The Order Structure of Continua

Athanassios Tzouvaras
Department of Mathematics
University of Thessaloniki
540 06 Thessaloniki, Greece
E-mail:tzouvara@ccf.auth.gr

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Abstract

A *continuum* is here a primitive notion intended to correspond precisely to a path-connected subset of the usual euclidean space. In contrast, however, to the traditional treatment, we treat here continua not as pointsets, but as irreducible entities equipped only with a partial ordering $\leq$ interpreted as parthood. Our aim is to examine what basic topological and geometric properties of continua can be expressed in the language of $\leq$, and what principles we need in order to prove elementary facts about them. Surprisingly enough $\leq$ suffices to formulate the very heart of continuity (=jumpless and gapless transitions) in a general setting. Further, using a few principles about $\leq$ (together with the axioms of **ZFC**), we can define points, joins, meets and infinite closeness. Most important, we can develop a dimension theory based on notions like path, circle, line (=one-dimensional continuum), simple line and surface (=two-dimensional continuum), recovering thereby in a rigorous way Poincaré's well-known intuitive idea that dimension expresses the ways in which a continuum can be torn apart. We outline a classification of lines according to the number of circles and branching points they contain.

The ordering $(C, \leq)$ is a topped and bottomed, atomic, almost dense and complete partial ordering, weaker than a lattice. Continuous transformations from C to C are also defined in a natural way and results about them are proved.

The key notions on which the dimension theory is based are the "minimal extensions of continua", or "joins", and the "splittings of continua over subcontinua".

0 Introduction.

Since the first decades of this century, there have been various attempts to build geometry and topology on notions other than that of point. The subject is often referred to as "geometry without points", though this heading is a bit misleading. In fact most of these alternative approaches originated with Lesniewski's *mereology*, i.e., the formal theory that treats the relation of *part to whole*. Lesniewski himself "*...suggested the problem of establishing the foundations of geometry of solids, understanding by this term a system of geometry destitute of such geometrical figures as points, lines, and surfaces, and admitting as figures only solids.*" (cf. Tarski (1926), p. 24). Responding to this problem, Tarski (1926) proposed a foundation using as primitive the notions of "sphere" and "proper part". Through a chain of definitions he arrives at the notion of "point" ("set of concentric spheres"), as well as at the notion of "equidistance of two points from a third one", from which all concepts of euclidean geometry can be defined.

Tarski's work had already been preceded by Huntington (1913), to which Tarski refers. Huntington was the first to use the notion of sphere as primitive for the foundation of three-dimensional point geometry.

Points in the above approach are not denied altogether, but only from being a *primitive* notion; they are recovered as derivative entities. The simplest way to recover points in every mereological theory based on a relation $\leq$ of parthood, is to postulate the existence of $\leq$-minimal nonvoid objects. These are precisely the points of the theory.

In the same trend one should also mention Menger (1940) that surveys various approaches to the foundation of topology not based on point as primitive.

Another option, more consistent with the non-point tradition, is to drop minimal objects altogether. This is for example the approach of Grzegorczyk (1960), where an axiomatization of topology of "spatial bodies" is proposed (the system T) based on two primitive relations, proper parthood $x \subset y$ and separation $x)(y$. Here neither a void body nor minimal ones (points) are supposed to exist.

More recent addresses on the subject are Gerla (1985), Gerla (1990) and Gerla (1995), in which the author proposes an approach to metric spaces based on the primitives: solid, inclusion, and distance between solids. Points in this work are just solids of diameter zero. The reader is also suggested to consult the excellent bibliography contained in Buekenhout (1995).

In contrast to the aforementioned approaches, in the present paper the objects of discource will be, intuitively, the "pieces of space" in which something can travel "continuously", with no restriction on their shape, size or metric properties. These objects, called here *continua*, are, of course, not new; in classical topology they are described as *path-connected subsets* of $\mathbf{R}^n$. What is new is their treatment as primitive, irreducible entities, structured only under the relation of parthood $\leq$. Our aim is to examine the expressive power of $\leq$ with respect to the behavior and classification of these objects. Namely we wish to address the following questions:

(a) Which geometric/topological properties of continua can be, first, defined, second, formulated as axioms and, third, proved as theorems by means of the language $\mathcal{L} = \{\leq, \mathbf{0}, \mathbf{1}\}$?

(b) Can the basic features of *continuity-connectedness* of these entities be grasped in terms of $\leq$?

(c) What about *dimension*? Can a satisfactory notion of dimension, distinguishing between points, lines, surfaces and so on, be defined in terms of $\leq$?

The results are encouraging. Some fundamental notions like "path", "circle", "continuity", "simple line", (i.e., line without loops or branches), as well as "line", in general, and "surface" can be defined by means of $\leq$. This means that at least the treatment of continua of dimensions 0, 1 and 2 falls completely within the capacity of the language $\mathcal{L} = \{\leq\}$, the complexity of the formulas being reasonably low. The other dimensions can also be treated, but the definitions become rather involved, so we will not proceed further. About 10 simple axioms suffice to capture a good deal of the non-metric properties of continua (including Jordan's theorem).

Some similarities can be detected in our axiomatic system with that of Grzegorczyk, though his scope, which is concerned with capturing closeness and continuity, is quite different. Since he uses a second primitive relation (separeteness) and a fairly complicated principle (A_4), we believe that our treatment is simpler.

The paper is organized as follows: In section 1 we set out a system of 8 axioms $\mathrm{C}_0 - \mathrm{C}_7$ in the language $\mathcal{L} = \{\leq, \mathbf{0}, \mathbf{1}\}$, with the appropriate motivation, concerning notions like points, meets, joins, distributivity, completeness, branching and adjacency.

In section 2 we consider the fundamental notion of “splitting continua” by extracting subcontinua, and the corresponding “analysis over a set of subcontinua ”. Concerning this notion and its relation to adjacency, two further axioms are introduced, C_8 and C_9, that correspond to the main properties of continuity, namely absence of jumps and absence of gaps, respectively.

Section 3 is the main one. Using tools from the previous sections, we define notions of dimension theory, like path and line (1-dimensional continuum), and further a classification of lines according to their “genus” or “splitting degree”. This sheds light on the notion of dimension itself as captured first by Poincaré, by giving a rigorous formulation to the inductive idea that the dimension of x is n, if x can be split using finitely many subcontinua of dimension $n-1$. Various results are proved, among them a Jordan-type theorem, most of them stating familiar facts, but which are proved only by means of the order-theoretic principles.

In section 4 we consider continuous transformations of continua in particular lines. The main result is that any two lines without loops and with the same end-conditions can be continuously transformed onto one another.

In section 5, finally, we define 2-dimensional continua, or surfaces, and try to extend the dimension theory on them. The situation is much more complicated and the results proved aim at showing that the definition is sound. It is proved for instance that if we split a surface by a line, then the pieces are surfaces again, and any two points on a surface can be joined by an infinity of paths lying on it. A final axiom, C_{10}, asserting the existence of surfaces, is added.

1 Structures of Continua.

We shall describe the structure of continua in the language $\mathcal{L} = \{\leq, \mathbf{0}, \mathbf{1}\}$, where $\leq$ is a binary predicate and $\mathbf{0}$, $\mathbf{1}$ are constants. This language is supposed to extend the usual language of set theory with equality. The lower-case variables $x, y, z, \ldots$ of $\mathcal{L}$ range over continua which, from a set-theoretic point of view, are *urelements*. Besides we have upper-case variables $X, Y, Z, A, B, \ldots$ ranging over pure sets or sets of continua. The meaning of $x \leq y$ will be: "x is a part of y". $\mathbf{0}$ denotes the *void continuum* and $\mathbf{1}$ denotes the *universal continuum*, or the *space*.

We make light use of set-theoretic notation and terminology and this is the standard one. $\mathbf{N}$ is the set of natural numbers and $|X|$ is the cardinality of X. X is always at most countable (i.e., countably infinite or finite). n, m, i, j usually range over elements of $\mathbf{N}$, but some of them appear also in other notations. For instance $m(\cdot, \cdot, \cdot)$, $j(\cdot, \cdot, \cdot)$ will be predicates for the "meet" and "join" respectively, l will denote a "line" etc. In any case the clarity of the context will help us to avoid confusion.

Using **ZFC** for our metatheory, we shall formulate in $\mathcal{L}$ a finite set $\mathbf{C}$ of principles called the "theory of continua". Our intuition draws mainly from the corresponding pointsets of $\mathbf{R}^3$ (or $\mathbf{R}^n$ if you prefer). The picture we have in mind when referring to a continuum is that of a *path-connected* subset of $\mathbf{R}^3$. Path-connectedness is the primary property of what we understand here as continuum. At first sight it looks like an "observable" macro-property. However, this is only an illusion. Path -connectedness stems from the *continuity* of transition, and continuity, in the sense of Dedekind, is a non-observable micro-property of how parts of a continuum "are glued together". The axioms we shall provide about $\leq$ intend to capture primarily these ideas.

By a *structure of continua* we understand a quadruple $\mathcal{C} = (C, \leq, \mathbf{0}, \mathbf{1})$ satisfying the axioms below, which we introduce step by step with the appropriate motivation. $x < y$ will denote *proper parthood*. The same symbol, however, will be used for the ordinary ordering of natural numbers. But since the letters denoting continua do not overlap with those denoting numbers, there is no danger of confusion.

Countability. The domain of discource C is assumed to be countably infinite. This is stated by the following

Axiom C_0*:* $|C| = \aleph_0$.

Therefore for any set X of continua occurring henceforth $|X| \leq \aleph_0$.

The partial ordering. Parthood is a partial ordering. Moreover $\mathbf{0}$ is the bottom element and $\mathbf{1}$ is the top element of this ordering.

Axiom C$_1$*:* $\leq$ *is a partial ordering and* $(\forall x)(\mathbf{0} \leq x \leq \mathbf{1})$.

Points. The structure $\mathcal{C}$ is *atomic* in the sense that every nonvoid continuum contains minimal nonvoid continua called *points*. That is, x is a point if $x \neq \mathbf{0}$ and $(\forall z)(z \leq x \Rightarrow z = x \ \vee \ z = \mathbf{0})$. We use the letters $p, q, r, \ldots$ to denote points.

Axiom C$_2$*: Every nonvoid continuum includes points.* In symbols

$$(\forall x \neq \mathbf{0})(\exists p)(p \leq x).$$

Points will be also referred to as *trivial* continua.

Completeness. A set X of continua is said to be *upward (downward) directed* if for any $x, y \in X$ there is a $z \in X$ such that $x \leq z$ and $y \leq z$ ($z \leq x$ and $z \leq y$). We will suppose that directed families have *limits*, called *suprema* and *infima*, respectively.

Axiom C$_3$ *(Completeness): Every upward or downward directed set* X *of continua has a supremum, denoted* $\bigvee X$*, and an infimum denoted* $\bigwedge X$*, respectively.*

Meets and Joins. Continua are either disjoint (i.e., they meet at $\mathbf{0}$), or they meet at various subcontinua whose (set-theoretic) union is not in general a continuum. On the left column of figure 1 we see lines meeting at points. On the right we cite the corresponding graph-theoretical pictures of the situation inside the structure $(C, \leq, \mathbf{0})$.

Figure 1

Thus given x, y, as a rule they do not have an infimum. Nevertheless they (should) have *maximal common parts* which we shall call *meets.* That is to say, z is a meet of x, y if

$$z \leq x \ \& \ z \leq y \ \& \ (\forall u)(u \leq x \ \& \ u \leq y \Rightarrow z \not< u).$$

We abbreviate the preceding formula by

$$m(z, x, y).$$

Dually, given continua x, y, there is no supremum in general for x, y. Instead there should always be *minimal extensions* for them, which we shall call *joins* (see figure 2). The fact that z is a join of x, y is written formally

$$x \leq z \ \& \ y \leq z \ \& \ (\forall u)(x \leq u \ \& \ y \leq u \Rightarrow u \not< z)$$

and we abbreviate it by

$$j(z, x, y).$$

Figure 2

Fortunately, existence of meets and joins follows from C_3 and Zorn's Lemma, so we do not have to introduce them through axioms.

Proposition 1.1 *Let x, y be given. Then for every $z \leq x, y$, there is a $z' \geq z$ such that $m(z', x, y)$. Similarly for every $u \geq x, y$, there is a $u' \leq u$ such that $j(u', x, y)$.*

Proof. Let $z \leq x, y$ and let $X = \{w : z \leq w \ \& \ w \leq x, y\}$. Clearly, $m(z', x, y)$ iff z' is a maximal element of X. By Zorn's Lemma, X has maximal elements provided it is closed under limits of increasing chains. Let $\{w_i : i \in I\} \subseteq X$ be such a chain. By C_3, $\bigvee_i w_i$ exists and is an element of X. This proves the first claim. For the other one consider the set $Y = \{w : x, y \leq w \leq z\}$ and simply apply Zorn's Lemma to the set $(Y, \geq)$ to get minimal elements. Any such u' satisfies $j(u', x, y)$. $\Box$

In general x, y may have a multitude of meets (as in figure 1). If it happens that $(\exists! z) m(z, x, y)$ (as in figure 2(c)), i.e.,

$$(\exists z)(\forall u)(m(u, x, y) \Rightarrow u = z),$$

then we write $x \wedge y = z$ and say that z is the *infimum* of x, y. In particular if $x \wedge y = \mathbf{0}$, x, y are said to be *disjoint*. Thus $x \wedge y \neq \mathbf{0}$ means $(\exists z)(m(z, x, y)$ & $z \neq \mathbf{0})$.

Similarly if $(\exists ! z) j(z, x, y)$ (as in figures 2(b), 2(c)), we write $x \vee y = z$ and say z is the *supremum* of x, y.

The corresponding infinitary (partial) operations are $\bigwedge$ and $\bigvee$ respectively, introduced in C_3.

Another elementary fact is the following: Any two intersecting continua x, y must form a unique new one. In our terminology, x, y must have a supremum. This is stated below.

Axiom C_4*: Any two intersecting continua form a (unique) third one.*

$$x \wedge y \neq \mathbf{0} \Rightarrow (\exists z)(z = x \vee y).$$

As a consequence of C_4, two continua may have a multitude of joins or a multitude of meets but not both. This is illustrated in figure 3. $(C, \leq)$ may contain subgraphs of the form (a) and (b) but not of the form (c).

Figure 3

Distributivity. In connection with completeness the question of distributivity of $\wedge$, $\vee$ with respect to each other arises. Since, according to our ruling intuition, whenever suprema and infima exist, they behave exactly as unions and intersections respectively, we should accept distributivity if and when all limits exist.

Axiom C_5 *(Distributivity):*

$$x \wedge (\bigvee_i u_i) = \bigvee_i (x \wedge u_i) \ \textit{and} \ \ x \vee (\bigwedge_i u_i) = \bigwedge_i (x \vee u_i),$$

provided all limits involved in the equations exist.

Notice that the existence of some limit does not guarantee the existence of others, so the above equalities fail even in very simple finite cases as in figure 4 below. Here $x \wedge y = p$ and $z \vee x$, $z \vee y$ exists too but $z \vee p$ and $(z \vee x) \wedge (z \vee y)$ do not. Thus the equality $z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y)$ does not make sense.

Figure 4

Proposition 1.2 *Let $\{u_i : i \in I\}$ be directed upward (resp. downward) and x be such that $x \wedge u_i = \mathbf{0}$ (resp. $x \vee u_i = \mathbf{1}$) for all $i \in I$. Then $x \wedge (\bigvee_i u_i) = \mathbf{0}$ (resp. $x \vee (\bigwedge_i u_i) = \mathbf{1}$).*

Proof. $\bigvee_i u_i$ exists by C_3. Let $x \wedge (\bigvee_i u_i) \neq \mathbf{0}$. By proposition 1.1, there is a $z \neq \mathbf{0}$ such that $m(z, x, \bigvee_i u_i)$. Thus $z \leq x$ and $z \leq \bigvee_i u_i$. The last relation implies $z \wedge (\bigvee_i u_i) = z$. On the other hand, $z \wedge u_i = \mathbf{0}$ since $z \leq x$ and $x \wedge u_i = 0$. Therefore, by C_3, $\bigvee_i (z \wedge u_i) = \mathbf{0}$. By C_5, $\mathbf{0} = \bigvee_i (z \wedge u_i) = z \wedge (\bigvee_i u_i) = z \neq \mathbf{0}$, which is a contradiction. The dual case is similar. □

Branching of Continua. If $y < x$, then x must include a part z disjoint from y. Even stronger, If x is not a part of y, x must have a part omitting y. This is postulated below.

Axiom C_6 *(Branching):* $x \not\leq y \Rightarrow (\exists z)(z \leq x \;\&\; z \neq \mathbf{0} \;\&\; z \wedge y = \mathbf{0})$.

Proposition 1.3 *For any x, y the following hold:*
(a) $x \leq y \iff (\forall z)(z \leq x \Rightarrow z \leq y) \iff (\forall p)(p \leq x \Rightarrow p \leq y)$.
(b) $x = y \iff (\forall z)(z \leq x \Leftrightarrow z \leq y) \iff (\forall p)(p \leq x \Leftrightarrow p \leq y)$.

Proof. It suffices to show (a). Consider the first equivalence

$$x \leq y \iff (\forall z)(z \leq x \Rightarrow z \leq y).$$

The $\Rightarrow$-part of this is clear. For the converse let $x \not\leq y$. By C_6, there is a $z \leq x$, $z \neq \mathbf{0}$ such that $z \wedge y = \mathbf{0}$. Hence $z \not\leq y$. Similarly we show the other equivalence using C_2. □

Let

$$\Pi(x) = \{y : y \leq x\}, \quad \Pi_0(x) = \{p : p \leq x\}$$

be the set of all parts and the set of points of x respectively. (Notice that these sets are defined without the use of the powerset axiom.) Then the equivalences of the preceding proposition can be rewritten as follows:

Proposition 1.4 *(a)* $x \leq y \iff \Pi(x) \subseteq \Pi(y) \iff \Pi_0(x) \subseteq \Pi_0(y)$.
(b) $x = y \iff \Pi(x) = \Pi(y) \iff \Pi_0(x) = \Pi_0(y)$.
(c) For all x, $x = \bigvee \Pi(x) = \bigvee \Pi_0(x)$.

Proof. All claims are immediate consequences of proposition 1.3. $\square$[1]

Adjacency. By Axiom C_4, if x, y meet, then their union forms a new continuum. The converse however is false: there are disjoint continua x, y which also have a supremum. For example an open disc and its boundary circle, or an open segment and its ends, or the continua of figure 2(b), are of this kind. This situation leads to the following definition.

Definition 1.5 Two continua x, y are said to be *adjacent,* written $Ad(x, y)$, if $x \vee y$ exists. Otherwise we call them *strongly disjoint.*

[1]The origin of the present mereological treatment of continua can be traced to Tzouvaras(1995) and (1993), where a formal theory of artifacts and their identity is presented. According to this, every artifact is analyzed to a finite set of atomic parts (i.e., parts not further decomposable), which is also denoted $\Pi_0(x)$, and if $\Pi_0(x) = \Pi_0(y)$ then the artifacts x, y are identical. It would perhaps be instructive to compare the role the set $\Pi_0(x)$ of points plays in the construction of continuum x, with the role the set $\Pi_0(x)$ of atomic parts plays in the construction of the artifact x.

If the atoms of the artifact x are not all copies of each other (a situation appearing only in trivial toy constructions), then they fit among them in a unique way, that is to say the particular structure x is already engraved on the atoms themselves. In this sense x can be identified with the set $\Pi_0(x)$. For whenever the elements of $\Pi_0(x)$ are given, x can be restored unabiguously. Given, for instance, all elementary parts forming a washing machine, you can assemble from them a unique artifact, namely the specific washing machine.

This is not the case, of course, with the atoms (=points) of a continuum. There is no notion of fitness between any two of them. They are all just identical abstractions satisfying the convention that we can't lessen them any further. Given any totality of points in separation, it makes no sense to try and guess which continuum they have been parts of. Thus in no sensible way can we claim that they "form" the continuum.

Clearly if x, y meet then they are adjacent (or, equivalently, if x, y are strongly disjoint then they are disjoint). Adjacency is the analog of "infinite closeness" or "nonseparation" of metric topology. For instance, the continua $[0, 1)$ and $[1, 2)$ are adjacent, while $[0, 1)$ and $(1, 2)$ are strongly disjoint. The latter are separated by the point $\{1\}$.

Proposition 1.6 *(a) Let $\{x_i : i \in I\}$ be a family of continua. Then $\bigvee_i x_i$ exists iff there is a y such that $\Pi_0(y) = \bigcup_i \Pi_0(x_i)$. The dual holds for infima.*

(b) In particular x, y are adjacent (i.e., $x \vee y$ exists) iff there is a z such that $\Pi_0(z) = \Pi_0(x) \cup \Pi_0(y)$. (Equivalently, x, y are strongly disjoint iff for every $z \geq x, y$, $\Pi_0(x) \cup \Pi_0(y) \subset \Pi_0(z)$.)

Proof. (a) "$\Rightarrow$". Suppose $\bigvee_i x_i$ exists and is equal to y. Then, clearly, $\bigcup_i \Pi_0(x_i) \subseteq \Pi_0(y)$. Let $p \notin \bigcup_i \Pi_0(x_i)$. Then $p \wedge x_i = \mathbf{0}$ for all $i \in I$. By proposition 1.2, $p \wedge (\bigvee_i x_i) = p \wedge y = \mathbf{0}$, hence $p \notin \Pi_0(y)$.

"$\Leftarrow$". Let $\Pi_0(y) = \bigcup_i \Pi_0(x_i)$ for some y. It follows from 1.4 that y is an upper bound for the family $\{x_i : i \in I\}$. Consider another upper bound z of x_i. Then $\bigcup_i \Pi_0(x_i) \subseteq \Pi_0(z)$, or $\Pi_0(y) \subseteq \Pi_0(z)$. From 1.4 it follows that $y \leq z$. Thus y is the least upper bound of the family $\{x_i : i \in I\}$, hence $y = \bigvee_i x_i$. $\square$

Proposition 1.7 *Let X be a set of continua such that $\bigvee X = u$ exists. If $x \in X$, $x \leq x'$ and $X' = X \cup \{x'\} - \{x\}$, then $\bigvee X'$ also exists and equals $u \vee x'$. In particular if $Ad(x, y)$ and $x \leq x'$, then $Ad(x', y)$.*

Proof. Clearly $u \wedge x' \neq \mathbf{0}$. It follows from C_4 that $u \vee x'$ exists. Using proposition 1.6 we easily verify that this is the supremum of X'. $\square$

Proposition 1.8 *Let X be a set such that for some $x_0 \in X$, $Ad(x, x_0)$ for all $x \in X$. Then $\bigvee X$ exists.*

Proof. For any two $x, y \in X$, $x \vee x_0$ and $y \vee x_0$ exist, and since the latter meet, $\bigvee\{x, y, x_0\}$ exists too. Inductively we can show that for every finite $\{x_1, \ldots, x_n\} \subseteq X$, $\bigvee\{x_1, \ldots, x_n, x_0\}$ exists. Recall that X is countable by C_0, and let $X = \{x_1, x_2, \ldots\}$ be an enumeration of its elements. Let

$$X' = \{\bigvee\{x_1, \ldots, x_n, x_0\} : n \in \mathbf{N}\}.$$

Then X' is upward directed, so $\bigvee X'$ exists by C_3. By 1.4, $\bigvee X' = \bigvee X$. $\square$

Proposition 1.9 *Let x, y meet and let z_1, z_2 be two distinct meets of them. Then z_1, z_2 are strongly disjoint.*

Proof. Suppose on the contrary they are adjacent. Then $z_1 \vee z_2$ exists and clearly $z_1 \vee z_2 \leq x, y$. But this contradicts the fact that z_1, z_2 are maximal common parts of x, y. $\square$

So far our axioms are compatible with the existence of nontrivial continua having only finitely many points, e.g. an x such that $x = p \vee q$. Since this is clearly incompatible with the intended meaning of continuum, we shall rule it out. To this effect it suffices to admit the following (combined with the subsequent Axioms C_8 or C_9):

Axiom C_7*: No two points are adjacent.* In symbols

$$(\forall p, q)(p \neq q \Rightarrow \neg Ad(p, q)).$$

C_7 is the analog of the topological "separation principle" stating that for any two points $p \neq q$ there is a neighborhood V such that $p \in V$ and $q \notin V$, or $q \in V$ and $p \notin V$.

2 Splitting Continua over Subcontinua. Continuity.

Given x, y such that $y \not\leq x$, we will set

$$\Pi(y \backslash x) = \{z : z \leq y \ \& \ z \wedge x = \mathbf{0}\}$$

(the set of parts of y disjoint from x). By Axiom C_6, $\Pi(y \backslash x)$ is always nonvoid.

Proposition 2.1 *Every element of $\Pi(y \backslash x)$ is contained in a maximal element of it.*

Proof. By Zorn's Lemma. Let $z \in \Pi(y \backslash x)$ and let $\{u_i : i \in I\}$ be a chain of elements of $\Pi(y \backslash x)$ that extend z. By C_3, $\bigvee_i u_i$ exists and it suffices for it to belong to $\Pi(y\backslash x)$, or $x \wedge (\bigvee_i u_i) = \mathbf{0}$. Since $x \wedge u_i = \mathbf{0}$ $\forall i \in I$, this follows from Prop. 1.2. $\Box$

Given $y \not\leq x$ let

$$An(y\backslash x) = \{z : z \text{ is a maximal element of } \Pi(y\backslash x)\}.$$

We shall call $An(y\backslash x)$ *the analysis of* y *over* x, and the elements of it *the components of* y *over* x. If $|An(y\backslash x)| > 1$, we say that y *splits over* x, or x *splits* y.

Proposition 2.2 *(a) The elements of* $An(y\backslash x)$ *are pairwise strongly disjoint.*

(b) If $x < y$ *then* $\bigvee(An(y\backslash x) \cup \{x\}) = y$.

Proof. (a) is immediate from the maximality of the elements of $An(y\backslash x)$ and (b) follows from Prop. 1.4. $\Box$

More generally, given y and a set $X \subseteq \Pi(y)$ of subcontinua of y, the set

$$\Pi(y\backslash X) = \{z \leq y : z \wedge x = \mathbf{0},\ \forall x \in X\}$$

is either empty, or each one of its elements is contained in a maximal one. The set

$$An(y\backslash X) = \{z : z \text{ is a maximal element of } \Pi(y\backslash X)\}$$

is called *the analysis of* y *over* X. Clearly, the elements of $An(y\backslash X)$ are pairwise strongly disjoint (p.s.d. henceforth). As before, if $|An(y\backslash X)| > 1$ we say that y *splits over* X, or that X *splits* y.

For finitely many $x_1, \ldots, x_n$ the above sets will be written $\Pi(y\backslash x_1, \ldots, x_n)$ and $An(y\backslash x_1, \ldots, x_n)$ respectively.

Path-connectedness now is just "continuity of transition", and, roughly,

Continuity=Absence of Jumps+Absence of Gaps.

For linearly ordered sets the above equation is equivalent to

Continuity=Density+Completeness.

(Typical examples of jumpless, gapless and both jumpless and gapless sets are the sets of rationals, integers and reals, respectively.) Let's see how these notions could be expressed here and what axioms they need.

A) Absence of Jumps. When we say that we can pass from x to y *without a jump*, we simply mean that $x \vee y$ exists, that is to say x, y are *adjacent*. Equivalently, we need to "jump" in order to pass from x to y if they are mutually incommunicado (in our terminology, *strongly disjoint*).

Suppose now we are given three p.s.d. x, y, z. Can $\bigvee\{x, y, z\}$ exist? If it does and $u = \bigvee\{x, y, z\}$, then x, y could be joined by a path inside u. But since x, y are not adjacent, the continuous transition would be possible only via z. That means that z would be adjacent to both x and y. Equivalently, the limit $\bigvee\{x, y, z\}$ exists only if one of the three is adjacent to each of the other two. Thus absence of jumps imposes the condition:

$$\text{If } \bigvee\{x, y, z\} \text{ exists then one of the three is adjacent to the other two.} \quad (1)$$

Nevertheless (1) does not follow from $\mathrm{C}_0 - \mathrm{C}_7$. Consider for example the situation of figure 5, (a) or (b), where $x < u$ and $An(u \backslash x) = \{y, z\}$.

Figure 5

Then we can easily find a graph-model satisfying $\mathrm{C}_0 - \mathrm{C}_7$, containing an element u split into three p.s.d. parts x, y, z, i.e., $u = \bigvee\{x, y, z\}$, and such that $An(u \backslash x) = \{y, z\}$. (Of course no $x \vee y$, $x \vee z$, $y \vee z$ exists; specifically u is a minimal extension (join) of all these pairs.) So the above requirement (1) fails in this model and we have to introduce it as a new principle. In fact (1) is a consequence of the following axiom: If we split a continuum y over a subcontinuum x, then x must be adjacent to every component of $An(y \backslash x)$. In fact x is the "gluing- together" continuum for the strongly disjoint components. Thus we admit the following:

Axiom C_8 *(Absence of Jumps)*: $(\forall z \in An(y \backslash x)) Ad(x, z)$.

We have seen that given any y and $X \subseteq \Pi(y)$, the components of $An(y\backslash X)$ are p.s.d. as maximal parts of $\Pi(y\backslash x)$. In fact they satisfy something stronger: If A is any subset of $An(y\backslash x)$ with more than one element, then $\bigvee A$ does not exist. Indeed, if $\bigvee A = z$ exists, then $x \wedge z = \mathbf{0}$, contradicting the fact that the elements of A are maximal disjoint from x. We shall call a family of continua having this propery *normal.*

Definition 2.3 A set of continua X is said to be *normal* if no set $A \subseteq X$ with at least two elements has a supremum. A *normalization* of X is a set X^* such that:

a) X^* is normal,

b) Every element of X is part of some element of X^*, and

c) $\Pi_0(X^*) = \Pi_0(X)$ (where $\Pi_0(X) = \bigcup\{\Pi_0(x) : x \in X\}$).

Notice that normality is stronger than pairwise strong disjointedness. For example every set of points is p.s.d., though its supremum may exist. However the following holds:

Proposition 2.4 *Every finite p.s.d. set is normal.*

Proof. By induction on $|X| = n$. Let n be the least number such that for some $X = \{x_1, \ldots, x_n\}$, X is p.s.d. and $\bigvee\{x_1, \ldots, x_n\} = u$ exists. Then the set $\{x_2, \ldots, x_n\}$ is clearly normal, and therefore $An(u\backslash x_1\} = \{x_2, \ldots, x_n\}$. By C_8, we must have $Ad(x_1, x_k)$ for every $k = 2, \ldots, n$, contrary to the assumption that X is p.s.d. □

Proposition 2.5 *For every nontrivial x,*

$$|\Pi_0(x)| = |\Pi(x)| = |\Pi(x) - \Pi_0(x)| = \aleph_0.$$

Proof. Suppose $x \neq p$ and $\Pi_0(x)$ is finite. By C_7, $\Pi_0(x)$ is p.s.d. and from 2.4 it follows that $\Pi_0(x)$ is normal. Hence $\bigvee \Pi_0(x)$ does not exist, contary to the fact, following from 1.4, that $\bigvee \Pi_0(x) = x$. Hence $\Pi_0(x)$ is infinite, and from countability assumption C_0 it follows $|\Pi_0(x)| = \aleph_0$. Now in order to show the existence of infinitely many nontrivial parts of x, we make use of C_6. Take some $p_0 < x$ (use C_2 for this purpose). By C_6 there is an $\mathbf{0} \neq x_0 < x$ such that $x_0 \wedge p_0 = \mathbf{0}$. By C_7, x_0 can be taken to be nontrivial. Choose some $p_1 < x_0$ and find similarly a nontrivial $x_1 < x_0$ omitting p_1, etc. This way we find an infinite sequence of nontrivial parts $\cdots < x_2 < x_1 < x_0 < x$. □

Proposition 2.6 *(a) Every set X of continua has a unique normalization X^*.*

(b) For every $X \subseteq \Pi(y)$, $An(y\backslash X)$ is a normal family. Specifically $An(y\backslash X)$ is the normalization of $\Pi(y\backslash X)$, i.e.,

$$An(y\backslash X) = \Pi(y\backslash X)^*.$$

(c) If $X \subseteq \Pi(y)$, then $X^* \subseteq \Pi(y)$ and $\Pi(y\backslash X) = \Pi(y\backslash X^*)$. Consequently, $An(y\backslash X) = An(y\backslash X^*)$.

Proof. (a) Given the set X let

$$Y = \{u : (\exists A \subseteq X)(\bigvee A \text{ exists and } \bigvee A = u)\}.$$

Using Zorn's Lemma we easily see that each element of Y is included in a maximal element of Y. Let X^* be the set of these maximal elements. Clearly, $\Pi_0(X^*) = \Pi_0(X)$, every $x \in X$ is extended by some $x' \in X^*$ and, by maximality, no set $A \subseteq X^*$ with $|A| > 1$ has a supremum. Thus X^* is a normalization of X. Now uniqueness follows from the fact that each normalization of X contains exactly the maximal suprema of subsets of X whenever they exist.

(b) Immediate from the definitions of $\Pi(y\backslash X)$ and normality.

(c) If $z \in \Pi(y\backslash X^*)$, then z misses all suprema of elements of X, hence it misses all elements of X. Thus $\Pi(y\backslash X^*) \subseteq \Pi(y\backslash X)$. Conversely, if z misses all elements of X and $\bigvee A$ exists for $A \subseteq X$, then, by 1.2, $z \wedge (\bigvee A) = \mathbf{0}$. Thus also $\Pi(y\backslash X) \subseteq \Pi(y\backslash X^*)$. $\square$

Proposition 2.7 *For normal $X \subseteq \Pi(y)$, $An(y\backslash X) = Y \iff An(y\backslash Y) = X$.*

Proof. Suppose $An(y\backslash X) = Y$. If $x \in X$, then $x \wedge z = \mathbf{0}$ $\forall z \in Y$, hence $x \in \Pi(y\backslash Y)$. To show that every element of X is a maximal element of $\Pi(y\backslash Y)$, assume $x \in X$ and $x < x' \in \Pi(y\backslash Y)$. Then $\Pi_0(x') \cap \Pi_0(Y) = \emptyset$, whereby $\Pi_0(x') \subseteq \Pi_0(X)$. It follows that x' is the supremum of some subfamily A of X such that $x \in A$ and $|A| > 1$ since $x < x'$. But this contradicts the normality of X. This proves that $X \subseteq An(y\backslash Y)$. Now since the last families contain the same points and are both normal, one is the normalization of the other, so, by 2.6(a), $X = An(y\backslash Y)$. This shows that if

X is normal, then $An(y\backslash X) = Y \Rightarrow An(y\backslash Y) = X$. The reverse implication follows immediately from the latter and the fact that $Y = An(y\backslash X)$ is also normal, according to 2.6(b). $\square$

Two normal families X, Y are said to be *complementary* if $\bigvee(X \cup Y)$ exists. In this case, if $\bigvee(X \cup Y) = u$, then $An(u\backslash X) = Y$ and $An(u\backslash Y) = X$ as follows from 2.7. For example, if $An(y\backslash x) = Y$, the families Y and $\{x\}$ are complementary. If in particular $An(u\backslash x) = \{y\}$, then the continua x, y are said to be *(relatively) complementary* and we write $x = u - y$ and $y = u - x$.

A natural question is this: Given two normal complementary families X, Y, is each element of the one necessarily adjacent to some element of the other? The answer is Yes if one of the families is finite, but No (in general) if they are both infinite. (For X non-normal the answer is negative even if $Y = An(u\backslash X)$ is a singleton. For instance take as u the interval $(0, 2)$ and as X the set of points of the interval $(0, 1)$. Then $An(u\backslash X) = \{[1, 2)\}$ and no point in X is adjacent to $[1, 2)$.) First a key lemma.

Lemma 2.8 *(a) Let X be normal and $An(u\backslash X) = Y = \{y_1, y_2, \ldots\}$. Let also $x \in X$ and $B = \{y_i : Ad(x, y_i)\}$. If $\bigvee(B \cup \{x\}) = v$, then*

$$An(u\backslash X - \{x\}) = (Y - B) \cup \{v\}.$$

(b) In particular, if $\neg Ad(x, y_i)\ \forall y_i \in Y$ (i.e., $B = \emptyset$),

$$An(u\backslash X - \{x\}) = Y \cup \{x\}.$$

Proof. It suffices to prove (a). That $\bigvee(B \cup \{x\})$ exists follows from 1.8. By the fact that X is normal it follows easily that

$$\Pi(u\backslash X - \{x\}) = \Pi(u\backslash X) \cup \Pi(x).$$

Therefore, by 2.6(a),

$$An(u\backslash X - \{x\}) = \Pi(u\backslash X - \{x\})^* = (\Pi(u\backslash X) \cup \Pi(x))^*.$$

So in order to show that $An(u\backslash X - \{x\}) = (Y - B) \cup \{v\}$, it suffices to prove that

$$(+) \qquad (Y - B) \cup \{v\} = (\Pi(u\backslash X) \cup \Pi(x))^*.$$

Now since the sets $(Y-B)\cup\{v\}$ and $\Pi(u\backslash X)\cup\Pi(x)$ contain exactly the same points, as we easily check, in order to prove (+) it suffices to show that $(Y-B)\cup\{v\}$ is normal. Clearly $Y-B$ is normal and assume to the contrary that for some $A\subseteq Y-B$, $\bigvee(A\cup\{v\})=w$ exists. Then $\bigvee(A\cup B\cup\{x\})=w$. But $A\cup B$ is normal as a subset of Y, hence $An(w\backslash x)=A\cup B$. Then, by C_8, $Ad(x,y_k)$ for all $y_k\in A$, contrary to the fact that $Ad(x,y_i)$ holds precisely for $y_i\in B$ and $A\cap B=\emptyset$. □

Proposition 2.9 *Let $X=\{x_1,\ldots,x_n\}$ be a finite normal subset of* $\Pi(u)$ *and let $An(u\backslash X)=Y$. Then*
(a) $(\forall y\in Y)(\exists x_i\in X)Ad(y,x_i)$.
(b) $(\forall x_i\in X)(\exists y\in Y)Ad(y,x_i)$.

Proof. (a) Suppose there is a $y\in Y$ such that $(\forall x_i)\neg Ad(y,x_i)$. Applying lemma 2.8 to $An(u\backslash X)$ for $x=x_1$, we get

$$An(u\backslash x_2,\ldots,x_n)=(Y-B_1)\cup\{v_1\},$$

where $B_1=\{y\in Y: Ad(x_1,y)\}$ and $v_1=\bigvee(B_1\cup\{x_1\})$, and $y\in Y-B_1$. Applying again 2.8, we find B_2 and v_2 such that

$$An(u\backslash x_3,\ldots,x_n)=(Y-B_2)\cup\{v_2\},$$

and $y\in Y-B_2$. Continuing this way we find B_{n-1} and v_{n-1} such that

$$An(u\backslash x_n)=(Y-B_{n-1})\cup\{v_{n-1}\},$$

and $y\in Y-B_{n-1}$. But then, by C_8, $Ad(y,x_n)$, which contradicts the assumption $(\forall x_i)\neg Ad(y,x_i)$.

(b) Assume again the contrary and suppose x_1, for example, is not adjacent to any element of Y. By 2.8(b), $An(u\backslash x_2,\ldots,x_n)=Y\cup\{x_1\}$. But then from (a) of this proposition it follows that $Ad(x_1,x_i)$ for some $i=2,\ldots,n$. This however contradicts the normality of X. □

Finally we show that 2.9 does not necessarily hold when both X,Y are infinite.

Proposition 2.10 *For every nontrivial x, there is a partition of* $\Pi_0(x)$ *into two normal complementary pointsets X and Y.*

Proof. By Prop. 2.5, we can write $\Pi_0(x) = \{p_1, p_2, \ldots\}$ and $\Pi(x) - \Pi_0(x) = \{x_1, x_2, \ldots\}$, for the set of points and the set of all nontrivial parts of x, respectively. Then using choice we can easily find by induction two sequences of points $q_1, q_2, \ldots$ and $r_1, r_2, \ldots$ such that:

(i) the sets $\{q_1, q_2, \ldots\}$, $\{r_1, r_2, \ldots\}$ are disjoint, and

(ii) $q_i, r_i < x_i$ for all $i \in \mathbf{N}$.

We set $X = \{q_1, q_2, \ldots\}$ and $Y = \Pi_0(x) - X$. Clearly X, Y are complementary. To see that they are normal, just observe that for every $y \leq x$, $\Pi_0(y)$ is included neither in X nor in Y, hence y cannot be the limit of any subset of X or Y. Thus no such subset has a limit. $\Box$

Some other consequences of C_8, combined with the rest axioms are examined below.

Proposition 2.11 *Let $Ad(x, y)$. Then for every z,*

$$Ad(z, x \vee y) \iff Ad(z, x) \textit{ or } Ad(z, y).$$

Proof. "$\Leftarrow$" follows from Prop. 1.7. For the converse suppose $Ad(z, x \vee y)$ and let $u = z \vee (x \vee y)$. Assume $\neg Ad(z, x)$. Then the family $\{x, z\}$ is normal, hence $An(u \backslash y) = \{x, z\}$. By C_8, $Ad(z, y)$. $\Box$

Proposition 2.12 *Let x, y, z be disjoint continua. Then $\bigvee\{x, y, z\}$ exists iff one of x, y, z is adjacent to the other two.*

Proof. Suppose one of them, say x, is adjacent to y, z. Then $x \vee y$ and $x \vee z$ exist. Since the last two continua meet, it follows from C_6 that $(x \vee y) \vee (x \vee z)$ exists too, and clearly this is $\bigvee\{x, y, z\}$.

Conversely, suppose none of the three is adjacent to the remaining two, while $\bigvee\{x, y, z\} = u$ exists. Equivalently, there is one of them, say x, such that $\neg Ad(x, y)$ and $\neg Ad(x, z)$. Since x, y, z are disjoint and $\{x, y\}$ is normal, we have $An(u \backslash z) = \{x, y\}$. By C_8, $Ad(x, z)$, which contradicts $\neg Ad(x, z)$. $\Box$

Proposition 2.13 *Let X be a normal family and y be a continuum. Then $X \cup \{y\}$ is normal iff $(\forall x \in X) \neg Ad(x, y)$.*

Proof. "$\Rightarrow$" is clear. For the converse suppose $X \cup \{y\}$ is not normal. We have to show that there is an $x \in X$ such that $Ad(x, y)$. From the hypothesis there is an $A \subseteq X$ such that $\bigvee(A \cup \{y\}) = u$ exists. A is normal too. If y meets some element of A the claim follows. Otherwise it is clear that $An(u \backslash y) = A$. By C_8, y is adjacent to every element of A and we are done. $\square$

Lemma 2.14 *Let $x < u$, $y < u$ and $\neg Ad(x, y)$. Then there is some $z \in An(u \backslash x, y)$ such that $Ad(x, z)$ and $Ad(y, z)$.*

Proof. Let $An(u \backslash x, y) = Z$. Then from Lemma 2.8 we get $An(u \backslash x) = (Z - B) \cup \{v\}$, where $B = \{z \in Z : Ad(y, z)\}$ and $v = \bigvee(B \cup \{y\})$. It suffices to show that for some $z \in B$, $Ad(x, z)$. By C_8, $Ad(x, v)$, hence $\bigvee(B \cup \{x, y\}) = w$ exists. Assume $\neg Ad(x, z)\ \forall z \in B$. Then by 2.13 $B \cup \{x\}$ would be normal. Therefore $An(w \backslash y) = B \cup \{x\}$. But C_8 implies $Ad(x, y)$, which contradicts the hypothesis. $\square$

Corollary 2.15 *Let x, y be strongly disjoint and let u be a join of them. Then there is a unique z such that $u = x \vee y \vee z$, $z \wedge x = z \wedge y = \mathbf{0}$ and $Ad(z, x)$, $Ad(z, y)$.*

Proof. Let x, y, u be as stated. From Lemma 2.14, there is a $z \in An(u \backslash x, y)$ such that $Ad(x, z)$ and $Ad(y, z)$. Therefore $x \vee y \vee z$ exists and is contained in u. Since u is a minimal extension of x, y, $An(u \backslash x, y) = \{z\}$, and thus $u = x \vee y \vee z$. $\square$

This unique z asserted by the last corollary, that corresponds to the join u of x, y, is said to be a *connection of* x, y (see figure 6).

Figure 6

Every join can be represented uniquely in this form. Clearly, x, y are adjacent iff the only connection between them is $\mathbf{0}$.

B) Absence of Gaps. Recall that absence of gaps in the real line is established by means of Dedekind's continuity principle which is formulated in terms of "cuts", i.e., splittings of the line into two nonempty disjoint segments A, B such that all elements of A precede all elements of B. Dedekind's principle then says that continuity is rendered possible always via some *point* that "glues" A and B together, and belongs either to A as a last element or to B as a first one. Whenever such a gluing point is missing, as is the case e.g. with cuts in the rational line, the partition is a "gap" and continuity fails. By extracting even a single point from the real line we create a gap at this place and continuity breaks down.

We can express this idea here very simply by postulating that whenever x, y are adjacent, then there is always a *point* of one of them which is adjacent to the other.

Axiom C_9 *(Absence of gaps):*

$$Ad(x, y) \Rightarrow (\exists p \leq x)Ad(p, y) \text{ } or \text{ } (\exists q \leq y)Ad(q, x).$$

Informally we shall often refer to the points p asserted by the axiom as "gluing points" between two adjacent continua. C_8 and C_9 express the two complementary aspects of continuity. Whereas C_8 is of a *macroscopic*, global character, having visible consequences on the behavior of continua, C_9 is a rather local principle, of *microscopic* character, referring to the invisible fine structure of the joins.

If x, y meet, C_9 holds trivially, so it gives real information only in the case that x, y are adjacent and disjoint. Recall that such x, y are said to be *relatively complementary* (r.c.). Let x, y be r.c. and $x \vee y = u$. A point $p < x$ such that $Ad(p, y)$ is said to be *boundary* with respect to y, and we set

$$BP_y(x) = \{p < x : Ad(p, y)\}.$$

C_9 says that if x, y are adjacent, then either $BP_y(x) \neq \emptyset$ or $BP_x(y) \neq \emptyset$. If $BP_y(x) = \emptyset$, x is said to be *open* with respect to y. Further, if $BP_y(x) \neq \emptyset$ then the *boundary of* x with respect to y is the normalization of the set $BP_y(x)$, i.e., the set of maximal continua formed by the boundary points. We denote this by $B_y(x)$, i.e.,

$$B_y(x) = BP_y(x)^*.$$

If now $x \vee y$ exists, $y \leq z$ and x, z are r.c., then $BP_y(x) \subseteq BP_z(x)$, and the boundary continua in $B_y(x)$ are parts of those in $B_z(x)$.

At the presence of the space $\mathbf{1}$ (greatest continuum), it is natural to call the analysis $An(\mathbf{1}\backslash x)$ *(absolute) complementary normal family of* x. The simplest kind of continua are those for which $|An(\mathbf{1}\backslash x)| = 1$.

Definition 2.16 A continuum such that $|An(\mathbf{1}\backslash x)| = 1$ is called a *simplex.*

Intuitively, a simplex is a continuum without "holes". For any simplex x (and only for them) there exists a unique y such that $x \vee y = \mathbf{1}$ and $x \wedge y = \mathbf{0}$. We call this y the *complement* of x and we denote it $-x$. In this case we denote $B(x)$ the boundary of x with respect to $-x$.

3 Towards Dimension: Lines.

The notion of dimension underlies the whole of geometric intuition, but it acquires a clear meaning only when it is defined through rather advanced algebraic or analytic tools. Without these tools we are unable to tell precisely what the difference between, say, an orbit and a surface is.

To go back to the origin of the idea, let us recall that dimension expresses, roughly, the "degrees of freedom" that an agent has within a space. If the space is a continuum and the agent is a point inside it moving across, then the degrees of freedom are the distinct "paths" that the agent can follow starting from any given position. The notion of path is fundamental as it represents the elementary *means of transition* within a continuum. The path itself is also a "thin" continuum with the property that if we remove any of its points except the end-ones, the path *splits*, hence the motion along it is disrupted.

To start an intuitive classification of continua, notice, first, that a single point includes no paths at all, therefore it has zero degrees of freedom. Next, a line is a continuum in which a moving point is allowed to go forth and back only, or at most finitely many directions if we consider lines with branching points. Alternatively, this can be put as follows: If we remove one point from each of the paths available, then the whole continuum splits and the motion along it becomes impossible. Thus we can define lines as those continua that split over *finitely many* points. Then we may go on inductively and define a surface as a continuum that splits over a finite number of lines, and so on.

Since we have defined quite precisely splittings of continua over subcontinua, this may provide a rigorous inductive definition of dimension without the help of any analytic or algebraic tools.

The idea set forth above is indeed very old. In the introduction of Hurewicz et al. (1948) we read the following quotations from Poincaré:

"...If to divide a continuum C, cuts which form one or several continua of one dimension suffice, we shall say that C is a continuum of *two dimensions*; if cuts which form one or several continua of at most two dimensions suffice, we shall say that C is a continuum of *three dimensions*; and so on.

"...This is just the idea given above: to divide space, cuts that are called surfaces are necessary; to divide surfaces, cuts that are called lines are necessary; to divide lines, cuts that are called points are necessary; we can go no further and a point can not be divided, a point not being a continuum. Then lines, which can be divided by cuts which are not continua, will be continua of one dimension; surfaces, which can be divided by continuous cuts of one dimension, will be called continua of two dimensions; and finally space, which can be divided by continuous cuts of two dimensions, will be a continuum of three dimensions" (*A. Poincaré, Revue de Métaphysique et de Morale, 1912, p. 486*).

The fact is that Poincaré's idea is vague. As W. Hurewicz and H. Wallman say "...Poincaré was concerned only with putting forth an intuitive concept of dimension and not an exact mathematical formulation" (Hurewicz et al. (1948), p. 4). For example he does not specify *how many* and *which* points are allowed to be used in order to divide a one-dimensional continuum. In the present treatment, such questions can be given precise answers. The fundamental notions in this approach will be (a) the notion of *path* and (b) the notion of *splitting continua by means of subcontinua*. Now a path can be defined simply as a *join* (i.e., a minimal extension) of two distinct points.

Definition 3.1 Given any two points p, q, any join x of them will be called a *(simple) closed path between* p, q. The connection c such that $x = p \vee c \vee q$ will ce called the corresponding *open path* between them. The points p, q are the *ends* of the paths in question. *Semi-open* paths are defined in the obvious way. Two paths joining p, q and q, r are said to be *consecutive* if the corresponding open paths are disjoint.

Proposition 3.2 *Let x be a path (open or closed) with ends p, q. Then for any point $r \neq p, q$ of x, $An(x \backslash r)$ contains exactly two components which are (open or semi-open) paths again with ends p, r and r, q.*

Proof. It suffices to argue for x closed. Consider the analysis $An(x\backslash r)$. If it contained a single element z, then $x = z \vee r$ and since $r \neq p, q$, z would be an extension of p, q strictly smaller than x. This contradicts the minimality of x as a join. Now let $An(x\backslash r)$ contain more than two components, among them, say z_1, z_2, z_3. Then $p < z_i$ and $q < z_j$ for some $i, j \in \{1, 2, 3\}$. Suppose without loss of generality that $p < z_1$ and $q < z_2$. z_1, z_2 are strongly disjoint and from C_8 it follows that $Ad(r, z_1)$ and $Ad(r, z_2)$. Thus $r \vee z_1 \vee z_2$ exists and contains p, q. This is strictly contained in x, which is a contradiction again. Therefore $|An(x\backslash r)| = 2$.

Let $An(x\backslash r) = \{x_1, x_2\}$. Clearly $p < x_1 \iff q < x_2$ (otherwise the first contradiction would reappear). Thus, if $p < x_1$, $\bar{x}_1 = x_1 \vee r$ is an extension of p, r. We also easily see that it is a minimal one, i.e., a join. Otherwise there would be a strictly smaller extension $y < \bar{x}_1$ of p, r and $y \vee x_2$ would be an extension of p, q properly contained in x. Thus x_1 is a path and similarly for x_2. □

Corollary 3.3 *(a) Let x be a path with ends p, q. For every $n \in \mathbf{N}$, any n points $p_1, \ldots, p_n$ of x, distinct from p, q and from one another, split x into precisely $n + 1$ consecutive subpaths.*

(b) If x is a path and $p, q < x$, then the subpath of x with ends p, q is the only path joining p, q which is contained in x.

Proof. (a) By induction on n. For $n = 1$ the claim follows from 3.2. Suppose it holds for $n - 1$ points and consider n points $p_1, \ldots, p_n$. By the hypothesis, the points $p_1, \ldots, p_{n-1}$ split x into the paths $x_1, \ldots, x_n$. Then clearly p_n belongs to some x_i and is distinct from its ends. By 3.2 again, p_n splits x_i into two paths x_i^0 and x_i^1, and $x_1, \ldots, x_i^0, x_i^1, \ldots, x_n$ is the required division.

(b) This is obvious from the minimality of x as a join. □

Recall from the preceding section that if x, y are strongly disjoint, then any join u of them can be uniquely written as $u = x \vee y \vee z$, where z is the connection of x, y, which is disjoint from x, y and adjacent to both of them.

Proposition 3.4 *Let x, y be strongly disjoint continua and z be a connection of them. Then z is a path.*

Proof. Since $Ad(x, z)$, by C_9, there is a point p such that either $p < x$ and $Ad(p, z)$, or $p < z$ and $Ad(p, x)$. And similarly there is a q such that either $q < y$ and $Ad(q, z)$, or $q < z$ and $Ad(q, y)$. These points p, q can be attached to x, y repectively, so z is obviously the open path joining them. □

We can also prove the following Jordan-type theorem:

Proposition 3.5 (Jordan-Type). *Let x, y be relatively complementary continua x, y (i.e., disjoint and adjacent) with $u = x \vee y$, and let $p < x$ and $q < y$. Then each path c joining p and q inside u, passes through some boundary point of either x or y. Conversely, for every boundary point r of x or y there is path joining p, q inside u and passing through r.*

Proof. Let c be a path inside u, with ends p, q, and consider the analyses $An(c\backslash x)$, $An(c\backslash y)$. It is routine to check that each one of them contains a single element c_1, c_2 respectively (which are subpaths of c), and $c = c_1 \vee c_2$. Then by C_9, c_1, c_2 have some boundary point r lying either in c_1 or in c_2. If $r < c_1$ then clearly $r < x$ and $r \in BP_y(x)$. Otherwise $r \in BP_x(y)$.

For the converse, let $r \in BP_y(x)$. Consider a path s inside x with ends p, r. Then remove r from x and attach it to y. Let $y' = y \vee r$. y' is a continuum disjoint from $x - r$. Take inside y' a path t joining r and q. Then, clearly, $c = s \vee t$ is a path inside u as required. □

Recall that a *simplex* (definition 2.16) is any continuum x which does not split the space **1**, hence it has a complement $-x$. For a simplex x, 3.5 yields:

Corollary 3.6 (Jordan's Theorem) *Let x be any nonvoid simplex and let $p < x$ and $q < -x$. Then any path joining p and q crosses the boundary between x and $-x$.*

Proposition 3.7 *(a) Every nontrivial part of a path is a path.*

(b) If c, d are paths, d is closed and $c < d$ then the end-points of c are contained in d.

(c) If two paths meet, then each meet is either a single point or a path.

Proof. (a) Let c be a path with ends p, q and $x < c$ be nontrivial. We claim that $An(c \backslash x)$ contains at most two components. Indeed, if it contains three, z_1, z_2, z_3, and $p < z_1$, $q < z_2$, then, by C_8, $Ad(x, z_1)$ and $Ad(z, x_2)$, hence $x \vee z_1 \vee z_2$ would be an extension of p, q smaller than c, contrary to the minimality of c as a join of p, q. Thus $An(c \backslash x)$ has either one or two elements, according to whether x contains some of the ends of c or not. From this we easily conclude that x is a path.

(b) Let $c < d$. Taking again $An(d \backslash c)$, we see as in (a) that $An(d \backslash c)$ contains one or two components which are paths or points. Then we easily conclude using C_9 and the closedness of d that the boundary points of c belong to d.

(c) Immediate from (a) (see figure 7(a)). □

Proposition 3.8 *(a) Let c be a path and r be a point not in c. Then r cannot be adjacent to c unless r is an end-point.*

(b) If the paths c, d are disjoint and adjacent, then they are adjacent by the help of at most four points, namely their end-points.

Proof. (a) It suffices to assume c closed, $c = [p, q]$, with end-points p, q, and to show that if $r \not< [p, q]$, r cannot be adjacent to c. Suppose the contrary, and let $c \vee r$ exists, where $c \wedge r = \mathbf{0}$. Then there are closed paths $[p, r]$ and $[r, q]$ inside $c \vee r$ joining p, q with r respectively. Obviously the semi-open paths $[p, r)$ and $(r, q]$ are parts of c, hence subpaths of it. Moreover, $[p, r) \vee (r, q] = c$, therefore $[p, r)$, $(r, q]$ have a common end-point $r' < c$. Since r, r' are distinct points of $c \vee r$, they are joined by a nontrivial path $[r, r']$ inside $c \vee r$. Then the open path (r, r') is nonvoid and $(r, r') < c$. But since c is a closed path, it follows from Prop. 3.7(b) that the end-points of (r, r') must also belong to c, hence $r < c$, which contradicts the hypothesis.

(b) It suffices to show that the paths c, d can be adjacent only via their end-points, as in figure 7(b).

Figure 7

Assume on the contrary that c, d with end-points p, q and r, s respectively, are adjacent via some point $t < c$ distinct from r, s. Then $Ad(t, d)$ and

$t \not< [r, s]$. But this is impossible by (a). $\square$

We proceed now to define one-dimensional continua. We shall use for them the general name *lines*.

Definition 3.9 A *line* is any nontrivial continuum satisfying the following two conditions:

(a) For any $p < x$ there is a finite number of disjoint paths in x having p as an end-point, and

(b) Any two points $p, q < x$ are joined with finitely many simple paths in x.

The number of disjoint paths in x having p as an end-point will be called the *rank of p in x* and will be denoted $rank(p; x)$. If there is a greatest number of paths n, joining any two points of x, this will be called the *genus of x* and will be denoted $gen(x)$. If no such n exists we write $gen(x) = \infty$. That is to say,

$$gen(x) = \begin{cases} max\{\text{number of paths between } p, q : p, q < x\}, \\ \infty \text{ otherwise.} \end{cases}$$

(By the "number of paths" in the above definition we mean the number of "independent paths", i.e., we do not count a path c which is a part of the supremum $\bigvee_i c_i$ of others.) The above is a fairly general notion of line intended to capture a large variety of 1-dimensional continua. Of course one might also consider lines with *infinitely* many branches at a point, or with an infinity of paths joining two points. Such continua tend to fill whole areas, forming thus an intermediate kind of entities, between lines and surfaces, and we will not include them in our consideration. But even the definition 3.9 is liberal enough to leave room for vague entities, lying between lines and surfaces. For example according to 3.9, the so called "comb space"

$$E = \bigcup\{[0, 1] \times \{1/n\} : n \geq 1\} \cup ([0, 1] \times \{0\})$$

(see Munkress (1975)), as well as the "dense comb"

$$D = \bigcup\{[0, 1] \times \{r\} : r \in \mathbf{Q}^+\}$$

(see figure 8 below), should be classified as lines, since they have no loops, i.e., $gen(E) = gen(D) = 1$, and $rank(p; E), rank(p; D) \leq 3$.

Figure 8

The continuum E is more on the side of lines rather, while D is more on the side of surfaces. If we wish to exclude them from the range of normal lines, it suffices to add a third requirement in the above definition saying:

(c) For any path $[p,q]$ of x, the number of branching points of $[p,q]$ is finite.

A line satisfying in addition (c) has indeed a particularly simple form: It is a *graph*, made of points and non-crossing edges (see definition 3.15 and the remark following it).

Our main concern here, however, will be about what we shall call *simple lines*. These correspond to the familiar loopless curves of analysis. First let us introduce the helpful notion of *splitting degree*.

Definition 3.10 Let x be a line. A point p is an *end-point* of x, if $x \vee p$ exists and there is no closed path $[q_1, q_2] < x$ such that $p < (q_1, q_2)$. The end-point may or may not belong to x.

Recall that x *splits over* $X \subseteq \Pi(x)$ if $|An(x \backslash X)| > 1$.

Proposition 3.11 *If p is an end-point of a line x, then x does not split over p.*

Proof. Suppose x splits over $p < x$, and let $y_1, y_2 \in An(x \backslash p)$. Then $Ad(p, y_1)$, $Ad(p, y_2)$ and y_1, y_2 are nontrivial. Hence $y_1 \vee p \vee y_2 = u$ exists. If we choose points $q_1 < y_1$ and $q_2 < y_2$ then by, 3.5, there is a path $[q_1, q_2]$ inside u passing through p. Therefore $p < (q_1, q_2)$, which means that p is not an end-point. □

Definition 3.12 The *splitting degree of x with respect to points (or simply the splitting degree),* denoted by $sd(x)$, is the least number n such that for any n non-end-points $p_1, \ldots, p_n < x$, x splits over $\{p_1, \ldots, p_n\}$. If no such n exists, $sd(x) = \infty$. That is,

$$sd(x) = \begin{cases} min\{n : (\forall p_1 \cdots p_n < x)(p_i \text{ non-end} \Rightarrow |An(x \backslash p_1, \ldots, p_n)| > 1)\}, \\ \infty \text{ otherwise.} \end{cases}$$

The vast majority of continua that we currently call lines have a finite splitting degree. Roughly, the latter counts the number of loops of the line (or self-crosses). For instance a loopless curve x has $sd(x) = 1$, i.e., every point of x splits x, unless it is an end-point (see figure 9(a)). To split a line containing a single loop we need at most two points (figure 9(b)), while to tear apart a line forming 4 loops we neeed at most 5 points (figure 9(c)). ("At most 5" means that *any 5 points* split the line, though there may exist less points at special places that also tear it apart.)

Figure 9

The splitting degree is an equivalent description of the genus of a line. Before showing this let us define a special kind of line.

Definition 3.13 A continuum x is a *(simple) circle* if it is the supremum of two paths $[p,q]^1$, $[p,q]^2$ having common end-points and disjoint interiors.

The proof of the following is easy and we leave it to the reader.

Proposition 3.14 *(a) A simple circle has no end-points.*

(b) If c is any circle and $p \neq q < c$, then $An(c \backslash p, q)$ has exactly two components which are open disjoint paths with end-points p, q.

(c) If $p < c$, then $An(c \backslash p)$ has a unique component, such that any two of its points are joined by a unique path.

Because of (c) above, we can think of a circle as a path $[p,p]$, with identical end-points (though this does not make sense according to the original definition of a path as a minimal extension).

Definition 3.15 A *graph* is a line consisting of a (possibly infinite) set of paths or circles $[p_i, p_j]$ (that is, we allow $p_i = p_j$), meeting at most at their ends, i.e., $(p_i, p_j) \wedge (p_k, p_l) = \mathbf{0}$. The ends p_i's are the *nodes* of the graph, while the paths are the *edges.*

Remark. It is worth noticing that if we adopt the definition of a line 3.9 with the additional constraint (c) introduced earlier, then it is not hard to show that *every line is a graph.* Indeed, in this case, for every point p there is a set of points *immediately joined with* p, as a consequence of the finitely many branching points of each path.

Proposition 3.16 *Given any line x, and any finite set of points $\{p_i : i \leq n\} \subseteq \Pi_0(x)$, there is a finite graph in x containing all paths and circles joining any two points p_i, p_j. This is denoted $\langle p_i : i \leq n\rangle^x$.*

Proof. Let

$$y = \bigvee\{c < x : c \text{ is a path joining some pair of } p_i\text{'s}\}.$$

By the condition of finiteness for the number of paths in a line joining two points, the number of paths in y is finite. Considering all the meets of these paths and all the disjoint open subpaths between them, we easily get a graph with edges the aforementioned subpaths and with nodes their crossing points. $\square$

Proposition 3.17 *(a) Suppose any two points of the line x are joined with at most n distinct paths. Then $sd(x) \leq n = gen(x)$.*

(b) Conversely, if $sd(x) = n$, then any $p, q < x$ are connected by at most n paths. That is to say, $gen(x) \leq sd(x)$.

(c) Therefore $gen(x) = sd(x)$

Proof. (a) Suppose the hypothesis $n = gen(x)$ holds and let $p_1, \ldots, p_n$ be n non-end-points of x. It suffices to show that $\{p_1, \ldots, p_n\}$ splits x. Consider the subgraph of x, $y = \langle p_1, \ldots, p_n\rangle^x$, formed by the totality of paths in x joining any two of the p_i's. Choose two points q, r distinct from all p_i's and lying in two distinct paths of y. Then any path between q, r in y contains some p_i. Let P be the set of paths connecting q, r in x. Clearly $|P| \leq n$.

Case 1. Suppose that each path $c \in P$ contains some point p_i. By removing all p_i's, x splits into at least two components containing p, q. Indeed, if x would not split there would be, according to 3.5, a path in x joining them. But every such path has been split, by 3.2, after removing a point p_i from each one of them.

Case 2. Suppose case 1 fails, i.e., some path in P contains none of the p_i's. Then we have at most $n-1$ paths in the graph y joining q, r, and n points on them. We show the following:

Claim. If there are k paths between q and r in y and $n > k$ points p_i on them, then for some path c and some pair p_i, p_j on it, the open path (p_i, p_j) on c is disjoint from all the other paths of y.

If this happens, then clearly, x splits over p_i, p_j, because the open path (p_i, p_j) is a component of $An(x \backslash p_i, p_j)$.

Proof of the claim. By induction on the number of paths k. For $k = 1$ and $n > 1$ the claim is obvious. Suppose it holds for k paths and any $n > k$, and assume we have $k+1$ paths between q, r and $n > k+1$ points on them. Clearly, there are two distinct points p_i, p_j on some path c. If the open path (p_i, p_j) meets no other path of y we are done. Suppose there is a path c' meeting (p_i, p_j). Clearly we can assume that no other point of the p_k's lies in (p_i, p_j). In view of 3.16, c' is formed in the graph y by a finite number of edges. Remove some suitable edge from this path, as well as one of the nodes p_i, p_j, and get the graph y'. This removal diminishes the total number of paths between q, r. Thus we have $k' < k$ paths with $n' > k'$ points on it. By the induction hypothesis, there are two points p_k, p_l such that (p_k, p_l) crosses no other path in y'. Then obviously (p_k, p_l) doesn't cross any other path of y either, and the claim and (a) are proved.

(b) The converse is easy. Suppose $sd(x) = n$, while $gen(x) = m$, i.e., there are points p, q in x with m independent paths $c_1, \ldots, c_m$ between them. By (a), $m \geq n$. To prove equality, assume $m > n$. Then we can find points $p_i < c_i$, for each $i = 1, \ldots, m$ such that $p_i \not< c_j$ for $i \neq j$. Then, any n of these points obviously do *not* split x, since p, q are still bridged by the remaining $m - n$ paths. This contradicts the fact that $sd(x) = n$.

(c) It follows from (a) and (b) above that $gen(x) = sd(x)$ if one of these is finite. On the othet hand, for every n, $gen(x) > n$ iff $sd(x) > n$, thus $gen(x) = \infty$ iff $sd(x) = \infty$. □

It follows from the preceding result that we can use the genus and the splitting degree of a line interchangeably. If $sd(x) = 1$, then x is a loopless tree-like line, where the rank of a point p determines the number of branches at that point. For instance it follows from 3.3 and 3.14 immediately that

Proposition 3.18 *(a) For any path x, $sd(x) = 1$ and $rank(p; x) \leq 2$.*
(b) For any circle x, $sd(x) = 2$ and $rank(p; x) = 1$.

The genus of a line and the rank of its points are not quite independent properties. For example, a tree-like line may have points of arbitrarily high rank, though it is always of genus 1. But if the rank is low, so is necessarily the genus. We say that x *branches at* p if $rank(p; x) \geq 3$. If x does not branch at a point, i.e., $rank(p; x) \leq 2$ for every $p < x$, then, intuitively, either x is a circle, or there is a single path between any two points of x (otherwise we should have branching somewhere). This intuitive guess can be proved.

Proposition 3.19 *Let x be a line such that $rank(p; x) \leq 2$. Then either x is a circle, or $gen(x) = 1$.*

Proof. Suppose $rank(p; x) \leq 2$ for every $p < x$ and that x is not a circle. Let also $p < x$ be a non-end-point. Since $gen(x) = sd(x)$, it suffices to show that x splits over p. Since p is not an end-point, there is a path $[q, r] < x$ such that $p < (q, r)$. Suppose x does not split over p, that is, $An(x \backslash p) = \{y\}$, where $y = (q, p) \vee z \vee (p, r)$. Then either z is a path with ends q, r or not. In the first case x is obviously a circle as a supremum of two disjoint paths with common end-points. In the other case z contains q, r without being a path between them. Therefore there is an open path c joining q, r and strictly contained in z. Consider $An(z \backslash c)$ which contains some nontrivial component w such that $q < w$. Let c' be a path inside w with end-point q. Now inside x there are at least three distinct paths in x with end-point q, namely (q, p), c and c'. This contradicts the hypothesis. $\square$.

As follows from the preceding result, if $rank(p; x) \leq 2$ for every $p < x$, the line takes the simplest possible form.

Definition 3.20 A line is said to be *simple* if $rank(p; x) \leq 2$ for every point $p < x$. If x is simple and not a circle, it will be said to be a *curve.*

Notice that the notion of a curve is only slightly more general than that of a path. Obviously every path is a curve as follows from corollary 3.3(b).

The only difference is that a path is determined by end-points, whereas a curve need not have end-points. It may be "open" - for instance in the case (if any) that it happens to be a maximal one (as a line).

Since for a curve x $sd(x) = 1$, if $p, q < x$, there exists a unique closed path in x joining p, q, and this is denoted $[p, q]^x$. Besides we have the corresponding open and semi-open paths $(p, q)^x$, $[p, q)^x$, $(p, q]^x$. Notice that the order of p, q in this notation is immaterial, that is $[p, q]^x$ and $[q, p]^x$ mean the same thing.

Proposition 3.21 *(a) Every nontrivial part y of a curve x is a curve, and for any $p, q < y$, $[p, q]^y = [p, q]^x$.*

(b) If two lines x, y meet, then each of their meets is either a point or a common subline.

(c) If c is a circle and $p < c$, then $An(c\backslash p)$ contains just one curve.

Proof. (a) Let x be a curve and $y \leq x$ be nontrivial. Then for any $p < y$, clearly, $rank(p; y) \leq rank(p; x) \leq 2$, hence $rank(p; y) \leq 2$. In order for y to be a curve it suffices that $sd(y) = 1$. But as easily checked, $y < x$ implies $sd(y) \leq sd(x)$. So, since $sd(x) = 1$, $sd(y) = 1$ too. Now for any $p, q < y$, there is a single path in y and a single path in x joining p, q, whence $[p, q]^y = [p, q]^x$.

(b) Immediate from (a).

(c) It follows from 3.14. □

A simple line x is said to be *maximal* if there is no simple line properly extending x. Maximality can be also relativized to any particular continuum that contains x.

Proposition 3.22 *(a) Every circle is a maximal simple line.*

(b) If $(x_i), i \in \mathbf{N}$, is a chain of simple lines, the $\bigvee_i x_i$ is a simple line.

(c) Every simple line on a continuum y can be extended to a maximal one on y.

Proof. (a) Let x be a circle and let y be a line such that $x < y$. It suffices to show that y is not simple. By an argument similar to 3.8 (a), we can show that if $p \not< x$, then $\neg Ad(x, p)$. Therefore there is a nontrivial component $z \in An(y\backslash x)$. Then $Ad(z, x)$ by C_8, and, by C_9, either $Ad(p, x)$ for some $p < z$, or $Ad(p, z)$ for some $p < x$. The former is impossible as we said previously (by the argument of 3.8(a)). Thus $Ad(p,z)$ for a $p < x$. Since p is not an end-point of x (in fact x has no end-points), there is a path $(q,r) < x$ containing p. Choose also a point s in z. Then in y the paths (p,q), (p,r), (p,s) are disjoint and start at p. Hence $rank(p;y) \geq 3$. This shows that y is not a simple line.

(b) Let $x = \bigvee_i x_i$. Since $x_i < x_j$ for $i < j$, the lines x_i are not maximal, hence they are curves by (a), and for any $p < x_i$, obviously $rank(p;x_i) = rank(p;x) \leq 2$. This proves the claim.

(c) Immediate from (b) and Zorn's Lemma. $\Box$

Proposition 3.23 *If x is a non-maximal curve, then there is at least one end-point for x.*

Proof. Suppose x is not maximal and let y be a simple line properly extending it. We can easily see that $An(y \backslash x)$ cannot contain more than two components since they must be disjoint subcurves of y. Let $z \in An(y \backslash x)$. Then either for some $p < x$ $Ad(p,z)$, or for some $q < z$ $Ad(q,x)$ (by C_8, C_9). In both cases it is not hard to see that p or q is an end-point of x. $\Box$.

The converse of 3.23 does not hold necessarily. There may exist maximal curves with end-points belonging to them. Suppose for example that $\mathbf{1}$ is the 2-dimensional ordinary plane and consider on it the double spiral of figure 10 converging to the points $P(0,0)$ and $Q(2,0)$ (the points P, Q included). Obviously the spiral is a maximal curve with two end-points.

Figure 10

4 Continuous Transformations.

A function $f : X \to Y$, $X, Y \subseteq C$, is said to be *continuous*, if it preserves all joins and meets whenever they exist, i.e., if $\bigvee A$ (resp. $\bigwedge A$) exists, then $\bigvee f''A$ (resp. $\bigwedge f''A$) exists too and $f(\bigvee A) = \bigvee f''A$ (resp. $f(\bigwedge A) = \bigwedge f''A$). Clearly, every continuous function is *monotonic*, i.e.,

$$x \leq y \Rightarrow f(x) \leq f(y),$$

for all $x, y \in X$.

We intend to consider transformations of continua under continuous mappings. Now if we restrict ourselves to the above condition only, then every continuum x can be mapped continuously to any other y, provided we put $f(p) = y$, $\forall p \in X$. To avoid such a triviality, it is reasonable to define that *f continuously transforms x onto y* if

(a) f is continuous, and

(b) f maps $\Pi(x)$ onto $\Pi(y)$.

Such an f is called a *continuous transformation* (c.t.). f is said to be *nondegenerate* if the range of f is not a point. It follows immediately from the definitions that

Proposition 4.1 *Let f be a c.t. Then*

(a) $Ad(x, y) \Rightarrow Ad(f(x), f(y))$, and

(b) f sends points to points.

Proposition 4.2 *Every nondegenerate c.t. sends paths to paths and simple lines to simple lines. However it may transform a curve to a circle and vice-versa.*

Proof. Let $[p, q]^x$ be a path on x. We show that $f([p, q]^x) = y$ is a path connecting $f(p), f(q)$. Indeed, clearly, y is an extension of the points $f(p), f(q)$. If it were not a minimal extension, then there would be a $z < y$ containing $f(p), f(q)$. By the definition of c.t., there would be a $u < [p, q]^x$ containing p, q such that $f(u) = z$. This contradicts the minimality of the join $[p, q]^x$.

Now it is an easy task to show that a c.t. does not raise the rank of a point, i.e., for any x and any $p < x$,

$$rank(f(p); f(x)) \leq rank(p; x).$$

From this we immediately infer that if x is a simple line, so is $f(x)$. Concerning the last claim we can easily imagine a path being continuously transformed to a circle by identifying its ends. Similarly, a circle can be transformed to a path by cutting it into two disjoint co-ended paths and shrinking them to a single one. However, a precise proof of the existence of such transformations will be given in 4.6 below. □

Disjoint paths or curves with common end-points are said to be *consecutive*. By definition 3.20 and proposition 3.21, any n distinct points of a curve, none of which is an end-point, split the curve into $n+1$ segments $z_i, i \leq n+1$, such that z_i and z_{i-1} are consecutive. This way a notion of consecutivity is assigned to the splitting points themselves, hence a linear ordering $\prec$ of the n points is induced in the obvious way. This ordering is essentially unique if we do not distinguish between $\prec$ and its inverse $\succ$.

Let x be a curve and let p, q, r be three points of x. A notion of *betweenness* for p, q, r with respect to x is defined as follows: q is said to be *between p and r on x*, in symbols $B_x(p, q, r)$, if $q < (p, r)^x$.

Proposition 4.3 *Let x be a curve and let three distinct points be on x. Then one and only one of them lies between the others.*

Proof. We can assume without loss of generality that none of the p, q, r is an end-point. By 3.3, x is divided into four segments z_1, z_2, z_3, z_4 such that z_i, z_{i+1} are consecutive for $i \leq 3$, and they induce by their ordering, an ordering $\prec$ of p, q, r. Suppose $q \prec r \prec p$. Then it is easy to verify that $B_x(q, r, p)$. If we consider the inverse $\succ$ of $\prec$, the relation of betweenness is preserved. The uniqueness of this ordering implies the uniqueness of the point lying between the other two. $\Box$

Given any curve x, it follows from 1.4 that x contains an infinity of points. From our basic countability hypothesis, the points can be written $q_0, q_1, \ldots$. If x contains end-points, we let q_0, q_1 be these ends. Assigning a conventional ordering $\prec$ to q_0, q_1, say $q_0 \prec q_1$, we can extend $\prec$ to a linear ordering on the whole set $\Pi(x)$ inductively as follows: Suppose $\prec$ is already defined on $\{q_0, \ldots, q_n\}$. These points split the line into $n + 2$ (or $n + 1$ if q_0 is an end-point) consecutive paths, according to 3.3, whose ends are ordered with respect to $\prec$ as follows:

$$q_{i_0} \prec q_{i_1} \prec \cdots \prec q_{i_n},$$

where the sequence $(i_k), k = 0, \ldots, n$, is a permutation of $0, \ldots, n$. Given q_{n+1}, we put $q_{n+1} \prec q_{i_k}$, for $k \leq n$, if q_{n+1} lies in some path "to the right" of q_{i_k}, and $q_{i_k} \prec q_{n+1}$ otherwise. Since there are no adjacent points, the above ordering will be necessarily *dense*, thus we have shown the following:

Proposition 4.4 *The points of every curve admit a unique (up to inversion) dense countable linear ordering with or without end-points.*

Let us denote by $\prec^x$ this unique linear ordering of the points of the curve x. By the preceding result, for any curve x, the set $(\Pi(x), \prec^x)$ is order-isomorphic either to an open interval, or to a closed interval, or to a semi-open interval of the set of rationals. Accordingly, x can be written either as $x = [p, q]$, or $x = \bigcup_n [p_n, q_n]$, or $x = \bigcup_n [p, q_n]$, or $x = \bigcup_n [p_n, q]$, where $(p_n), (q_n), n \in \mathbf{N}$, are sequences of points of x such that for all $n \in \mathbf{N}$,

$$p_{n+1} \prec^x p_n \text{ and } q_n \prec^x q_{n+1}.$$

Given the curves x, y, a mapping $f : \Pi_0(x) \to \Pi_0(y)$, onto, which preserves the orderings $\prec^x$ and $\prec^y$, is said to be a *similarity* between x and y.

Proposition 4.5 *Given a similarity f between x, y, for every $u \leq x$, the supremum $\bigvee f''\Pi_0(u)$ exists and is a subline of y. Moreover, if we put*

$$\bar{f}(u) = \bigvee f''\Pi_0(u),$$

then $\bar{f}$ is a c.t. of x onto y.

Proof. Let $u \leq x$ be nontrivial. By corollary 3.3, u is a subline of x, hence it has one of the above mentioned forms being either closed or open etc. It suffices to consider the case where u is a closed path $u = [p, q]^x$; the treatment of the other cases is reduced to that. Now let $v = [f(p), f(q)]^y$. Suppose $p \prec^x q$. Then for every point $r < (p, q)^x$, $p \prec^x r \prec^x q$, thus $f(p) \prec^y f(r) \prec^y f(q)$, whence $f''\Pi_0(u) \subseteq \Pi_0(v)$. Moreover it is easy to check that $f''\Pi_0(u) = \Pi_0(v)$. It follows that $\bigvee f''\Pi_0(u) = \bigvee \Pi_0(v) = v$.

Clearly $\bar{f}$ maps $\Pi(x)$ onto $\Pi(y)$. To check continuity, let $Z \subseteq \Pi(x)$ be such that $\bigvee Z = u$ exists and let us verify that $\bigvee \bar{f}''Z = \bar{f}(u)$. By 1.10, it suffices to show that $\Pi_0(\bigvee \bar{f}''Z) = \Pi_0(\bar{f}(u))$. But $\Pi_0(\bar{f}(u)) = f''\Pi_0(u)$, as follows from the definition of $\bar{f}$. Thus it suffices to see that

$$\Pi_0(\bigvee \bar{f}''Z) = f''\Pi_0(u) = \{f(p) : p < u\}.$$

But this is fairly obvious from the fact that $\bigvee Z = u$ and the definition of $\bar{f}$. $\square$

Let us say that two curves have the *same end-conditions* if they both have either two or one or no end-points.

Proposition 4.6 *(a) Any two curves x, y with the same end-conditions are continuously transformed onto one another.*

(b) Similarly for any two circles.

(c) For any circle x and any curve y with at least one end-point, there is a continuous f such that $f(y) = x$. If y has two end-points, also $g(x) = y$ for some continuous g.

(d) If however f is continuous and 1-1, then f maps curves to curves and circles to circles.

Proof. (a) If x, y are both curves with the same end-conditions, then clearly there is a similarity f between $(\Pi_0(x), \prec^x)$ and $(\Pi_0(y), \prec^y)$. By 4.5, this can be extended to a continuous mapping $\bar{f}$ from x to y.

(b) If x, y are two circles, we just cut each one of them into a pair of disjoint paths $[p, q]^1$, $(p, q)^2$ and $[r, s]^1$, $(r, s)^2$ respectively, and then we transform continuously $[p, q]^1$ to $[r, s]^1$ and $(p, q)^2$ to $(r, s)^2$ according to (a).

(c) If x is a circle and y is $[p, q]$ or $[p, q)$, we cut x at the point r and map either $[p, q]$ to $[r, r]$ or $[p, q)$ to $[r, r)$. In the case that $y = [p, q]$, we cut the circle at two points r, s and map the two co-ended paths continuously onto $[p, q]$.

(d) Obvious from the fact that in this case distinct end-points are mapped to distinct end-points. $\square$

5 Hints about Surfaces.

We do not intend to provide here an analytic trearment of surfaces to the extend we did it for lines. We shall rather provide some clue-definitions and hints, along the general guide-lines exposed above.

Lines are to 2-dimensional continua, roughly, what points are to 1 - dimensional ones: They can be used to tear surfaces apart. But the analogy is far from strict. For instance there is a vast variety of lines with distinct behaviors, while all points behave the same. Also a line can be split to an infinity of sublines, whereas this is false for points, etc. Therefore we should not expect the treatment of surfaces to be a *mot à mot* translation of the corresponding treatment of lines.

Another reason for this asymmetry is that there is no clear analog of the elementary notion of "path", which has been the key notion in the study of lines. The 2- dimensional counterpart of path would be "region", but we have no simple definition for this. (Recall that the path was defined as a minimal extension of two points; however a minimal extension of two paths is not a surface area but a line.)

Instead of working by analogy, we can try to define surfaces as the kind of continua properly interpolated between lines and solids. To be specific we shall isolate two principles, a "separation" and a "non-separation" one, which distinguish surfaces from solids on the one hand, and lines on the other, respectively. These principles are formulated in terms of splittings. Nevertheless, since "separation" will be used repeatedly below, we shall isolate it in advance. It amounts to slightly more than splitting.

Definition 5.1 Let $x, x_1, x_2 < y$. We say that x *separates* x_1, x_2 *in* y, if there are $z_1, z_2 \in An(y \backslash x)$ such that $x_1 < z_1$ and $x_2 < z_2$.

Clearly, if x separates some pair of continua in y, then x splits y. But for a given pair $x_1, x_2 < y$, x may split y without separating x_1, x_2. The simplest case is the separation of points. We come now to the principles.

1. Separation Principle (or: Surfaces are no solids). Whenever on a surface x we have a line l and a point p such that $\neg Ad(l, p)$, we should be able to separate them by cutting the surface with a simple line (i.e., a circle or a curve) into two pieces, one containing the line and one containing the point. In our terminology introduced in 5.1, l, p should be *separated* by a simple line. More strictly the following holds:

(Sep) *For every line $l < x$ and every point $p < x$ such that $\neg Ad(l, p)$, there is a simple line $c < x$ separating l and p.*

This principle is supposed to capture the fact that a surface, though richer than a line, is not a solid.

2. Non-Separation Principle (or: Surfaces are no lines). Suppose now we have a path $[p, q]$ on the surface x and suppose we cut x along the *open* path (p, q). What will happen? It is possible for x to split as we can easily see by an example (see figure 11 below), but this splitting does *not* separate p, q, i.e., p, q belong to the same component of $An(x \backslash (p, q))$. If p, q were separated by the extraction of (p, q), that would mean that every transition from p to q across x is made possible only through the path (p, q), or that there is no path between p, q disjoint from (p, q). This is quite an unnatural fact for a surface. Similarly, if p, q are points of a circle c lying on x and we remove the two open paths joining them along c, we cannot expect that p, q will separate, even if x is going to split.

Figure 11

To generalize this fact we simply have to consider any finite number of open paths joining p, q and postulate that p, q are not separated by the extraction of all of them. This principle is formulated as follows:

(Non-Sep) *For every $p, q < x$ and every finite number of open paths $l_1, \ldots, l_n$ of x joining p, q, the latter are not separated after the extraction of $l_1, \ldots, l_n$, i.e., p, q belong to the same component of $An(x \backslash l_1, \ldots, l_n)$.*

If a continuum x violates Non-Sep, then there exist two points $p, q < x$ which communicate via only finitely many paths, hence the piece of x "extending between p and q" is just a line. Therefore we can think of the principle as capturing the fact that no part of the surface degenerates to a line. Thus we propose the following definition of a surface:

Definition 5.2 A *surface* is any nontrivial continuum x satisfying the principles Sep and Non-Sep.

In view of the Jordan-type result 3.5, Sep can also be stated equivalently as follows:

(Sep′) *For any line $l < x$ and any $p < x$ such that $\neg Ad(l, p)$, there is a simple line $c < x$ such that any path joining p with any point q of l meets c.*

It is by no means provable from the axioms introduced so far that surfaces exist. For instance it is perfectly possible for the space **1** to be just a line. But even if our space is *not* a line, this simply means (according to definition 3.9) that either there exists a point p from which infinitely many disjoint paths start, or there are two points p, q joined by infinitely many paths. Neither of these facts imply the existence of a surface. Therefore what we need to postulate first is the following:

C_{10}*: Surfaces exist.*

Let us make sure that:

Proposition 5.3 *No line is a surface.*

Proof. Let x be a line and $p, q < x$. By definition there are finitely many open paths connecting p, q. Therefore, removing them all, p, q are separated in x. This means that x violates Non-Sep. □

Proposition 5.4 *Let x be a surface. For any line $l < x$, every component $y \in An(x \backslash l)$ is again a surface. The same is true for $y \vee l$.*

Proof. Let $y \in An(x \backslash l)$. We have to show that y satisfies Sep and Non-Sep. The last claim follows then easily.

Sep: Let $m, p < y$ be a line and a point respectively such that $\neg Ad(p, m)$. Applying Sep to x we find a simple line $c < x$ separating m, p in x. Let $c' = c \wedge y$. Clearly, c' is a simple line in y and it suffices to verify that c' separates m, p in y. Indeed, assume the contary. This means that m, p are connected in y by a path that omits c'. But then this path omits also c in x, which contradicts the fact that c separates m, p in x.

Non-Sep: Let $p, q < y$ and let $m_i, i \leq k$, be k open paths in y joining p, q. Suppose p, q separate in y after removing the paths m_i. This means that every path in y joining p, q crosses some m_i. All the remaining paths joining p, q in x obviously cross l. Call X the set of all paths which join p, q and cross l, and let $X = \{c_n : n \in \mathbf{N}\}$ be an enumeration of X. For each n, it is easy to see that there is a maximal subpath $d_n < c_n \wedge y$, such that $p < d_n$ and $d_n \wedge l = \mathbf{0}$, as well as a maximal subpath $e_n < c_n \wedge y$ such that $q < e_n$ and $e_n \wedge l = \mathbf{0}$. Also, d_n, e_n have end-points p_n, q_n on l. If the set $\{d_n, e_n : n \in \mathbf{N}\}$ is infinite, we can easily choose (using choice) suitable points $r_n < y, n \in \mathbf{N}$, either on (p, p_n) or on (q, q_n), and find via them an infinity of distinct paths $(p, r_n] \vee [r_n, q)$ inside y. This however would contradict the hypothesis that y contains only the paths m_i joining p, q. Therefore the set $\{d_n, e_n : n \in \mathbf{N}\}$ is finite, and so is also the set $\{p_n, q_n : n \in \mathbf{N}\}$ of the points where the paths meet l. Each such pair of points p_i, p_j determines a path $[p_i, q_j]$ on l as well as a path $o_{ij} = (p, p_i) \vee [p_i, q_j] \vee (q_j, q)$ joining p, q along some part of l. If we now remove m_i, as well as the finitely many open paths o_{ij}, from x it is clear that p, q will be separated in x. This contradicts the hypothesis about x and proves the claim. □

Proposition 5.5 *(a) Any two points of a surface are joined by infinitely many disjoint paths.*

(b) From any two points on a surface there passes a circle.

Proof. Both claims are immediate consequences of Non-Sep. □

If l is a line on a surface x and x does not split over l, we say that l is a *boundary line* of x. For instance, if x splits over l, then for each $y \in An(x \backslash l)$, $y \vee l$ is a surface having l as a boundary line. Suppose c is a circle on x and x splits into two parts x_1, x_2 over c. Then, intuitively, two, one or none of the x_1, x_2 are "bounded" surfaces (as in (a), (b) and (c), respectively of figure 12). We would like to call a bounded such piece a "region".

Figure 12

But with the means available so far, capturing boundedness seems infeasible. This is because we cannot distinguish among the decompositions (a), (b), (c) above. This inability is closely connected with the inability to tell what "inside" and "outside" of a circle means. For instance this is impossible to tell in the cases (a) and (c) of the figure 12, even when we are given the usual metric of the space. The notions of "inside" and "outside" are attributed only to splittings in which the two parts x_1, x_2, first, are *measured* in some way and, second, their measures are found to be "very unequal", e.g. one having finite and the other infinite measure. Then the "inside" part is just the one with the small measure. Without such an *asymmetry* of the parts, the inside-outside distinction loses its sense.

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