the clopen ones.

Let $l(X) := -v(-X)$ be the *dual* of v. l is intended to mean "a little". Intuitively, if X is the predicate "hot", then $v(-X)$, "very not hot", is something like "cool", hence $l(X)$ is "not cool", i.e., something like "a little hot". By F1, $v(X) \subseteq X \subseteq l(X)$. For every X call *boundary* of X the class

$$\partial X = l(X) \backslash v(X).$$

Definition 3.2 We say that X is *crisp* (with respect to v) if $v(X) = X$. We say that X is *hereditarily crisp* if X is crisp and every element of $Tc(X)$ (the transitive closure of X) is crisp. A set x is *weakly crisp* if $v(\{x\}) = \{x\}$. X is said to be *totally fuzzy* if $X \neq \emptyset$ and $v(X) = \emptyset$.

Proposition 3.3 *i) For every X, $\partial X = \partial - X$.*

ii) X is crisp iff $v(X) = X = l(X)$ iff $\partial X = \emptyset$.

iii) Let v satisfy F5. *If X is non-crisp, then ∂X is the union of two totally fuzzy sets.*

Proof. i) and ii) are easy. iii) Let X be fuzzy, i.e., $v(X) \subset X$. $\partial X = l(X) \backslash v(X) = (l(X) \backslash X) \cup (X \backslash v(X)) \neq \emptyset$. Let $Y = l(X) \backslash X$ and $Z = X \backslash v(X)$. It suffices to show that Y and Z are totally fuzzy. First $Y, Z \neq \emptyset$. Because, by F4, $Y = \emptyset$ iff $Z = \emptyset$, hence if one of them were empty, then ∂X would be empty, which contradicts our assumption. Then, by F5, $v(Y) = v(-v(-X)) \cap v(-X) = \emptyset$, since $-v(-X) \cap v(-X) = \emptyset$. Similarly $v(Z) = \emptyset$. ⊣

Let

$$Cr(v) = \{x : v(x) = x\},$$

$$WCr(v) = \{x : v(\{x\}) = \{x\}\}, \quad HCr(v) = \{x : \ x \text{ is hereditarily crisp}\}.$$

We write simply Cr, HCr etc., instead of $Cr(v)$, $HCr(v)$, if v is understood.

Proposition 3.4 (KM_v^-)

i) For any family of crisp classes X_i, $i \in I$, $\cup_{i\in I} X_i$ and $\bigcap_{i\in I} X_i$ are crisp.

ii) $WF \subseteq Cr$ (WF is the class of well-founded sets). More generally, $X \subseteq WF \Rightarrow v(X) = X$. In particular, for every X, $v(X) \supseteq X \cap WF$.

iii) The class HCr is an inner model of ZF^-.

Proof. i) By the monotonicity of v, $v(\cup_{i\in I} X_i) \supseteq \cup_{i\in I} v(X_i)$. Since X_i are crisp, $v(X_i) = X_i$. Hence $v(\cup_{i\in I} X_i) \supseteq \cup_{i\in I} X_i$, therefore $v(\cup_{i\in I} X_i) = \cup_{i\in I} X_i$. Thus $\cup_{i\in I} X_i$ is crisp. Further, by F4, $v(\bigcap_{i\in I} X_i) = \bigcap_{i\in I} X_i$ iff $v(-\bigcap_{i\in I} X_i) = -\bigcap_{i\in I} X_i$, or $v(\bigcup_{i\in I} -X_i) = \bigcup_{i\in I} -X_i$. But since $-X_i$ are crisp, the latter holds as proved above. Therefore also $v(\bigcap_{i\in I} X_i) = \bigcap_{i\in I} X_i$, i.e., $\bigcap_{i\in I} X_i$ is crisp.

ii) Follows from Corollary 2.2 and (i) above using F4.

iii) HCr is transitive and universal i.e., $x \subseteq HCr \Rightarrow x \in HCr$. But it is easy to verify that a transitive universal class is a model of ZF^-. (See e.g. [6], p. 24, where almost universal classes are examined. For an almost universal class to be a model of FZ^- it needs in addition to be closed under the Gödel operations, but for a universal class this is obviously true.) $\dashv$

Each very-operator v gives rise to a natural very-operator abs_v (the "absolute core") which is idempotent on sets and such that $Cr(v) = Cr(abs_v)$. We shall define first the α-iterate v^α of v for every ordinal α. In order to do this *inside* KM_v^-, we do the following: For every class X of pairs let $dom(X) = \{x : \exists y\ (x, y) \in X\}$ and for every $x \in dom(X)$, let $X_{(x)}$ be the usual coding device for families of classes, i.e., $X_{(x)} = \{y : (x, y) \in X\}$. Then consider the formula:

$\phi(x, \alpha, X)$=:

$$(\exists Y)[Y_{(0)} = X \ \&\ x \in Y_{(\alpha)} \ \&\ (\forall \beta \leq \alpha)(\text{if } \beta = \gamma + 1, \text{ then } Y_{(\beta)} = v(Y_{(\gamma)})$$
$$\text{and if } \beta \text{ is limit, then } Y_{(\beta)} = \bigcap_{\gamma<\beta} Y_{(\gamma)})]. \qquad (1)$$

By the comprehension scheme of KM_v^-, for every class X and every ordinal α, there is a unique class Y such that $Y = \{x : \phi(x, \alpha, X)\}$. We call this Y, the *α-th iterate of v on X* and denote it by $v^\alpha(X)$. (The existence of these classes was in fact the reason that we chose to work in KM_v^- rather than in GB_v^-. Note that the formula ϕ contains an existential *class* quantifier. Therefore the classes $v^\alpha(X)$ cannot be shown to exist in GB_v.)

The following is easy to verify by induction on α.

Lemma 3.5 *For every X and α:*
i) $v^{\alpha+1}(X) = v(v^{\alpha}(X))$.
ii) $v^{\alpha}(X) = \bigcap_{\beta<\alpha} v^{\beta}(X)$, for limit α.

Finally, the *absolute core* of X (with respect to v) is the class

$$abs_v(X) = \bigcap_{\alpha \in On} v^{\alpha}(X).$$

(Its existence follows again from the comprehension scheme of KM_v^- and the above uniform definition of the iterates $v^{\alpha}(X)$.)

For simplicity we write *abs* instead of abs_v if there is no danger of confusion. Intuitively, if X is, say, the predicate "hot", then $abs(X)$ is the predicate "absolutely hot".

Lemma 3.6 (KM_v^-)
i) (AC) *For every set x, $abs^2(x) = abs(x)$, i.e., abs is idempotent on sets. If in addition v satisfies* F5*, *then for every class X, $abs(X) = \bigcap_{n<\omega} v^n(X)$, and $abs^2(X) = abs(X)$.*
ii) $abs(X) = X \iff v(X) = X$.
Suppose v satisfies F5*. *Then*
iii) abs satisfies properties F0-F5. *Hence abs is a very-operator such that $Cr(abs) = Cr(v)$.*
iv) $abs(X)$ is the greatest crisp subclass of X.
(If v does not satisfy F5*, *then (iii), (iv) above hold for sets only.)*

Proof. i) It suffices to show that for every set x there is $\alpha \in On$, such that $v^{\alpha+1}(x) = v^{\alpha}(x)$. Assume on the contrary that there is an x such that $v^{\alpha+1}(x) \subset v^{\alpha}(x)$. Let $|x| = \kappa$. If $\lambda > \kappa$, using AC we can find a λ-sequence $(y_\alpha)_{\alpha<\lambda}$, of elements of x such that $y_\alpha \in v^{\alpha}(x) \backslash v^{\alpha+1}(x)$, which is a contradiction. (Note that this need not be true for a proper class X.) Now suppose v satisfies F5*. Then:

$$v^{\omega+1}(X) = v(v^{\omega}(X)) = v(\bigcap_{n<\omega} v^n(X)) = \bigcap_{n<\omega} v^{n+1}(X) = v^{\omega}(X),$$

hence $abs(X) = v^{\omega}(X) = \bigcap_{n<\omega} v^n(X)$.

To show that $abs^2 = abs$, clearly it suffices to show that $v(abs(X)) = abs(X)$, but this has just been proved.

ii) $abs(X) \subseteq v(X) \subseteq X$, hence $abs(X) = X$ implies $X \subseteq v(X)$, and $v(X) = X$. Conversely, $v(X) = X$ implies $v^\alpha(X) = X$ for every ordinal α, hence $abs(X) = X$.

For the rest properties suppose v satisfies F5*.

iii) This is easy to check.

iv) For every X, $abs(X)$ is crisp since $abs^2 = abs$. To show that $abs(X)$ is the greatest crisp subclass, let Y be any crisp subclass of X. Then $v(Y) \subseteq v(X)$, or $Y \subseteq v(X)$. Inductively we see that $Y \subseteq v^n(X)$ for every $n \in \mathbb{N}$, hence $Y \subseteq abs(X)$.

If F5* fails for v, then we just use (i) to show the other clauses for the case of sets. $\dashv$

4 Realizations of "very"

In this section we give some concrete examples of definable very-operators.

Example 1. In KM$^-$, put
$v(X) = X$ if $X \subseteq WF$ or $-X \subseteq WF$, and
$v(X) = X \cap WF$ otherwise.

Proposition 4.1 *Let $V \neq WF$. Then v satisfies* F0-F5*. Moreover $Cr = HCr = WCr = WF$ and $abs(X) = v(X)$.*

Proof. Again F1 is obvious. F4: $v(X) = X$ iff $X \subseteq WF$ or $-X \subseteq WF$, hence $v(X) = X$ iff $v(-X) = -X$. F3: Let $v(x) = x$. Then $x \subseteq WF$ (because $-x \subseteq WF$, i.e., $-WF \subseteq x$ is not possible. This is because, since by assumption $-WF \neq \emptyset$, $-WF$ is a proper class. Indeed, let a be non-well-founded. Then $\{x \cup \{a\} : x \in WF\} \subseteq -WF$ and for $x, y \in WF$ such that $x \neq y$, clearly $x \cup \{a\} \neq y \cup \{a\}$. Therefore $-WF$ is a proper class). Then clearly, $x \in WF$, or $\{x\} \subseteq WF$, therefore $v(\{x\}) = \{x\}$.

F5, F2: Let $\mathcal{B}$ be the collection of classes X such that $X \subseteq WF$ or $-X \subseteq WF$. Of course $\mathcal{B}$ is an informal object not belonging to the KM$_v$ universe. We write $X \in \mathcal{B}$ simply as an abbreviation of the cumbersome formula "$X \subseteq WF \ \vee \ -X \subseteq WF$". It is easy to see that $\mathcal{B}$ is a complete boolean algebra. Indeed, let $X_i, i \in I$, be any family of elements of $\mathcal{B}$. If some of them is $\subseteq WF$, then $\bigcap_{i\in I} X_i \subseteq WF$, hence $\bigcap_{i\in I} X_i \in \mathcal{B}$. Suppose for all $i \in I$, $-X_i \subseteq WF$. Then $-(\bigcap_{i\in I} X_i) = \bigcup_{i\in I} -X_i \subseteq WF$. But $-(\bigcap_{i\in I} X_i) \in \mathcal{B}$ and hence $\bigcap_{i\in I} X_i \in \mathcal{B}$.

We show now F5.

Case 1. $X, Y \in \mathcal{B}$. Then $X \cap Y \in \mathcal{B}$, therefore, $v(X \cap Y) = X \cap Y = v(X) \cap v(Y)$.

Case 2. $X \notin \mathcal{B}$ and $Y \notin \mathcal{B}$.

Case 2a. $X \cap Y \notin \mathcal{B}$. Then $v(X \cap Y) = (X \cap Y) \cap WF = (X \cap WF) \cap (Y \cap WF) = v(X) \cap v(Y)$.

Case 2b. $X \cap Y \in \mathcal{B}$. Then either $X \cap Y \subseteq WF$, or $-(X \cap Y) \subseteq WF$. In the first case $v(X \cap Y) = (X \cap Y) \cap WF = (X \cap WF) \cap (Y \cap WF) = v(X) \cap v(Y)$. The other case is impossible, because if $-(X \cap Y) \subseteq WF$, then $X \supseteq X \cap Y \supseteq -WF$, hence $-X \subseteq WF$, i.e., $X \in \mathcal{B}$, a contradiction.

Case 3. $X \in \mathcal{B}$ and $Y \notin \mathcal{B}$.

Case 3a. $X \subseteq WF$ and $Y \notin \mathcal{B}$. Then $X \cap Y \subseteq WF$, hence $v(X \cap Y) = X \cap Y = X \cap (Y \cap WF) = v(X) \cap v(Y)$.

Case 3b. $-X \subseteq WF$ and $Y \notin \mathcal{B}$. We show that $X \cap Y \notin \mathcal{B}$. Assume the contrary. Then either $X \cap Y \subseteq WF$, or $-(X \cap Y) \subseteq WF$. Suppose first that $X \cap Y \subseteq WF$. Then $-(X \cap Y) = -X \cup -Y \supseteq -WF$. By assumption $X \supseteq -WF$. The last two relations imply that $-WF \subseteq -Y$, or $Y \subseteq WF$, whence $Y \in \mathcal{B}$, a contradiction. Suppose now that $-(X \cap Y) \subseteq WF$ or $X \cap Y \supseteq -WF$. But then $Y \supseteq -WF$, and hence $Y \in \mathcal{B}$, a contradiction again. Therefore $X \cap Y \notin \mathcal{B}$ and $v(X \cap Y) = X \cap Y \cap WF = \emptyset$. On the other hand, $v(X) = X$, and $v(Y) = Y \cap WF$, hence $v(X) \cap v(Y) = X \cap Y \cap WF = \emptyset$. It follows $v(X) \cap v(Y) = v(X \cap Y)$.

The class Cr of crisp sets with respect to v are the sets x such that $x \subseteq WF$, (since $x \supseteq -WF$ is impossible by the argument in the beginning of the proof). Since also $x \subseteq WF \iff x \in WF \iff \{x\} \in WF$, it follows that $Cr = HCr = WCr = WF$. Again $v^2 = v$, hence $abs(X) = v(X)$. $\dashv$

Example 2. Let vN be the von Neumann axiom of choice, $|V| = |On|$. In $\mathrm{KM}^- + \mathrm{vN}$, let us a fix a well-ordering $\preceq$ of V making $(V, \preceq)$ isomorphic $(On, \leq)$, and for every nonempty class X let $\min(X)$ be the $\preceq$-least element of X. Set

$v(X) = X$ if $X \in \mathcal{B}$, and

$v(X) = X\backslash\{\min(X\backslash WF)\}$ otherwise,
where $\mathcal{B}$ is the collection defined in the previous example.

Proposition 4.2 *v satisfies axioms* F0-F4. *Here too, $Cr = HCr = WCr = WF$.*

Proof. As before we easily see that F1, F4, F3 hold true.

F2: Let $X \subseteq Y$. If $Y \in \mathcal{B}$, then $v(X) \subseteq X \subseteq Y = v(Y)$, so the claim holds. Let $Y \notin \mathcal{B}$. Let $X \in \mathcal{B}$. Then either $X \subseteq WF$ or $X \supseteq -WF$. The latter is impossible because if $X \supseteq -WF$, then $-WF \subseteq X \subseteq Y$, hence $Y \in \mathcal{B}$, contradicting our assumption. So $X \subseteq WF$ and clearly, by the definition of v, $X = v(X) \subseteq v(Y)$.

Suppose now that $X \notin \mathcal{B}$. We have to show that

$$v(X) = X\backslash\{\min(X\backslash WF)\} \subseteq v(Y) = Y\backslash\{\min(Y\backslash WF)\}.$$

The preceding formula holds true provided: if $\min(Y\backslash WF) \in X$, then $\min(Y\backslash WF) = \min(X\backslash WF)$. But if $\min(Y\backslash WF) \in X$, then clearly $\min(Y\backslash WF) \in X\backslash WF$. Since $X \subseteq Y$, $X\backslash WF \subseteq Y\backslash WF$, so if $\min(Y\backslash WF) \in X\backslash WF$, obviously $\min(Y\backslash WF) = \min(X\backslash WF)$. This proves that F2 is true.

However, we easily see that F5 fails. For example we can have $X, Y \subset -WF$ such that $\min(X), \min(Y) \notin X \cap Y$. Then $v(X) \cap v(Y) = X \cap Y \supset v(X \cap Y)$. As a compensation v is not idempotent. We easily see that the operators $v^n, n \in \mathbb{N}$, are all distinct. $\dashv$

As for the absolute core of X we have:

Lemma 4.3 *For every X, $abs(X) = X$ if $X \subseteq WF$ or $X \supseteq -WF$, and $abs(X) = X \cap WF$ otherwise. That is, abs is the "very" operator of Example 1.*

Proof. For $X \subseteq WF$ or $X \supseteq -WF$, $v(X) = X$, hence also $abs(X) = X$. Otherwise X contains elements of $-WF$ and

$$v^{\alpha+1}(X) = v^{\alpha}(X)\backslash\{\min(v^{\alpha}(X)\backslash WF)\}.$$

Since the well-ordering of V is such that every initial segment is a set, it follows immediately that eventually, $abs(X) = X \cap WF$. $\dashv$

Example 3. Let our theory be KM^- plus some antifoundation axiom, like AFA of [1] (in order to make sure that non-well-founded sets exist). An *$\in$-cycle of length n*, or just an *n-cycle*, is a sequence of the form $x \in x_{n-1} \in \cdots \in x_1 \in x$. An *infinite cycle* is an infinite sequence of the form $\cdots \in x_{n+1} \in x_n \in \cdots \in x_1 \in X_0$. Of course every n-cycle gives rise to an infinite cycle in the obvious way, so we shall use the term *proper infinite* cycle for an infinite cycle not produced by a finite one. A class X is said to be *non-well-founded* if there is an n-cycle, for some $n \in \mathbb{N}$, or a proper infinite cycle below X. Let X have a finite cycle below it and $x \in X$. We say that x has *cycle depth k*, if k is the least integer such that starting from x we can trace an n-cycle, for $n < k$, in k steps. That is, there is a sequence

$$y \in y_{k-1} \in \cdots \in y_{k-n} \in y \in y_{k-n-2} \in \cdots \in y_1 \in x \in X.$$

The cycle depth of an element is, in a sense, a measure of its non-well-foundedness. The smaller the cycle depth of x, the more unfounded x is supposed to be. To give another picture, suppose we are given a class X, and want to find a relative "regularization" of X by throwing away the "most unfounded" of its elements. Then we can define this regularization $v(X)$ to be X minus its elements of smallest cycle depth. For any X, let $cd(X)$ denote the cycle depth of X. Namely

$$cd(X) = \min\{m : \text{we can trace a cycle in } m \text{ steps below } X\}.$$

If there is no finite cycle below X, we just set $cd(X) = \infty$. Let us call a set x *weakly well-founded* if there is no finite $\in$-cycle below it. Let

$$WWF = \{x : x \text{ is weakly well-founded}\} = \{x : cd(x) = \infty\}.$$

Define $v(X) = X$, if $X \subseteq WWF$ or $X \supseteq -WWF$, and
$v(X) = X \backslash \{x \in X : cd(x) \text{ is smallest}\}$, otherwise.

Proposition 4.4 *i) v satisfies* F0-F4 *and $HCr = Cr = WWF$.*
ii) For every X, $abs(X) = X$ if $X \subseteq WWF$ or $X \supseteq -WWF$. Otherwise, $abs(X) = X \cap WWF$.

Proof. i) It is easy to see that the classes X such that X or $-X$ has no finite cycles below it, form a boolean algebra $\mathcal{B}^* \supseteq \mathcal{B}$ and $v(X) = X$ iff

$X \in \mathcal{B}^*$, so F1, F4 hold trivially. F3 is also clear. If $x \in \mathcal{B}^*$, then obviously $\{x\} \in \mathcal{B}^*$.

F2: Let $X \subseteq Y$. If $Y \in \mathcal{B}^*$, then $v(X) \subseteq X \subseteq Y = v(Y)$. So let $Y \notin \mathcal{B}^*$. If $X \in \mathcal{B}^*$, then clearly $X \subseteq v(Y)$, hence $X = v(X) \subseteq v(Y)$. Let $X \notin \mathcal{B}^*$. If $x \in Y$, $cd(x)$ is smallest in Y and $x \in X$, then $cd(x)$ is clearly smallest in X, since $X \subseteq Y$. Therefore

$$v(X) = X \backslash \{x \in X : cd(x) \text{ is smallest}\} \subseteq Y \backslash \{x \in Y : cd(x) \text{ is smallest}\} = v(Y).$$

However as in the preceding example we easily see that F5 fails, and that the mappings v^n, $n \in \mathbb{N}$, are all distinct.

By the definition of v in this example we have $Cr = WWF$. Moreover, since WWF is transitive, $Cr = HCr = WWF$. Also, since $x \in Cr$ iff $\{x\} \in Cr$, we get $Cr = HCr = WCr = WWF$.

ii) Clearly for $X \subseteq WF$ or $X \supseteq -WF$, $abs(X) = X$. Otherwise X contains finite cycles below it and in each application of v we remove the elements of least cycle depth. Since the cycle depths are finite, all such elements will be removed in ω steps, therefore $abs(X) = X \cap WWF$. $\dashv$

It follows immediately from 4.4 and 3.4 (iv), that

Corollary 4.5 *WWF is an inner model of* ZF$^-$.

5 Dropping the lifting axiom. Sets of urelements. Recovering quantitative sets from the very-operator

One may wonder whether in the formalism of the preceding sections there is room for "ordinary" fuzzy sets, like the set of "big" natural numbers and similar fuzzy subsets of $\mathbb{N}$. The answer is Yes, provided we dispense with the lifting axiom F3, or (which essentially amounts to the same thing) with the cumulative treatment of $\mathbb{N}$. Indeed in everyday life no one believes that e.g. 3 *is* the set $\{0, 1, 2\}$. Rather we treat $\mathbb{N}$, $\mathbb{R}$ etc. as sets of *urelements* (atoms). We can work in a variant of set theory (like ZFU) which postulates the existence of urelements right from the beginning (see e.g [2]), or, even more simply, we can *ignore* the set structure of the objects in question. This is for example the way T. Lindström in [8] treats the elements of a set S in order to define the superstructure $V(S)$ on S. In such a case v need not be defined throughout the whole universe but only on a set or class of urelements. Therefore in the sequel we shall use only axioms F1, F4, F5 (since F2 follows from F5).

So let A be a set (or class) of urelements, and let us confine ourselves to subsets of A, i.e., to elements of $\mathcal{P}(A)$. Let $v : \mathcal{P}(A) \to \mathcal{P}(A)$ be a very-operator satisfying F1, F4, F5. For example every interior (=topological) clopen operator i on A satisfies F1, F4, F5. (i is clopen if in the produced topology all open sets are also closed.)

The proof of the following is easy and left to the reader.

Lemma 5.1 *Let $i_1, \ldots, i_n$ be operators on A satisfying* F1, F4, F5. *Then $i_1 \circ \cdots \circ i_n$ satisfies* F1, F4, F5.

Note that $i_1 \circ \cdots \circ i_n$ in the preceding lemma is in general non-idempotent. We show now that fuzzy sets of the traditional type can be (partly) recovered from those defined by means of a very-operator.

Definition 5.2 For any $X \subseteq A$, its *quantitative representation* $\overline{X} : A \to [0, 1]$ is defined follows:
i) $\overline{X}(a) = 0$ if $a \notin X$,
ii) $\overline{X}(a) = 1$ if $a \in \bigcap_{n \geq 0} v^n(X)$ (i.e. $a \in abs(X)$ if F5* holds),
iii) $\overline{X}(a) = n/(n+1)$ if $a \in v^{n-1}(X) \backslash v^n(X)$.

$\overline{X}$ is a traditional fuzzy set, but with a restricted spectrum of membership degrees, namely the latter are among 0 and $n/(n+1)$, $n > 0$. The intuitive idea of the above definition is that, the closer an element $a \in X$ is to the absolute core of X, the greater is its membership degree. And if $a \in \bigcap_{n \geq 0} v^n(X)$ (which equals $abs(X)$ if v satisfies F5*), then the degree of a is 1. The least positive membership degree assigned by $\overline{X}$ happens to be 1/2. Therefore a class X such that $\overline{X}(a) = 1/2$ for all $a \in X$, should be totally fuzzy. This is indeed the case as we see in the next proposition.

Recall that $\overline{X} \subseteq \overline{Y}$ if $\overline{X}(a) \leq \overline{Y}(a)$ for all $a \in A$, and $(\overline{X} \cap \overline{Y})(a) = \min\{\overline{X}(a), \overline{Y(a)}\}$. The following establishes a good relation between a structurally fuzzy set X and its quantitative representation.

Proposition 5.3 (Assuming F5*) *For any* $X, Y \subseteq A$,

i) $X \subseteq Y \iff \overline{X} \subseteq \overline{Y}$.

ii) $\overline{X \cap Y} = \overline{X} \cap \overline{Y}$.

iii) X *is crisp iff* $\overline{X}$ *is crisp (i.e.,* $\overline{X}(a) = 0$ *or* 1*, for all* $a \in A$*).*

vi) X *is totally fuzzy iff* $\overline{X}(a) = 1/2$ *for all* $a \in X$.

Proof. i) "$\Rightarrow$": Let $X \subseteq Y$ and $a \in A$. We have to show that $\overline{X}(a) \leq \overline{Y}(a)$. If $a \notin X$, $\overline{X}(a) = 0$, so this is obvious. Let $a \in abs(X)$. Then $\overline{X}(a) = 1$, but $X \subseteq Y \Rightarrow abs(X) \subseteq abs(Y)$ (see 3.6), hence $a \in abs(Y)$ and therefore $\overline{Y}(a) = 1$.

Finally, let $a \in v^{n-1}(X) \backslash v^n(X)$. Then $\overline{X}(a) = n/(n+1)$. By monotonicity of v, hence of v^n, $v^{n-1}(X) \subseteq v^{n-1}(Y)$. Therefore $a \in v^{n-1}(Y)$, which means that $\overline{Y}(a) \geq n/(n+1)$, i.e., $\overline{Y}(a) \geq \overline{X}(a)$.

"$\Leftarrow$": Let $\overline{X} \subseteq \overline{Y}$ and let $a \notin Y$. Then $\overline{Y}(a) = 0$. By the assumption, $\overline{X}(a) = 0$ too, hence $a \notin X$. Thus $X \subseteq Y$.

ii) We have to show that $\overline{X \cap Y}(a) = \min\{\overline{X}(a), \overline{Y}(a)\}$.

Case 1. $a \notin X \cap Y$. Then $\overline{X \cap Y} = 0$ and either $a \notin X$ or $a \notin Y$, therefore either $\overline{X}(a) = 0$ or $\overline{Y}(a) = 0$, hence $\overline{X \cap Y}(a) = 0 = \min\{\overline{X}(a), \overline{Y}(a)\}$.

Case 2. Let $a \in abs(X \cap Y)$. Then $\overline{X \cap Y}(a) = 1$. Since $abs(X \cap Y) = abs(X) \cap abs(Y)$, it follows that $a \in abs(X)$ and $a \in abs(Y)$, i.e., $\overline{X}(a) = \overline{Y}(a) = 1$, hence $\overline{X \cap Y}(a) = 1 = \min\{\overline{X}(a), \overline{Y}(a)\}$.

Case 3. Let $a \in v^{n-1}(X) \backslash v^n(X)$, $a \in v^{m-1}(Y) \backslash v^m(Y)$ and $m \leq n$. Then $v^{n-1}(X) \subseteq v^{m-1}(X)$, therefore $a \in v^{m-1}(X) \cap v^{m-1}(Y) = v^{m-1}(X \cap Y)$. Moreover, since $a \notin v^m(Y)$, $a \notin v^m(X \cap Y)$. Thus $a \in v^{m-1}(X \cap Y) \backslash v^m(X \cap Y)$, whence $\overline{X \cap Y}(a) = m/(m+1) = \min\{m/(m+1), n/(n+1)\} = \min\{\overline{X}(a), \overline{Y}(a)\}$.

iii) X is crisp iff $v(X) = X$. Then $abs(X) = X$, hence $\overline{X}(a) = 1$ for all $a \in X$. So $\overline{X}(a) = 1$ or 0 for every $a \in A$, which means that $\overline{X}$ is crisp.

iv) Recall that X is totally fuzzy if $X \neq \emptyset$ and $v(X) = \emptyset$. So if X is totally fuzzy, $a \in X$ iff $a \in X \backslash v(X)$ and so for every $a \in X$, $\overline{X}(a) = 1/2$. Conversely, let $\overline{X}(a) = 1/2$ for every $a \in X$. This means that $a \in X \ \Rightarrow \ a \in X \backslash v(X)$, hence $v(X) = \emptyset$. Therefore X is totally fuzzy. $\dashv$

The functions $\overline{X}$ however do not respect the usual law of negation, since in general $\overline{(-X)}(a) \neq 1 - \overline{X}(a)$. This is because we defined $\overline{X}(a) = 0$ for every $a \notin X$. To repair this consider for every X the mapping $\overline{\overline{X}} : A \to [0, 1]$ defined as follows:

i) $\overline{\overline{X}}(a) = \overline{X}(a)$, if $a \in X$, and
ii) $\overline{\overline{X}}(a) = 1 - \overline{(-X)}(a)$, if $a \in -X$.

Proposition 5.4 *For any $X, Y \subseteq A$,*
i) $X \subseteq Y \iff \overline{\overline{X}} \subseteq \overline{\overline{Y}}$.
ii) $\overline{\overline{(-X)}}(a) = 1 - \overline{\overline{X}}(a)$, for every $a \in A$.
iii) X is crisp iff $\overline{\overline{X}}$ is crisp.
iv) X is totally fuzzy iff $\overline{\overline{X}}(a) \leq 1/2$ for all $a \in A$.

Proof. i) "$\Rightarrow$": Let $X \subseteq Y$. If $a \in X$, then, by (i) of 5.3, $\overline{\overline{X}}(a) = \overline{X}(a) \leq \overline{Y}(a) = \overline{\overline{Y}}(a)$. If $a \notin X$, then $\overline{\overline{X}}(a) = 1 - \overline{(-X)}(a)$. Since by assumption $-X \supseteq -Y$, by 5.3, $\overline{(-X)}(a) \geq \overline{(-Y)}(a)$, or $1 - \overline{(-X)}(a) \leq 1 - \overline{(-Y)}(a)$, hence $\overline{\overline{X}}(a) \leq \overline{\overline{Y}}(a)$. The converse is similar.

ii) Immediate from the definition of $\overline{\overline{X}}$.

iii) If X is crisp, so is $-X$, hence $\overline{\overline{X}}(a) = 1$ for $a \in X$ and $\overline{\overline{X}}(a) = 0$ for $a \in -X$.

iv) If X is totally fuzzy, then $\overline{\overline{X}}(a) = 1/2$ for $a \in X$, and $\overline{\overline{X}}(a) = 1 - \overline{(-X)}(a)$ for $a \in -X$. But since $\overline{(-X)}(a) \geq 1/2$, $\overline{\overline{X}}(a) \leq 1/2$. $\dashv$

$\overline{\overline{X}}$ now does not satisfy clause (ii) of 5.3. So both $\overline{X}$ and $\overline{\overline{X}}$ partially represent X in their basic behavior.

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