Affine Dual Frames and Extension Principles

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Abstract

In this work we provide three new characterizations of affine dual frames constructed from refinable functions. The first one is similar to [10, Proposition 5.2] but without any decay assumptions on the generators of a pair of affine systems. The second one reveals the geometric significance of the Mixed Fundamental function and the third one shows that the Mixed Oblique Extension Principle actually characterizes dual framelets. We also extend recent results on the characterization of affine Parseval frames obtained in [28, Theorem 2.3].

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1. Introduction

Extension Principles were first proposed by Ron and Shen [26, 27] and were subsequently extended by Daubechies *et al.* [10] in the form of the Oblique Extension Principle. OEP relaxes the requirements for the construction of wavelets arising from a pair of refinable functions or from a single refinable function extending thus Mallat's construction of wavelets from orthonormal

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scaling functions. Extension Principles are important because they can be used to construct wavelets from refinable functions which may not be scaling functions (in the sense that their integer translates may not form a frame but only a Bessel system) with desirable properties such as symmetry and antisymmetry, smoothness or compact support.

In this paper we study the geometric structure associated with bi-framelets arising from pairs of refinable functions. We also show how the Mixed Fundamental function arises from a weak form of a reduction of redundancy between the spaces generated by the integer translates of the pair of refinable functions and those spanned by the detail spaces of scales $j \geq 0$. More details about the significance of Extension Principles can be found at [2, 5, 10, 26, 27]. We also mention the earliest pioneering works [13, 14] on the construction of affine dual frames using Oblique Extension Principle. Apart from Extension Principles various design strategies have been developed for constructing multiscale representations with desirable properties such as good spatial localization, high regularity, arbitrary smoothness, see [1, 3, 18, 19, 20, 21, 22, 23, 24] and references therein. We end this brief discussion of the Extension Principles literature and of related constructions with the pioneering ϕ-transform of Frazier *et al.* [17] generalized in the form of dual families of pseudoframes of translates [20].

We begin with some necessary notation. Let $L_2 := L_2(\mathbb{R}^s)$ be the Hilbert space of all measurable square integrable functions on $\mathbb{R}^s$ with usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_2$. We define the Fourier transform of an integrable function $f : \mathbb{R}^s \to \mathbb{C}$ by

$$\widehat{f}(\gamma) = \int_{\mathbb{R}^s} f(x) e^{-2\pi i x \cdot \gamma} dx \ (\gamma \in \mathbb{R}^s),$$

where $x \cdot \gamma$ is the usual inner product on $\mathbb{R}^s$ and we extend the Fourier transform on L_2 as usual. We say that a matrix A of order $s \times s$ is *expansive* if it has integer entries and the eigenvalues of A are bigger than one in modulus. By A^* we denote the Hermitian transpose of A. We define the dilation operator on L_2 with respect to an expansive matrix A by $D_A f = |\det A|^{1/2} f(A\cdot)$. The shift operator on L_2 is defined by $\tau_k f = f(\cdot - k),\ k \in \mathbb{Z}^s$.

Throughout this paper we assume that ϕ and ϕ^d are two functions in L_2 with the following properties:

(i) the functions $\widehat{\phi}$ and $\widehat{\phi^d}$ are continuous in a neighborhood of the origin and $\lim\limits_{\gamma \to 0} \widehat{\phi}(\gamma) = \lim\limits_{\gamma \to 0} \widehat{\phi^d}(\gamma) = 1$,

(ii) the $\mathbb{Z}^s$-periodic functions $\Phi = \sum_{k\in\mathbb{Z}^s} |\widehat{\phi}(\cdot+k)|^2$ and $\Phi^d = \sum_{k\in\mathbb{Z}^s} |\widehat{\phi}^d(\cdot+k)|^2$ belong in $L_\infty(\mathbb{T}^s)$, the space of all measurable essentially bounded functions on $\mathbb{T}^s = [0,1)^s$ and

(iii) the functions ϕ and ϕ^d are *refinable* with respect to an expansive matrix A, i.e. there exist two $\mathbb{Z}^s$-periodic functions H_0 and H_0^d in $L_2(\mathbb{T}^s)$ (the space of all square integrable functions on $\mathbb{T}^s$) called *low pass filters* or *refinement masks* such that

$$\widehat{\phi}(A^*\gamma) = H_0(\gamma)\widehat{\phi}(\gamma) \text{ and } \widehat{\phi}^d(A^*\gamma) = H_0^d(\gamma)\widehat{\phi}^d(\gamma)$$

up to a null set with respect to the Lebesgue measure on $\mathbb{R}^s$. For the above definition of Φ and Φ^d we denote the *spectrum* of ϕ and ϕ^d by

$$\sigma(\phi) = \{\gamma \in \mathbb{T}^s : \Phi(\gamma) \neq 0\} \text{ and } \sigma(\phi^d) = \{\gamma \in \mathbb{T}^s : \Phi^d(\gamma) \neq 0\}$$

and we denote by V_0 and V_0^d the closed linear span of the sets $\{\phi(\cdot - n) : n \in \mathbb{Z}^s\}$ and $\{\phi^d(\cdot - n) : n \in \mathbb{Z}^s\}$ respectively.

We also consider two finite sets of refinable functions in L_2 whose elements are called *wavelets*, namely $\Psi = \{\psi_i : i = 1, ..., m\}$ and $\Psi^d = \{\psi_i^d : i = 1, ..., m\}$ such that

$$\widehat{\psi}_i(A^*\gamma) = H_i(\gamma)\widehat{\phi}(\gamma) \text{ and } \widehat{\psi_i^d}(A^*\gamma) = H_i^d(\gamma)\widehat{\phi}^d(\gamma) \text{ a.e. } \gamma \in \mathbb{R}^s,$$

where H_i and H_i^d are $\mathbb{Z}^s$-periodic functions in $L_2(\mathbb{T}^s)$ called *high pass filters* or *wavelet masks*. For the above selection of wavelets $\psi \in \Psi$ and $\psi^d \in \Psi^d$ we denote the set

$$X_\Psi = \{\psi_{i,j,k} = D_A^j \tau_k \psi_i : j \in \mathbb{Z},\ k \in \mathbb{Z}^s, i = 1, ..., m\}.$$

The corresponding notation for the set X_{Ψ^d} is similar. The set X_Ψ is called a *homogeneous wavelet family* or *affine family* generated from the *mother wavelets* $\psi \in \Psi$. If there exist two positive constants c and C such that for any $f \in L_2$ we have

$$c\|f\|_2^2 \leq \sum_{\psi \in X_\Psi} |\langle f, \psi\rangle|^2 \leq C\|f\|_2^2,$$

then we say that X_Ψ is an *affine frame* or *a homogeneous wavelet frame* for L_2 and the elements of X_Ψ are called *framelets*. If $c = C$ then X_Ψ is a *tight* *frame* and if $c = C = 1$ then X_Ψ is a *Parseval frame*. On the other hand if only the right hand side of the above double inequality holds then we say that X_Ψ is a *Bessel system*. If both X_Ψ and X_{Ψ^d} are Bessel systems and for any $f \in L_2$ we have the reconstruction formula

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^s} \sum_{i=1}^{m} \langle f, \psi^d_{i,j,k}\rangle \psi_{i,j,k}$$

in the L_2-sense, then we say that X_{Ψ^d} is an *affine dual frame* of X_Ψ (and vice versa) or we simply say that (X_Ψ, X_{Ψ^d}) is a pair of *dual framelets*. We note that in the study of affine dual frames the Bessel property of a wavelet family is important [13, Theorem 2.3]. We also remark that the previous equation implies that each one of the two wavelet families is a frame for L_2 [27, Proposition 1]. If X_Ψ is a Riesz basis of L_2 then the unique dual Riesz basis of X_Ψ may not be a wavelet family [6, 8, 9]. Therefore the construction of an affine wavelet family which is dual to another affine wavelet family is not automatic.

On the other hand let φ, φ^d be two functions in L_2 (not necessarily equal to ϕ and ϕ^d) and $\Psi,\ \Psi^d$ be two sets of wavelet families as above. For any $j_0 \in \mathbb{Z}$ we denote the set:

$$X^{(j_0)}_{\varphi,\Psi} = \{D^j_A \tau_k \psi : j \geq j_0,\ k \in \mathbb{Z}^s,\ \psi \in \Psi\} \cup \{D^{j_0}_A \tau_k \varphi : k \in \mathbb{Z}^s\}. \tag{1.1}$$

The corresponding notation for the set $X^{(j_0)}_{\varphi^d,\Psi^d}$ is similar. The set $X^{(j_0)}_{\varphi,\Psi}$ is called a *nonhomegeneous wavelet family*. If there exist two positive constants c and C such that for any $f \in L_2$ we have

$$c\|f\|_2^2 \leq \sum_{j=j_0}^{\infty} \sum_{k\in\mathbb{Z}^s} \sum_{i=1}^{m} |\langle f, \psi_{i,j,k}\rangle|^2 + \sum_{k\in\mathbb{Z}^s} |\langle f, D^{j_0}_A \tau_k \varphi\rangle|^2 \leq C\|f\|_2^2,$$

then we say that the set $X^{(j_0)}_{\varphi,\Psi}$ is a *nonhomogeneous wavelet frame* for L_2. If both $X^{(j_0)}_{\varphi,\Psi}$ and $X^{(j_0)}_{\varphi^d,\Psi^d}$ are Bessel systems and for any $f \in L_2$ we have the reconstruction formula

$$f = \sum_{j=j_0}^{\infty} \sum_{k\in\mathbb{Z}^s} \sum_{i=1}^{m} \langle f, \psi^d_{i,j,k}\rangle \psi_{i,j,k} + \sum_{k\in\mathbb{Z}^s} \langle f, D^{j_0}_A \tau_k \varphi^d\rangle \tau_k \varphi$$

in the L_2-sense, then we say that $(X^{(j_0)}_{\varphi,\Psi}, X^{(j_0)}_{\varphi^d,\Psi^d})$ is a pair of *nonhomogeneous dual wavelet frames* for L_2. Nonhomogeneous dual wavelet frames are important because they are associated with filter banks and they have natural connections with refinable structures as noted in [24] where this type of wavelet frames was first introduced. Most notably, Bin Han was the one who coined the term 'nonhomogeneous' for this type of wavelet frames and who extensively studied them and under more general assumptions for dilations and refinable masks, in L_2 and in the space of distributions [15, 16]. In particular, Han proves that if $(X^{(j_0)}_{\varphi,\Psi}, X^{(j_0)}_{\varphi^d,\Psi^d})$ is a pair of nonhomogeneous dual wavelet frames for L_2, then (X_Ψ, X_{Ψ^d}) is a pair of affine dual frames for L_2 and he also establishes a connection between the Mixed Oblique Extension Principle and the former type of frames (see Remark 1). The connection between nonhomogeneous Parseval wavelet frames and their homogeneous counterparts in L_2, where $\varphi = \phi$, was first established in [28, Theorem 2.3]. The main result of this paper generalizes Theorem 2.3 of [28] to the fullest extend for pairs of dual homogeneous affine wavelet frames.

The following Theorem is one of the main results in [10] and is generalized by Theorem 2.1.

Proposition 1.1. *Let ϕ, ϕ^d be refinable functions as above with spectrum $\sigma(\phi)$ and $\sigma(\phi^d)$ respectively satisfying a mild decay condition on the Fourier domain, namely* (4.6) *in [26]. Let the sets Ψ and Ψ^d be defined above, their corresponding wavelet families X_Ψ and X_{Ψ^d} be Bessel systems and the masks $H_i, H_i^d,\ i = 0, ..., m$ be bounded on $\mathbb{T}^s$. Define a $\mathbb{Z}^d$-periodic function Θ_{M} called Mixed Fundamental function associated with the pair (X_Ψ, X_{Ψ^d}) by*

$$\Theta_{\mathrm{M}} = \sum_{j=0}^{\infty}\sum_{i=1}^{m} H_i(A^{*j}\cdot)\overline{H_i^d(A^{*j}\cdot)}\prod_{k=0}^{j-1} H_0(A^{*k}\cdot)\overline{H_0^d(A^{*k}\cdot)}. \tag{1.2}$$

Then the following conditions are equivalent:

(a) (X_Ψ, X_{Ψ^d}) *is a pair of dual framelets.*

(b) The Mixed Fundamental function Θ_{M} satisfies the following conditions:

(i) $\lim_{j\to-\infty} \Theta_{\mathrm{M}}(A^{*j}\gamma) = 1,$ *a.e.* $\gamma \in \mathbb{R}^s$ *and*

(ii) $\Theta_{\mathrm{M}}(A^*\gamma)H_0(\gamma)\overline{H_0^d(\gamma+n)} + \sum_{i=1}^m H_i(\gamma)\overline{H_i^d(\gamma+n)} = 0,$ *a.e.* $\gamma \in \sigma(\phi)\cap\sigma(\phi^d)$ *and for any $n \neq 0$ in $(A^{*-1}\mathbb{Z}^s)/\mathbb{Z}^s$ such that $\gamma + n \in \sigma(\phi)\cap\sigma(\phi^d)$.*

In case where $\Psi = \Psi^d$ then (1.2) becomes a non-negative function

$$\Theta = \sum_{j=0}^{\infty} \sum_{i=1}^{m} |H_i(A^{*j} \cdot)|^2 \prod_{k=0}^{j-1} |H_0(A^{*k} \cdot)|^2, \tag{1.3}$$

called *Fundamental function* of the set X_Ψ. Here we mention that Θ (resp. Θ^d) is finite for a.e $\gamma \in \sigma(\phi)$ (resp. $\sigma(\phi^d)$), see the proof of Lemma 2.1 below. Since $|\Theta_{\mathrm{M}}| \leq \sqrt{\Theta}\sqrt{\Theta^d}$ we deduce that Θ_{M} is a.e. finite on $\sigma(\phi) \cap \sigma(\phi^d)$. To avoid measurability problems we define $\Theta_{\mathrm{M}} = 0$ on $\mathbb{R}^s \backslash \sigma(\phi) \cap \sigma(\phi^d)$.

Proposition 1.1 is the cornerstone for the derivation of the Mixed Oblique Extension Principle [10, Corollary 5.3].

Proposition 1.2. MOEP *Suppose that ϕ, ϕ^d are refinable functions and X_Ψ, X_{Ψ^d} are wavelet families as in Proposition 1.1. If there exists a $\mathbb{Z}^s$-periodic function θ satisfying the following conditions:*

(i) θ is essentially bounded, continuous at the origin and $\theta(0) = 1$,

(ii) $\theta(A^*\gamma)H_0(\gamma)\overline{H_0^d(\gamma)} + \sum_{i=1}^m H_i(\gamma)\overline{H_i^d(\gamma)} = 1$ for a.e. $\gamma \in \sigma(\phi) \cap \sigma(\phi^d)$ and

(ii) $\theta(A^*\gamma)H_0(\gamma)\overline{H_0^d(\gamma + q)} + \sum_{i=1}^m H_i(\gamma)\overline{H_i^d(\gamma + q)} = 0$ for a.e. $\gamma \in \sigma(\phi) \cap \sigma(\phi^d)$ and for any $q \neq 0$ in $(A^{*-1}\mathbb{Z}^s)/\mathbb{Z}^s$ such that $\gamma + q \in \sigma(\phi) \cap \sigma(\phi^d)$,

then (X_Ψ, X_{Ψ^d}) is a pair of dual framelets.

The statements of Propositions 1.1 and 1.2 do not provide insight how to select Θ_{M}. However our generalization of Proposition 1.1 shows how to derive Θ_{M}.

Our main contribution is the generalization (Theorem 2.1) of the characterization of affine dual frames arising from a pair of refinable functions first proposed in [10] and in [27] and of the MOEP. More precisely, we prove that any pair of affine dual frames for L_2 gives rise to a pair of nonhomogeneous dual frames for L_2 and vice versa. In addition we establish under the most general assumptions how the Mixed Fundamental function Θ_{M} gives rise to a pair $(X_{\widetilde{\phi},\Psi}, X_{\widetilde{\phi}^d,\Psi^d})$ of nonhomogeneous dual frames for L_2 whenever (X_Ψ, X_{Ψ^d}) is a pair of affine dual frames for L_2, thus demonstrating the geometric significance of Θ_{M}. In fact, we will see below that Θ_{M} prescribes a suitable (though not unique in general) selection of $\widetilde{\phi} \in V_0$ and $\widetilde{\phi}^d \in V_0$, modifying the geometry of V_0 and V_0^d in a proper way so that perfect and stable reconstruction in L_2 can be obtained by using a suitable pair of dual frames of the type (1.1). This is important for practical implementations because when we perform a multiscale decomposition of an image there is always a "coarse-scale" residual. Without the knowledge of this residual we can not reconstruct our image from the various detail outputs. We also establish that the Mixed Oblique Extension Principle actually characterizes dual framelets. Notice that in Proposition 1.2 only the " $\Rightarrow$ " part was proved under a mild assumption on the decay rate of the generators ϕ, ϕ^d in the frequency domain. This assumption is removed here. Last but not least we generalize the characterization of affine Parseval frames [28, Theorem 2.3].

Structure of this paper: In Section 2 we provide new characterizations of dual framelets, see Theorem 2.1 for details. The first characterization (see equivalence (1)$\leftrightarrow$(2) of Theorem 2.1) is a generalization of Proposition 1.1. The second characterization (see equivalence (1)$\leftrightarrow$(3) of Theorem 2.1) reveals the geometric significance of the Mixed Fundamental function Θ_{M} associated with a pair of wavelet families (X_Ψ, X_{Ψ^d}) and generalizes the characterization of affine dual frames arising from a pair of refinable functions proved in [10] and in [27] as well as the MOEP (equivalence (1)$\leftrightarrow$(4) of Theorem 2.1). In Section 3 we generalize the characterization of affine Parseval wavelets arising from a refinable function proved in [28].

2. Main results

Let the sets X_Ψ and $X_{\varphi,\Psi}^{(j_0)}$ be defined in the previous section. Throughout the paper we use some well known results related to characterization of dual frames for L_2. In order for the paper to be self-contained we state and discuss these results below.

The quasi-affine analysis developed in [26] (see also [7]) connects tight framelets with tight shift invariant frames (i.e. frames being invariant under all integer translates of a countable set of generators) via the following characterization: X_Ψ is a tight frame for L_2 if and only if the set

$$X_\Psi^q = \{D_A^j \tau_k \psi : j \geq 0, k \in \mathbb{Z}^s, \psi \in \Psi\} \cup \{|\det A|^{j/2} \tau_k D_A^j \psi : j < 0, k \in \mathbb{Z}^s, \psi \in \Psi\} \tag{2.1}$$

(which is shift invariant under a proper selection of a countable set of generators) is a tight frame for L_2. A similar result is obtained in [7, 27] for dual framelets. This equivalence combined with another important characterization of dual shift invariant frames via their Mixed Gramian matrix [27, Lemma 1] leads to the following important characterization for dual framelets:

Proposition 2.1. *Let X_Ψ and X_{Ψ^d} be a pair of Bessel systems as above. Then (X_Ψ, X_{Ψ^d}) is a pair of dual framelets if and only if for a.e. $\gamma \in \mathbb{R}^s$ and for every $n \in \mathbb{Z}^s$ we have*

$$\sum_{j=\kappa(n)}^{\infty} \sum_{i=1}^{m} \widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi_i^d}(A^{*j}(\gamma+n))} = \delta_{0,n}, \tag{2.2}$$

where $\kappa : \mathbb{Z}^s \to \mathbb{Z}_- \cup \{0\} : \ \kappa(n) = \inf\{j \le 0 : \ A^{*j}n \in \mathbb{Z}^s\}$.

We note that Proposition 2.1 was proved in [12] and [27] under a mild decay condition on the elements of X_Ψ and X_{Ψ^d} in the frequency domain and then in [4] it was proved in full generality.

We will use the following relation between a pair of affine dual frames and a pair of nonhomogeneous dual frames for L_2:

Proposition 2.2. [16, Proposition 5] *If $(X^{(j_0)}_{\varphi,\Psi}, X^{(j_0)}_{\varphi^d,\Psi^d})$ is a pair of dual frames for L_2 for some $j_0 \in \mathbb{Z}$ then $(X^{(l)}_{\varphi,\Psi}, X^{(l)}_{\varphi^d,\Psi^d})$ is a pair of dual frames for L_2 for any $l \in \mathbb{Z}$. In particular (X_Ψ, X_{Ψ^d}) is a pair of dual framelets.*

The following Proposition provides a useful characterization of a pair of nonhomogeneous dual frames for L_2:

Proposition 2.3. *Let $(X^{(0)}_{\varphi,\Psi}, X^{(0)}_{\varphi^d,\Psi^d})$ be a pair of Bessel systems and let $\kappa : \mathbb{Z}^s \to \mathbb{Z}_- \cup \{0\} : \ \kappa(n) = \inf\{j \le 0 : \ A^{*j}n \in \mathbb{Z}^s\}$. Then $X^{(0)}_{\varphi^d,\Psi^d}$ is a dual frame of $X^{(0)}_{\varphi,\Psi}$ for L_2 if and only if for a.e. $\gamma \in \mathbb{R}^s$ and for every $n \in \mathbb{Z}^s$ we have*

$$\sum_{j=\kappa(n)}^{0} \sum_{i=1}^{m} \widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi_i^d}(A^{*j}(\gamma+n))} + \widehat{\varphi}(\gamma)\overline{\widehat{\varphi}^d(\gamma+n)} = \delta_{0,n}. \tag{2.3}$$

Proposition 2.3 is proved in [16, Theorems 9,11] for a pair of frequency based nonhomogeneous and non-stationary dual wavelet frames in the space of distributions under more general assumptions, i.e. convergence of (2.3) for

$n=0$ is in the sense of distributions. However, the Bessel assumption on the pair $(X^{(0)}_{\varphi,\Psi}, X^{(0)}_{\varphi^d,\Psi^d})$ guarantees that the series $\sum_{j=-\infty}^{0}\sum_{i=1}^{m}\widehat{\psi}_i(A^{*j}\cdot)\overline{\widehat{\psi}^d_i(A^{*j}\cdot)}$ is absolutely convergent for a.e. $\gamma\in\mathbb{R}^s$ to a bounded function, say g, a.e. If we denote by g_J ($J<0$) the partial sums of the left-hand side of (2.3) and by $G_J := g_J + \widehat{\varphi}(\cdot)\overline{\widehat{\varphi}^d(\cdot)}$ then $\lim_J G_J(\gamma) = g(\gamma)+\widehat{\varphi}(\gamma)\overline{\widehat{\varphi}^d(\gamma)}$ absolutely a.e. Hence, the function $G = g+\widehat{\varphi}(\cdot)\overline{\widehat{\varphi}^d(\cdot)}$ is locally integrable on $\mathbb{R}^s$. But, (2.3) indicates that the sequence $\{G_J\}_J$ converges to the constant function equal to one in the sense of distributions. If γ_0 is a Lebesgue point of G, then for every $r>0$, we can find a sequence of test functions h_n converging to $\chi_{B(\gamma_0,r)}$ pointwise. Moreover, we can select $0\le h_n\le \chi_{B(\gamma_0,r)}$. So, $\lim_J\int G_J h_n = \int h_n$ for every n and by Lebesgue's Dominated Convergence Theorem we derive $\lim_J \int G_J h_n = \int G h_n$ for every n. Therefore, $\int G h_n = \int h_n$ for every n, which again due to Lebesgue's Dominated Convergence Theorem implies $\int G\chi_{B(\gamma_0,r)} = \int \chi_{B(\gamma_0,r)}$. Since γ_0 is a Lebesgue point of G we infer, $G(\gamma_0)=1$. Combining this with the fact that every point of G is a Lebesgue point a.e. we obtain that $G=1$ pointwise a.e. Now, the proof of Proposition 2.3 follows easily from [16, Theorems 9,11].

The following Lemma is crucial for the proof of our main Theorem 2.1:

Lemma 2.1. *Let ϕ,ϕ^d be a pair of refinable functions as in Section 1 with spectrum $\sigma(\phi)$ and $\sigma(\phi^d)$ respectively and let (X_Ψ, X_{Ψ^d}) be a pair of wavelet Bessel systems with their corresponding Mixed Fundamental function Θ_{M} be given in (1.2). Consider the spaces V_0 and V_0^d as in Section 1. Then there exists a pair (μ,μ^d) of measurable and a.e. finite $\mathbb{Z}^s$-periodic functions such that*

- *(i)* $\Theta_{\mathrm{M}} = \mu\overline{\mu^d}$ *on* $\sigma(\phi)\cap\sigma(\phi^d)$,
- *(ii) the functions $\widetilde{\phi}$ and $\widetilde{\phi}^d$ defined by $\widehat{\widetilde{\phi}} = \mu\widehat{\phi}$ and $\widehat{\widetilde{\phi}^d} = \mu^d\widehat{\phi}^d$ belong in V_0 and V_0^d respectively and*
- *(iii) the shift invariant sets $\{\tau_k\widetilde{\phi} : k\in\mathbb{Z}^s\}$ and $\{\tau_k\widetilde{\phi}^d : k\in\mathbb{Z}^s\}$ form Bessel systems.*

Proof. Since X_Ψ is a Bessel system then X^q_Ψ (see (2.1)) is a Bessel system too [7], so for any $f\in L_2$ there exists a positive constant C such that

$$\sum_{j=1}^{\infty}\sum_{k}\sum_{i=1}^{m}|\langle f, |\det A|^{-j}\psi_i(A^{-j}(\cdot-k))\rangle|^2 \le C\|f\|_2^2,$$

or equivalently

$$\sum_{j=1}^{\infty}\sum_{i=1}^{m}\int_{\mathbb{T}^s}\Big|\sum_k \widehat{f}(\gamma+k)\overline{\widehat{\psi}_i(A^{*j}(\gamma+k))}\Big|^2 d\gamma \le C\|f\|_2^2.$$

Using the refinement equations on $\widehat{\psi}_i$ and then on $\widehat{\phi}$ the above inequality becomes

$$\int_{\mathbb{T}^s}\Theta(\gamma)\Big|\sum_k \widehat{f}(\gamma+k)\overline{\widehat{\phi}(\gamma+k)}\Big|^2 d\gamma \le C\|f\|_2^2, \tag{2.4}$$

where $\Theta = \sum_{j=0}^{\infty}\sum_{i=1}^{m}|H_i(A^{*j}\cdot)|^2\prod_{k=0}^{j-1}|H_0(A^{*k}\cdot)|^2$ is the Fundamental function of the set X_Ψ as in (1.3). The above inequality implies that the set $\{\phi^\dagger(\cdot-k) : k\in\mathbb{Z}^s\}$ is a Bessel system where $\widehat{\phi}^\dagger = \sqrt{\Theta}\widehat{\phi}$. Therefore $\Theta\sum_k|\widehat{\phi}(\cdot-k)|^2 \in L_\infty(\mathbb{T}^s)$. If Θ^d is the Fundamental function of the set X_{Ψ^d} then we may show that $\Theta^d\sum_k|\widehat{\phi}^d(\cdot-k)|^2\in L_\infty(\mathbb{T}^s)$ in a similar manner. Taking into account that $\sum_k|\widehat{\phi}(\cdot+k)|^2\in L_\infty(\mathbb{T}^s)$ (an assumption made at the beggining of Section 1) we obtain $\Theta<\infty$ a.e. $\gamma\in\sigma(\phi)$ otherwise $\Theta\sum_k|\widehat{\phi}(\cdot-k)|^2$ would be infinite on a set of positive measure, contradiction. Using the same arguments we can prove that $\Theta^d<\infty$ a.e. $\gamma\in\sigma(\phi^d)$. Since the Mixed Fundamental function Θ_{M} of the pair (X_Ψ, X_{Ψ^d}) satisfies $|\Theta_{\mathrm{M}}|\le\sqrt{\Theta}\sqrt{\Theta^d}$ we have $|\Theta_{\mathrm{M}}|<\infty$ a.e. $\gamma\in\sigma(\phi)\cap\sigma(\phi^d)$, hence Θ_{M} is a.e. finite on $\sigma(\phi)\cap\sigma(\phi^d)$ and it is also measurable on $\sigma(\phi)\cap\sigma(\phi^d)$ as the pointwise limit of measurable functions. We define now a pair (μ,μ^d) of $\mathbb{Z}^s$-periodic functions by

$$\mu(\gamma)=\begin{cases}\Theta_{\mathrm{M}}(\gamma)(\Theta^d(\gamma))^{-1/2}, & \gamma\in\sigma(\phi)\cap\sigma(\phi^d)\\ 0, & \gamma\in\mathbb{T}^s\backslash\sigma(\phi)\cap\sigma(\phi^d)\end{cases} \tag{2.5}$$

and

$$\mu^d(\gamma)=\begin{cases}(\Theta^d(\gamma))^{1/2}, & \gamma\in\sigma(\phi)\cap\sigma(\phi^d)\\ 0, & \gamma\in\mathbb{T}^s\backslash\sigma(\phi)\cap\sigma(\phi^d)\end{cases}. \tag{2.6}$$

We note that if $\Theta^d(\gamma_0)=0$ for some $\gamma_0\in\sigma(\phi)\cap\sigma(\phi^d)$ then necessarily $\Theta_{\mathrm{M}}(\gamma_0)=0$. In this case we use the convention $0/0=0$ in (2.5). From the above observations we deduce that the functions μ and μ^d are measurable and a.e. finite on $\mathbb{T}^s$. Also (i) is obviously satisfied. Let the functions $\widetilde{\phi},\widetilde{\phi}^d$ be defined by $\widehat{\widetilde{\phi}}=\mu\widehat{\phi}$ and $\widehat{\widetilde{\phi}}^d=\mu^d\widehat{\phi}^d$ respectively. If $E=\cup_{n\in\mathbb{Z}^s}(\sigma(\phi)\cap\sigma(\phi^d)+n)$ then by (2.5) we have

$$\int_{\mathbb{R}^s}|\widehat{\widetilde{\phi}}(\gamma)|^2 d\gamma = \int_{\mathbb{R}^s}|\mu(\gamma)\widehat{\phi}(\gamma)|^2 d\gamma \le \int_E \Theta(\gamma)|\widehat{\phi}(\gamma)|^2 d\gamma<\infty,$$

because $\Theta\sum_k|\widehat{\phi}(\cdot-k)|^2 \in L_\infty(\mathbb{T}^s)$ as we showed above and so $\Theta|\widehat{\phi}|^2 \in L_1(\mathbb{R}^s)$. Therefore $\widetilde{\phi} \in L_2$ and by using [11, Theorem 2.14] we obtain $\widetilde{\phi} \in V_0$. The proof for $\widetilde{\phi}^d$ is similar. Finally we consider the set $\{\tau_k\widetilde{\phi} : k \in \mathbb{Z}^s\}$. Then for any $f \in L_2$ we have

$$\begin{aligned}
&\sum_k |\langle f, \widetilde{\phi}(\cdot-k)\rangle|^2 = \int_{\mathbb{T}^s} \big|\mu(\gamma)\big|^2 \Big|\sum_k \widehat{f}(\gamma+k)\overline{\widehat{\phi}(\gamma+k)}\Big|^2 d\gamma \\
= &\int_{\sigma(\phi)\cap\sigma(\phi)^d} |\Theta_{\mathrm{M}}(\gamma)|^2|\Theta^d(\gamma)|^{-1}\Big|\sum_k \widehat{f}(\gamma+k)\overline{\widehat{\phi}(\gamma+k)}\Big|^2 d\gamma \\
\leq &\int_{\sigma(\phi)\cap\sigma(\phi)^d} \Theta(\gamma)\Big|\sum_k \widehat{f}(\gamma+k)\overline{\widehat{\phi}(\gamma+k)}\Big|^2 d\gamma \leq C\|f\|_2^2
\end{aligned}$$

as a result of (2.4). Hence the set $\{\tau_k\widetilde{\phi} : k \in \mathbb{Z}^s\}$ is a Bessel system and so (iii) is satisfied. We can show that the set $\{\tau_k\widetilde{\phi}^d : k \in \mathbb{Z}^s\}$ is a Bessel system too in a similar manner. We omit the proof here. $\square$

We are now ready to state our main Theorem involving new characterizations of affine dual frames constructed from refinable functions. Specifically, we will show that:

(i) any pair of homogeneous dual wavelet frames gives rise to a pair of nonhomogeneous dual wavelet frames and vice versa,

(ii) the Mixed Oblique Extension Principle characterizes dual framelets.

Theorem 2.1. *Let ϕ, ϕ^d be refinable functions and X_Ψ, X_{Ψ^d} be wavelet families as in Section 1. Then the following conditions are equivalent:*

(1) (X_Ψ, X_{Ψ^d}) is a pair of dual framelets.

(2) The Mixed Fundamental function Θ_{M} associated with a pair (X_Ψ, X_{Ψ^d}) of wavelet Bessel systems satisfies the following conditions:

(i) $\lim_{j\to-\infty} \Theta_{\mathrm{M}}(A^{*j}\gamma) = 1$ a.e. $\gamma \in \mathbb{R}^s$ and

(ii) $\Theta_{\mathrm{M}}(A^*\gamma)H_0(\gamma)\overline{H_0^d(\gamma+n)}+\sum_{i=1}^m H_i(\gamma)\overline{H_i^d(\gamma+n)} = 0$ for a.e. $\gamma \in \sigma(\phi)\cap\sigma(\phi^d)$ and for any $n \neq 0$ in $(A^{*-1}\mathbb{Z}^s)/\mathbb{Z}^s$ such that $\gamma+n \in \sigma(\phi)\cap\sigma(\phi^d)$.

(3) There exists a pair (μ, μ^d) of measurable and a.e. finite $\mathbb{Z}^s$-periodic functions such that

(i) the functions $\widetilde{\phi}$ *and* $\widetilde{\phi}^d$ *defined by* $\widehat{\widetilde{\phi}} = \mu\widehat{\phi}$ *and* $\widehat{\widetilde{\phi}^d} = \mu^d\widehat{\phi}^d$ *belong in* V_0 *and* V_0^d *respectively and*

(ii) $(X^{(0)}_{\widetilde{\phi},\Psi}, X^{(0)}_{\widetilde{\phi}^d,\Psi^d})$ *is a pair of nonhomogeneous dual frames for* L_2*.*

(4) (X_Ψ, X_{Ψ^d}) *is a pair of Bessel systems and there exists a measurable* $\mathbb{Z}^s$*-periodic function* θ *such that the sequence* $\left\{\|\theta\, \widehat{\phi}\, \overline{\widehat{\phi}^d(\cdot - k)}\|_{L_1} : k \in \mathbb{Z}^s\right\}$ *is bounded and* θ *satisfies the following conditions:*

(a) $\theta(\gamma) = \theta(A^*\gamma)H_0(\gamma)\overline{H_0^d(\gamma)} + \sum_{i=1}^m H_i(\gamma)\overline{H_i^d(\gamma)}$ *a.e.* $\gamma \in \sigma(\phi) \cap \sigma(\phi^d)$*,*

(b) $\theta(A^*\gamma)H_0(\gamma)\overline{H_0^d(\gamma + q)} + \sum_{i=1}^m H_i(\gamma)\overline{H_i^d(\gamma + q)} = 0$ *for a.e.* $\gamma \in \sigma(\phi) \cap \sigma(\phi^d)$ *and for any* $q \neq 0$ *in* $(A^{*-1}\mathbb{Z}^s)/\mathbb{Z}^s$ *such that* $\gamma + q \in \sigma(\phi) \cap \sigma(\phi^d)$ *and*

(c) $\lim\limits_{j\to-\infty} \theta(A^{*j}\gamma) = 1$ *a.e.* $\gamma \in \mathbb{R}^s$*.*

Before we proceed with the proof of Theorem 2.1 we discuss the above characterizations of dual framelets. The equivalence (1) $\leftrightarrow$ (2) provides a generalization of Proposition 1.1 originally appeared in [10, Proposition 5.2]. In this statement a mild decay assumption imposed on the refinable functions ϕ and ϕ^d in the Fourier domain was used in [25] to prove that X_Ψ is a frame (Bessel system) if and only if the set X^q_Ψ in (2.1) is a frame (Bessel system). However as we noted at the beginning of Section 2 this equivalence holds without this extra condition and indeed in equivalence (1) $\leftrightarrow$ (2) we prove that Proposition 1.1 holds without this condition as well. We recall that we need to amend the definition of the Mixed Fundamental function Θ_{M} so that $\Theta_{\mathrm{M}} = 0$ outside $\sigma(\phi) \cap \sigma(\phi^d)$. We need this technical modification because we don't assume any decay conditions on the generators ϕ and ϕ^d.

On the other hand the equivalence (1) $\leftrightarrow$ (3) of Theorem 2.1 reveals the nature of the Mixed Fundamental function Θ_{M}. Specifically in the proof (1) $\rightarrow$ (3) we show how Θ_{M} gives rise to the construction of two auxiliary refinable functions which generate a proper coarse scale residual when decomposing an input image into various scales. Those auxiliary functions are determined by (2.5) and (2.6).

Finally the equivalence (1) $\leftrightarrow$ (4) of Theorem 2.1 shows that the Mixed Oblique Extension Principle [10, Corollary 5.3] actually characterizes dual framelets.

Remark 1. We remark that the equivalence (3) $\leftrightarrow$ (4) is proved by Han [15, Theorem 2] for dimension 1 and [16, Theorem 17] for high dimensions in the space of distributions under the assumption that there exists a pair (μ, μ^d) of $\mathbb{Z}^s$-periodic functions generating the pair of distributions $(\widetilde{\phi}, \widetilde{\phi}^d)$ which define the pair $(X_{\widetilde{\phi},\psi}, X_{\widetilde{\phi}^d,\psi^d})$ of nonhomegeneous wavelet families. With this assumption Han defines $\theta = \mu\overline{\mu^d}$. The essence of our statement (4) $\rightarrow$ (3) is to show that any function θ satisfying the assumptions of (4) admits a factorization $\theta = \mu\overline{\mu^d}$ which gives rise to a pair (μ, μ^d) of $\mathbb{Z}^s$-periodic functions generating the pair of auxiliary functions $(\widetilde{\phi}, \widetilde{\phi}^d)$ in the spaces V_0 and V_0^d respectively which define the pair $(X_{\widetilde{\phi},\psi}, X_{\widetilde{\phi}^d,\psi^d})$ of nonhomegeneous wavelet families.

Proof. (1) $\leftrightarrow$ (2): Let Θ_{M} be the Mixed Fundamental function associated to a pair (X_Ψ, X_{Ψ^d}) of wavelet families. The direct (in case (2) holds) or implicit (in case (1) holds) Bessel assumption on the pair (X_Ψ, X_{Ψ^d}) implies that the sequence of functions $\{f^{(n)}(\cdot):\ n \in \mathbb{Z}^s\}$ defined by

$$f^{(n)}(\cdot) = \Theta_{\mathrm{M}}(\cdot)\widehat{\phi}(\cdot)\overline{\widehat{\phi}^d(\cdot + n)}$$

belongs in $L_1 \cap L_\infty(\mathbb{R}^s)$. In fact we have

$$\|f^{(n)}\|_{L_1} \leq \|\sqrt{\Theta}\widehat{\phi}\|_{L_2}\|\sqrt{\Theta^d}\widehat{\phi}^d\|_{L_2} < \infty,$$

because $\Theta \sum_k |\widehat{\phi}(\cdot - k)|^2 \in L_\infty(\mathbb{T}^s)$ and $\Theta^d \sum_k |\widehat{\phi}^d(\cdot - k)|^2 \in L_\infty(\mathbb{T}^s)$ as we showed in the proof of Lemma 2.1. We also have

$$|f^{(n)}(\cdot)| \leq |\Theta_{\mathrm{M}}(\cdot)| \sum_k |\widehat{\phi}(\cdot - k)\overline{\widehat{\phi}^d(\cdot - k + n)}|$$

and from the Cauchy-Schwartz inequality we obtain that $\{f^{(n)}(\cdot):\ n \in \mathbb{Z}^s\}$ is a sequence of functions in $L_\infty(\mathbb{R}^s)$.
We now consider an $s \times s$ expansive matrix A and we define the function $\kappa : \mathbb{Z}^s \to \mathbb{Z}_- \cup \{0\} :\ \kappa(n) = \inf\{j \leq 0 :\ A^{*j}n \in \mathbb{Z}^s\}$. Given any integer n we define a sequence of functions $\{f_j^{(n)}(\cdot):\ j \geq \kappa(n) + 1\}$ by

$$f_j^{(n)}(\cdot) = f^{(A^{*j}n)}(A^{*j}\cdot) = \Theta_{\mathrm{M}}(A^{*j}\cdot)\widehat{\phi}(A^{*j}\cdot)\overline{\widehat{\phi}^d(A^{*j}(\cdot + n))},$$

where the functions $f^{(k)} \in L_1 \cap L_\infty(\mathbb{R}^s)$ are as above. Then it is easy to derive the following equality for a.e. $\gamma \in \mathbb{R}^s$ and for any $j \geq \kappa(n)+1$:

$$\sum_{i=1}^{m} \widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi}_i^d(A^{*j}(\gamma+n))} = \sum_{\lambda=j}^{\infty}\sum_{i=1}^{m} \widehat{\psi}_i(A^{*\lambda}\gamma)\overline{\widehat{\psi}_i^d(A^{*\lambda}(\gamma+n))}$$
$$- \quad \sum_{\lambda=j+1}^{\infty}\sum_{i=1}^{m} \widehat{\psi}_i(A^{*\lambda}\gamma)\overline{\widehat{\psi}_i^d(A^{*\lambda}(\gamma+n))} = f_{j-1}^{(n)}(\gamma) - f_j^{(n)}(\gamma). \tag{2.7}$$

From Proposition 2.1 it is clear that (1) is equivalent to (2.2). We take $n=0$ in (2.2). Then $\kappa(0) = -\infty$ and by (2.7) we obtain

$$\sum_{j=-\infty}^{\infty}\sum_{i=1}^{m} \widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi}_i^d(A^{*j}\gamma)} = \lim_{N,M\to+\infty}\sum_{j=-M}^{N}(f_{j-1}^{(0)}(\gamma) - f_j^{(0)}(\gamma))$$
$$= \quad \lim_{N,M\to+\infty}(f_{-M-1}^{(0)}(\gamma) - f_N^{(0)}(\gamma)) = \lim_{M\to+\infty} f_{-M-1}^{(0)}(\gamma),$$

because $\lim_{N\to+\infty} f_N^{(0)}(\gamma) = \lim_{N\to+\infty} f^{(0)}(A^{*N}\gamma) = 0$ since $f^{(0)} \in L_1(\mathbb{R}^s)$ as we showed above. From assumption (a) in Section 1 we infer $\lim_{M\to+\infty} f_{-M-1}^{(0)}(\gamma) = 1$ if and only if $2(i)$ holds.

If $n \neq 0$, then $\kappa(n)$ is finite and by working as above we obtain

$$\sum_{j=\kappa(n)}^{\infty}\sum_{i=1}^{m} \widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi}_i^d(A^{*j}(\gamma+n))} = \sum_{i=1}^{m} \widehat{\psi}_i(A^{*\kappa(n)}\gamma)\overline{\widehat{\psi}_i^d(A^{*\kappa(n)}(\gamma+n))}$$
$$+ \quad f_{\kappa(n)}^{(n)}(\gamma) = \Big(\sum_{i=1}^{m} H_i(A^{*(\kappa(n)-1)}\gamma)\overline{H_i^d(A^{*(\kappa(n)-1)}(\gamma+n))}$$
$$+ \quad \Theta_{\mathrm{M}}(A^{*\kappa(n)}\gamma)H_0(A^{*(\kappa(n)-1)}\gamma)\overline{H_0^d(A^{*(\kappa(n)-1)}(\gamma+n))}\Big)$$
$$\times \quad \widehat{\phi}(A^{*(\kappa(n)-1)}\gamma)\overline{\widehat{\phi}^d(A^{*(\kappa(n)-1)}(\gamma+n))}, \quad \text{a.e. } \gamma. \tag{2.8}$$

Since $q = A^{*(\kappa(n)-1)}n \in A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s - \{0\}$ (recall the definition of $\kappa(n)$), if $2(ii)$ holds then it is easy to see that (2.8) is equal to zero for a.e. $\gamma \in \mathbb{R}^s$. On the other hand let us assume that (2.8) is equal to zero for every $n \neq 0$ and for a.e. $\gamma \in \mathbb{R}^s$. Then for any $q \neq 0$ in $A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s$ and for every $r \in \mathbb{Z}$ we take $n = A^*(r+q)$. For this selection of n we have $\kappa(n) = 0$ and so (2.8) can be written by

$$\Big(\sum_{i=1}^{m} H_i(A^{*-1}\gamma)\overline{H_i^d(A^{*-1}\gamma+q))}+\Theta_{\mathrm{M}}(\gamma)H_0(A^{*-1}\gamma)\overline{H_0^d(A^{-1}\gamma+q)}\Big)$$
$$\times \quad \widehat{\phi}(A^{*-1}\gamma)\overline{\widehat{\phi}^d(A^{*-1}\gamma+r+q)}=0, \text{ a.e. } \gamma\in\mathbb{R}^s \text{ and } r\in\mathbb{Z}^s.$$

Let $A^{*-1}\gamma \in \sigma(\phi)\cap\sigma(\phi^d)$ such that $A^{*-1}\gamma+q \in \sigma(\phi)\cap\sigma(\phi^d)$ for any $q\neq 0$ in $A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s$. Then there exists a pair (k_0,λ_0) of integers such that $\widehat{\phi}(A^{*-1}\gamma+k_0)\overline{\widehat{\phi}^d(A^{*-1}\gamma+\lambda_0+q)}\neq 0$, otherwise the left hand side of the above equality is equal to zero. By substituting $A^{*-1}\gamma$ with $A^{*-1}\gamma+k_0$ and r with λ_0-k_0 we obtain $2(ii)$. Therefore (2.8) is equal to zero for a.e. $\gamma\in\mathbb{R}^s$ if and only if $2(ii)$ holds and so (2.2) is true if and only if both $2(i)$ and $2(ii)$ are satisfied. Therefore the proof $(1)\leftrightarrow(2)$ is complete.

$(1)\leftrightarrow(3)$: If (1) holds then from (2.2) the following equality holds for a.e. $\gamma\in\mathbb{R}^s$ and for any $n\in\mathbb{Z}^s$:

$$\sum_{j=\kappa(n)}^{0}\sum_{i=1}^{m}\widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi}_i^d(A^{*j}(\gamma+n))}+\sum_{j=1}^{\infty}\sum_{i=1}^{m}\widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi}_i^d(A^{*j}(\gamma+n))}=\delta_{0,n}. \tag{2.9}$$

We work with the second term of (2.9). Define the set $E=\cup_{n\in\mathbb{Z}^s}\big(\sigma(\phi)\cap\sigma(\phi^d)+n\big)$ and take $j\geq 1$. Then for a.e. $\gamma\in\mathbb{R}^s\backslash E$ we have either $\widehat{\psi}_i(A^{*j}\gamma)=\ldots=\widehat{\psi}_i(A^{*}\gamma)=\widehat{\phi}(\gamma)=0$ or $\widehat{\psi}_i^d(A^{*j}(\gamma+n))=\ldots=\widehat{\psi}_i^d(A^{*}(\gamma+n))=\widehat{\phi}^d(\gamma+n)=0$ otherwise the refinement equations would not hold on a set of positive measure. On the other hand for a.e. $\gamma\in E$ we use the refinement equations and we obtain

$$\sum_{j=1}^{\infty}\sum_{i=1}^{m}\widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi}_i^d(A^{*j}(\gamma+n))}=\Theta_{\mathrm{M}}(\gamma)\widehat{\phi}(\gamma)\overline{\widehat{\phi}^d(\gamma+n)},$$

where Θ_{M} is the Mixed Fundamental function associated with the pair of wavelet families (X_Ψ, X_{Ψ^d}) as in (1.2) and it is well defined for a.e. $\gamma\in E$ as we showed in Lemma 2.1. Combining these two observations we obtain

$$\sum_{j=1}^{\infty}\sum_{i=1}^{m}\widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi}_i^d(A^{*j}(\gamma+n))}=\begin{cases}\Theta_{\mathrm{M}}(\gamma)\widehat{\phi}(\gamma)\overline{\widehat{\phi}^d(\gamma+n)}, & \text{a.e.}\gamma\in E\\ 0, & \text{a.e.}\gamma\in\mathbb{R}^s\backslash E\end{cases}.$$

Using Lemma 2.1 the above equality can be written by

$$\sum_{j=1}^{\infty}\sum_{i=1}^{m}\widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi}_i^d(A^{*j}(\gamma+n))}=\widehat{\widetilde{\phi}}(\gamma)\overline{\widehat{\widetilde{\phi}}^d(\gamma+n)} \tag{2.10}$$

for a.e. $\gamma\in\mathbb{R}^s$, where $\widetilde{\phi}$ and $\widetilde{\phi}^d$ are two functions in V_0 and V_0^d respectively defined by $\widehat{\widetilde{\phi}}=\mu\widehat{\phi}$ and $\widehat{\widetilde{\phi}}^d=\mu^d\widehat{\phi}^d$ for some pair (μ,μ^d) of measurable $\mathbb{Z}^s$-periodic functions as in (2.5) and (2.6). Furthermore in Lemma 2.1 we proved that the sets $\{\tau_k\widetilde{\phi}:k\in\mathbb{Z}^s\}$ and $\{\tau_k\widetilde{\phi}^d:k\in\mathbb{Z}^s\}$ are Bessel systems and so the sets $X^{(0)}_{\widetilde{\phi},\Psi}$ and $X^{(0)}_{\widetilde{\phi}^d,\Psi^d}$ (which are of type (1.1)) are Bessel systems too. Substituting (2.10) in (2.9) we get

$$\sum_{j=\kappa(n)}^{0}\sum_{i=1}^{m}\widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi}_i^d(A^{*j}(\gamma+n))}+\widehat{\widetilde{\phi}}(\gamma)\overline{\widehat{\widetilde{\phi}}^d(\gamma+n)}=\delta_{0,n}$$

for a.e. $\gamma\in\mathbb{R}^s$ and $n\in\mathbb{Z}^s$ and the result is obtained as a direct application of (2.3). On the other hand if (3) holds then (1) holds as a result of Proposition 2.2. Therefore the proof of the equivalence (1) $\leftrightarrow$ (3) is complete.

(1) $\leftrightarrow$ (4): If (1) holds then we will show that (4) is satisfied by $\theta=\Theta_{\mathrm{M}}$. We already know from Lemma 2.1 that the Mixed Fundamental function Θ_{M} is measurable and well defined for a.e. $\gamma\in\sigma(\phi)\cap\sigma(\phi^d)$. Then Θ_{M} satisfies $4(a)$ (see [10, eq. 5.1]) and $4(b)$ and $4(c)$ by equivalence (1) $\leftrightarrow$ (2). Finally we showed in the proof of equivalence (1) $\leftrightarrow$ (2) that the sequence $\left\{\|\Theta_{\mathrm{M}}\,\widehat{\phi}\,\overline{\widehat{\phi}^d(\cdot-k)}\|_{L_1}:\ k\in\mathbb{Z}^s\right\}$ is bounded.
On the other hand suppose that (4) holds. If we are able to prove that $\theta=\Theta_{\mathrm{M}}$ for a.e. $\gamma\in\sigma(\phi)\cap\sigma(\phi^d)$ then the proof will be complete due to equivalence (1) $\leftrightarrow$ (2). This equality has been already proved implicitly in [10, Corollary 5.3] under the assumption that θ is bounded on $\mathbb{T}^s$ and in [28] under the assumption that $\theta\in L_1(\mathbb{T}^s)$. Using arguments from these two proofs we see below that the same result holds for a more general assumption on θ, namely the sequence $\left\{\|\theta\,\widehat{\phi}\,\overline{\widehat{\phi}^d(\cdot-k)}\|_{L_1}:k\in\mathbb{Z}^s\right\}$ is bounded.
Let $j\geq 1$ and $\gamma\in\sigma(\phi)\cap\sigma(\phi^d)$. Using assumption $4(a)$ we obtain by induction the equality

$$\theta(\gamma)=\theta(A^{*j}\gamma)\prod_{l=0}^{j-1}H_0(A^{*l}\gamma)\overline{H_0^d(A^{*l}\gamma)}+\Theta_{\mathrm{M},j}(\gamma), \tag{2.11}$$

where $\Theta_{\mathrm{M},j}(\gamma) = \sum_{n=0}^{j-1}\sum_{i=1}^{m} H_i(A^{*n}\cdot)\overline{H_i^d(A^{*n}\cdot)}\prod_{k=0}^{n-1} H_0(A^{*k}\cdot)\overline{H_0^d(A^{*k}\cdot)}$ provided that $A^{*k}\gamma \in \sigma(\phi)\cap\sigma(\phi^d)$ for any $0 < k \leq j-1$.
If $A^{*j}\gamma \notin \sigma(\phi)\cap\sigma(\phi^d)$, then necessarily $H_0(A^{*j-1}\gamma)\overline{H_0^d(A^{*j-1}\gamma)} = 0$ and so (2.11) becomes $\theta(\gamma) = \Theta_{\mathrm{M},j}(\gamma) = \Theta_{\mathrm{M}}(\gamma)$.
We consider the case $A^{*j}\gamma \in \sigma(\phi)\cap\sigma(\phi^d)$ for any $j \geq 0$. Obviously there exists a pair of integers (k_0, m_0) such that $\widehat{\phi}(\gamma+k_0) \neq 0$ and $\widehat{\phi}^d(\gamma+m_0) \neq 0$. For this selection of k_0, m_0 we have

$$\begin{aligned} & \Big(\theta(A^{*j}\gamma)\prod_{l=0}^{j-1} H_0(A^{*l}\gamma)\overline{H_0^d(A^{*l}\gamma)}\Big)\widehat{\phi}(\gamma+k_0)\overline{\widehat{\phi}^d(\gamma+m_0)} \\ = \ & \theta(A^{*j}\gamma)\widehat{\phi}(A^{*j}(\gamma+k_0))\overline{\widehat{\phi}^d(A^{*j}(\gamma+m_0))}. \end{aligned} \tag{2.12}$$

But

$$\begin{aligned} & \int_{\mathbb{R}^s} |\theta(A^{*j}\omega)\widehat{\phi}(A^{*j}(\omega+k_0))\overline{\widehat{\phi}^d(A^{*j}(\omega+m_0))}|d\omega \\ = \ & \frac{1}{|\mathrm{det}A|^j}\int_{\mathbb{R}^s} |\theta(\omega)\widehat{\phi}(\omega)\overline{\widehat{\phi}^d(\omega+A^{*j}(m_0-k_0))}|d\omega \leq \frac{C}{|\mathrm{det}A|^j} \end{aligned} \tag{2.13}$$

due to the assumption that the sequence $\big\{\|\theta\ \widehat{\phi}\ \overline{\widehat{\phi}^d(\cdot-k)}\|_{L_1} : k \in \mathbb{Z}^s\big\}$ is bounded. By letting $j \to +\infty$ we have

$$\|\theta(A^{*j}\cdot)\widehat{\phi}(A^{*j}(\cdot+k_0))\overline{\widehat{\phi}^d(A^{*j}(\cdot+m_0))}\|_{L_1} \to 0$$

and from the monotone convergence theorem we conclude that

$$\lim_{j\to+\infty} \theta(A^{*j}\omega)\widehat{\phi}(A^{*j}(\omega+k_0))\overline{\widehat{\phi}^d(A^{*j}(\omega+m_0))} = 0$$

pointwise a.e. on $\mathbb{R}^s$. Using this fact, by taking limits in both sides of (2.12) we obtain $\lim\limits_{j\to+\infty} \theta(A^{*j}\gamma)\prod_{l=0}^{j-1} H_0(A^{*l}\gamma)\overline{H_0^d(A^{*l}\gamma)} = 0$ a.e. $\gamma \in\ \sigma(\phi)\cap\sigma(\phi^d)$ and by substituting this in (2.11) we get $\theta(\gamma) = \Theta_{\mathrm{M}}(\gamma)$ a.e. $\gamma \in \sigma(\phi)\cap\sigma(\phi^d)$. Therefore the proof of equivalence (1) $\leftrightarrow$ (4) is complete. $\square$

An immediate result emerges naturally from this Theorem.

Corollary 2.1. *(X_Ψ, X_{Ψ^d}) is a pair of dual framelets if and only if there exists a pair (μ, μ^d) of measurable and a.e. finite $\mathbb{Z}^s$-periodic functions such that the functions $\widehat{\widetilde{\phi}} = \mu\widehat{\phi}$ and $\widehat{\widetilde{\phi}}^d = \mu^d\widehat{\phi}^d$ belong in V_0 and V_0^d respectively and for some/any $j_0 \in \mathbb{Z}$ the pair $(X^{(j_0)}_{\widetilde{\phi},\Psi}, X^{(j_0)}_{\widetilde{\phi}^d,\Psi^d})$ is a pair of dual frames for L_2.*

Proof. Combine the equivalence (1) $\leftrightarrow$ (3) of Theorem 2.1 with Proposition 2.2. $\square$

From Proposition 2.2 we know that if $(X^{(0)}_{\varphi,\Psi}, X^{(0)}_{\varphi^d,\Psi^d})$ is a pair of dual frames for L_2 then (X_Ψ, X_{Ψ^d}) is a pair of dual framelets. If (X_Ψ, X_{Ψ^d}) is a pair of dual framelets and φ, φ^d are equal to ϕ, ϕ^d respectively, what can we infer about the sets $X^{(0)}_{\phi,\Psi}$ and $X^{(0)}_{\phi^d,\Psi^d}$? Below we provide a sufficient condition so that the sets $X^{(0)}_{\phi,\Psi}$ and $X^{(0)}_{\phi^d,\Psi^d}$ form frames for L_2 (but not necessarily dual to each other) provided that (X_Ψ, X_{Ψ^d}) is a pair of dual framelets.

Corollary 2.2. *If (X_Ψ, X_{Ψ^d}) is a pair of dual framelets and if the $\mathbb{Z}^s$-periodic functions μ and μ^d as in* (3) *of Theorem 2.1 are essentially bounded then the sets $X^{(0)}_{\phi,\Psi}$ and $X^{(0)}_{\phi^d,\Psi^d}$ form frames for L_2.*

Proof. By the equivalence (1) $\leftrightarrow$ (3) of Theorem 2.1 there exists a pair $(\widetilde{\phi}, \widetilde{\phi}^d)$ of functions in L_2 such that $(X^{(0)}_{\widetilde{\phi},\Psi}, X^{(0)}_{\widetilde{\phi}^d,\Psi^d})$ is a pair of dual frames for L_2. Since $X^{(0)}_{\widetilde{\phi},\Psi}$ is a Bessel system and the function $\sum_k |\widehat{\phi}(\cdot + k)|^2$ is essentially bounded on $\mathbb{T}^s$ we infer that $X^{(0)}_{\phi,\Psi}$ is a Bessel system too. On the other hand for any $f \in L_2$ we have

$$
\begin{aligned}
\sum_{\psi \in X_\Psi} |\langle f, \psi\rangle|^2 + \sum_n |\langle f, \tau_n \phi\rangle|^2 &= \sum_{\psi \in X_\Psi} |\langle f, \psi\rangle|^2 \\
+ & \int_{\sigma(\phi)} \big| \sum_n \widehat{f}(\gamma+n)\overline{\widehat{\phi}(\gamma+n)}\big|^2 d\gamma \\
\geq & \sum_{\psi \in X_\Psi} |\langle f, \psi\rangle|^2 + \frac{1}{\|\mu\|^2_\infty} \int_{\sigma(\phi)} \big| \sum_n \widehat{f}(\gamma+n)\overline{\widehat{\phi}(\gamma+n)}\big|^2 |\mu(\gamma)|^2 d\gamma \\
\geq & \min\big\{1, \frac{1}{\|\mu\|^2_\infty}\big\}\Big(\sum_{\psi \in X_\Psi} |\langle f, \psi\rangle|^2 + \sum_n |\langle f, \tau_n \widetilde{\phi}\rangle|^2\big) \geq c\|f\|_2^2
\end{aligned}
$$

for some positive constant c, because the set $X^{(0)}_{\widetilde{\phi},\Psi}$ is a frame for L_2 by assumption. The proof for $X^{(0)}_{\phi^d,\Psi^d}$ is similar, hence omitted. $\square$

3. Affine Parseval frames and further discussion

In this section we discuss the case of affine Parseval frames. We note that one cannot derive the characterization of affine Parseval frames of Ron and

Shen [26, Theorem 6.5] directly from equivalence (1) $\leftrightarrow$ (2) of Theorem 2.1 because Theorem 2.1 requires X_Ψ to be a Bessel system an assumption not included in the statement of Theorem 6.5 in [26]. However this result and [28, Theorem 2.3] as well can be generalized by a slight modification of the steps of the proof of Theorem 2.1. Moreover the next Proposition reveals the geometric significance of the Fundamental function Θ of a wavelet family X_Ψ to its full extent.

Proposition 3.1. *Let ϕ be a refinable function and Ψ be a set of wavelets as in Section 1. Then the following conditions are equivalent:*

(1) X_Ψ is a Parseval frame for L_2.

(2) The Fundamental function Θ of X_Ψ satisfies the following conditions:

(a) $\lim_{j\to-\infty} \Theta(A^{*j}\gamma) = 1$ a.e. $\gamma \in \mathbb{R}^s$,

(b) $\Theta(A^*\gamma)H_0(\gamma)\overline{H_0(\gamma+q)} + \sum_{i=1}^m H_i(\gamma)\overline{H_i(\gamma+q)} = 0$ for a.e. $\gamma \in \sigma(\phi)$ and for all $q \neq 0$ in $(A^{*-1}\mathbb{Z}^s)/\mathbb{Z}^s$ such that $\gamma + q \in \sigma(\phi)$ and

(c) $\Theta|\widehat{\phi}|^2 \in L_1(\mathbb{R}^s)$.

(3) There exists a measurable and a.e. finite $\mathbb{Z}^s$-periodic function μ such that the function $\widetilde{\phi}$ defined by $\widehat{\widetilde{\phi}} = \mu\widehat{\phi}$ belongs in the space $V_0 = \overline{span}\{\phi(\cdot - n) : n \in \mathbb{Z}^s\}$ and $X_{\widetilde{\phi},\Psi}$ is a Parseval frame for L_2.

(4) There exists a measurable $\mathbb{Z}^s$-periodic function θ such that

$$\theta(\gamma) = \theta(A^*\gamma)|H_0(\gamma)|^2 + \sum_{i=1}^m |H_i(\gamma)|^2 \quad a.e. \ \gamma \in \sigma(\phi) \tag{3.1}$$

and θ satisfies the above properties (a)-(c).

Proof. (1) $\leftrightarrow$ (2): We recall that

$$\sum_{j=\kappa(n)}^{\infty} \sum_{i=1}^{m} \widehat{\psi}_i(A^{*j}\gamma)\overline{\widehat{\psi}_i(A^{*j}(\gamma+n))} = \delta_{0,n} \text{ a.e. } \gamma \in \mathbb{R}^s \tag{3.2}$$

if and only if X_Ψ is a Parseval frame for L_2, see [28, Proposition 2.2] or [26]. If (2) holds then (3.2) holds too (modify the proof of equivalence (1) $\leftrightarrow$ (2)

of Theorem 2.1 for the case $\Psi = \Psi^d$ and $\phi = \phi^d$) and so X_ψ is a Parseval frame for L_2. On the other hand, if X_ψ is a Parseval frame for L_2 then (3.2) is satisfied and furthermore $\Theta|\widehat{\phi}|^2 \in L_1(\mathbb{R}^s)$ by Lemma 2.1. From these two observations we derive that $2(a)$ and $2(b)$ are satisfied by following the steps of the proof of equivalence (1) $\leftrightarrow$ (2) of Theorem 2.1 once again.

(1) $\leftrightarrow$ (3): It is a special case of the proof of equivalence (1) $\leftrightarrow$ (3) of Theorem 2.1 for the case $\Psi = \Psi^d$, $\phi = \phi^d$ and $\mu = \mu^d = \Theta^{1/2}$.

(1) $\leftrightarrow$ (4): If (4) holds then we observe that (3.1) and the assumption $\Theta|\widehat{\phi}|^2 \in L_1(\mathbb{R}^s)$ ensure that $\theta = \Theta$ on $\sigma(\phi)$ (see (2.12) and (2.13) for the case $\phi = \phi^d$ and $k_0 = m_0$). The rest follow by the proof of equivalence (1) $\leftrightarrow$ (4) of Theorem 2.1 for the case $\phi = \phi^d$ and $\Psi = \Psi^d$. $\square$

We note that the equivalence (1) $\leftrightarrow$ (2) under an additional smoothness assumption on the decay rate of the elements of Ψ was shown in [26, Theorem 6.5]. We remark that part (3) of Proposition 3.1 as noted in [28] shows that the Fundamental function Θ is associated with a weak form of an orthogonalization process in case where the affine framelets are constructed from a refinable function ϕ. We note that [26, Theorem 6.5] is not more general than the equivalence (II)$\leftrightarrow$(III) of [28, Theorem 2.3]. But [28, Theorem 2.3] is the first work to directly show the geometric significance of Extension Principles.

Closing this work we discuss a possibly technical point. Lemma 2.1 states that there exist $\mathbb{Z}^s$-periodic measurable functions μ, μ^d such that $\Theta_{\mathrm{M}} = \mu\overline{\mu^d}$ on $\sigma(\phi) \cap \sigma(\phi^d)$ and provides a pair of such functions in its proof. How many different choices of μ, μ^d can we have though?

Let $\sigma(\phi) \cap \sigma(\phi^d) \subset \mathbb{T}^s$. From the definition of μ, μ^d in (2.5) and (2.6) and the proof of Lemma 2.1 we can see that we are free to define μ, μ^d on $\mathbb{T}^s \backslash \sigma(\phi) \cap \sigma(\phi^d)$ as long as

(i) measurability is maintained and

(ii) the sets $\{\widetilde{\phi}(\cdot - k)\}_k$ and $\{\widetilde{\phi}^d(\cdot - k)\}_k$ are Bessel systems.

We denote by $\mathcal{M}$ (resp. $\mathcal{M}^d$) the class of functions μ (resp. μ^d) such that

$$\mu(\gamma) = \begin{cases} \Theta_{\mathrm{M}}(\gamma)(\Theta^d(\gamma))^{-1/2}, & \gamma \in \sigma(\phi) \cap \sigma(\phi^d) \\ \alpha(\gamma), & \gamma \in \mathbb{T}^s \backslash \sigma(\phi) \cap \sigma(\phi^d) \end{cases}$$

and

$$\mu^d(\gamma) = \begin{cases} (\Theta^d(\gamma))^{1/2}, & \gamma \in \sigma(\phi) \cap \sigma(\phi^d) \\ \alpha^d(\gamma), & \gamma \in \mathbb{T}^s \backslash \sigma(\phi) \cap \sigma(\phi^d) \end{cases},$$

where $\alpha, \alpha^d \in L_\infty(\mathbb{T}^s \backslash \sigma(\phi) \cap \sigma(\phi^d))$. Then every pair $(\mu, \mu^d) \in \mathcal{M} \times \mathcal{M}^d$ satisfies the above assumptions (i) and (ii) and moreover we have

$$|\mu|^2 \sum_k |\widehat{\phi}(\cdot + k)|^2 \in L_\infty(\mathbb{T}^s) \text{ and } |\mu^d|^2 \sum_k |\widehat{\phi}^d(\cdot + k)|^2 \in L_\infty(\mathbb{T}^s), \qquad (3.3)$$

because the sets $\{\widetilde{\phi}(\cdot - k) : k \in \mathbb{Z}^s\}$ and $\{\widetilde{\phi}^d(\cdot - k) : k \in \mathbb{Z}^s\}$ are Bessel systems as we can see from a slight modification of the proof of Lemma 2.1. In addition if $\mu \in \mathcal{M}_+$ and $\mu^d \in \mathcal{M}_+^d$ (meaning that both μ and μ^d do not vanish on $\sigma(\phi)$ and $\sigma(\phi^d)$ respectively), then their corresponding function $\widetilde{\phi}$ and $\widetilde{\phi}^d$ are generators for the spaces $V_0 = \overline{span}\{\phi(\cdot - k)\}$ and $V_0^d = \overline{span}\{\phi^d(\cdot - k)\}$ but they are not necessarily stable generators. However if the shifts of ϕ and ϕ^d form frames for the spaces V_0 and V_0^d then both such functions μ, μ^d belong to $L_\infty(\mathbb{T}^s)$ as a result of (3.3) and moreover if $1/\mu, 1/\mu^d$ are essentially bounded on $\sigma(\phi)$ and $\sigma(\phi^d)$ respectively then the functions $\widetilde{\phi}, \widetilde{\phi}^d$ are stable generators for the spaces V_0 and V_0^d.

If $\sigma(\phi) \cap \sigma(\phi^d) = \mathbb{T}^s$ which is satisfied in most examples of interest then the functions μ, μ^d are uniquely defined by (2.5) and (2.6) up to a null set. In this case if the shifts of ϕ, ϕ^d form a Riesz basis for V_0, V_0^d respectively then the functions $\mu \in \mathcal{M}$ and $\mu^d \in \mathcal{M}^d$ belong to $L_\infty(\mathbb{T}^s)$ and so $\Theta_{\mathrm{M}} \in L_\infty(\mathbb{T}^s)$. In addition if $\mu \in \mathcal{M}_+$ and $\mu^d \in \mathcal{M}_+^d$ and if $1/\mu, 1/\mu^d \in L_\infty(\mathbb{T}^s)$ then the shifts of their corresponding functions $\widetilde{\phi}, \widetilde{\phi}^d$ form a Riesz basis for the spaces V_0 and V_0^d.

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References

[1] A. Aldroubi, M. Papadakis, Characterization and parametrization of multiwavelet bases, in: A. Aldroubi, E.B. Lin, (Eds.), Wavelets, Multiwavelets and Applications, Contemp. Math., 11, (1998), 97–116.

[2] J.J. Benedetto, O.M. Treiber., Wavelet frames: Multiresolution Analysis and Extension Principles, in: Lokenath Debnath (Ed.), Wavelet

Transforms and Time-Frequency Signal Analysis, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2001, pp. 3–36.

[3] J.J. Benedetto and S. Li, The theory of multiresolution frames and applications to filter banks, Appl. Comput. Harmon. Anal. 5 (1998) 389–427.

[4] M. Bownik, A Characterization of Affine Dual Frames in $L_2(\mathbb{R}^n)$, Appl. Comput. Harmon. Anal. 8 (2) 203–221 (2000).

[5] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.

[6] C.K. Chui and X.L. Shi, Inequalities of Littlewood-Paley Type for Frames and Wavelets, SIAM J. Math. Anal. 24 263–277 (1993).

[7] C.K. Chui, X. Shi and J. Stöckler, Affine frames, quasi-affine frames and their duals, Adv. Comput. Math. 8 (1-2) (1998) 1–17.

[8] I. Daubechies, The Wavelet Transform, Time-Frequency Localization and Signal Analysis, IEEE Trans. Inform. Theory 36 (5) (1990) 961–1005.

[9] I. Daubechies, B. Han, The Canonical Dual Frame of a Wavelet Frame, Applied Comput. Harmon. Anal. 12 (3) (2002) 269–285.

[10] I. Daubechies, B. Han, A. Ron and Z. Shen, Framelets: MRA based constructions of wavelet frames, Applied Comput. Harmon. Anal. 14 (1) (2003) 1–46.

[11] C. De Boor, R.A. DeVore and A. Ron, Approximation From Shift-Invariant Subspaces of $L_2(\mathbb{R}^d)$, Trans. Amer. Math. Soc. 341 (2) (1994) 787–806.

[12] B. Han, On Dual Wavelet Tight Frames, Appl. Comput. Harmon. Anal. 4 (4) (1997) 380–413.

[13] B. Han, Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix, J. Comput. Appl. Math. 155 (1) (2003) 43–67.

[14] I. Daubechies, B. Han, Pairs of dual wavelet frames from any two refinable functions, Constr. Approx. 20 (3) (2004) 325–352.

[15] B. Han, Pairs of frequency-based nonhomogeneous dual wavelet frames in the distribution space, Appl. Comput. Harmon. Anal. 29 (3) (2010) 330–353.

[16] B. Han, Nonhomogeneous wavelet systems in high dimensions, Appl. Comput. Harmon. Anal. 32 (2) (2012) 169–196.

[17] M. Frazier, B. Jawerth, and G. Weiss. Littlewood-Paley Theory and the study of functions spaces, Volume 79 of NSF-CBMS Regional Conf. ser. in Mathematics. Amer. Math. Soc., Providence, RI, 1991.

[18] S. Jain, M. Papadakis, E. Dussaud, Explicit schemes in seismic migration and isotropic multiscale representations. Radon transforms, geometry, and wavelets, Contemp. Math., 464, 177–200, Amer. Math. Soc., Providence, RI, 2008.

[19] S. Li. A theory of generalized multiresolution structure and pseudoframes of translates, J. Fourier Anal. Appl. 7 (1) (2001) 23–40.

[20] S. Li and H. Ogawa, Pseudoframes for subspaces with applications, J. Fourier Anal. Appl. 10 (4) (2004) 409–432.

[21] M. Paluszyńnski, H. Šikiċ, G. Weiss and S. Xiao, Generalized low pass filters and MRA frame wavelets, J. Geom. Anal. 11 (2001) 311-342.

[22] M. Papadakis. Frames of translates and the generalized frame multiresolution analysis, in: K. Kopotun, T. Lyche and M. Neamtu (Eds.) Trends in Approximation Theory, Innovations in Applied Mathematics, Vanderbilt University Press, Nashville, TN, 2001, pp. 353–362.

[23] Richard A. Zalik, Bases of translates and multiresolution analyses, Applied Comput. Harmon. Anal. 8 (1) (2008) 41–57.

[24] J. Romero, S. Alexander, S. Baid, S. Jain, and M. Papadakis, The geometry and the analytic properties of isotropic Multiresolution Analysis, Adv. Comput. Math. 31 (1-3) (2009) 283–328.

[25] A. Ron and Z. Shen, Frames and stable bases for shift invariant subspaces of $L_2(\mathbb{R}^d)$, Canad. J. Math. 47 (5) (1995) 1051–1094.

[26] A. Ron and Z. Shen, Affine Systems in $L_2(\mathbb{R}^d)$: The Analysis of the Analysis Operator, J. Funct. Anal. 148 (1997), 408–447.

[27] A. Ron and Z. Shen, Affine Systems in $L_2(\mathbb{R}^d)$ II: Dual systems, J. Fourier Anal. Appl. 3 (5) (1997) 617–637.

[28] T. Stavropoulos, The Geometry of Extension Principles, Houston J. Math. 38 (3) (2012) 833–853.