Sets with dependent elements: A formalization of Castoriadis' notion of magma

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Abstract

We present a formalization of collections that Cornelius Castoriadis calls "magmas", especially the property which mainly characterizes them and distinguishes them from the usual cantorian sets. It is the property of their elements to *depend* on other elements, either in a one-way or a two-way manner, so that one cannot occur in a collection without the occurrence of those dependent on it. Such a dependence relation on a set A of atoms (or urelements) can be naturally represented by a pre-order relation $\preccurlyeq$ of A with the extra condition that it contains no minimal elements. Then, working in a mild strengthening of the theory ZFA, where A is an infinite set of atoms equipped with a primitive pre-ordering $\preccurlyeq$, the class of magmas over A is represented by the class $LO(A,\preccurlyeq)$ of nonempty open subsets of A with respect to the lower topology of $\langle A,\preccurlyeq\rangle$. The non-minimality condition for $\preccurlyeq$ implies that all sets of $LO(A,\preccurlyeq)$ are infinite and none of them is $\subseteq$-minimal. Next the pre-ordering $\preccurlyeq$ is shifted (by a kind of simulation) to a pre-ordering $\preccurlyeq^+$ on $\mathcal{P}(A)$, which turns out to satisfy the same non-minimality condition as well, and which, happily, when restricted to $LO(A,\preccurlyeq)$ coincides with $\subseteq$. This allows us to define a hierarchy $M_\alpha(A)$, along all ordinals $\alpha \geq 1$, the"magmatic hierarchy", such that $M_1(A) = LO(A,\preccurlyeq)$, $M_{\alpha+1}(A) = LO(M_\alpha(A),\subseteq)$, and $M_\alpha(A) = \bigcup_{\beta<\alpha} M_\beta(A)$, for a limit ordinal α. For every $\alpha \geq 1$, $M_\alpha(A) \subseteq V_\alpha(A)$, where $V_\alpha(A)$ are the levels of the universe $V(A)$ of ZFA. The class $M(A) = \bigcup_{\alpha\geq 1} M_\alpha(A)$ is the "magmatic universe above A." The axioms of Powerset and Union (the latter in a restricted version) turn out to be true in $\langle M(A),\in\rangle$. Besides it is shown that three of the five principles about magmas that Castoriadis proposed in his writings (mildly modified and adapted to the needs of formalization),

$M2^*$, $M3^*$ and $M5^*$, are true of $M(A)$. A selection of excerpts from these writings, in which the concept of magma was first introduced and elaborated, is presented in the Introduction.

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1 Introduction

In his seminal book [2] Cornelius Castoriadis[1] is concerned, among many other things, with the way collections of things are presented to (or created by) the human mind. In particular he aims to make us rethink the "self-evident" belief of western rationalistic tradition that any collection of things needs to be a *cantorian* collection, that is a totality of distinct, definite and ontologically independent elements. His favorite (counter) examples are the "totality of meanings" of a natural language, the "totality of one's memories", and the like. The elements of such totalities are not entirely definite, and not fully differentiated and independent from one another, so one could hardly call them "sets" in the ordinary sense of the word, and thus include them in the cantorian universe. If, for instance, a is a particular meaning or memory, one could not fully separate it from related meanings or memories, respectively, in the sense that whenever we think of a as an element of some collection, a inevitably brings to mind other similar meanings or memories as members of the same collection, and as a result a cannot exist in isolation. A typical consequence of this is that the one-element set $\{a\}$ can hardly make sense, as the cantorian tradition requires. Since such collections abound around us, it would be natural to try to accommodate and comprehend them in the framework of a theory that differs from that of Cantor.

C. Castoriadis seems to be the first thinker who felt the need for some kind of theory that would embrace, even without full rigor, the study of

[1]Cornelius Castoriadis (1922-1997) was a prominent social and political thinker of 20th century. Born in Greece he spent most of his life in Paris. From 1979 until his death he was Director of Studies at the École des Hautes Études en Sciences Sociales (EHESS). His monograph [2] is widely considered his main work. Although a humanitarian philosopher by training, he had got an impressive solid background in science, especially in economics, mathematics and theoretical physics.

such collections. He believes that the specific cantorian tradition, which requires collections to comply with the rules of standard set theory and rejects those that do not as non-existent, is rather accidental and due to the early adoption by the western thought of what he calls "identitary-ensemblistic" logic (roughly, the two-valued classical logic interacting with naive set theory of predicate extensions).

> "For the past 25 centuries, Greco-Western thinking has constituted, developed, amplified and refined itself on the basis of this thesis: being is being something determined (einai ti), speaking is saying something determined (ti legein). And, of course, speaking the truth is determining speaking and what is said by the determinations of being or else determining being by the determinations of speaking, and, finally, observing that both are but one and the same. This evolution, instigated by the requirements of one dimension of speaking and amounting to the domination or the autonomization of this dimension, was neither accidental nor inexorable; it corresponded to the institution by the West of thinking as Reason. I call the logic described above identitary logic and also, aware of the anachronism and the stretching of words involved here, set-theoretical logic, for reasons that will soon be apparent. (...)
>
> The logical rudiments of set-theory are important in this respect for, regardless of what may happen in the future from the perspective of mathematics, they condense, clarify, and exemplify in a pure manner what, all the while, was underlying identitary logic, and what, long before this logic was sketched out, constituted an essential and unexpungible dimension of all activity and all social life. These rudiments, indeed, posit and constitute explicitly both the type of logic, in its greatest generality, required by identitary logic and the relations necessary and almost sufficient for this logic to function unhampered and without limit." ([2], pp. 221, 222, 223)

Castoriadis believes that identitary-ensemblistic logic affects greatly our grasping of the reality through the "creation" of sets out of something preexistent and rather undifferentiated. This undifferentiated reality out of which the identitary-ensemblistic logic generates sets, classes, objects and properties, is, roughly, what he calls *magma*.

> "What we seek to understand is the mode of being of what

gives itself before identitary or ensemblist logic is imposed; what gives itself in this way in this mode of being, we are calling a *magma*. It is obviously not a question of giving a formal definition of it in received language or in any language whatsoever. The following statement, however, may not be unhelpful:

> A magma is that from which one can extract (or in which one can construct) an indefinite number of ensemblist organizations but which can never be reconstituted (ideally) by a (finite or infinite) ensemblist composition of these organizations." ([2], p. 343)

As mentioned above, Castoriadis' favorite examples of magmas are the "multiplicity of meanings/significations" of a natural language and the "multiplicity of one's representations."

> "Let us try then, by means of an accumulation of contradictory metaphors, to give an intuitive description of what we mean by magma (the best intuitive support the reader can present to himself is to think of 'all the significations of the English language' or 'all the representations of his life'). We have to think of a multiplicity which is not one in the received sense of the term but which we mark out as such, and which is not a multiplicity in the sense that we could actually or virtually enumerate what it 'contains' but in which we could mark out in each case terms which are not absolutely jumbled together. (...)
>
> And we have to think of the operations of identitary logic as simultaneous, multiple dissections which transform or actualize these virtual singularities, these components, these terms into distinct and definite elements, solidifying the pre-relation of referral into relation as such, organizing the holding together, the being-in, the being-on, the being-proximate into a system of determined and determining relations (identity, difference, belonging, inclusion), differentiating what they distinguish in this way into 'entities' and 'properties' , using this differentiation to constitute 'sets' and 'classes."' ([2], p. 344)

Objects and properties of the world seem to be the outcome of the ability of our mind for separation, partitioning and individuation. Sets and classes are also products of this very mental mechanism. Castoriadis mentions time and again Cantor's well-known "definition" of set: "A set is a collection into a whole of definite and separate objects of our intuition or our thought.

These objects are called 'elements' of the set." Any specification of a set is clearly an act of separation and individuation. When we say "let X be the set of all x such that...," we focus on a specific part of the reality, we individualize it and cut it off as a separate object by an act of saying, that is, a linguistic construct (a formula). On the other hand, the basic quality of magma, which differentiates it from ordinary sets and classes, is the fact that its "elements" are neither fully determined nor fully distinguishable and separable from one another.

> "As a magma, the significations of a language are not the elements of an ensemble subject to determinacy as their mode and their criterion of being. A signification is indefinitely determinable (and the 'indefinitely' is obviously essential) without thereby being determined. It can always be marked out, provisionally assigned as an identitary element to an identitary relation with another identitary element (this is the case in designation), and as such be 'a something' as the starting point for an open series of successive determinations. These determinations, however, in principle never exhaust it. What is more, they can, and always do, force us to reconsider the initial 'something' and lead us to posit it as 'something else,' overturning by this very fact, or in order to bring it about, the relations by means of which the initial determination had been made." ([2], p. 346)

It is remarkable that a very similar position is expressed by John Searle about mental states and the content of our consciousness in general:

> "One has conscious states such as pains and thoughts only as a part of living a conscious life, and each state has the identity it has only in relation to other such states. My thought, for example, about a ski race I ran long ago, is only that very thought because of its position in a complex network of other thoughts, experiences, and memories. My mental states are internally related to each other in the sense that in order for a mental state to be that state with that character it has to stand in certain relation to the real world." ([5], p. 42)

The above speculations about magma are clearly vague. However in [3] Castoriadis devotes a whole chapter to the subject entitled "The logic of magmas and the problem of autonomy." He starts the chapter with a quotation from a letter of G. Cantor to R. Dedekind: "Every multiplicity is either an inconsistent multiplicity or it is a set." On this Castoriadis comments:

> "To say of a multiplicity that it is inconsistent obviously implies that this multiplicity *is,* it is in a certain fashion that remains to be specified and that Cantor does not specify. Clearly, we are not dealing here with an empty set, which is a set in full right, with its place in set theory.
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> It is toward these inconsistent multiplicities - inconsistent from the standpoint of a logic that claims to be consistent or rigorous - that I turned, starting from the moment, in 1964-1965, when the importance of what I have called the radical imaginary in the human world became apparent to me. Noting that the human psychism cannot be 'explained' by biological factors or considered as a logical automaton of no-matter-what richness and complexity. (...)
>
> After various terminological peregrinations - cluster, conglomerate, and others - for this mode of being, as well as the logico-ontological organization it bears, I have ended up with the term *magma.* I was later to discover that from 1970 on the editions of Nicolas Bourbaki's *Algèbre* utilized the term with an acceptation that bears no relation at all to the one I have tried to give it and that is, of course, strictly ensemblistic-identitary in character. As the term, by its connotations, admirably lends itself to what I want to express, and as, dare I say, its utilization by Bourbaki seems to me both rare and superfluous, I have decided to retain it." ([3], pp. 366-368)

A little later, after recalling the definition of magma, Castoriadis says:

> "I note in passing that Jean-Pierre Dupuy remarked to me that the 'definition' cited above is unsatisfactory, for it would cover just as well what, to avoid Russell's Paradox, has been called in mathematics a 'class.' The objection is formally correct. It does not trouble me much, for I have always thought, and still think, that the 'class,' in this acceptation of the word, is a logical artifact constructed *ad hoc* to get around Russell's Paradox, and that it succeeds in doing so only by means of an infinite regress. Rather than comment on this 'definition,' however, we are going to try here to illuminate other aspects of the idea of magma by exploring the paths (and the impasses) of a more 'formal' language. For this, one must introduce a primitive (indefinable and undecomposable) term/relation: the marking

(repérer) term/relation, whose valence is at once unary and binary. So, let us suppose that the reader unambiguously understands the expressions: 'to mark X;' 'X marks Y;' 'to mark X in Y' (to mark a dog; the collar marks the dog; to mark or locate the dog in the field). In using this term/relation, I 'define' a magma by the following properties:

M1: If M is a magma, one can mark, in M, an indefinite number of ensembles.

M2: If M is a magma, one can mark, in M, magmas other than M.

M3: If M is a magma, M cannot be partitioned into magmas.

M4: If M is a magma, every decomposition of M into ensembles leaves a magma as residue.

M5: What is not a magma is an ensemble or is nothing." ([3], pp. 379-380)

Unfortunately, the meaning of "marking" in M1 and M2 is unclear. However we guess that M1 most likely means: for every magma M there is an indefinite number of sets x such that $x \subseteq M$. While M2 means: for every magma M there is a magma $N \neq M$ such that $N \subseteq M$. If we interpret M1 this way, then we easily see that M1 and M4 are contradictory. This is pointed out by Castoriadis himself ([3], p. 383): given a magma M, let X be the union of all sets contained in M. By M4, $M \backslash X$ is a magma. But then, by M1, there is a set x such that $x \subseteq M \backslash X$. This contradicts the fact that X is the union of all sets contained in M.

I think that the problem with the principles M1 and M4 arises from the fact that they both relate magmas to *sets*, which by definition are collections of a different kind, and in doing so they contradict each other. In contrast M2 and M3 describe how magmas relate to other magmas *alone*, specifically their submagmas. As for M5, I construe it as saying: "What is not a magma is an ensemble and nothing but an ensemble." In other words, it suggests that the classes of magmas and sets exhaust the content of the universe and are complementary. However in the real world, as well as in that of ZFA, which will be used below, there exist objects/atoms that are non-collections. Therefore M5 could more realistically be reformulated as follows: "What is not a magma is a set or an atom."

Castoriadis thinks that M3 is the most crucial of the above properties of magmas. He says:

> "The third property (M3) is undoubtedly the most decisive. It expresses the impossibility of applying here the schema/operator

of separation - and, above all, its irrelevance in this domain. In the magma of my representations, I cannot rigorously separate out those that 'refer to my family' from the others. (In other words, in the representations that at first sight 'do not refer to my family,' there always originates at least one associative chain that, itself, leads to 'my family.' This amounts to saying that a representation is not a 'distinct and well-defined being,' but is everything that it brings along with it.) In the significations conveyed by contemporary English, I cannot rigorously separate out those that (not in my representation, but in this tongue [langue] itself) refer in any way at all to mathematics from the others." ([3], p. 381)

Let me sum up. The description of magma by Castoriadis through the principles M1-M5 is not sufficiently clear and, most important, two of these principles, namely M1 and M4, are straightforwardly contradictory. Nevertheless there is an aspect of the idea that deserves further elaboration. This is the real fact that we often come across collections, let us call them "unusual", that differ considerably from the "usual" ones. The latter include all finite collections of things we deal with in our everyday life, like the collection of students in a class, the collection of books at our bookcase, etc, but also infinite collections of abstract mathematical entities, e.g. the collections of integers, rational numbers, real numbers, etc. The main characteristic of all usual collections is that each one of their elements occurs in absolute (existential) independence from all the rest, so it can be separated and removed from a collection, or added to that, without affecting the occurrence of the others. In contrast, the members of unusual collections come up not in full separation from one another, but rather as unbreakable *chains* or *bunches* of *dependent* objects, so that one cannot add or subtract one element without adding or subtracting the elements depending on it. This dependence is vividly described in the last excerpt above through the example of things "that refer to my family", on the one hand, and "those that do not", on the other, and our inability to completely separate one kind from the other.

It is exactly this type of *collections with dependent elements* that we are going to consider and formalize in this paper. As for M1-M5, I propose that, firstly, M1 and M4 be left out of consideration because of their inconsistency, and, secondly, the rest principles M2, M3, M5 be slightly reformulated as follows:

*M2**: If M is a magma, there is a magma $N \neq M$ such that $N \subseteq M$.

*M3**: If M is a magma, there is no partition of M into submagmas M_1, M_2.

*M5**: What is not a magma is a set or an atom.

It turns out that the formalization of magmas that we develop below succeeds in capturing M2*, M3* and M5*.

2 Formalizing dependence of objects in an extension of ZFA

In all Castoriadis's examples of magmas (the collection of meanings of a natural language, the collection of one's mental representations, etc), everyone of their members shows a clear "ontological" dependence on other members: everyone of them cannot occur in one's mind without the simultaneous occurrence of others. We find this notion of dependence interesting and challenging, and it is our purpose in this paper to try to capture it mathematically. The idea, very roughly, is to work in the theory ZFA, which consists of the axioms of ZF plus a set A of non-sets, called "urelements" or "atoms", which throughout will be referred to for simplicity just as *atoms* (see [4, p. 250] for the formal treatment of this theory). Magmas are going to be represented by certain subsets of A after the latter will be endowed with a dependence relation.

Now a first basic question before going further is whether magmas should be treated as finite or infinite collections. As we saw in the Introduction, the examples of magmas given by Castoriadis - the collection of meanings of a natural language, the totality of memories of a single person and so on - are all collections that contain elements of some kind of *human resources*, and as such cannot be infinite, at least in the strict sense of infinity as we use it in mathematics. That is, we cannot prove e.g. that if A is the collection of memories, then there is an 1-1 function $f : \mathbb{N} \to A$. But it isn't finite either in the strict sense, because there is no $n \in \mathbb{N}$ for which we can prove that $|A| = n$. The best we can say is that A is *potentially infinite*, i.e., a non-infinite collection and yet without specific finite cardinality, unfinished and uncompleted, for which we can potentially discover new elements. Many physical collections of the world are of this kind: the collections of animals, of plants, of species and so on. Now if for some reason, we want to treat such collections with mathematical means, we cannot treat them as finite, because in such a case we should assign them a specific cardinality n. But if

$|A| = n$, it would mean that A is a *completed* totality, without the possibility to reveal new elements, and this contradicts the previously described nature of A. Thus necessarily we must treat them as infinite, since in mathematics there is no intermediate state between finite and infinite.[2]

So we start with an infinite set A of atoms (in ZFA) and equip it with a binary relation which can adequately capture the most basic properties of dependence, which are just two: reflexivity (every object a depends on a) and transitivity (if a depends on b and b depends on c, then a depends on c). A binary relation with these two properties is a very familiar mathematical object, is called a *pre-order relation*, or just a *pre-ordering*, and is usually denoted $\preccurlyeq$.[3] So we assume that A comes up with such a relation $\preccurlyeq$. The intended meaning of $a \preccurlyeq b$ is: "a depends on b", or "b points to a," or "b reminds a," all of which practically mean that every occurrence of b is followed by the occurrence of a.

It follows by the preceding discussion, that given the relation $\preccurlyeq$ on A, "magmas over A" (with respect to $\preccurlyeq$) are just the collections $x \subseteq A$ that are *downward closed under* $\preccurlyeq$, i.e., have the property:

$$(\forall a, b)(a \preccurlyeq b \wedge b \in x \Rightarrow a \in x). \tag{1}$$

Definition 2.1 Given a pre-ordered set $\langle A, \preccurlyeq \rangle$, the class $m(A)$ of *magmas over* A (with respect to $\preccurlyeq$) consists of the nonempty subsets of A having property (1), namely,

$$m(A) = \{x \subseteq A : x \neq \emptyset \wedge (\forall a, b \in A)(a \in x \wedge b \preccurlyeq a \to b \in x)\}.$$

In my view, the following three conditions should be met in the treatment of magmas. Firstly, magmas must coexist together with ordinary sets, as well as with atoms, in a "mixed" universe. (This after all was explicitly stated as property M5* above.) Secondly, the magmas of the bottom level of the

[2]Notice also that treating physical collections in general as finite sets may lead to real paradoxes. To give an example (not of a magma but of a potentially infinite physical collection), let X be the collection of human beings since their appearance on earth, and let $<$ denote the genealogical order relation on X, where $a < b$ means "a is the father of the father of the father.... of the father of b", where we take finitely many applications of the operation "father of". If X were typically finite, then every descending chain $\cdots < a_3 < a_2 < a_1$ should have a first element a_n. But then a_n either should be fatherless, or should have a father who is not human. Clearly both conclusions are absurd. So necessarily X has to be treated as infinite.

[3]Of course a more general relation of dependence is sensible, where an object a depends not on a single element b, but rather on a group of elements $\{b_1, \ldots, b_n\}$, but such a relation does not fit to our context.

universe must consist exclusively of *atoms*, not sets. (Nevertheless, magmas of higher ranks can be constructed inductively, having as elements magmas of lower ranks.) Thirdly, the dependence relation $\preccurlyeq$ of atoms should be a *primitive* one, that is, not definable from or reducible to other relations of the ground theory.[4]

Below we shall treat magmas along the lines of the above three conditions. So starting with an *infinite* set of atoms A and a primitive pre-ordering $\preccurlyeq$ on it, we shall build the *class of magmas* $M(A)$ *above* $\langle A, \preccurlyeq\rangle$, as a subclass of the universe $V(A)$ of the theory ZF with atoms, ZFA.

Recall that the language of ZFA is $L = \{\in, S(\cdot), A(\cdot)\}$, where $S(\cdot)$ and $A(\cdot)$ are the unary predicates (sorts) for sets and atoms, respectively.

Definition 2.2 Given the set A of atoms, *the universe* $V(A)$ *of* ZFA is the class $V(A) = \bigcup_{\alpha\in Ord} V_\alpha(A)$, where:
$V_0(A) = A$,
$V_{\alpha+1}(A) = V_\alpha(A) \cup \mathcal{P}(V_\alpha(A))$, and
$V_\alpha(A) = \bigcup_{\beta<\alpha} V_\beta(A)$, for limit α.

Here, in addition, we need A to carry a pre-ordering, so we introduce $\preccurlyeq$ as a new primitive binary relation symbol, besides $\in$, and extend L to $L(\preccurlyeq) = L \cup \{\preccurlyeq\}$. As usual, in order to avoid using the sorts $S(\cdot)$ and $A(\cdot)$, we use variables $a, b, c, \ldots$ for atoms, variables $x, y, z, \ldots$ for sets, and also variables u, v, $w, \ldots$ that range over both sets and atoms. The atomic formulas of $L(\preccurlyeq)$ are those of L, plus the formulas $a \preccurlyeq b$. For every $a \in A$, let $pr(a)$ denote the set of its *predecessors,*

$$pr(a) = \{b \in A : b \preccurlyeq a\}.$$

In view of the preceding discussion, $pr(a)$ represents the set of elements of A which *depend* on a. It is well-known that over every pre-ordered set $\langle A, \preccurlyeq\rangle$, the sets $pr(a)$, $a \in A$, form the basis of one of the natural topologies induced by $\preccurlyeq$, usually called "lower topology" for obvious reasons (the corresponding "upper topology" has as basis the sets $suc(a) = \{b : a \preccurlyeq b\}$). A set $x \subseteq A$ is said to be *open* w.r.t. the lower topology, or *lower open*, if for every $a \in x$, $pr(a) \subseteq x$. (It is easy to check that a $x \subseteq A$ is a lower open set if and only if $A \backslash x$ is an upper open set.) By the transitivity of $\preccurlyeq$, for every $b \in pr(a)$, $pr(b) \subseteq pr(a)$, so every $pr(a)$ is open, and we refer to them as "basic open"

[4]The third condition, concerning non-definability of $\preccurlyeq$, could be possibly skipped if we assumed that $\preccurlyeq$ can be constructed on A by *choice*, but in this case we should work in ZFCA rather than ZFA. However we think that working in ZFA, even mildly augmented, is a simpler and more natural option.

sets, or b.o. sets for short. Let $LO(A, \preccurlyeq)$ denote the set of all *nonempty* lower open subsets of A. It is obvious that for every family $(x_i)_{i\in I}$ of elements of $LO(A, \preccurlyeq)$, $\cup_{i\in I} x_i$ belongs to $LO(A, \preccurlyeq)$, and so does also $\cap_{i\in I} x_i$ whenever it is nonempty. In particular $A \in LO(A, \preccurlyeq)$. A moment's inspection shows that the sets in $LO(A, \preccurlyeq)$ are exactly those of the class $m(A)$ defined in Definition 2.1, that is,

$$m(A) = LO(A, \preccurlyeq) = \{x \subseteq A : x \neq \emptyset \wedge (\forall a \in x)(pr(a) \subseteq x)\}.$$

Note that there can be $a \neq b$ in A such that $a \preccurlyeq b$ and $b \preccurlyeq a$. This is in accordance with the intuitive meaning of $\preccurlyeq$, and expresses the fact that a, b are *mutually dependent*. We write then $a \sim b$, $\sim$ is an equivalence relation on A and we denote by $[a]_\sim$, or just $[a]$, the equivalence class of a. Obviously, $[a] = [b]$ if and only if $pr(a) = pr(b)$.

A set $x \in LO(A, \preccurlyeq)$ is said to be $\subseteq$-*minimal* or just *minimal*, if there is no $y \in LO(A, \preccurlyeq)$ such that $y \subsetneq x$.

Lemma 2.3 *Let $x \in LO(A, \preccurlyeq)$. The following are equivalent.*
(i) x is minimal.
(ii) $(\forall a \in x)(x = pr(a))$.
(iii) $(\forall a \in x)(x = [a])$.

Proof. (i) $\Rightarrow$ (ii) Let x be open and minimal and assume that for some $a \in x$, $x \neq pr(a)$. Since, by openness $pr(a) \subseteq x$, it means that $pr(a) \subsetneq x$, which contradicts the minimality of x.

(ii) $\Rightarrow$ (iii). Assume (ii) is true, and let $a \in x$. Since $[a] \subseteq pr(a) \subseteq x$, it is always true that $[a] \subseteq x$. For the other inclusion, pick $a \in x$. If $x = \{a\}$, obviously $pr(a) = \{a\} = [a] = x$. If there is $b \in x$ such that $b \neq a$, by (ii) $x = pr(b) = pr(a)$, so $x = [a] = [b]$.

(iii) $\Rightarrow$ (i). We show the contrapositive. Suppose (i) is false, that is there is an open y such that $y \subsetneq x$. Picking $a \in y$, we have $[a] \subseteq pr(a) \subseteq y \subsetneq x$, therefore $[a] \neq x$, so (iii) is false. $\dashv$

Now for an arbitrary pre-ordering $\preccurlyeq$ it is clear that we may have $pr(a) = \{a\}$, for some a. In that case $\{a\}$ should be included in the class of magmas over A, a fact which is counter intuitive according to our previous discussion. But even if $pr(a) \neq \{a\}$ but $pr(a)$ is *minimal*, and hence $pr(a) = [a]$, according to Lemma 2.3, we shall have the same problem later, with respect to the pre-ordering $\preccurlyeq^+$, which will be defined in the next section on $LO(A, \preccurlyeq)$. Namely, in that case the singleton $\{pr(a)\}$ will be open in the lower topology induced by $\preccurlyeq^+$. So it is necessary to avoid the existence of minimal open sets not only in $LO(A,\preccurlyeq)$, but also in all topologies induced by the shiftings of $\preccurlyeq$ to the higher levels of the magmatic hierarchy.

Lemma 2.4 *If $pr(a)$ is finite, then it contains minimal open subsets.*

Proof. Let $pr(a)=\{b_1,\ldots,b_n\}$ for some $n\geq 1$. For each $i=1,\ldots,n$, $pr(b_i)\subseteq pr(a)$ is finite, so we can pick one $pr(b_k)$ with the *least number* of elements. Then $pr(b_k)$ is minimal. For otherwise $pr(b_k)$ should contain a b_j such that $pr(b_j)\subsetneq pr(b_k)$. But then $|pr(b_j)|<|pr(b_k)|$, a contradiction. $\dashv$

It follows from the preceding lemma that in order to avoid existence of minimal open sets, it is necessary to impose a condition to $\preccurlyeq$ which implies that every b.o. set $pr(a)$ is infinite. Given a pre-ordering $\preccurlyeq$ on A, let us define

$$a\prec b \text{ iff } (a\preccurlyeq b \ \wedge\ b\not\preccurlyeq a)\Leftrightarrow(a\preccurlyeq b\ \wedge\ a\not\sim b).$$

A reasonable condition which will guarantee the absence of minimal open sets in $LO(A,\preccurlyeq)$ is the following:

(*) $\qquad (\forall a\in A)(\exists b\in A)(b\prec a).$

Proposition 2.5 *(i) If $\preccurlyeq$ satisfies (*), $LO(A,\preccurlyeq)$ does not contain minimal sets. A fortiori, in view of Lemma 2.4, all $pr(a)$ are infinite, and hence all sets in $LO(A,\preccurlyeq)$ are infinite.*

(ii) The converse is also true. That is, if (*) fails, then $LO(A,\preccurlyeq)$ has minimal open sets.

Proof. (i) Suppose $\preccurlyeq$ satisfies (*). It suffices to show that no b.o. set $pr(a)$ is minimal. Take a set $pr(a)$. By (*) there is b such that $b\prec a$. Then $b\in pr(a)$, so $pr(b)\subseteq pr(a)$, while $a\notin pr(b)$, because $a\not\preccurlyeq b$, so $a\in pr(a)\backslash pr(b)$, and therefore $pr(b)\subsetneq pr(a)$.

(ii) Assume that (*) fails, i.e. there is a such that $(\forall b)(b\not\prec a)$, or

$$(\forall b)(b\preccurlyeq a\rightarrow a\preccurlyeq b).$$

The latter means that $(\forall b)(b\in pr(a)\rightarrow b\in[a])$, or that $pr(a)\subseteq[a]$. Since always $[a]\subseteq pr(a)$, it means that for this specific a, $pr(a)=[a]$. Thus $(\forall c\in pr(a))(pr(a)=[c])$, i.e., condition (iii) of Lemma 2.3 holds, and therefore $pr(a)$ is minimal. $\dashv$

It follows that condition (*) for $\preccurlyeq$ is necessary and sufficient in order for $LO(A,\preccurlyeq)$ not to contain minimal open sets. It is exactly such pre-orderings, satisfying (*), that we are going to deal with below. Moreover this property will be incorporated in the formal system we shall adopt for the treatment of magmas. Namely, the formal system in which we work below differs from ZFA in the following points:

(a) We strengthen the schemes of Separation and Replacement of ZFA so that they hold for the formulas of $L(\preccurlyeq)$ rather than just L. We denote this system $\mathrm{ZFA}_{\preccurlyeq}$. (Notice that without this strengthening, we could not guarantee, for example, that the collections $pr(a) = \{b \in A : b \preccurlyeq a\}$ and $LO(A, \preccurlyeq)$ are sets.)

(b) We add to the axioms of $\mathrm{ZFA}_{\preccurlyeq}$ the following statements about $\preccurlyeq$:

(D_1) $(\forall a)(a \preccurlyeq a)$ (reflexivity).
(D_2) $(\forall a, b, c)(a \preccurlyeq b \wedge b \preccurlyeq c \to a \preccurlyeq c)$ (transitivity).
(D_3) $(\forall a)(\exists b)(b \prec a)$ (no minimal elements).

Two Remarks about axiom D_3. 1) D_3 suggests that *every* element of A depends on other elements, whereas one may object that, at least in *some* magmas, there may exist *independent* elements, i.e., $a \in A$ such that $(\forall b \neq a)(a \not\preccurlyeq b \wedge b \not\preccurlyeq a)$. This is a reasonable objection, provided we agree that such elements are rather exceptions to the rule, and that the subset of A that consists of the *dependent* elements is still infinite. But then it suffices simply to take as set of atoms the set $A' = \{a \in A : (\exists b \neq a)(a \preccurlyeq b \vee b \preccurlyeq a)\}$ instead of A, and work with it as before.

2) By condition (*) and Proposition 2.5, axiom D_3 is responsible for the nonexistence of minimal magmas, and thus, in a sense, the non-trivialization of the magmatic hierarchy. So one might suspect that D_3 is a rather ad hoc axiom. But it is not. Suppose for instance that $\langle A, \preccurlyeq\rangle$ represents the magmatic collection of meanings of a natural language and assume that D_3 is false for $\preccurlyeq$. Then $\langle A, \preccurlyeq\rangle$ contains at least one element a such that $(\forall b)(b \preccurlyeq a \to a \preccurlyeq b)$. It means that a is a linguistic meaning (word) such that from *every* meaning (word) b that descends from a, i.e., is generated from a, we recover a. But this is clearly false and unnatural. We can hardly think of a natural language that contains such minimal, that is, *terminal* meanings, which in the long run do not generate other meanings that *strictly descend* from them. One can easily be convinced about that by remembering that every natural language is a living organism constantly changing and creating new meanings.

So the theory we shall be working in below is

$$\mathrm{ZFA}_D = \mathrm{ZFA}_{\preccurlyeq} + \{D_1, D_2, D_3\}.$$

In this theory we shall define the class of magmas $M(A)$ as a subclass of the universe $V(A)$. And as the universe of ZFA is made of levels $V_\alpha(A)$, for $\alpha \in Ord$, the class $M(A)$ will be made also of levels $M_\alpha(A) \subseteq V_\alpha(A)$. As first level we take the set

$$M_1(A) = LO(A, \preccurlyeq), \tag{2}$$

so indeed $M_1(A) \subseteq V_1(A)$.

Remarks 2.6 (i) Since $A \in M_1(A)$, A itself is a magma. If $(x_i)_{i\in I}$ is any family of magmas, then so is $\bigcup_i x_i$, as well as $\bigcap_i x_i$ if it is nonempty.

(ii) In view of axiom D_3, which is identical to condition (*), and Proposition 2.5 (i), all sets of $M_1(A)$ are infinite.

(iii) Every open set x is *saturated* with respect to the equivalence relation $\sim$ induced by $\preccurlyeq$. That is, for every $a \in x$ $[a]_\sim \subseteq x$. This follows from the fact that the b.o. sets $pr(a)$ are saturated.

Our next step is to define higher levels $M_\alpha(A)$, for $\alpha \geq 1$, of the magmatic hierarchy. For this task we need to shift the pre-ordering $\preccurlyeq$ of A to the levels $\mathcal{P}^\alpha(A)$ appropriately.

3 Shiftings of pre-orderings to powersets

Given a pre-ordered set $\langle A, \preccurlyeq\rangle$, a natural way to shift $\preccurlyeq$ to the set $\mathcal{P}(A)$ is by means of the relation $\preccurlyeq^+$ of "simulation" defined as follows: For $x, y \subseteq A$, let

$$x \preccurlyeq^+ y :\Leftrightarrow (\forall a \in x)(\exists b \in y)(a \preccurlyeq b). \tag{3}$$

We call $\preccurlyeq^+$ *exponential shifting*, or just *shifting*, of $\preccurlyeq$.[5]

Lemma 3.1 *(i) If $\preccurlyeq$ is a pre-ordering on A, $\preccurlyeq^+$ is a pre-ordering on $\mathcal{P}(A)$. (However, if $\preccurlyeq$ is an order, $\preccurlyeq^+$ need not be so.)*

(ii) If $\preccurlyeq$ is a total pre-order, i.e., $a \preccurlyeq b$ or $b \preccurlyeq a$ for all $a, b \in A$, so is $\preccurlyeq^+$.

[5] I borrowed the notation $\preccurlyeq^+$ from [1] although, given a relation $R \subseteq X \times X$, Aczel denotes by R^+ the relation on $\mathcal{P}(X)$ defined by:

$$xR^+y \Leftrightarrow (\forall u \in x)(\exists v \in y)(uRv) \ \& \ (\forall v \in y)(\exists u \in x)(uRv).$$

If in addition X is a transitive set and $R \subseteq R^+$, R^+ is said to be a "bisimulation." So our definition of $\preccurlyeq^+$ is "half" of that of R^+. This is because the above definition of R^+ is appropriate for symmetric relations R, especially equivalences, while $\preccurlyeq$ is a nonsymmetric relation.

Proof. (i) It is straightforward that $\preccurlyeq^+$ is reflexive and symmetric. (As a counterexample for the case of orderings, take $\preccurlyeq$ to be a total order on A, and let x_1, x_2 be distinct cofinal subsets of $(A, \preccurlyeq)$, i.e., $(\forall a \in A)(\exists b \in x_i)(a \preccurlyeq b)$, for $i = 1, 2$. Then clearly $x_1 \preccurlyeq^+ x_2$ and $x_2 \preccurlyeq^+ x_1$, while $x_1 \neq x_2$.)

(ii) Assume $\preccurlyeq$ is total and that for $x, y \in \mathcal{P}(A)$, $x \not\preccurlyeq^+ y$. Then $(\exists a \in x)(\forall b \in y)(a \not\preccurlyeq b)$. Since $\preccurlyeq$ is total, it follows that $(\exists a \in x)(\forall b \in y)(b \preccurlyeq a)$. But then, by logic alone, $(\forall b \in y)(\exists a \in x)(b \preccurlyeq a)$, so $y \preccurlyeq^+ x$. ⊣

Since $\preccurlyeq^+$ is a pre-ordering on $\mathcal{P}(A)$, the sets

$$pr^+(x) = \{y \subseteq A : y \preccurlyeq^+ x\}$$

form the b.o. sets of the lower topology $LO(\mathcal{P}(A), \preccurlyeq^+)$. Recall that $LO(A, \preccurlyeq) \subseteq \mathcal{P}(A)$, so the pre-ordering $\preccurlyeq^+$ applies also to the elements of $LO(A, \preccurlyeq)$. Then the following remarkable relation holds between the sets $pr^+(x)$ and the powersets of x.

Proposition 3.2 *(i) For any $x, y \in \mathcal{P}(A)$, $y \subseteq x \Rightarrow y \preccurlyeq^+ x$, so $\mathcal{P}(x) \subseteq pr^+(x)$.*

(ii) If $x \in LO(A, \preccurlyeq)$, then the converse of (i) holds, i.e., for every $y \in \mathcal{P}(A)$, $y \preccurlyeq^+ x \Rightarrow y \subseteq x$, so $pr^+(x) \subseteq \mathcal{P}(x)$.

(iii) Therefore for every $x \in LO(A, \preccurlyeq)$, $pr^+(x) = \mathcal{P}(x)$.

(iv) In particular, $\preccurlyeq^+ \restriction LO(A, \preccurlyeq) = \subseteq$, and

$$LO(LO(A, \preccurlyeq), \preccurlyeq^+) = LO(LO(A, \preccurlyeq), \subseteq).$$

Proof. (i). Let $y \subseteq x$ and pick $a \in y$. Then $a \in x$ and since $a \preccurlyeq a$, it means that there is $b \in x$ such that $a \preccurlyeq b$. Therefore $y \preccurlyeq^+ x$.

(ii) Suppose x is open, let $y \preccurlyeq^+ x$ and pick $a \in y$. Since $y \preccurlyeq^+ x$, there is $b \in x$ such that $a \preccurlyeq b$, i.e., $a \in pr(b)$. But $pr(b) \subseteq x$, since x is open, so $a \in x$ and therefore $y \subseteq x$.

(iii) and (iv) follow immediately from (i) and (ii). ⊣

Corollary 3.3 *If $\preccurlyeq$ satisfies D_3, i.e., does not contain minimal elements, then so does $\preccurlyeq^+$ on $LO(A, \preccurlyeq)$.*

Proof. Let $x \in LO(A, \preccurlyeq)$. We have to show that there is $y \in LO(A, \preccurlyeq)$ such that $y \prec^+ x$, i.e., $y \preccurlyeq^+ x$ and $x \not\preccurlyeq^+ y$. By Proposition 3.2 (iv), this amounts to finding an open y such that $y \subseteq x$ and $x \not\subseteq y$, or equivalently $y \subsetneq x$. Pick an $a \in x$. Then $pr(a) \subseteq x$. By D_3, there is a b such that $b \prec a$, hence $pr(b) \subsetneq pr(a) \subseteq x$. Therefore $pr(b) \subsetneq x$ and $pr(b)$ is open. ⊣

Proposition 3.2 is crucial for the construction of the levels $M_\alpha(A)$ of the magmatic hierarchy, for every $\alpha \geq 1$. It says that, whatever the starting pre-ordering $\preccurlyeq$ of A is, the restriction of $\preccurlyeq^+$ to $LO(A, \preccurlyeq)$ is $\subseteq$. That is, if we set $M_1 = LO(A, \preccurlyeq)$, then $LO(M_1, \preccurlyeq^+) = LO(M_1, \subseteq)$. And for the same reason, if we set $M_2 = LO(M_1, \subseteq)$, the shifting $\subseteq^+$ of the relation $\subseteq$ of M_1 to the sets of M_2 is $\subseteq$ again. And so on with $\subseteq^{++}$, $\subseteq^{+++}$ etc, for all subsequent levels M_3, $M_4, \ldots$, which are defined similarly. As a result, every such level consists of infinite sets only, ordered by $\subseteq$ with no minimal element. Moreover, when we reach a limit ordinal α, we can take as M_α just the union of all previous levels, ordered again by $\subseteq$. And then we can proceed further by setting $M_{\alpha+1} = LO(M_\alpha, \subseteq)$. This way M_α is defined for every ordinal $\alpha \geq 1$.

4 The magmatic universe

Fix a set A of atoms and a pre-ordering $\preccurlyeq$ of it which satisfies the axioms of ZFA_D, in particular $\preccurlyeq$ has no minimal element. In view of Proposition 3.2 and the remarks at the end of the last section, we can define the levels $M_\alpha(A)$ of the *magmatic hierarchy above* A, by induction on α as follows. (For simplicity we write M_α instead of $M_\alpha(A)$.)

Definition 4.1 $M_1 = LO(A, \preccurlyeq)$.
$M_{\alpha+1} = LO(M_\alpha, \subseteq)$, for every $\alpha \geq 1$.
$M_\alpha = \bigcup_{1 \leq \beta < \alpha} M_\beta$, if α is a limit ordinal.
$M = M(A) = \bigcup_{\alpha \geq 1} M_\alpha$.
M is said to be the *magmatic universe above* A (with respect to the pre-ordering $\preccurlyeq$).

Here are some basic facts about M_α's and M.

Lemma 4.2 *(i)* $M_\alpha \subseteq V_\alpha(A)$, $M_1 \subseteq \mathcal{P}(A)\backslash\{\emptyset\}$, $M_{\alpha+1} \subseteq \mathcal{P}(M_\alpha)\backslash\{\emptyset\}$ *and* $M_\alpha \in M_{\alpha+1}$, *for every* $\alpha \geq 1$.
(ii) The b.o. sets of the space $M_{\alpha+1} = LO(M_\alpha, \subseteq)$ *are the sets*

$$pr_\alpha(x) = \{y \in M_\alpha : y \subseteq x\} = \mathcal{P}(x) \cap M_\alpha.$$

Therefore:

$$x \in M_{\alpha+1} \;\Leftrightarrow\; x \subseteq M_\alpha \wedge x \neq \emptyset \wedge (\forall y \in x)(\mathcal{P}(y) \cap M_\alpha \subseteq x).$$

(iii) The class M is "almost" transitive, in the sense that for every $x \in M_\alpha$ for $\alpha \geq 2$, $x \subseteq M$. However for $x \in M_1$, $x \subseteq A$. So $M \cup A$ is transitive.

(iv) M is a proper subclass of $V(A)$, while $M \cap V = \emptyset$ where V is the subclass of pure sets of $V(A)$ (that is of x's such that $TC(x) \cap A = \emptyset$).

(v) The inductive definition of M_α's does not reach any fixed point, that is for every $\alpha \geq 1$, $M_{\alpha+1} \neq M_\alpha$.

(vi) All sets of M are infinite.

(vii) There is no $\subseteq$-minimal set in M.

Proof. (i), (ii) and (iii) follow immediately from the definitions.

(iv) Let $rank(x)$ be the usual rank of sets in $V(A)$, i.e. $rank(x) = \min\{\alpha : x \in V_{\alpha+1}(A)\}$. It is easy to see by induction that for every $\alpha \geq 1$, $rank(M_\alpha) = \alpha$. Concerning the other claim, let $x \in V \cap M$ be a pure set of least rank α. Then $x \in V_{\alpha+1} \cap M$. If $\alpha \geq 1$, $x \subseteq V_\alpha \cap M$, and $x \neq \emptyset$. But then there is $y \in x$, $y \in V \cap M$ and $rank(y) < rank(x)$, a contradiction. So $rank(x) = 0$, which means that $x = \emptyset$ and $x \in V_1 \cap M$, a contradiction again since $\emptyset \notin M$.

(v) If $M_{\alpha+1} = M_\alpha$, then $M_\alpha = LO(M_\alpha, \subseteq)$, hence by (i) $M_\alpha \in M_\alpha$, a contradiction. (A bit differently: if $M_{\alpha+1} = M_\alpha$, then $M_\beta = M_\alpha$ for every $\beta > \alpha$, and hence $M = M_\alpha$, so M would be a set, which contradicts (iv).)

(vi) and (vii) follow from the fact that the initial pre-ordering $\preccurlyeq$ on A does not contain minimal elements (by D_3), so by Corollary 3.3 so does the relation $\preccurlyeq^+ = \subseteq$ on M_1, and by induction, so does $\subseteq$ on every M_α. Therefore, by Proposition 2.5 for every $\alpha \geq 0$, (a) all sets of $M_{\alpha+1}$ are infinite, and (b) $M_{\alpha+1}$ does not contain $\subseteq$-minimal sets. ⊣

Remark 4.3 *Notice that as follows from Lemma 4.2 (i), $M_\alpha \subseteq V_\alpha(A) \cap M$. However in general $M_\alpha \neq M \cap V_\alpha(A)$.*

Proof. Consider, for example, the levels M_1, M_2 of M which are disjoint (see Lemma 4.8 below). Then $M_1 \subseteq \mathcal{P}(A)$ and $M_2 \subseteq \mathcal{P}(M_1) \subseteq \mathcal{P}^2(A)$. Since $\mathcal{P}(A) \cup \mathcal{P}^2(A) \subseteq V_2(A)$, we have $M_1 \cup M_2 \subseteq M \cap V_2(A)$. However $M_1 \cup M_2 \not\subseteq M_2$ because $M_1 \cap M_2 = \emptyset$. Therefore $M_2 \neq M \cap V_2(A)$. ⊣

By the next three results it is shown that the collection $\mathcal{P}(x) \cap M$ of submagmas of a magma x is a magma too, that occurs exactly at the next level of that of x.

Lemma 4.4 *(i) If $\mathcal{P}(x) \cap M_\alpha \neq \emptyset$, then $\mathcal{P}(x) \cap M_\alpha \in M_{\alpha+1}$.*

(ii) For every $x \in M$, there is a limit ordinal β such that $\mathcal{P}(x) \cap M = \mathcal{P}(x) \cap M_\beta$.

Proof. (i) Notice that if $x \in M_\alpha$, then, by Lemma 4.2 (ii), $\mathcal{P}(x) \cap M_\alpha = pr_\alpha(x)$, which is a b.o. set of $LO(M_\alpha, \subseteq) = M_{\alpha+1}$. However we can prove the claim without this assumption. So let $u = \mathcal{P}(x) \cap M_\alpha \neq \emptyset$. Then $u \subseteq M_\alpha$, so it suffices to show that $u \in LO(M_\alpha, \subseteq)$, i.e., $(\forall z \in u)(\mathcal{P}(z) \cap M_\alpha \subseteq u)$. Pick a $z \in u$. Then $z \subseteq x$, so $\mathcal{P}(z) \subseteq \mathcal{P}(x)$, and therefore $\mathcal{P}(z) \cap M_\alpha \subseteq \mathcal{P}(x) \cap M_\alpha = u$, as required.

(ii) M is a (definable) subclass of $V(A)$ and $\mathcal{P}(x)$ is a set, so by the Separation scheme of ZFA_D, $\mathcal{P}(x) \cap M$ is a set. For each $y \in \mathcal{P}(x) \cap M$, let $\beta_y = \min\{\gamma \in Ord : y \in M_\gamma\}$. If $X = \{\beta_y : y \in \mathcal{P}(x) \cap M\}$, X is a set of ordinals (by the Replacement axiom), so $\sup X$ exists, and let β be the first limit ordinal such that $\sup X \leq \beta$. Then $M_\beta = \bigcup_{1 \leq \gamma < \beta} M_\gamma$, and therefore $\mathcal{P}(x) \cap M \subseteq M_\beta$. But then also $\mathcal{P}(x) \cap M \subseteq \mathcal{P}(x) \cap M_\beta$, so finally $\mathcal{P}(x) \cap M = \mathcal{P}(x) \cap M_\beta$ since the reverse inclusion holds trivially. ⊣

Proposition 4.5 *(i) If $x \in M_1$, then $\mathcal{P}(x) \cap M \subseteq M_1$, and hence $\mathcal{P}(x) \cap M = \mathcal{P}(x) \cap M_1$. In particular $\mathcal{P}(A) \cap M = M_1$.*

(ii) If $x \in M_{\alpha+1}$, then $\mathcal{P}(x) \cap M \subseteq M_{\alpha+1}$, for every $\alpha \geq 1$, and hence $\mathcal{P}(x) \cap M = \mathcal{P}(x) \cap M_{\alpha+1}$. In particular $\mathcal{P}(M_\alpha) \cap M = M_{\alpha+1}$.

Proof. (i) Let $x \in M_1$ and pick a $y \in \mathcal{P}(x) \cap M$. Then $y \subseteq x \subseteq A$. If $y \notin M_1$, then $y \in M_{\alpha+1}$ for some $\alpha \geq 1$. So $y \subseteq M_\alpha$, and hence $A \cap M_\alpha \neq \emptyset$, which contradicts the fact that $A \cap M = \emptyset$. For the case of $x = A$, we have $A \in M_1$, so $\mathcal{P}(A) \cap M \subseteq M_1$, but besides $M_1 = LO(A, \preccurlyeq) \subseteq \mathcal{P}(A)$, therefore $\mathcal{P}(A) \cap M = M_1$.

(ii) Let $x \in M_{\alpha+1}$, and fix a $y_0 \in \mathcal{P}(x) \cap M$. We have to show that $y_0 \in M_{\alpha+1}$, i.e., $y_0 \subseteq M_\alpha$ and y_0 is open in M_α. Since $y_0 \subseteq x \subseteq M_\alpha$, already $y_0 \subseteq M_\alpha$. Towards a contradiction, suppose that y_0 is not open in M_α. It means that the following holds:

(a) $(\exists z \in y_0)(\mathcal{P}(z) \cap M_\alpha \not\subseteq y_0)$.

Now by Lemma 4.4 (ii), there is limit $\beta > \alpha+1$ such that $\mathcal{P}(x) \cap M = \mathcal{P}(x) \cap M_\beta$, so by Lemma 4.4 (i), $\mathcal{P}(x) \cap M \in M_{\beta+1}$. It follows that $y_0 \in M_{\beta+1}$, i.e. y_0 is an open subset of M_β. So

(b) $(\forall z \in y_0)(\mathcal{P}(z) \cap M_\beta \subseteq y_0)$.

But since $\beta > \alpha + 1$ and β is limit, $M_\alpha \subseteq M_\beta$, so

(c) $(\forall z \in y_0)(\mathcal{P}(z) \cap M_\alpha \subseteq \mathcal{P}(z) \cap M_\beta)$.

Obviously (b) and (c) contradict (a), and this proves the claim.

In particular for $x = M_\alpha$, $\mathcal{P}(M_\alpha) \cap M \subseteq M_{\alpha+1}$, since $M_\alpha \in M_{\alpha+1}$, but also $M_{\alpha+1} = LO(M_\alpha, \subseteq) \subseteq \mathcal{P}(M_\alpha)$, so $\mathcal{P}(M_\alpha) \cap M = M_{\alpha+1}$. ⊣

Corollary 4.6 *(i) For every $\alpha \geq 0$, if $x \in M_{\alpha+1}$ then $\mathcal{P}(x) \cap M \in M_{\alpha+2}$.*

(ii) If $x \in M$ and $x \subseteq M_\alpha$, then $x \in M_{\alpha+1}$ (that is, every subset of M_α that belongs to M, is an open subset of M_α).

Proof. (i) By Proposition 4.5 (ii), for every $\alpha \geq 0$, if $x \in M_{\alpha+1}$, then $\mathcal{P}(x) \cap M = \mathcal{P}(x) \cap M_{\alpha+1}$. In addition, by Lemma 4.4 (i), $\mathcal{P}(x) \cap M_{\alpha+1} \in M_{\alpha+2}$.

(ii) It follows again from clause (ii) of Proposition 4.5, in particular from the equality $\mathcal{P}(M_\alpha) \cap M = M_{\alpha+1}$. ⊣

Proposition 4.5 helps us also to compare the levels of the form $M_{\alpha+n}$, where α is a limit ordinal and $n \geq 0$.

Lemma 4.7 *For all limit ordinals α, β such that $\alpha < \beta$, and all $n \geq 0$, $M_{\alpha+n} \subseteq M_{\beta+n}$.*

Proof. That $M_\alpha \subseteq M_\beta$ follows from the definition of M_α for limit α. We prove the claim for $n = 1$, i.e., $M_{\alpha+1} \subseteq M_{\beta+1}$. Since $M_{\alpha+1} = LO(M_\alpha, \subseteq)$ and $M_{\beta+1} = LO(M_\beta, \subseteq)$, it suffices to show that every b.o. subset of M_α, $pr_\alpha(x)$ for some $x \in M_\alpha$, is also a b.o. subset of M_β. Pick a $x \in M_\alpha$, so $x \in M_\beta$ too. Since α is a limit ordinal, $x \in M_{\gamma+1}$ for some $\gamma < \alpha$, and by Proposition 4.5 (ii), $\mathcal{P}(x) \cap M = \mathcal{P}(x) \cap M_{\gamma+1}$. Then $M_{\gamma+1} \subseteq M_\alpha \subseteq M_\beta$, which implies that $\mathcal{P}(x) \cap M = \mathcal{P}(x) \cap M_\alpha = \mathcal{P}(x) \cap M_\beta$, so

$$pr_\alpha(x) = \mathcal{P}(x) \cap M_\alpha = \mathcal{P}(x) \cap M_\beta = pr_\beta(x). \tag{4}$$

This proves that every set $pr_\alpha(x)$ is also an open subset of M_β, and proves that $M_{\alpha+1} \subseteq M_{\beta+1}$. That $M_{\alpha+n} \subseteq M_{\beta+n}$, for every $n \geq 1$, is shown by an easy induction and the help of Proposition 4.5 (ii) as before. ⊣

We give next some results concerning the non-limit levels of M. First notice that the levels M_n, for finite n, have the peculiarity to be pairwise disjoint.

Lemma 4.8 *$M_n \cap M_m = \emptyset$, for all $n \neq m$.*

Proof. Let us set $\mathcal{P}^*(X) = \mathcal{P}(X) \backslash \{\emptyset\}$, for any set X. By Lemma 4.2 (i), $M_1 \subseteq \mathcal{P}^*(A)$, and $M_2 \subseteq \mathcal{P}^*(M_1) \subseteq \mathcal{P}^*(\mathcal{P}^*(A)) = \mathcal{P}^{*2}(A)$. So by induction, for every $n \geq 1$,

$$M_n \subseteq \mathcal{P}^{*n}(A). \tag{5}$$

On the other hand, $A \cap \mathcal{P}^{*k}(A) = \emptyset$, for every $k > 0$, since A contains non-sets, therefore $\mathcal{P}^{*n}(A) \cap \mathcal{P}^{*n+k}(A) = \emptyset$, for every n and every $k > 0$, or for all $n \neq m$,

$$\mathcal{P}^{*n}(A) \cap \mathcal{P}^{*m}(A) = \emptyset. \tag{6}$$

By (5) and (6), $M_n \cap M_m = \emptyset$ for all $n \neq m$. ⊣

We can give now a uniform characterization of the sets of $M_{\alpha+1} = LO(M_\alpha, \subseteq)$, for any limit ordinal α. First observe that if α is limit and $x \subseteq M_\alpha = \bigcup_{1 \le \beta < \alpha} M_\beta$, then x can be written $x = \bigcup_{\beta<\alpha}(x \cap M_{\beta+1})$, because every $y \in x$ belongs to some level of the form $M_{\beta+1}$, for a $\beta < \alpha$.

Proposition 4.9 *Let α be a limit ordinal. Let $\emptyset \neq x \subseteq M_\alpha$, $I = \{\beta < \alpha : x \cap M_{\beta+1} \neq \emptyset\}$, and $x_{\beta+1} = x \cap M_{\beta+1}$, for every $\beta \in I$. Then $x \in M_{\alpha+1}$ if and only if*

$$I \neq \emptyset \ \wedge \ x = \bigcup_{\beta \in I} x_{\beta+1} \ \wedge \ (\forall \beta \in I)(x_{\beta+1} \in M_{\beta+2}). \tag{7}$$

Proof. Given $\emptyset \neq x \subseteq M_\alpha$, clearly $I \neq \emptyset$ and $x = \bigcup_{\beta\in I} x_{\beta+1}$. So in order to prove (7) it suffices to prove the equivalence

$$x \in M_{\alpha+1} \Leftrightarrow (\forall \beta \in I)(x_{\beta+1} \in M_{\beta+2}). \tag{8}$$

$\Rightarrow$ of (8): Assume $x \in M_{\alpha+1}$, let $\beta \in I$, and pick $y \in x_{\beta+1}$. We have to show that $\mathcal{P}(y) \cap M_{\beta+1} \subseteq x_{\beta+1}$. Now $y \in x$ and the assumption $x \in M_{\alpha+1}$ implies $\mathcal{P}(y) \cap M_\alpha \subseteq x$. The last inclusion yields $\mathcal{P}(y) \cap M_\alpha \cap M_{\beta+1} \subseteq x \cap M_{\beta+1}$, which is identical to the required inclusion $\mathcal{P}(y) \cap M_{\beta+1} \subseteq x_{\beta+1}$.

$\Leftarrow$ of (8): Assume $(\forall \beta \in I)(x_{\beta+1} \in M_{\beta+2})$, where $x_{\beta+1} = x \cap M_{\beta+1}$. It means that for every $\beta \in I$, $(\forall y \in x_{\beta+1})(\mathcal{P}(y) \cap M_{\beta+1} \subseteq x_{\beta+1})$. Pick a $y \in x$. We have to show that $\mathcal{P}(y) \cap M_\alpha \subseteq x$. Now $y \in x_{\beta+1}$, for some $\beta \in I$, hence $y \in M_{\beta+1}$. By Proposition 4.5 (ii), $y \in M_{\beta+1}$ implies that $\mathcal{P}(y) \cap M \subseteq M_{\beta+1}$, so $\mathcal{P}(y) \cap M_\alpha = \mathcal{P}(y) \cap M_{\beta+1}$, and since by assumption $\mathcal{P}(y) \cap M_{\beta+1} \subseteq x_{\beta+1}$, and $x_{\beta+1} \subseteq x$, it follows $\mathcal{P}(y) \cap M_\alpha \subseteq x$ as required. This completes the proof of (8) and the Proposition. ⊣

Corollary 4.10 *Let α be a limit ordinal. Then:*

(i) For all $1 \le i \le n$, $M_i \cap M_{\alpha+n} = \emptyset$. That is, $(\bigcup_{1\le i\le n} M_i) \cap M_{\alpha+n} = \emptyset$.

(ii) For every $n \ge 1$, $M_\alpha \backslash (\bigcup_{1\le i\le n} M_i) = \bigcup_{n+1\le\beta<\alpha} M_\beta \subseteq M_{\alpha+n}$.

(iii) For all $n \ge 0$, $M_{\alpha+n} \not\subseteq M_{\alpha+n+1}$.

Proof. (i) It suffices to show that given limit α, for all $\kappa \geq 1$ and $n \geq 0$, $M_k \cap M_{\alpha+n+k} = \emptyset$. By induction on k. Clearly $A \cap M_{\alpha+n} = \emptyset$, so $\mathcal{P}(A) \cap \mathcal{P}(M_{\alpha+n}) = \emptyset$. Since $M_1 \subseteq \mathcal{P}(A)$ and $M_{\alpha+n+1} \subseteq \mathcal{P}(M_{\alpha+n})$ we have $M_1 \cap M_{\alpha+n+1}$, so the claim holds for $k = 1$. Assume $M_k \cap M_{\alpha+n+k} = \emptyset$. Then $M_{k+1} \subseteq \mathcal{P}(M_k)$ and $M_{\alpha+n+k+1} \subseteq \mathcal{P}(M_{\alpha+n+k})$. By the assumption the larger sets in these two relations are disjoint, thus so are the smaller ones.

(ii) The claim holds for $n = 1$, because every $x \in M_\beta$, for some $2 \leq \beta < \alpha$, is a subset of M_α that satisfies condition (7) of Proposition 4.9, so $x \in M_{\alpha+1}$. Then we can continue using induction on n. For simplicity, for $n \geq 1$, assume $M_{n+1} \subseteq M_{\alpha+n}$, and prove that $M_{n+2} \subseteq M_{\alpha+n+1}$. Let $x \in M_{n+2}$. Then $x \subseteq M_{n+1}$, and $(\forall y \in x)(\mathcal{P}(y) \cap M_{n+1} \subseteq x)$. Now $y \in x$ implies $y \in M_{n+1}$, so by Proposition 4.5 (ii), $\mathcal{P}(y) \cap M \subseteq M_{n+1}$. By this fact and the induction assumption $x \subseteq M_{n+1} \subseteq M_{\alpha+n}$, it follows that $x \in M_{\alpha+n+1}$.

(iii) By (i) and (ii) above, $M_\alpha \not\subseteq M_{\alpha+1}$ because $M_1 \subseteq M_\alpha$ while $M_1 \cap M_{\alpha+1} = \emptyset$, and $M_{\alpha+n} \not\subseteq M_{\alpha+n+1}$ because $M_{n+1} \subseteq M_{\alpha+n}$, while $M_{n+1} \cap M_{\alpha+n+1} = \emptyset$. ⊣

It follows from clause (iii) of the preceding corollary that, in contrast to the limit levels M_α, which, by definition, contain the elements of all lower levels, the successor levels do not behave cumulatively. Each level $M_{\alpha+n}$, for a limit α and $n \geq 1$, always omits elements of lower levels. One may infer from the comparison given above that $M_{\alpha+n}$ and $M_{\alpha+n+1}$ differ only with respect to elements of the initial levels M_n of the hierarchy, i.e. elements of M_ω. But this is not true. For example pick a $x \in M_2$ and a $y \in M_3$. Then clearly $x \cup y \in M_{\omega+1}$, according to the characterization given in Proposition 4.9, while $x \cup y \notin M_\omega$, so $x \cup y \in M_{\omega+1} \backslash M_\omega$. But also $x \cup y \notin M_{\omega+2}$ either, because otherwise $x \cup y \subseteq M_{\omega+1}$. Since $x \subseteq M_1$, that would mean that $M_1 \cap M_{\omega+1} \neq \emptyset$, which is false according to clause (i) of the preceding Corollary. This shows that $M_{\omega+1} \backslash M_\omega \not\subseteq M_{\omega+2}$.

We turn now to another more standard point of view from which the class M could be looked at: the point of view from which M is seen as an $\in$-structure and so questions are raised as to what set-theoretic properties of the language $L_0 = \{\in\}$ could be satisfied in $\langle M, \in \rangle$. Such a question about the truth of some of the axioms of ZF in $\langle M, \in \rangle$ is reasonable. There is however a technical difficulty with sentences of L_0 in M, because of the lack of transitivity due to the bottom level M_1, since for every $x \in M_1$, $x \cap M = \emptyset$. In view of this, given $x, y \in M_1$, the truth of simple properties like $x = y$ and $x \subseteq y$ cannot be expressed inside M by the usual formulas. The problem is fixed if we add to M the atoms and work in $M^* = M \cup A$ rather than M, with language $L = \{\in, S(\cdot), A(\cdot)\}$, or with sorted variables.

Still, of course, we do not expect M^* to satisfy many of the closure properties expressed through the axioms of ZFA. Of these axioms Extensionality and Foundation do hold in $\langle M^*, \in\rangle$ because the latter is a transitive substructure of $\langle V(A), \in\rangle$. But since by construction $\emptyset \notin M$, the Emptyset axiom fails. So does the Pairing axiom because every $x \in M$ is an infinite set. The Infinity axiom fails too, since the "measure" by which we decide infinity of a set is $\omega = \{0, 1, \ldots\}$, and this is not a resident of M^*. Finally the truth of Separation and Replacement in M^* is obviously out of the question. Nevertheless, the rest two axioms, Powerset (Pow) and Union (Un), are indeed true (the second one not quite). We begin with Pow which is a direct consequence of a result established previously.

Proposition 4.11 *$M^* \models Pow$. Intuitively, the collection of submagmas of a magma is again a magma (of the next higher rank.)*

Proof. We have to show that $M^* \models (\forall x)(\exists y)(y = \mathcal{P}(x))$, or $(\forall x \in M)(\exists y \in M)(y = \mathcal{P}(x) \cap M)$. However this follows immediately from Corollary 4.6 (i): if $x \in M_{\alpha+1}$, then $\mathcal{P}(x) \cap M \in M_{\alpha+2}$. ⊣

In contrast to the powerset operation, which "goes upward", the union operation "goes downward" and may lead out of M if $\cup x$ hits the bottom level that consists of atoms. For example if $x \in M_1$ then $x \subseteq A$, so according to the formal definition of $\cup x$, $\cup x = \emptyset \notin M$. Perhaps one may guess that this concerns the elements of M_1 only and that $(\forall x \in M \backslash M_1)(\cup x \in M)$. But still this is not true. For example, as follows from Proposition 4.9, $M_1 \cup M_2 \in M_{\omega+1}$ while $\cup(M_1 \cup M_2) = (\cup M_1) \cup (\cup M_2) = A \cup M_1$. This does not belong to M either, because on the one hand clearly $A \cup M_1 \notin M_1$, and on the other if we assume $A \cup M_1 \in M_{\alpha+1}$, for some $\alpha \geq 1$, then $A \cup M_1 \subseteq M_\alpha \subseteq M$, which is false because $A \cap M = \emptyset$.

In fact, since the elements of every $x \notin M_1$ are open sets, and every union of open sets of the *same space* $LO(M_\alpha, \subseteq)$ is open again, it follows that if $x \subseteq LO(A, \preccurlyeq)$, or $x \subseteq LO(M_\alpha, \subseteq)$, for some $\alpha \geq 1$, then $\cup x \in LO(A, \preccurlyeq)$, or $\cup x \in LO(M_\alpha, \subseteq)$, respectively. This situation occurs exactly when $x \in M_{\alpha+2}$, for some $\alpha \geq 0$, and gives a sufficient condition in order for $\cup x$ to belong to M. It turns out that this condition is also necessary. In fact the next proposition describes precisely, with two equivalent conditions, the elements of M whose unions belong to M.

Proposition 4.12 *Let $x \in M$. The following conditions are equivalent.*
(i) $x \in M_{\alpha+2}$, for some $\alpha \geq 0$.
(ii) $\cup x \in M$.
(iii) $x \subseteq M_1$ or $x \cap M_1 = \emptyset$.

Proof. (i)⇒(ii) Assume first that $\alpha \neq 0$ and $x \in M_{\alpha+2}$. Then $x \subseteq M_{\alpha+1} = LO(M_\alpha, \subseteq)$, so x is a family of open subsets of M_α. The union $\cup x$ of this family is open too, so $\cup x \in M_{\alpha+1}$. If $\alpha = 0$, then $x \in M_2$, or $x \subseteq M_1 = LO(A, \preccurlyeq)$, so $\cup x \in M_1$. In both cases $\cup x \in M$.

(ii)⇒ (iii): We prove the contrapositive. Suppose $x \not\subseteq M_1$ and $x \cap M_1 \neq \emptyset$. If $x_1 = x \cap M_1$, then $x = x_1 \cup x_2$, where $x_1, x_2 \neq \emptyset$, $x_1 \subseteq M_1$ and $x_2 \subseteq M \backslash M_1$. Therefore $\cup x = (\cup x_1) \cup (\cup x_2)$, where $\cup x_1 \subseteq \cup M_1 = A$ and $\cup x_2 \cap A = \emptyset$. So if $\cup x \in M$, necessarily $\cup x \subseteq M_\alpha$ for some $\alpha \geq 1$, and hence $\cup x_1 \subseteq M_\alpha$. But then $A \cap M_\alpha \neq \emptyset$, a contradiction.

(iii)⇒ (i): Let $x \subseteq M_1$. By Corollary 4.6 (ii), $x \in M$ and $x \subseteq M_1$ imply $x \in M_2$, so we are done. Let now $x \cap M_1 = \emptyset$. Then $x \subseteq M \backslash M_1$, and if $x \in M_{\alpha+1}$, then $x \subseteq M_\alpha \backslash M_1$. Without loss of generality we may take α to be a limit ordinal. Then by Corollary 4.10 (ii), $M_\alpha \backslash M_1 \subseteq M_{\alpha+1}$, so $x \subseteq M_{\alpha+1}$. By Corollary 4.6 (ii) again, $x \in M_{\alpha+2}$. ⊣

On the other hand every level of the form $M_{\alpha+1}$, for limit α, contains sets x such that $\cup x \notin M$.

Fact 4.13 *If $\alpha = 0$, or α is a limit ordinal, there exists $x \in M_{\alpha+1}$ such that $\cup x \notin M$.*

Proof. We saw above that for every $x \in M_1$, $\cup x \notin M$. Also if, for example, $x = M_1 \cup M_2$, then $x \in M_{\omega+1}$ and $\cup x \notin M$. By Lemma 4.7 $M_{\omega+1} \subseteq M_{\alpha+1}$, for every limit α, so $M_1 \cup M_2 \in M_{\alpha+1}$ too. ⊣

We close here the description of the technical features of the magmatic universe and come to the question about the extent to which the class M actually satisfies some, or all, of Castoriadis' intuitive principles M1-M5, which he proposed as main properties of magmas. Recall however (see Introduction) that we decided to leave out M1 and M4 as inconsistent, and reformulate slightly the rest of them into M2*, M3*, M5*. Given this adjustment, the answer to the question is yes, M2*, (a weak form of) M3* and M5* are true in M. In our formalization some magmas can be called *basic*, if they generate all the rest of the same level. They are just the b.o. sets $pr(a) \in M_1$, for $a \in A$, and $pr_\alpha(x) \in M_{\alpha+1}$, for $x \in M_\alpha$ and $\alpha \geq 1$.

Proposition 4.14 *The following statements are true about M:*

(i) (M2): For every magma x there is a magma $y \neq x$ such that $y \subseteq x$.*

(ii) (weak M3): If x is a basic magma, there is no finite partitioning of x into submagmas.*

(iii) (M5): What is not a magma is a set or an atom.*

Proof. (i) Let $x \in M$ be a magma, and let $x \in M_{\alpha+1} = LO(M_\alpha, \subseteq)$, for some $\alpha \geq 1$, or $x \in M_1 = LO(A, \preccurlyeq)$. We know (see Lemma 4.2) that none of these topologies contains minimal open sets, so there is always a $y \in M_{\alpha+1}$ such that $y \subsetneq x$. Such a y is a proper submagma of x.

(ii) Given a basic magma $pr_\alpha(x)$, assume for simplicity that $pr_\alpha(x) = y_1 \cup y_2$, where y_1, y_2 are disjoint submagmas, that is disjoint open sets y_1, y_2. But then either $x \in y_1$ or $x \in y_2$, hence either $pr_\alpha(x) \subseteq y_1$, or $pr_\alpha(x) \subseteq y_2$, both of which contradict the fact that $pr_\alpha(x) = y_1 \cup y_2$ and $y_1 \cap y_2 = \emptyset$.

(iii) This claim follows simply from the fact that the class $M = M(A)$ of magmas above A is a subclass of the set-theoretic universe with atoms $V(A)$. So if $u \notin M(A)$, then necessarily either u is a classical set of $V(A)$, or $u \in A$. ⊣

Conclusions and some possibilities for future work. A reviewer wrote that "Castoriadis himself would probably reject this attempt of formalizing magmas. In particular, he seems to think that set theory might not be the right framework for such a formalization". I largely agree. The above presented formalization in no way should be construed as an attempt to capture or interpret mathematically Castoriadis' thoughts and intuitions in some "authentic" fashion; not even to capture his theory of magmas *in all of its aspects*. After all I think Castoriadis did not keep secret his low appreciation for the pair "classical logic/classical set theory", to which he, disdainfully enough, referred throughout his writings (among them in some excerpts cited in the Introduction) as "identitary-ensemblistic logic" – as if there were plenty of *superior* logics and theories of collections around.

The concept of magma was of interest to me simply because it challenged the fundamental assumption of classical set theory that every object behaves *independently* of all the rest with respect to $\in$. And while this is unquestionable for *pure sets*, it is not that obvious for atoms (urelements) and perhaps for sets containing atoms. So my interest in magmas stemmed primarily from the relation of *dependence* that underlies their elements, and the challenge to build a structure for an inclusive class of sets with dependent elements. Towards this aim the only *secure* tools available to me were just the old good "identitary-ensemblistic logic". Nevertheless I would be very much interested to see other, alternative approaches to the concept, through different logical and/or set-theoretical environments.

Concerning the possibilities for future work I can see two directions:

(a) To try to construct in M "magmatic analogs", or "magmatic codes" of some standard set-theoretical objects, like ordered pair, function, natural number, $\mathbb{N}$, etc. The starting point could be the observation that the magmatic analogs of the simplest sets $\{a\}$ of ZFA, for $a \in A$, are the basic open sets $pr(a)$ of M ($pr(a)$ is the smallest magma that contains a). More generally the analog of the n-element set $\{a_1, \ldots, a_n\}$ is the magma $pr(a_1) \cup \cdots \cup pr(a_n)$. I cannot say how far this magmatic coding can go and how complicated and intuitively attractive would be. I just wonder how our mathematical intuition would be if we were living not in the present "separative" world of independent entities, but in a "non-separative" world of dependent entities and "thick" magmatic objects.

(b) The other possible direction is the *syntactic* approach to magmas, i.e., an attempt to axiomatize their basic properties by a set of axioms T in a suitable language and then look for models of T. Normally the language must include, except $\in$, a primitive binary relation $D(x, y)$ for dependence. However to this direction, of help might be an old paper of H. Skala, "An alternative way of avoiding set-theoretical paradoxes", *Zeitsch. f. math. Logik und Grundlagen d. Math.* 20 (1974), pp. 233-237 (the former title of *Mathematical Logic Quarterly*), which, strangely enough, is related to sets with dependent elements. This is because Skala's axioms, which overlap with those of ZF, without forming either a subsystem or an extension of ZF, do not include Pairing, and leave room for objects $a \neq b$ such that $(\forall x)(a \in x \rightarrow b \in x)$. If we denote the last relation between a and b by $b \preccurlyeq a$, then $\preccurlyeq$ is a pre-ordering, and hence a dependence relation.

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