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EXTENSION PRINCIPLES FOR DUAL MULTIWAVELET FRAMES OF $L_2(\mathbb{R}^s)$ CONSTRUCTED FROM MULTIREFINABLE GENERATORS

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Abstract. In this work we prove that any pair of homogeneous dual multiwavelet frames of $L_2(\mathbb{R}^s)$ constructed from a pair of refinable function vectors gives rise to a pair of nonhomogeneous dual multiwavelet frames and vice versa. We also prove that the Mixed Oblique Extension Principle characterizes dual multiwavelet frames. Our results extend recent characterizations of affine dual frames derived from scalar refinable functions obtained in [3].

Keywords: Framelets, Refinable functions vectors, Multiwavelets, Mixed Fundamental function, Extension Principles.

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1. Introduction

Among a variety of multiscale representations, multiwavelet constructions started appearing in mid 1990's, e.g. see [1, 2, 25, 40, 50, 16,

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17, 18, 5, 9, 20, 53, 47, 38] and references therein. Multiwavelets are generalizations of orthonormal wavelets associated with multiresolution analysis (MRA). They are useful because they possess properties not simultaneously shared by scalar wavelets [17, 18, 24, 45, 46, 47]. Orthonormal multiwavelets, in particular, first arise in the context of MRA in multidimensions ([22, 39, 7, 16, 27, 13, 4]). In the case of an MRA generated by a single orthonormal scaling function with respect to some dilation matrix A, a precise number of $(\mathrm{Det}(A) - 1)$ orthonormal multiwavelets are required ([25, 34, 36]). Non-orthonormal multiwavelets arising from a function which may only be refinable are formally introduced in the work of Ron and Shen [50] and independently by Chui, He and Stockler [18] who proposed a general construction strategy of multiwavelets. Recall that a refinable function may only satisfy a two-scale relation, whereas a scaling function is refinable and its integer shifts form at least a frame sequence for their closed linear span.

The construction of refinable function vectors (which are finite sets of jointly refinable functions) from matrix-valued low pass filters was studied in [24, 30, 55, 16, 45, 46]. On the other hand, the construction of multiwavelets from refinable functions (or refinable function vectors) in higher dimensions appears to be much more elaborate and technical than in dimension 1 (see the discussion in [7]), especially for non-separable multiwavelets. Non separable designs in multidimensions arise from the need to address the spatial organization of singularities along curves or planes which tensor products of one dimensional wavelets fail to do and to avoid the reconstruction errors in non-preferred orientations [7]. Non separable designs such as directional transforms of the sort of Curvelets [12], Shearlets [29] and more generally α-molecules [28] are mostly carried out in the frequency domain resulting in wavelets and other similar atoms with unbounded support in space. This property is less desirable because it reduces the accuracy of reconstructions.

One of the available tools for (non-separable) wavelet designs in any number of dimensions is Extension Principles. Extension Principles were first proposed by Ron and Shen [50, 51] and subsequently were extended by Daubechies *et al.* [20] in the form of the Oblique Extension Principle. They are mainly used for the construction of dual affine multiwavelet systems (framelets) arising from a pair of refinable functions (see [22, 9, 14, 17, 18, 21, 32, 34, 50, 51]). Extension Principles provide great flexibility allowing the designed framelets to often combine several desirable properties. Although, there are still only a few examples of affine multiwavelet frames genuinely constructed via Extension Principles, the generality of the assumptions and the need to start only from refinable functions allows room for more classes of this type. This is our motivation to study the problem of Extension Principles in the most general context:

Starting from a pair of finite length, square integrable, refinable function vectors $(\phi_1, ..., \phi_r)$ and $(\phi_1^d, ..., \phi_r^d)$ (we give the precise definition of refinability later in this section), where $\phi_i, \phi_i^d \in L^2(\mathbb{R}^s)$, we expect to have even more freedom to design framelets with desirable properties rather than starting from a pair of refinable functions. This anticipated higher degree of freedom is based on the fact that these functions are not necessarily individually refinable, but they are jointly refinable. *The goal of this paper is to develop the theory of the refinable vector valued Extension Principles in the most general setting.* Thus, we are mainly concerned with the development of the theory of the refinable vector valued Extension Principles and we demonstrate a few basic non-separable multiwavelet examples in order to illustrate the proposed Extension Principles.

More specifically, in this paper we establish the equivalence between, homogeneous and nonhomogeneous dual affine multiwavelet frames (both, derived from the same pair of finite length refinable function vectors) and the Mixed Oblique Extension Principles. We do that by producing a suitable matrix-valued factorization of the (Mixed) Fundamental function which was introduced by Ron and Shen [50] but it also appears in [18] as the Vanishing Moment Recovery function compensating for the loss of vanishing moments in tight framelet designs. This factorization "adjusts" the original pair $(\phi_1, ..., \phi_r), (\phi_1^d, ..., \phi_r^d)$ and yields a "dual" pair of "coarse" scale residuals in the multiscale decomposition of an arbitrary square-integrable function with respect to the dual nonhomogeneous affine multiwavelet frames. To the best of our knowledge, we are the first to establish the equivalence between Oblique Extension Principles (for refinable function vectors) with homogeneous and nonhomogeneous dual affine multiwavelet frames of $L^2(\mathbb{R}^s)$ arising from $(\phi_1, ..., \phi_r), (\phi_1^d, ..., \phi_r^d)$. We also highlight the role of the (Mixed) Fundamental function in the geometry of the dual MRA structure by showing that the geometry of the multiscale structure with the coarse residual(s) in the analog domain defined by the nonhomogeneous (dual) multi-wavelet frames of $L^2(\mathbb{R}^s)$ faithfully reflects the geometry of the exact reconstruction filter bank defined by the low and high pass masks of the refinement equations. We are not the first to propose the use of refinable function vectors. In fact, our work Theorem 2 generalizes

Theorem 3.4 in [6] and also the main results of [44]. Similar existence results but not as general as those in [6] can be found in [15, 8]. In the cases considered in both of these papers the Fundamental function is identically equal to one. Our viewpoint is in line with [3]. We focus on extending the results of [3] for refinable function vectors thus underscoring the geometric role of the Fundamental and Mixed Fundamental functions.

We launch the presentation of our work with some necessary notation. Let E be a measurable subset of $\mathbb{R}^s$ and $\mathbb{C}^{m\times n}(E)$ be the space of all measurable $m \times n$ matrix valued functions defined on E, i.e.

$$\mathbb{C}^{m\times n}(E) = \big\{ f = (f_{ij}) : \ i = 1, ..., m, \ j = 1, ..., n : \quad f_{ij} : E \to \mathbb{C} \big\}.$$

From now on we write $\mathbb{C}^{m\times 1}(E) := \mathbb{C}^m(E)$ for brevity. For any $1 \leq p \leq \infty$ we define the Banach space

$$L_p^{m\times n}(E) = \big\{ f = (f_{ij}) : \ i = 1, ..., m, \ j = 1, ..., n : \quad f_{ij} \in L_p(E) \big\}$$

with norm

$$\|f\|^p_{L_p^{m\times n}(E)} = \sum_{i=1}^{m} \sum_{j=1}^{n} \|f_{ij}\|^p_{L_p(E)},$$

where $L_p(E)$ is the Banach space of all p-integrable functions on E with usual norm $\|\cdot\|_{L_p(E)}$. When $E = \mathbb{R}^s$ we write $L_p^{m\times n} := L_p^{m\times n}(\mathbb{R}^s)$ for brevity. We also denote by $\ell_p(I)$ the Banach space of all p-summable sequences on an index set $I \subseteq \mathbb{Z}$ with usual norm $\|\cdot\|_{\ell_p(I)}$.

The Fourier transform of a function $f \in L_1^{m\times n}(\mathbb{R}^s)$ is defined by

$$\widehat{f}(\gamma) = \int_{\mathbb{R}^s} f(x) e^{-2\pi i x\cdot\gamma} dx = \begin{pmatrix} \widehat{f}_{11}(\gamma) & \cdots & \widehat{f}_{1n}(\gamma) \\ \vdots & \ddots & \vdots \\ \widehat{f}_{m1}(\gamma) & \cdots & \widehat{f}_{mn}(\gamma) \end{pmatrix},$$

where $x\cdot\gamma$ is the usual inner product on $\mathbb{R}^s$ and $\widehat{f_{ij}}$ is the usual Fourier transform of $f_{ij} \in L_1(\mathbb{R}^s)$.

Let A be an $n \times n$ matrix over $\mathbb{C}$. We write $\mathcal{N}_A$ (or $\mathcal{R}_A$) for the kernel (or range) of A in case where the matrix A is considered as a linear operator on the Euclidean space $\mathbb{C}^n$ consisting of all complex sequences of length n. By A^T, A^* and $A^\dagger$ we denote the transpose, Hermitian transpose and partial inverse matrix of A respectively. If $A = A^*$ then we say that A is Hermitian. If A is Hermitian, then we say that A is positive semi-definite (or positive definite) whenever $c^*Ac \geq 0$ (or $c^*Ac > 0$) for any column vector $c \in \mathbb{C}^n$. We say that A is *expansive* if it has integer entries and the modulus of all of the eigenvalues of

A is bigger than one. We define the dilation operator on $L_2(\mathbb{R}^s)$ with respect to an $s \times s$ expansive matrix A by $D_A f = |\det A|^{1/2} f(A\cdot)$. The shift operator on $L_2(\mathbb{R}^s)$ is defined by $\tau_k f = f(\cdot - k),\ k \in \mathbb{Z}^s$.

For any $f, g \in L_2 := L_2(\mathbb{R}^s)$ we define the *bracket product* $[f, g]$ by

$$[f, g] : \mathbb{T}^s \to \mathbb{C} : \ [f, g](\gamma) = \sum_{k \in \mathbb{Z}^s} f(\gamma + k)\overline{g(\gamma + k)}.$$

If $f \in L_2$, $h \in L_2^{r\times 1}$ we define the *bracket product function vector* $[[f, h]]$ by

$$[[f, h]] : \mathbb{T}^s \to \mathbb{C}^r : \quad [[f, h]](\gamma) = \big([f, h_1](\gamma), ..., [f, h_r](\gamma)\big)^T.$$

Clearly $[f, g] \in L_1(\mathbb{T}^s)$ and $[[f, h]] \in L_1^{r\times 1}(\mathbb{T}^s)$, where $\mathbb{T}^s = [0, 1)^s$.

Throughout this paper let $I_r = \{1, ..., r\}$ be a finite set and $\Phi = (\phi_1, ..., \phi_r)^T$, $\Phi^d = (\phi_1^d, ..., \phi_r^d)^T$ be two function vectors in $L_2^{r\times 1}$ such that:

(i) their Fourier transforms $\widehat{\Phi} = (\widehat{\phi}_1, \ldots, \widehat{\phi}_r)^T$ and $\widehat{\Phi}^d = (\widehat{\phi}_1^d, \ldots, \widehat{\phi}_r^d)^T$ are continuous in a neighborhood of the origin and do not vanish at the origin (i.e. $\widehat{\Phi}(0) \neq \mathbf{0}$ and $\widehat{\Phi}^d(0) \neq \mathbf{0}$),

(ii) the $\mathbb{Z}^s$-periodic functions

$$\mathbf{\Phi} = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^s} |\widehat{\phi}(\cdot + k)|^2 \text{ and } \mathbf{\Phi}^d = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^s} |\widehat{\phi}^d(\cdot + k)|^2$$

belong in $L_\infty(\mathbb{T}^s)$, the space of all measurable essentially bounded functions on $\mathbb{T}^s = [0, 1)^s$,

(iii) Φ, Φ^d are *refinable function vectors* with respect to an $s \times s$ expansive matrix A, i.e. there exist two $\mathbb{Z}^s$-periodic matrix valued functions $H_0, H_0^d \in L_2^{r\times r}(\mathbb{T}^s)$ called *low pass filters* or *refinement masks* or *symbols* such that the following matrix equalities

$$\widehat{\Phi}(A^*\gamma) = H_0(\gamma)\widehat{\Phi}(\gamma) \text{ and } \widehat{\Phi}^d(A^*\gamma) = H_0^d(\gamma)\widehat{\Phi}^d(\gamma)$$

are satisfied up to a null set with respect to the Lebesgue measure on $\mathbb{R}^s$. For the above definition of $\mathbf{\Phi}$ and $\mathbf{\Phi}^d$ we denote the *spectrum* of Φ and Φ^d by

$$\sigma_\Phi = \{\gamma \in \mathbb{T}^s : \mathbf{\Phi}(\gamma) \neq 0\} \text{ and } \sigma_{\Phi^d} = \{\gamma \in \mathbb{T}^s : \mathbf{\Phi}^d(\gamma) \neq 0\}$$

and we denote by V_0 and V_0^d the closed linear span of the sets $\{\tau_k\phi : \ k \in \mathbb{Z}^s, \phi \in \Phi\}$ and $\{\tau_k\phi^d : \ k \in \mathbb{Z}^s, \phi^d \in \Phi^d\}$ respectively.

(iv) The value $G(\gamma)$ of the *Gram* matrix

$$G = \left([\widehat{\phi}_i, \widehat{\phi}_j]\right)_{j,i=1,\dots,r} \tag{1.1}$$

is non singular for a.e. $\gamma \in \sigma_\Phi$. Here we note that the value $G(\gamma)$ at $\gamma \in \sigma_\phi \subseteq \mathbb{T}^s$ is the $r \times r$ Hermitian matrix

$$G(\gamma) = \left([\widehat{\phi}_i, \widehat{\phi}_j](\gamma)\right)_{j,i=1,\dots,r}.$$

The operator norm of $G(\gamma)$ is

$$\|G(\gamma)\| = \Lambda_G(\gamma),$$

where $\Lambda_G(\gamma)$ (resp. $\lambda_G(\gamma)$) is the largest (resp. smallest) eigenvalue of $G(\gamma)$. We note that $\{\tau_k\phi : \ k \in \mathbb{Z}^s, \phi \in \Phi\}$ is a Bessel set for L_2 if and only if the above condition (ii) holds or equivalently

$$\|\Lambda_G\|_{L_\infty(\sigma_\Phi)} < \infty,$$

i.e. G is bounded as an operator from $L_2^{r\times 1}(\mathbb{T}^s)$ to $L_2^{r\times 1}(\mathbb{T}^s)$, [49].

Given a finite natural number m, we also consider two sets of refinable function vectors in $L_2^{m\times 1}$ called *multiwavelets*, namely $\Psi = (\psi_1, ..., \psi_m)^T$ and $\Psi^d = (\psi_1^d, ..., \psi_m^d)^T$ such that the following matrix equalities

$$\widehat{\Psi}(A^*\gamma) = H_1(\gamma)\widehat{\Phi}(\gamma) \text{ and } \widehat{\Psi}^d(A^*\gamma) = H_1^d(\gamma)\widehat{\Phi}^d(\gamma) \text{ a.e. } \gamma \in \mathbb{R}^s$$

hold for another two $\mathbb{Z}^s$-periodic matrix valued functions $H_1, H_1^d \in L_2^{m\times r}(\mathbb{T}^s)$ called *high pass filters* or *wavelet masks*. For the above selection of the set Ψ we define its corresponding *homogeneous wavelet family* or *affine family* X_Ψ by

$$X_\Psi = \{\psi_{i,j,k} = D_A^j \tau_k \psi_i : j \in \mathbb{Z},\ k \in \mathbb{Z}^s, i = 1, ..., m\} \tag{1.2}$$

with a similar notation for the set X_{Ψ^d}. If there exist two positive constants c and C such that for any $f \in L_2$ we have

$$c\|f\|_2^2 \le \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^s} \sum_{i=1}^m |\langle f, \psi_{i,j,k}\rangle|^2 \le C\|f\|_2^2,$$

then we say that X_Ψ is an *affine frame* or *a homogeneous wavelet frame* for L_2 and the elements of X_Ψ are called *framelets*. If $c = C$ then X_Ψ is a *tight frame* and if $c = C = 1$ then X_Ψ is a *Parseval frame*. On the other hand if only the right hand side of the above double inequality holds then we say that X_Ψ is a *Bessel system*. If both X_Ψ and X_{Ψ^d} are Bessel systems and for any $f \in L_2$ we have the reconstruction formula

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^s} \sum_{i=1}^{m} \langle f, \psi^d_{i,j,k} \rangle \psi_{i,j,k}$$

in the L_2-sense then we say that X_{Ψ^d} is an *affine dual frame* of X_Ψ (and vice versa) or we simply say that (X_Ψ, X_{Ψ^d}) is a pair of *dual framelets*. We also remark that the previous equation implies that each one of the two wavelet families is a frame for L_2 [51, Proposition 1].

On the other hand let $\widetilde{\Phi} = (\varphi_1, ..., \varphi_r)^T$ and $\widetilde{\Phi}^d = (\varphi^d_1, ..., \varphi^d_r)^T$ be refinable function vectors (not necessarily equal to Φ and Φ^d) and Ψ, Ψ^d be wavelet sets as above. For any $j_0 \in \mathbb{Z}$ we denote a *nonhomogeneous wavelet family* $X^{(j_0)}_{\widetilde{\Phi},\Psi}$ by

$$\begin{aligned} X^{(j_0)}_{\widetilde{\Phi},\Psi} &= \{D^j_A \tau_k \psi_i : j \geq j_0,\ k \in \mathbb{Z}^s,\ i = 1, ..., m\} \\ &\cup\ \{D^{j_0}_A \tau_k \varphi_l : k \in \mathbb{Z}^s,\ l = 1, ..., r\} \end{aligned} \tag{1.3}$$

and we use a similar notation for the set $X^{(j_0)}_{\widetilde{\Phi}^d,\Psi^d}$. Nonhomogeneous dual wavelet frames are naturally related with filter banks and refinable structures [33, 34, 48]. Bin Han was the first who used the term *nonhomogeneous* for this type of wavelet systems and who extensively studied them and under more general assumptions for dilations and refinable masks, in L_2 and in the space of distributions [33, 34]. In particular, Han proved that if $(X^{(j_0)}_{\widetilde{\Phi},\Psi}, X^{(j_0)}_{\widetilde{\Phi}^d,\Psi^d})$ is a pair of nonhomogeneous dual wavelet frames for L_2, then (X_Ψ, X_{Ψ^d}) is a pair of affine dual frames for L_2. He also established a connection between the Mixed Oblique Extension Principle and the former type of frames. Note that an equivalence between nonhomogeneous Parseval wavelet frames and their homogeneous counterparts in $L_2(\mathbb{R})$ for a scalar refinable function $\varphi = \phi$ was first established in [52, Theorem 2.3] and then was extended in [3]. Our main Theorem 1 stated below provides characterizations of affine dual frames generalizing these aforementioned results.

Theorem 1. *Let Φ, Φ^d be $r \times 1$ refinable function vectors with lowpass filters $H_0, H^d_0 \in L_2^{r\times r}(\mathbb{T}^s)$ satisfying the above conditions $(i)-(iv)$. Consider a pair (X_Ψ, X_{Ψ^d}) of homogeneous wavelet families with masks $H_1, H^d_1 \in L_2^{m\times r}(\mathbb{T}^s)$ as in (1.2). Define a $\mathbb{Z}^d$-periodic $r \times r$ matrix valued function Θ_{M} called Mixed Fundamental function associated with* *the pair* (X_Ψ, X_{Ψ^d}) *by*

$$\Theta_{\mathrm{M}} : \mathbb{T}^s \to \mathbb{C}^{r\times r} : \quad \Theta_{\mathrm{M}}(\gamma) = \sum_{j=0}^{\infty} \theta_j^{d*}(\gamma)\theta_j(\gamma), \tag{1.4}$$

where $\theta_j : \mathbb{T}^s \to \mathbb{C}^{m\times r} : \quad \theta_j(\gamma) = H_1(A^{*j}\gamma)\prod_{k=0}^{j-1} H_0(A^{*j-1-k}\gamma)$ *for any* $j \geq 0,$ (1.5)

θ_j^d *is defined in a similar manner and with the conventions* $\prod_{k=0}^{-1} H_0(A^{*-1-k}\gamma) = I_r$ *(*I_r *is the* $r\times r$ *identity matrix) and* $\theta_j^{d*}(\gamma) := \left(\theta_j^d(\gamma)\right)^*$ *(from now this convention will be used several times). Then the following conditions are equivalent:*

(a) (X_Ψ, X_{Ψ^d}) *is a pair of dual framelets.*
(b) (X_Ψ, X_{Ψ^d}) *is a pair of wavelet Bessel systems whose Mixed Fundamental function* Θ_{M} *satisfies the following conditions:*
  (i) $\lim_{j\to-\infty} \widehat{\Phi}^{d*}(A^{*j}\gamma)\Theta_{\mathrm{M}}(A^{*j}\gamma)\widehat{\Phi}(A^{*j}\gamma) = 1$ *a.e.* $\gamma \in \mathbb{R}^s$ *and*
  (ii) $H_0^{d*}(\gamma+q)\Theta_{\mathrm{M}}(A^*\gamma)H_0(\gamma) + H_1^{d*}(\gamma+q)H_1(\gamma) = \mathbf{0}$, *for a.e.* $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ *such that* $\gamma + q \in \sigma_\Phi \cap \sigma_{\Phi^d}$ *for any* $q \neq \mathbf{0}$ *in* $A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s$.
(c) *Let the spaces* V_0 *and* V_0^d *be the closed linear span of the sets* $\{\tau_k\phi : \ k \in \mathbb{Z}^s, \phi \in \Phi\}$ *and* $\{\tau_k\phi^d : \ k \in \mathbb{Z}^s, \phi^d \in \Phi^d\}$ *respectively. Then there exists a pair* (μ, μ^d) *of* $\mathbb{Z}^s$*-periodic* $r \times r$ *matrix valued functions with measurable and a.e. finite entries such that*
  (i) *the* $r \times 1$ *function vectors* $\widetilde{\Phi}$ *and* $\widetilde{\Phi}^d$ *defined by* $\widehat{\widetilde{\Phi}} = \mu\widehat{\Phi}$ *and* $\widehat{\widetilde{\Phi}^d} = \mu^d\widehat{\Phi}^d$ *belong in* V_0 *and* V_0^d *respectively and*
  (ii) $(X^{(0)}_{\tilde{\Phi},\Psi}, X^{(0)}_{\tilde{\Phi}^d,\Psi^d})$ *(see (1.3)) is a pair of nonhomogeneous dual frames for* L_2.
(d) (X_Ψ, X_{Ψ^d}) *is a pair of wavelet Bessel systems and there exists a* $\mathbb{Z}^s$*-periodic* $r \times r$ *matrix valued function* θ *with measurable entries such that the sequence* $\left\{\left\|\widehat{\Phi}^{d*}(\cdot - k)\theta(\cdot)\widehat{\Phi}(\cdot)\right\|_{L_1} : k \in \mathbb{Z}^s\right\}$ *is bounded and* θ *satisfies the conditions:*
  (i) $\lim_{j\to-\infty} \widehat{\Phi}^{d*}(A^{*j}\gamma)\theta(A^{*j}\gamma)\widehat{\Phi}(A^{*j}\gamma) = 1$ *a.e.* $\gamma \in \mathbb{R}^s$ *and*
  (ii) $H_0^{d*}(\gamma+q)\theta(A^*\gamma)H_0(\gamma) + H_1^{d*}(\gamma+q)H_1(\gamma) = \delta_{q,\mathbf{0}}\,\theta(\gamma)$ *for a.e.* $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ *such that* $\gamma + q \in \sigma_\Phi \cap \sigma_{\Phi^d}$ *for any* q *in* $A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s$. *Here,* $\delta_{q,\mathbf{0}} = 1$ *whenever* $q = \mathbf{0}$ *and* $\delta_{q,\mathbf{0}} = 0$ *for* $q \in A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s - \{\mathbf{0}\}$.

  In this case $\Theta_M = \theta$ *a.e. on* $\sigma_\Phi \cap \sigma_{\Phi^d}$.

If $\Psi = \Psi^d$ then (1.4) becomes the $r \times r$ matrix valued function

$$\Theta : \mathbb{T}^s \to \mathbb{C}^{r\times r} : \quad \Theta(\gamma) = \sum_{j=0}^{\infty} \theta_j^*(\gamma)\theta_j(\gamma), \tag{1.6}$$

called *Fundamental function* of the set X_Ψ. Here, we mention that $\Theta(\gamma)$ (resp. $\Theta^d(\gamma)$) is a well defined Hermitian and positive semi-definite matrix for a.e $\gamma \in \sigma_\Phi$ (resp. σ_{Φ^d}), see the proof of Proposition 1 below. Consequently, as we also show in the proof of Proposition 1, the matrix $\Theta_{\mathrm{M}}(\gamma)$ is well defined for a.e. γ in $\sigma_\Phi \cap \sigma_{\Phi^d}$. To avoid measurability problems, from now on we set $\Theta_{\mathrm{M}}(\gamma) = 0$ for a.e. $\gamma \in \mathbb{R}^s \backslash (\sigma_\Phi \cap \sigma_{\Phi^d})$.

In contrast to [20, Proposition 5.2], the equivalence $(a) \leftrightarrow (b)$ of Theorem 1 holds without any decay assumptions on the refinable function vectors Φ, Φ^d on the Fourier domain. On the other hand, in the proof of equivalence $(a) \leftrightarrow (c)$ we will show how the Mixed Fundamental function Θ_{M} gives rise to the construction of two auxiliary function vectors which generate a proper coarse scale residual when decomposing an input function into various scales. This is important for practical implementations because when we perform a multiscale decomposition of an image there is always a "coarse-scale" residual. Without the knowledge of this residual we cannot reconstruct our image from the various detail outputs. Those auxiliary function vectors are determined by (2.13) and (2.14). Finally, the equivalence $(a) \leftrightarrow (d)$ of Theorem 1 shows that the Mixed Oblique Extension Principle characterizes dual framelets. We mention that a weaker version of the equivalence $(c) \leftrightarrow (d)$ in multidimensions and in the space of distributions is provided by Han in [34, Theorem 17], under the additional assumption that there exists a pair (μ, μ^d) of $\mathbb{Z}^s$-periodic functions generating the pair of distributions $(\widetilde{\Phi}, \widetilde{\Phi}^d)$ which define the pair $(X_{\widetilde{\Phi},\Psi}, X_{\widetilde{\Phi}^d,\Psi^d})$ of nonhomegeneous wavelet families. With this assumption Han defines $\Theta_{\mathrm{M}} = \mu^{d*}\mu$. In our equivalence $(c) \leftrightarrow (d)$ this assumption is removed.

We conclude with an outline of the structure of this paper. In Section 2 we prove Theorem 1. As a byproduct of Theorem 1, we obtain characterizations of affine Parseval frames arising from refinable function vectors. In Section 3 we provide examples of compactly supported multiwavelets derived from compactly supported refinable function vectors.

2. Main results

Let X_Ψ be as in (1.2). We define its corresponding *quasi-affine* system X_Ψ^q by

$$X_\Psi^q = \{D_A^j \tau_k \psi : j \geq 0, k \in \mathbb{Z}^s, \psi \in \Psi\} \cup \{|\det A|^{j/2} \tau_k D_A^j \psi : j < 0, k \in \mathbb{Z}^s, \psi \in \Psi\}. \tag{2.1}$$

The notation for $X_{\Psi^d}^q$ is defined in a similar way. Then, the following characterizations hold true:

Lemma 1. *Let $\Psi, \Psi^d \in L_2^{m\times 1}$ be two sets of multiwavelets whose corresponding pair of homogeneous wavelet Bessel families (X_Ψ, X_{Ψ^d}) be given in* (1.2)*. Define the function*

$$\kappa : \mathbb{Z}^s \to \mathbb{Z}_- \cup \{0\} : \ \kappa(n) = \inf\big\{j \leq 0 : \ A^{*j} n \in \mathbb{Z}^s\big\},$$

where A is an $s \times s$ expansive matrix corresponding to the dilation operator D_A. Then:

(a) *(X_Ψ, X_{Ψ^d}) is a pair of affine dual frames for L_2 if and only if $(X_\Psi^q, X_{\Psi^d}^q)$ is a pair of quasi-affine dual frames for L_2.*
(b) *(X_Ψ, X_{Ψ^d}) is a pair of affine dual frames for L_2 if and only if for a.e. $\gamma \in \mathbb{R}^s$ and for every $n \in \mathbb{Z}^s$ we have*

$$\sum_{j=\kappa(n)}^{\infty} \widehat{\Psi}^{d*}(A^{*j}(\gamma + n))\widehat{\Psi}(A^{*j}\gamma) = \delta_{0,n}. \tag{2.2}$$

(c) *If $\widetilde{\Phi}, \widetilde{\Phi}^d \in L_2^{r\times 1}$ and if $(X^{(0)}_{\widetilde{\Phi},\Psi}, X^{(0)}_{\widetilde{\Phi}^d,\Psi^d})$ is a pair of nonhomogeneous wavelet Bessel systems as in* (1.3)*, then $X^{(0)}_{\widetilde{\Phi}^d,\Psi^d}$ is a dual frame of $X^{(0)}_{\widetilde{\Phi},\Psi}$ for L_2 if and only if for a.e. $\gamma \in \mathbb{R}^s$ and for every $n \in \mathbb{Z}^s$ we have*

$$\sum_{j=\kappa(n)}^{0} \widehat{\Psi}^{d*}(A^{*j}(\gamma + n))\widehat{\Psi}(A^{*j}\gamma) + \widehat{\widetilde{\Phi}^{d*}}(\gamma + n)\widehat{\widetilde{\Phi}}(\gamma) = \delta_{0,n}. \tag{2.3}$$

We note that part (a) of Lemma 1 was proved in [51, Theorem 1] under a mild decay condition imposed on the Fourier transforms of the elements of the pair (X_Ψ, X_{Ψ^d}). Subsequently, in [19] (see also [11]) the same statement was proved in full generality. Under the same mild assumption imposed on the elements of the pair (X_Ψ, X_{Ψ^d}), part (b) of Lemma 1 was proved in [31] and [51] and subsequently in [11] the said mild decay condition was removed. Part (c) of Lemma 1 is proved in [34, Theorems 9, 11] for a pair of frequency based nonhomogeneous and non-stationary dual wavelet frames in the space of distributions under more general assumptions, i.e. convergence of (2.3) for $n = 0$

is in the sense of distributions. By exploiting the Bessel assumption on the pair $(X^{(0)}_{\widetilde{\Phi},\Psi}, X^{(0)}_{\widetilde{\Phi}^d,\Psi^d})$ and when $\widetilde{\Phi}$ is a singleton, we proved in [3] that (2.3) holds pointwise a.e. if and only if $(X^{(0)}_{\widetilde{\Phi},\Psi}, X^{(0)}_{\widetilde{\Phi}^d,\Psi^d})$ is a pair of nonhomogeneous wavelet Bessel systems. The same arguments apply to the more general case of the non-singleton $\widetilde{\Phi}$, establishing thus the validity of (c) above.

So far we have used duality notation to Φ and Φ^d without having invoked implicitly of explicitly any duality condition between the Bessel sequences $\{\tau_k\phi_i : k \in \mathbb{Z}^s, i = 1, 2, \dots, r\}$ and $\{\tau_k\phi^d_i : k \in \mathbb{Z}^s, i = 1, 2, \dots, r\}$. The only reason for which we have used that notation is that these two families give rise to the dual homogeneous wavelet systems (X_Ψ, X_{Ψ^d}).

The following two propositions are the key tools for the proof of Theorem 1. The first one of them is the cornerstone of our results and it guarantees that the Fundamental functions Θ and Θ^d are well-defined and measurable and, as a consequence of this fact, the Mixed Fundamental function Θ_{M} associated with the pair (X_Ψ, X_{Ψ^d}) is also well-defined. Then, Proposition 2 facilitates the connection between the pair of homogeneous affine dual frames (X_Ψ, X_{Ψ^d}) and their corresponding nonhomogeneous dual frames $(X^{(0)}_{\tilde{\Phi},\Psi}, X^{(0)}_{\tilde{\Phi}^d,\Psi^d})$.

Proposition 1. *Let Φ, Φ^d be a pair of $r \times 1$ refinable function vectors satisfying the conditions $(i)-(iv)$ of section 1. Assume that (X_Ψ, X_{Ψ^d}) is a pair of homogeneous wavelet Bessel systems whose corresponding pair (Θ, Θ^d) of Fundamental functions be given by (1.6). Let Θ_{M} be the $\mathbb{Z}^s$-periodic $r \times r$ matrix valued Mixed Fundamental function associated with the pair (X_Ψ, X_{Ψ^d}) as in (1.4). Then:*

(i) *The $r \times r$ matrix-valued function Θ (or Θ^d) is measurable and the entries of Θ (or Θ^d) are finite for a.e. γ on the spectrum σ_Φ (or σ_{Φ^d}) of Φ (or Φ^d).*

(ii) *The $r \times r$ matrix-valued function Θ_{M} is measurable and its entries are finite for a.e. $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$. Moreover the operator norm of $\Theta_{\mathrm{M}}(\gamma)$ satisfies*

$$\|\Theta_{\mathrm{M}}(\gamma)\| \le r\|\Theta(\gamma)\|\|\Theta^d(\gamma)\|, \quad a.e.\ \gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}.$$

(iii) *Let $\Theta^{\frac{1}{2}}$ be the positive square root of the linear operator Θ on $\mathbb{C}^r(\sigma_\Phi)$. Then for a.e. $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ and for any column vector function $c \in L_2^{r\times 1}(\mathbb{T}^s)$ we have*

$$\left\|\left(\Theta_{\mathrm{M}}(\gamma)(\Theta^{\frac{1}{2}}(\gamma))^\dagger\right)^* c(\gamma)\right\|^2_{\ell_2(I_r)} \le r\left(c^*(\gamma)\Theta^d(\gamma)c(\gamma)\right), \qquad (2.4)$$

where the superscript † denotes the partial inverse of a matrix (in this case it denotes the partial inverse of $\Theta^{\frac{1}{2}}(\gamma)$*) defined via the Polar Decomposition Theorem.*

Proof. (i) Since X_Ψ is a Bessel system, then by part (a) of lemma 1 its quasi-affine counterpart X_Ψ^q (see (2.1)) is also a Bessel system, so for any $f \in L_2$ there exists a positive constant C such that

$$\begin{aligned} &\sum_{j=1}^{\infty}\sum_k\sum_{i=1}^{m} |\langle f, |\det A|^{-j}\psi_i(A^{-j}(\cdot - k))\rangle|^2 \le C\|f\|_{L_2}^2 \\ \Rightarrow\ &\sum_{j=1}^{\infty}\int_{\mathbb{T}^s}\sum_{i=1}^{m}\Big|\sum_k \widehat{f}(\gamma+k)\overline{\widehat{\psi}_i(A^{*j}(\gamma+k))}\Big|^2 d\gamma \le C\|f\|_{L_2}^2, \end{aligned}$$

or equivalently

$$\sum_{j=1}^{\infty}\int_{\mathbb{T}^s}[[\widehat{f},\widehat{\Psi}(A^{*j}\cdot)]]^*(\gamma)\ [[\widehat{f},\widehat{\Psi}(A^{*j}\cdot)]](\gamma)d\gamma \le C\|f\|_{L_2}^2, \tag{2.5}$$

where $\widehat{\Psi} = (\widehat{\psi}_1, ..., \widehat{\psi}_m)^T$, $[[\widehat{f},\widehat{\Psi}(A^{*j}\cdot)]]$ is an $m \times 1$ function vector defined in section 1 and $[[\widehat{f},\widehat{\Psi}(A^{*j}\cdot)]]^*$ is its Hermitian transpose $1\times m$ function vector. From the refinement equations on $\widehat{\Psi}$ and then on $\widehat{\Phi}$ we obtain

$$\widehat{\Psi}(A^{*j}(\gamma+k)) = \theta_{j-1}(\gamma)\widehat{\Phi}(\gamma+k), \ \text{ a.e. } \gamma\in\mathbb{T}^s \text{ and } j\ge 1, \tag{2.6}$$

where $\widehat{\Phi} = (\widehat{\phi}_1, ..., \widehat{\phi}_r)^T$ and θ_j is a $\mathbb{Z}^s$-periodic $m\times r$ matrix valued function as in (1.5). By substituting (2.6) in (2.5) we obtain

$$\sum_{j=1}^{\infty}\int_{\mathbb{T}^s}[[\widehat{f},\widehat{\Phi}]]^*(\gamma)\theta_{j-1}^*(\gamma)\theta_{j-1}(\gamma)[[\widehat{f},\widehat{\Phi}]](\gamma)d\gamma \le C\|f\|_{L_2}^2. \tag{2.7}$$

Let G_i be the i-column of the Gram matrix G as in (1.1) corresponding to the refinable function vector Φ. If we take $\widehat{f} = \widehat{\phi}_1, ..., \widehat{f} = \widehat{\phi}_r$ in the above inequality and then take the sum of the resulting inequalities, we derive

$$\begin{aligned}
& \sum_{i=1}^{r}\sum_{j=1}^{\infty}\int_{\mathbb{T}^s} [[\widehat{\phi}_i,\widehat{\Phi}]]^*(\gamma)\ \theta_{j-1}^*(\gamma)\theta_{j-1}(\gamma)\ [[\widehat{\phi}_i,\widehat{\Phi}]](\gamma)d\gamma \le C\sum_{i=1}^{r}\|\phi_i\|_{L_2}^2 \\
\Rightarrow\ & \sum_{i=1}^{r}\sum_{j=1}^{\infty}\int_{\mathbb{T}^s} G_i^*(\gamma)\ \theta_{j-1}^*(\gamma)\theta_{j-1}(\gamma)\ G_i(\gamma)d\gamma \le C\|\Phi\|^2_{L_2^{r\times 1}} \\
\Rightarrow\ & \sum_{i=1}^{r}\sum_{j=1}^{\infty}\int_{\mathbb{T}^s}\sum_{k=1}^{m}\Big|\sum_{\mu=1}^{r}\big(\theta_{j-1}(\gamma)\big)_{k,\mu}\big(G(\gamma)\big)_{\mu,i}\Big|^2 \le C\|\Phi\|^2_{L_2^{r\times 1}} \\
\Rightarrow\ & \sum_{i=1}^{r}\sum_{j=1}^{\infty}\int_{\mathbb{T}^s}\sum_{k=1}^{m}\Big|\Big(\theta_{j-1}(\gamma)G(\gamma)\Big)_{k,i}\Big|^2 d\gamma \le C\|\Phi\|^2_{L_2^{r\times 1}}.
\end{aligned}$$

From the last inequality we obtain

$$\sum_{j=0}^{\infty}\big\|\theta_j(\gamma)G(\gamma)\big\|_F^2 < \infty, \quad \text{a.e. } \gamma\in\mathbb{T}^s, \tag{2.8}$$

where $\|\theta_j(\gamma)G(\gamma)\|_F$ is the Frobenius (or trace) norm of the matrix $\theta_j(\gamma)G(\gamma)$. We remark that the notation $\mathrm{Tr}(B)$ is used to denote the trace of a matrix B. Let $K_j = \theta_j^*\theta_j$, $(j\ge 0)$ be a sequence of $r\times r$ matrix valued functions defined on $\mathbb{T}^s$ with measurable entries and $\lambda_{K_j(\gamma)}$ (resp. $\Lambda_{K_j(\gamma)}$) be the smallest (resp. largest) eigenvalue of $K_j(\gamma)$. Then λ_{K_j} and Λ_{K_j} are non negative measurable functions on $\mathbb{T}^s$ [49, Lemma 2.3.5]. Since we have assumed that $G(\gamma)$ is non singular for a.e. $\gamma\in\sigma_\Phi$ (see condition (iv) in section 1) and since $\{\phi(\cdot-k):\ k\in\mathbb{Z}^s, \phi\in\Phi\}$ is a Bessel sequence (see condition (ii) in section 1), we have $0<\lambda_{G(\gamma)}\le\|\Lambda_{G(\gamma)}\|_{L_\infty}$ for a.e. $\gamma\in\sigma_\Phi$ and so

$$\begin{aligned}
& \sum_{j=0}^{\infty}\big\|\theta_j(\gamma)G(\gamma)\big\|_F^2 = \sum_{j=0}^{\infty}\mathrm{Tr}\big(G^*(\gamma)K_j(\gamma)G(\gamma)\big) \ge \sum_{j=0}^{\infty}\|G^*(\gamma)K_j(\gamma)G(\gamma)\| \\
= \ & \sum_{j=0}^{\infty}\frac{\|G^{*-1}(\gamma)\|}{\|G^{*-1}(\gamma)\|}\|G^*(\gamma)K_j(\gamma)G(\gamma)\|\frac{\|G^{-1}(\gamma)\|}{\|G^{-1}(\gamma)\|} \\
\ge \ & \sum_{j=0}^{\infty}\frac{1}{\|G^{*-1}(\gamma)\|}\|K_j(\gamma)\|\frac{1}{\|G^{-1}(\gamma)\|} = \sum_{j=0}^{\infty}\frac{\|K_j(\gamma)\|}{\|G^{-1}(\gamma)\|^2} \\
= \ & \sum_{j=0}^{\infty}\frac{\|K_j(\gamma)\|}{\lambda_{G(\gamma)}^{-2}} = \lambda_{G(\gamma)}^2\sum_{j=0}^{\infty}\Lambda_{K_j(\gamma)}.
\end{aligned}$$

By combining the above inequality with (2.8) we conclude

$$\lambda^2_{G(\gamma)} \sum_{j=0}^{\infty} \Lambda_{K_j(\gamma)} < \infty \ \text{ a.e. } \gamma \in \mathbb{T}^s.$$

From this we obtain

$$\sum_{j=0}^{\infty} \Lambda_{K_j(\gamma)} < \infty \ \text{ a.e. } \gamma \in \sigma_\Phi, \tag{2.9}$$

otherwise, $\lambda_{G(\gamma)} = 0$ on a subset of σ_Φ of positive measure, which clearly contradicts the assumption on the non-singularity of $G(\gamma)$. If $U_j(\gamma)$ and $D_j(\gamma)$ are a unitary matrix and a diagonal matrix respectively both with measurable entries such that $K_j(\gamma) = U_j(\gamma)D_j(\gamma)U_j^*(\gamma)$ for a.e. γ on σ_Φ [49, lemma 2.3.5], then by (2.9) and for any $k, n = 1, ..., r$ we obtain

$$\begin{aligned}\sum_{j=0}^{\infty} |(K_j(\gamma))_{k,n}| &= \sum_{j=0}^{\infty} \left|\left(U_j(\gamma)D_j(\gamma)U_j^*(\gamma)\right)_{k,n}\right| \\ \leq \ \sum_{j=0}^{\infty} \Lambda_{K_j(\gamma)} &= \sum_{j=0}^{\infty} \Lambda_{\theta_j^*(\gamma)\theta_j(\gamma)} < \infty, \ \text{ a.e. } \gamma \in \sigma_\Phi, \end{aligned}\tag{2.10}$$

so every entry of the matrix $\sum_{j=0}^{\infty} K_j(\gamma) = \sum_{j=0}^{\infty} \theta_j^*(\gamma)\theta_j(\gamma)$ converges pointwise to a measurable and finite function, say $\Theta_{k,n}(\gamma)$ for a.e. γ on σ_Φ. Hence $\Theta = \sum_{j=0}^{\infty} \theta_j^*\theta_j$ is an $r \times r$ matrix valued function whose values $\Theta(\gamma)$ are Hermitian and positive semi-definite matrices for a.e. γ on σ_Φ. We use the same arguments to prove that all entries of Θ^d are measurable and finite functions on σ_{Φ^d}. Therefore, the proof of (i) is complete.

(ii) Let $\gamma \in \mathbb{T}^s$ and $\mu, n = 1, ..., r$. Using the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}|(\Theta_{\mathrm{M}}(\gamma))_{\mu,n}| \leq \sum_{j=0}^{\infty}\sum_{k=1}^{m} \left|(\theta_j^{d*}(\gamma))_{\mu,k}(\theta_j(\gamma))_{k,n}\right| \leq \sum_{j=0}^{\infty} \left(\sum_{k=1}^{m} |(\theta_j^{d*}(\gamma))_{\mu,k}|^2\right)^{1/2} \\ \left(\sum_{k=1}^{m} |(\theta_j(\gamma))_{k,n}|^2\right)^{1/2} \leq r^2 \sum_{j=0}^{\infty} \|\theta_j^d(\gamma)\|\|\theta_j(\gamma)\| \leq r^2 \left(\sum_{j=0}^{\infty} \|\theta_j^d(\gamma)\|^2\right)^{1/2} \\ \left(\sum_{j=0}^{\infty} \|\theta_j(\gamma)\|^2\right)^{1/2} < \infty, \ \text{ a.e } \gamma \in \sigma_\Phi \cap \sigma_{\Phi^d},\end{aligned}$$

where the last inequality is obtained from (2.10). Therefore the entries of Θ_{M} are measurable and finite for a.e. $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$. Now, let

$c \in L_2^{r\times 1}(\mathbb{T}^s)$ and $I_r = \{1,2,\dots,r\}$. Then, using the Triangle and Cauchy-Schwartz inequalities we derive

$$\begin{aligned}
&\|\Theta_{\mathrm{M}}(\gamma)c(\gamma)\|^2_{\ell_2(I_r)} = \Big\|\sum_{j=0}^{\infty}\theta_j^{d*}(\gamma)\theta_j(\gamma)c(\gamma)\Big\|^2_{\ell_2(I_r)} \le \Big(\sum_{j=0}^{\infty}\|\theta_j^{d*}(\gamma)\theta_j(\gamma)c(\gamma)\|_{\ell_2(I_r)}\Big)^2 \\
\le\ & \Big(\sum_{j=0}^{\infty}\big\|\theta_j^{d}(\gamma)\big\|^2\Big)\Big(\sum_{j=0}^{\infty}\big\|\theta_j(\gamma)c(\gamma)\big\|^2_{\ell_2(I_r)}\Big) \le \Big(\sum_{j=0}^{\infty}\big\|\theta_j^{d}(\gamma)\big\|^2_F\Big)\Big(\sum_{j=0}^{\infty}\big\|\theta_j(\gamma)c(\gamma)\big\|^2_{\ell_2(I_r)}\Big) \\
=\ & \sum_{j=0}^{\infty}\big(\mathrm{Tr}\,\big(\theta_j^{d*}(\gamma)\theta_j^{d}(\gamma)\big)\big)\,\big(c^*(\gamma)\Theta(\gamma)c(\gamma)\big) \\
=\ & \mathrm{Tr}\,\big(\Theta^d(\gamma)\big)\,\big(c^*(\gamma)\Theta(\gamma)c(\gamma)\big) < \infty \ \text{ a.e. } \gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}.
\end{aligned} \tag{2.11}$$

(iii) By *(i)*, the $r\times r$ matrix valued function Θ has measurable and finite entries for a.e. $\gamma\in\sigma_\Phi$. Since $\Theta(\gamma)$ is positive semi-definite we can define the positive square root of Θ by

$$\Theta^{\frac12} : \sigma_\Phi \to \mathbb{C}^{r\times r} : \ \ (\Theta^{\frac12}(\gamma))^2 = \Theta(\gamma)\ . \tag{2.12}$$

We remark that $\Theta^{\frac12}$ is measurable on σ_Φ. If $\big(\Theta^{\frac12}(\gamma)\big)^\dagger$ is the partial inverse of $\Theta^{\frac12}(\gamma)$ then for a.e. $\gamma\in\sigma_\Phi\cap\sigma_{\Phi^d}$ we have

$$\begin{aligned}
&\Big\|\Big(\Theta_{\mathrm{M}}(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Big)^* c(\gamma)\Big\|^2_{\ell_2(I_r)} = \big\|c^*(\gamma)\Theta_{\mathrm{M}}(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\big\|^2_{\ell_2(I_r)} \\
=\ & \Big\|\sum_{j=0}^{\infty}\big(c^*(\gamma)\theta_j^{d*}(\gamma)\big)\Big(\theta_j(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Big)\Big\|^2_{\ell_2(I_r)} \\
=\ & \sum_{i=1}^{r}\Big|\sum_{j=0}^{\infty}\sum_{\mu=1}^{m}\big(c^*(\gamma)\theta_j^{d*}(\gamma)\big)_\mu\Big(\theta_j(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Big)_{\mu,i}\Big|^2 \\
\le\ & \sum_{i=1}^{r}\Big(\sum_{j=0}^{\infty}\sum_{\mu=1}^{m}\big|(c^*(\gamma)\theta_j^{d*}(\gamma))_\mu\big|^2\Big)\Big(\sum_{j=0}^{\infty}\sum_{\mu=1}^{m}\Big|\Big(\theta_j(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Big)_{\mu,i}\Big|^2\Big) \\
=\ & \Big(\sum_{j=0}^{\infty}\big\|c^*(\gamma)\theta_j^{d*}(\gamma)\big\|^2_{\ell_2(I_m)}\Big)\Big(\sum_{j=0}^{\infty}\big\|\theta_j(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\big\|^2_F\Big) \\
=\ & \Big(\sum_{j=0}^{\infty}c^*(\gamma)\theta_j^{d*}(\gamma)\theta_j^{d}(\gamma)c(\gamma)\Big)\sum_{j=0}^{\infty}\mathrm{Tr}\Big(\big(\Theta^{\frac12}(\gamma)\big)^\dagger\theta_j^*(\gamma)\theta_j(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Big) \\
=\ & \big(c^*(\gamma)\Theta^d(\gamma)c(\gamma)\big)\,\mathrm{Tr}\Big(\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Theta(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Big) \\
=\ & \big(c^*(\gamma)\Theta^d(\gamma)c(\gamma)\big)\Big\|\Theta^{\frac12}(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Big\|^2_F \le r\big(c^*(\gamma)\Theta^d(\gamma)c(\gamma)\big).
\end{aligned}$$

□

Next, we present the construction of a new pair of refinable function vectors $(\widetilde{\Phi}, \widetilde{\Phi}^d)$ giving rise to the nonhomogeneous dual frame pair $(X^{(0)}_{\tilde{\Phi},\Psi}, X^{(0)}_{\tilde{\Phi}^d,\Psi^d})$.

Proposition 2. *Let Φ, Φ^d be a pair of $r \times 1$ refinable function vectors as in section 1. Let Θ_{M} be the Mixed Fundamental function of a pair (X_Ψ, X_{Ψ^d}) of wavelet Bessel systems as in (1.4). Consider the spaces V_0 and V_0^d as in section 1. Then there exists a pair (μ, μ^d) of $\mathbb{Z}^s$-periodic $r \times r$ matrix valued functions with measurable entries such that*

(i) $\Theta_{\mathrm{M}}(\gamma) = \mu^{d*}(\gamma)\mu(\gamma)$, *a.e.* $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$,
(ii) *the $r\times 1$ function vectors $\widetilde{\Phi} = (\widetilde{\phi}_1, ..., \widetilde{\phi}_r)^T$ and $\widetilde{\Phi}^d = (\widetilde{\phi}^d_1, ..., \widetilde{\phi}^d_r)^T$ defined by $\widehat{\widetilde{\Phi}} = \mu\widehat{\Phi}$ and $\widehat{\widetilde{\Phi}^d} = \mu^d\widehat{\Phi}^d$ belong in V_0 and V_0^d respectively,*
(iii) *the shift invariant sets $\{\tau_k\widetilde{\phi}_i : k \in \mathbb{Z}^s,\ i = 1, ..., r\}$ and $\{\tau_k\widetilde{\phi}^d_i : k \in \mathbb{Z}^s,\ i = 1, ..., r\}$ are Bessel sequences.*

Proof. (i) Let $\Theta^{\frac12}$ be as in (2.12) and the Mixed fundamental function Θ_{M} be finite for a.e. $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ as a result of part (ii) of Proposition 1. We define a pair (μ, μ^d) of $\mathbb{Z}^s$-periodic $r \times r$ matrix valued functions by

$$\mu : \mathbb{T}^s \to \mathbb{C}^{r\times r} : \quad \mu(\gamma) = \begin{cases} \Theta^{\frac12}(\gamma), & \gamma \in \sigma_\Phi \cap \sigma_{\Phi^d} \\ \mathbf{0}, & \gamma \in \mathbb{T}^s \backslash \sigma_\Phi \cap \sigma_{\Phi^d} \end{cases} \tag{2.13}$$

and

$$\mu^d : \mathbb{T}^s \to \mathbb{C}^{r\times r} : \quad \mu^d(\gamma) = \begin{cases} \Big(\Theta_{\mathrm{M}}(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Big)^*, & \gamma \in \sigma_\Phi \cap \sigma_{\Phi^d} \\ \mathbf{0}, & \gamma \in \mathbb{T}^s \backslash \sigma_\Phi \cap \sigma_{\Phi^d} \end{cases}, \tag{2.14}$$

where $\big(\Theta^{\frac12}(\gamma)\big)^\dagger$ is the partial inverse matrix of $\Theta^{\frac12}(\gamma)$. Let $c \in \mathbb{C}^r(\sigma_\Phi \cap \sigma_{\Phi^d})$. If $c(\gamma) \in \mathcal{N}_{\Theta^{1/2}(\gamma)}$ then by (2.13) and (2.14) we have

$$\Theta_{\mathrm{M}}(\gamma)c(\gamma) = \mathbf{0} = \mu^{d*}(\gamma)\mu(\gamma)c(\gamma),$$

because $\mathcal{N}_{\Theta^{1/2}(\gamma)} = \mathcal{N}_{\Theta(\gamma)} \subseteq \mathcal{N}_{\Theta_{\mathrm{M}}(\gamma)}$, see (2.11). If $c(\gamma) \in \mathcal{N}^\perp_{\Theta^{1/2}(\gamma)}$, then we use the fact that $\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Theta^{\frac12}(\gamma)$ is the unique orthogonal projection onto $\mathcal{N}^\perp_{\Theta^{1/2}(\gamma)}$ and so

$$\Theta_{\mathrm{M}}(\gamma)c(\gamma) = \Theta_{\mathrm{M}}(\gamma)\big(\Theta^{\frac12}(\gamma)\big)^\dagger\Theta^{\frac12}(\gamma)c(\gamma) = \mu^{d*}(\gamma)\mu(\gamma)c(\gamma).$$

Therefore (i) is satisfied.

(ii) Let us define the $r \times 1$ function vectors $\widehat{\widetilde{\Phi}} = \mu\widehat{\Phi}$ and $\widehat{\widetilde{\Phi}^d} = \mu^d\widehat{\Phi}^d$. If $E = \cup_{n\in\mathbb{Z}^s}((\sigma_\Phi \cap \sigma_{\Phi^d}) + n)$, then we have

$$\|\widetilde{\Phi}\|^2_{L_2^{r\times 1}} = \|\widehat{\widetilde{\Phi}}\|^2_{L_2^{r\times 1}} = \int_E \|\Theta^{\frac{1}{2}}(\gamma)\widehat{\Phi}(\gamma)\|^2_{\ell_2(I_r)} d\gamma = \int_E \widehat{\Phi}^*(\gamma)\Theta(\gamma)\widehat{\Phi}(\gamma)d\gamma$$

$$= \sum_{i=1}^r \sum_{j=1}^\infty \int_E |\widehat{\psi}_i(A^{*j}\gamma)|^2 d\gamma \leq \sum_{j=1}^\infty |\mathrm{det}A|^{-j} \|\widehat{\Psi}\ \|^2_{L_2^{r\times 1}} < \infty \tag{2.15}$$

because A is an expansive matrix. We may obtain a similar estimate for $\widetilde{\Phi}^d$. Indeed

$$\|\widetilde{\Phi}^d\|^2_{L_2^{r\times 1}} = \int_E \left\| \Big(\Theta_{\mathrm{M}}(\gamma)\big(\Theta^{\frac{1}{2}}(\gamma)\big)^\dagger\Big)^* \widehat{\Phi}^d(\gamma)\right\|^2_{\ell_2(I_r)} d\gamma$$

$$\leq r\int_E \widehat{\Phi}^{d*}(\gamma)\Theta^d(\gamma)\widehat{\Phi}^d(\gamma)d\gamma < \infty,$$

where we used (2.4) to derive the semi-final inequality and an argument similar to (2.15) to derive the final inequality. Therefore $\widehat{\widetilde{\Phi}}, \widehat{\widetilde{\Phi}^d} \in L_2^{r\times 1}$. Now, [49, Result 2.3.1] implies $\widetilde{\Phi} \in V_0$ and $\widetilde{\Phi}^d \in V_0^d$ respectively.

(iii) Using (2.14), (2.4) and (2.7) we deduce that for every $f \in L_2$ we have

$$\begin{aligned}\sum_{i=1}^r \sum_k |\langle f, \widetilde{\phi}_i^d(\cdot - k)\rangle_{L_2}|^2 &= \int_{\sigma_\Phi \cap \sigma_\Phi^d} \left\| [[\widehat{f}, \widehat{\Phi}^d]]^*(\gamma)\Theta_{\mathrm{M}}(\gamma)\big(\Theta^{\frac{1}{2}}(\gamma)\big)^\dagger \right\|^2_{\ell_2(I_r)} d\gamma \\ &\leq r\int_{\sigma_\Phi \cap \sigma_\Phi^d} [[\widehat{f}, \widehat{\Phi}^d]]^*(\gamma)\Theta^d(\gamma)[[\widehat{f}, \widehat{\Phi}^d]](\gamma)d\gamma \\ &\leq rC\|f\|^2_{L_2}\ .\end{aligned}$$

Hence, $\{\tau_k\widetilde{\phi}^d :\ k \in \mathbb{Z}^s,\ \widetilde{\phi}^d \in \widetilde{\Phi}^d\}$ is a Bessel system. Similarly, we show that $\{\tau_k\widetilde{\phi} :\ k \in \mathbb{Z}^s,\ \widetilde{\phi} \in \widetilde{\Phi}\}$ is a Bessel system too. Indeed, by following the previous arguments and by using (2.7) we obtain

$$\sum_{i=1}^r \sum_k |\langle f, \widetilde{\phi}_i(\cdot{-}k)\rangle_{L_2}|^2 = \int_{\sigma_\Phi \cap \sigma_\Phi^d} [[\widehat{f}, \widehat{\Phi}]]^*(\gamma)\Theta(\gamma)[[\widehat{f}, \widehat{\Phi}]](\gamma)d\gamma \leq C\|f\|_2^2.$$

Now the proof is complete. □

Remark 1. Proposition 2 states that the Mixed Fundamental function Θ_{M} can be always factorized by $\Theta_{\mathrm{M}}(\gamma) = \mu^{d*}(\gamma)\mu(\gamma)$ so that

(a) these two auxiliary matrix valued functions μ and μ^{d*} give rise to a pair of functions $\widetilde{\Phi}$ and $\widetilde{\Phi}^d$ in V_0 and V_0^d respectively and
(b) the integer shifts of $\widetilde{\Phi}$ and $\widetilde{\Phi}^d$ form Bessel systems.

Notice that we are free to choose any other factorization $\Theta_{\mathrm{M}} = \nu^{d*}\nu$ of Θ_{M} (rather than strictly use (2.13) and (2.14)) provided that the factors ν and ν^{d*} satisfy the above conditions (a) and (b). For example, if $\Theta_{\mathrm{M}} = I_r$, then we may define $\nu = \nu^d = I_r$. Then $\widetilde{\Phi} = \Phi$ and $\widetilde{\Phi}^d = \Phi^d$.

We are now ready to prove our main result, Theorem 1 which characterizes the affine dual frames constructed from refinable function vectors and their nonhomogeneous counterparts. The reader may wish to recall the convention $A^{d*}(\gamma) = (A^d(\gamma))^*$ which will be used several times in the proof of Theorem 1.

Proof of Theorem 1. $(a) \leftrightarrow (b)$: Given a pair (Φ, Φ^d) of $r \times 1$ refinable function vectors satisfying the conditions $(i) - (iv)$ of section 1, we define the sequence of functions $\{f^{(n)} : \ n \in \mathbb{Z}^s\}$ by

$$f^{(n)} : \mathbb{R}^s \to \mathbb{C} : f^{(n)}(\gamma) = \widehat{\Phi}^{d*}(\gamma + n)\Theta_{\mathrm{M}}(\gamma)\widehat{\Phi}(\gamma). \tag{2.16}$$

Since (X_Ψ, X_{Ψ^d}) is a pair of wavelet Bessel systems either if (a) or (b) holds, the Mixed Fundamental function Θ_{M} is well defined on the set $E = \cup_{n \in \mathbb{Z}^s}((\sigma_\Phi \cap \sigma_{\Phi^d}) + n)$ as a result of part (ii) of proposition 1. Also, recall that $\Theta_{\mathrm{M}} = 0$ on $\mathbb{R}^s \backslash E$ in order to avoid measurability issues. Therefore, $f^{(n)}$ is well-defined on $\mathbb{R}^s$ and $f^{(n)} = 0$ on $\mathbb{R}^s \backslash E$. In addition, $f^{(n)} \in L_1$ for any $n \in \mathbb{Z}^s$. In fact, by part (i) of proposition 2 we have

$$f^{(n)}(\gamma) = \widehat{\Phi}^{d*}(\gamma + n)\mu^{d*}(\gamma)\mu(\gamma)\widehat{\Phi}(\gamma) = \widehat{\widetilde{\Phi}^{d*}}(\gamma + n)\widehat{\widetilde{\Phi}}(\gamma), \quad \text{a.e. } \gamma \in \mathbb{R}^s, \tag{2.17}$$

where $\widetilde{\Phi}^d, \widetilde{\Phi} \in L_2^{r \times 1}$ are defined in part (ii) of Proposition 2. Therefore

$$\|f^{(n)}\|_{L_1} = \int_{\mathbb{R}^s} \left| \widehat{\widetilde{\Phi}^{d*}}(\gamma + n)\widehat{\widetilde{\Phi}}(\gamma) \right| d\gamma \leq \sum_{i=1}^{r} \|\widehat{\widetilde{\phi}}{}_i^d\|_{L_2} \|\widehat{\widetilde{\phi}}_i\|_{L_2} < \infty. \tag{2.18}$$

Recall, from Lemma 1 that $\kappa(n) = \inf\{j \leq 0 : \ A^{*j}n \in \mathbb{Z}^s\}$. Combining, (1.4), (1.5), the two-scale conditions defining $\widehat{\Psi}, \widehat{\Psi}^d$, the refinability conditions satisfied by $\widehat{\Phi}, \widehat{\Phi}^d$ and after a necessary change of variables, we conclude that, for every $j \geq \kappa(n) + 1$ we have,

$$\begin{aligned} f^{(A^{*j-1}n)}(A^{*j-1}\gamma) &= \widehat{\Phi}^{d*}(A^{*j-1}(\gamma + n))\Theta_{\mathrm{M}}(A^{*j-1}\gamma)\widehat{\Phi}(A^{*j-1}\gamma) \\ &= \sum_{\lambda=j}^{\infty} \widehat{\Psi}^{d*}(A^{*\lambda}(\gamma + n))\widehat{\Psi}(A^{*\lambda}\gamma) \, . \end{aligned} \tag{2.19}$$

In order to establish the equivalence $(a) \leftrightarrow (b)$, first, observe that (a) is equivalent to (2.2), due to Lemma 1. Thus it remains to show that

(b) is equivalent to (2.2), as well. To accomplish this task we consider two cases:

Case I: First, take $n=0$ in (2.2). Then $\kappa(0)=-\infty$. By using (2.19) for $n=0$, the left hand side of eq. (2.2) can be expressed as

$$\sum_{\lambda=-\infty}^{\infty}\widehat{\Psi}^{d*}(A^{*\lambda}\gamma)\widehat{\Psi}(A^{*\lambda}\gamma)=\lim_{\lambda\to-\infty}f^{(0)}(A^{*\lambda-1}\gamma).$$

By combining (2.16) for $n=0$ with the right hand side of the last equality and by recalling condition (i) on $\widehat{\Phi},\widehat{\Phi}^{d}$ (see section 1) we conclude that (2.2) holds if and only if $b(i)$ holds.

Case II: Now, take $n\neq 0$ in (2.2). In this case $\kappa(n)$ is finite and by applying (2.19) for $j=\kappa(n)+1$ and using (2.16) and again the two-scale relations defining $\widehat{\Psi},\widehat{\Psi}^{d}$, the refinability of $\widehat{\Phi},\widehat{\Phi}^{d}$, the left hand side of (2.2) is now transformed into

$$\sum_{j=\kappa(n)}^{\infty}\widehat{\Psi}^{d*}(A^{*j}(\gamma+n))\widehat{\Psi}(A^{*j}\gamma)=\widehat{\Psi}^{d*}(A^{*\kappa(n)}(\gamma+n))\widehat{\Psi}(A^{*\kappa(n)}\gamma)+f^{(A^{*\kappa(n)}n)}(A^{*\kappa(n)}\gamma)$$

$$\begin{aligned}= &\ \ \widehat{\Phi}^{d*}(A^{*\kappa(n)-1}(\gamma+n))\Big(H_1^{d*}(A^{*\kappa(n)-1}(\gamma+n))H_1(A^{*\kappa(n)-1}\gamma)\\ + &\ \ H_0^{d*}(A^{*\kappa(n)-1}(\gamma+n))\Theta_{\mathrm{M}}(A^{*\kappa(n)}\gamma)H_0(A^{*\kappa(n)-1}\gamma)\Big)\widehat{\Phi}(A^{*\kappa(n)-1}\gamma),\end{aligned}$$

a.e. $\gamma\in\mathbb{R}^s$. From the definition of $\kappa(n)$ we infer $A^{*\kappa(n)-1}n=\lambda+q$, where $\lambda\in\mathbb{Z}^s$ and $q\in A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s-\{\mathbf{0}\}$. Using this observation and by a change of variables $\omega=A^{*\kappa(n)-1}\gamma$ the above equality is equivalent to

$$\sum_{j=1}^{\infty}\widehat{\Psi}^{d*}(A^{*j}(\omega+A^{*\kappa(n)-1}n))\widehat{\Psi}(A^{*j}\omega)=\widehat{\Phi}^{d*}(\omega+\lambda+q)\Big(H_1^{d*}(\omega+q)H_1(\omega)+H_0^{d*}(\omega+q)$$

$$\times\Theta_{\mathrm{M}}(A^{*}\omega)H_0(\omega)\Big)\widehat{\Phi}(\omega),\ \ \text{a.e. }\omega\in\mathbb{R}^s,\ \lambda\in\mathbb{Z}^s,\ q\in A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s-\{\mathbf{0}\}.\tag{2.20}$$

If the right-hand side of (2.2) is equal to zero then the right-hand side of (2.20) is equal to zero as well. Therefore,

$$
\begin{aligned}
& \widehat{\Phi}^{d*}(\omega+\mu+\lambda+q)\Big(H_1^{d*}(\omega+q)H_1(\omega)+H_0^{d*}(\omega+q) \\
\times\quad & \Theta_{\mathrm{M}}(A^*\omega)H_0(\omega)\Big)\widehat{\Phi}(\omega+\mu)=0, \quad \text{a.e. } \omega\in\mathbb{T}^s,\ \lambda,\mu\in\mathbb{Z}^s,\ q\in A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s-\{\mathbf{0}\} \\
\Rightarrow\quad & \widehat{\Phi}^{d}(\omega+\mu+\lambda+q)\widehat{\Phi}^{d*}(\omega+\mu+\lambda+q)\Big(H_1^{d*}(\omega+q)H_1(\omega) \\
+\quad & H_0^{d*}(\omega+q)\Theta_{\mathrm{M}}(A^*\omega)H_0(\omega)\Big)\widehat{\Phi}(\omega+\mu)\widehat{\Phi}^{*}(\omega+\mu)=\mathbf{0} \\
\Rightarrow\quad & \sum_{m\in\mathbb{Z}^s}\Big(\widehat{\Phi}^{d}(\omega+m+q)\widehat{\Phi}^{d*}(\omega+m+q)\Big)\Big(H_1^{d*}(\omega+q)H_1(\omega) \\
+\quad & H_0^{d*}(\omega+q)\Theta_{\mathrm{M}}(A^*\omega)H_0(\omega)\Big)\sum_{\mu\in\mathbb{Z}^s}\Big(\widehat{\Phi}(\omega+\mu)\widehat{\Phi}^{*}(\omega+\mu)\Big)=\mathbf{0} \\
\Rightarrow\quad & \overline{G^d}(\omega+q)\Big(H_1^{d*}(\omega+q)H_1(\omega)+H_0^{d*}(\omega+q)\Theta_{\mathrm{M}}(A^*\omega)H_0(\omega)\Big)\overline{G}(\omega)=\mathbf{0} \\
\Rightarrow\quad & H_1^{d*}(\omega+q)H_1(\omega)+H_0^{d*}(\omega+q)\Theta_{\mathrm{M}}(A^*\omega)H_0(\omega)=\mathbf{0}
\end{aligned}
$$

for a.e. $\omega,\omega+q\in\sigma_\Phi\cap\sigma_{\Phi^d}$, where in the above derivation we used the assumption that the Gram matrices $G(\omega)$ and $G(\omega+q)$ are non-singular a.e. on $\sigma_\Phi\cap\sigma_{\Phi^d}$. Consequently, $b(ii)$ is true. Note that $\overline{G}$ (or $\overline{G^d}$) denotes the conjugate matrix of G (or G^d), not the conjugate transpose. Inversely, if $b(ii)$ holds, then the right-hand side of (2.20) is equal to zero for a.e. $\omega,\omega+q\in E=\cup_{n\in\mathbb{Z}^s}((\sigma_\Phi\cap\sigma_{\Phi^d})+n)$. If $\omega\in\mathbb{R}^s\backslash E$ or $\omega+q\in\mathbb{R}^s\backslash E$ then $\widehat{\Phi}(\omega)=\mathbf{0}$ or $\widehat{\Phi}^d(\omega+\lambda+q)=\mathbf{0}$ for any λ and so, once again, the right-hand side of (2.20) (and so (2.2)) is equal to zero. Hence, the right-hand side of (2.20) (and so (2.2)) is equal to zero for any $n\neq 0$ and a.e. $\omega\in\mathbb{R}^s$ if and only if $b(ii)$ holds. By combining the above cases I and II we infer that (2.2) (and so (a)) holds if and only if (b) holds. Therefore the equivalence $(a)\leftrightarrow(b)$ has been established.

$(a)\leftrightarrow(c)$: If (a) holds then by (2.2) the following equality holds for a.e. $\gamma\in\mathbb{R}^s$ and for any $n\in\mathbb{Z}^s$:

$$
\sum_{j=\kappa(n)}^{0}\widehat{\Psi}^{d*}(A^{*j}(\gamma+n))\widehat{\Psi}(A^{*j}\gamma)+\sum_{j=1}^{\infty}\widehat{\Psi}^{d*}(A^{*j}(\gamma+n))\widehat{\Psi}(A^{*j}\gamma)=\delta_{0,n}. \tag{2.21}
$$

Using (2.19), first, and then (2.17) we obtain

$$\sum_{j=1}^{\infty} \widehat{\Psi}^{d*}(A^{*j}(\gamma+n))\widehat{\Psi}(A^{*j}\gamma) = f^{(n)}(\gamma) = \widehat{\widetilde{\Phi}^{d*}}(\gamma+n)\widehat{\widetilde{\Phi}}(\gamma) \qquad (2.22)$$

for a.e. $\gamma \in \mathbb{R}^s$ and $n \in \mathbb{Z}^s$, where $\widetilde{\Phi}$ and $\widetilde{\Phi}^d$ are two functions in V_0 and V_0^d defined in part (ii) of proposition 2. Furthermore in part (iii) of the same proposition we showed that $\{\tau_k\widetilde{\phi} : \ k \in \mathbb{Z}^s, \ \widetilde{\phi} \in \widetilde{\Phi}\}$ and $\{\tau_k\widetilde{\phi}^d : \ k \in \mathbb{Z}^s, \ \widetilde{\phi}^d \in \widetilde{\Phi}^d\}$ are Bessel sequences, so, the sets $X^{(0)}_{\widetilde{\Phi},\Psi}$ and $X^{(0)}_{\widetilde{\Phi}^d,\Psi^d}$ (which are of type (1.3)) are also Bessel systems. By substituting (2.22) in (2.21) we derive

$$\sum_{j=\kappa(n)}^{0} \widehat{\Psi}^{d*}(A^{*j}(\gamma+n))\widehat{\Psi}(A^{*j}\gamma) + \widehat{\widetilde{\Phi}^{d*}}(\gamma+n)\widehat{\widetilde{\Phi}}(\gamma) = \delta_{0,n}$$

for a.e. $\gamma \in \mathbb{R}^s$ and $n \in \mathbb{Z}^s$ and so, from (2.3), we infer that $(X^{(0)}_{\widetilde{\Phi},\Psi}, X^{(0)}_{\widetilde{\Phi}^d,\Psi^d})$ is a pair of dual frames for L_2. On the other hand if (c) holds, then (a) holds as a result of [34, Proposition 5].

$(a) \leftrightarrow (d)$: If (a) holds, then (d) is satisfied by $\theta = \Theta_{\mathrm{M}}$. To see this, notice first, that, as a result of part (ii) of proposition 2, Θ_{M} is measurable and well defined for a.e. $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$. Moreover, in the course of proving the equivalence between (a) and (b) we showed that $d(i)$ is true and $d(ii)$ is true for any $q \in A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s - \{\mathbf{0}\}$. Using (1.4) and (1.5) it is easy to see that Θ_{M} satisfies $d(i)$ whenever $q = \mathbf{0}$. Finally, from (2.16) and (2.18) we infer that the sequence $\Big\{\big\|\widehat{\Phi}^{d*}(\cdot - k)\Theta_{\mathrm{M}}(\cdot)\widehat{\Phi}(\cdot)\big\|_{L_1} : \ k \in \mathbb{Z}^s\Big\}$ is bounded.

On the other hand if (d) holds true, then the already established equivalence between (a) and (b) shows that the proof of $(d) \to (a)$ will be completed once we prove that $\theta = \Theta_{\mathrm{M}}$ for a.e. $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$. In fact, from assumption $d(ii)$ for $q = \mathbf{0}$, for any $j \geq 1$ and for a.e. $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ such that $A^{*l}\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ for all $0 \leq l \leq j-1$, we obtain iteratively using $d(ii)$ that

$$\theta(\gamma) = \Big(\prod_{l=0}^{j-1} H_0^{d*}(A^{*l}\gamma)\Big)\theta(A^{*j}\gamma)\Big(\prod_{l=0}^{j-1} H_0(A^{*j-1-l}\gamma)\Big) + \Theta_{\mathrm{M},j}(\gamma), \quad (2.23)$$

where $\Theta_{\mathrm{M},j}(\gamma) = \sum_{n=0}^{j-1}\theta_n^{d*}(\gamma)\theta_n(\gamma)$. The hypothesis that for a $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ we have $A^{*l}\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ for all $0 \leq l \leq j-1$ is not necessarily valid for almost all $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$. However, it is valid for $j = 1$ as $d(ii)$ shows. Loosely speaking, our final effort is aimed in showing that the first term in the right-hand side of (2.23) tends to zero as $j \to \infty$. However, if the orbit $\{A^{*j}\gamma\}$ jumps outside $\sigma_\Phi \cap \sigma_{\Phi^d}$ then we cannot continue using $d(ii)$ iteratively, because the latter equation is valid only on $\sigma_\Phi \cap \sigma_{\Phi^d}$. However, as we will immediately show this does not become a problem. Indeed, pick $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ such that for some j we have $A^{*j}\gamma \notin \sigma_\Phi \cap \sigma_{\Phi^d}$. In fact, we assume that this j is the smallest $j \geq 1$ satisfying the said property. We need this because for up to $j-1$ we need to be able to use $d(ii)$ iteratively. Without any loss of generality we assume $A^{*j}\gamma \notin \sigma_\Phi$. Then,

$$\mathbf{0} = \widehat{\Phi}(A^{*j}\gamma) = \prod_{l=0}^{j-1} H_0(A^{*j-1-l}\gamma)\widehat{\Phi}(\gamma)$$

$$\Rightarrow \prod_{l=0}^{j-1} H_0(A^{*j-1-l}\gamma)\widehat{\Phi}(\gamma+n) = \mathbf{0}, \quad \text{for any } n \in \mathbb{Z}^s$$

$$\Rightarrow \prod_{l=0}^{j-1} H_0(A^{*j-1-l}\gamma)\Big(\sum_{n\in\mathbb{Z}^s} \widehat{\Phi}(\gamma+n)\widehat{\Phi}^*(\gamma+n)\Big) = \mathbf{0}$$

$$\Rightarrow \prod_{l=0}^{j-1} H_0(A^{*j-1-l}\gamma)\overline{G(\gamma)} = \mathbf{0} \Rightarrow \prod_{l=0}^{j-1} H_0(A^{*j-1-l}\gamma) = \mathbf{0},$$

because the Gram matrix $G(\gamma)$ is non singular a.e. on $\sigma_\Phi \cap \sigma_{\Phi^d}$. From the previous equality and (2.23) we obtain

$$\theta(\gamma) = \Theta_{\mathrm{M},j}(\gamma) = \Theta_{\mathrm{M}}(\gamma).$$

If $A^{*j}\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ for any $j \geq 0$, then (2.23) is valid for every $j \geq 1$ and so for any $k, m \in \mathbb{Z}^s$ we have

$$\begin{aligned} &\widehat{\Phi}^{d*}(\gamma+k)\Big(\prod_{l=0}^{j-1} H_0^{d*}(A^{*l}\gamma)\Big)\theta(A^{*j}\gamma)\Big(\prod_{l=0}^{j-1} H_0(A^{*j-1-l}\gamma)\Big)\widehat{\Phi}(\gamma+m) \\ = \;& \widehat{\Phi}^{d*}(A^{*j}(\gamma+k))\theta(A^{*j}\gamma)\widehat{\Phi}(A^{*j}(\gamma+m)). \end{aligned} \tag{2.24}$$

Since we assume a priori that the sequence $\Big\{\big\|\widehat{\Phi}^{d*}(\cdot - k)\theta(\cdot)\widehat{\Phi}(\cdot)\big\|_{L_1} : k \in \mathbb{Z}^s\Big\}$ is bounded, we infer

$$\begin{aligned} &\int_{\mathbb{R}^s} \big|\widehat{\Phi}^{d*}(A^{*j}(\gamma+k))\theta(A^{*j}\gamma)\widehat{\Phi}(A^{*j}(\gamma+m))\big| d\gamma \\ = \;& \frac{1}{|\mathrm{det}A|^j} \int_{\mathbb{R}^s} \big|\widehat{\Phi}^{d*}(\gamma + A^{*j}(k-m))\theta(\gamma)\widehat{\Phi}(\gamma)\big| d\gamma \\ \leq \;& \frac{C}{|\mathrm{det}A|^j}\ , \end{aligned} \tag{2.25}$$

for some positive constant C. The previous inequality holds for all k, $m \in \mathbb{Z}^s$. Now, let $\mathcal{G}_j$ be the $r \times r$-matrix valued function defined by

$$\mathcal{G}_j(\gamma) := \Big(\prod_{l=0}^{j-1} H_0^{d*}(A^{*l}\gamma)\Big)\theta(A^{*j}\gamma)\Big(\prod_{l=0}^{j-1} H_0(A^{*j-1-l}\gamma)\Big) .$$

The a.e. non-singularity of the Gram matrices G and G^d implies that for a.e. $\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d}$ each one of the sets of vectors $\{\widehat{\Phi}(\gamma+m) : m \in \mathbb{Z}^s\}$ and $\{\widehat{\Phi}^d(\gamma+k) : k \in \mathbb{Z}^s\}$ spans $\mathbb{C}^r$. Consequently, if $\mathcal{I}$ is the set of all finite subsets of $\mathbb{Z}^s$ with r elements, then

$$\sigma_\Phi \cap \sigma_{\Phi^d} \subseteq \cup_{I,J \in \mathcal{I}} Y_{I,J}$$

where the previous inclusion is with respect to null sets and

$$Y_{I,J} := \{\gamma \in \sigma_\Phi \cap \sigma_{\Phi^d} : \mathbb{C}^r = span\{\widehat{\Phi}(\gamma+m) : m \in I\} = span\{\widehat{\Phi}^d(\gamma+k) : k \in J\}\}.$$

But, due to (2.24), we may write

$$\widehat{\Phi}^{d*}(A^{*j}(\gamma+k))\theta(A^{*j}\gamma)\widehat{\Phi}(A^{*j}(\gamma+m)) = \widehat{\Phi}^{d*}(\gamma+k)\mathcal{G}_j(\gamma)\widehat{\Phi}(\gamma+m) .$$

So, if v, w are two arbitrary vectors in $\mathbb{C}^r$, then the previous equation shows that if γ belongs to $Y_{I,J}$, then there exist coefficients c_m and c'_k such that

$$w^* \cdot \mathcal{G}_j(\gamma) \cdot v = \sum_{k' \in J}\sum_{m \in I} \overline{c'_k} c_m \widehat{\Phi}^{d*}(\gamma+k)\mathcal{G}_j(\gamma)\widehat{\Phi}(\gamma+m) .$$

Combining the previous equality with (2.25) we obtain

$$\begin{aligned} \int_{Y_{I,J}} |w^* \cdot \mathcal{G}_j(\gamma) \cdot v| d\gamma &= \int_{Y_{I,J}} \left| \sum_{k \in J}\sum_{m \in I} \overline{c'_k} c_m \widehat{\Phi}^{d*}(\gamma+k)\mathcal{G}_j(\gamma)\widehat{\Phi}(\gamma+m) \right| d\gamma \\ \leq \quad & \frac{C}{|\det A|^j} \sum_{k \in J}\sum_{m \in I} |c'_k||c_m|. \end{aligned} \tag{2.26}$$

Now, fix the vectors v and w of the previous argument. Then, inequality (2.26) together with Fatou's lemma imply that for a.e. γ there exists a subsequence $j_q(\gamma)$, where the inclusion in the notation of γ intends to underscore that the indices $j_q(\gamma)$ may change as γ varies in $Y_{I,J}$ such that

$$\lim_{q\to\infty} (w^* \cdot \mathcal{G}_{j_q(\gamma)}(\gamma) \cdot v) = 0 .$$

Combining this fact with the a.e. pointwise convergence of $\Theta_{\mathrm{M},j}(\gamma)$ to $\Theta_{\mathrm{M}}(\gamma)$ (see the proof of item (ii) of proposition 1), which gives

$$\lim_{q\to\infty} (w^* \cdot \Theta_{M,j_q(\gamma)}(\gamma) \cdot v) = (w^* \cdot \Theta_M(\gamma) \cdot v)$$

and with (2.23), we conclude that for a.e. $\gamma \in Y_{I,J}$ we have

$$\Theta_{\mathrm{M}}(\gamma) = \theta(\gamma).$$

Since $\{Y_{I,J} : I, J \in \mathcal{I}\}$ is a cover of $\sigma_\Phi \cap \sigma_{\Phi^d}$ modulo null sets, we conclude $\Theta_{\mathrm{M}}(\gamma) = \theta(\gamma)$ a.e. on $\sigma_\Phi \cap \sigma_{\Phi^d}$. $\square$

By slightly modifying the arguments in the proof of Theorem 1 we obtain the following result providing necessary and sufficient conditions for X_Ψ to be a Parseval frame. This result generalizes [3, Proposition 3.1] and [52, Theorem 2.3].

Theorem 2. *Let Φ be an $r \times 1$ refinable function vector with low-pass filter $H_0 \in L_2^{r\times r}(\mathbb{T}^s)$ satisfying the conditions $(i) - (iv)$ of section 1. Consider a wavelet family X_Ψ with mask $H_1 \in L_2^{m\times r}(\mathbb{T}^s)$. Then the following conditions are equivalent:*

(1) *X_Ψ is a Parseval frame for L_2.*
(2) *The $\mathbb{Z}^s$ periodic $r \times r$ matrix valued Fundamental function Θ of X_Ψ satisfies the following conditions:*
   (a) $\lim_{j\to-\infty} \widehat{\Phi}^*(A^{*j}\gamma)\Theta(A^{*j}\gamma)\widehat{\Phi}(A^{*j}\gamma) = 1$ *a.e.* $\gamma \in \mathbb{R}^s$,
   (b) $H_0^*(\gamma + q)\Theta(A^*\gamma)H_0(\gamma) + H_1^*(\gamma + q)H_1(\gamma) = \mathbf{0}$ *for a.e.* $\gamma \in \sigma_\Phi$ *such that* $\gamma + q \in \sigma_\Phi$ *for any* $q \neq \mathbf{0}$ *in* $A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s$.
(3) *There exists a $\mathbb{Z}^s$ periodic $r \times r$ matrix valued function μ with measurable entries such that the function $\widetilde{\Phi}$ defined by $\widehat{\widetilde{\Phi}} = \mu\widehat{\Phi}$ belongs in the space $V_0 = \overline{span}\{\phi(\cdot - n) :\ n \in \mathbb{Z}^s,\ \phi \in \Phi\}$ and $X_{\widetilde{\Phi},\Psi}$ is a Parseval frame for L_2.*
(4) *There exists a $\mathbb{Z}^s$ periodic $r \times r$ positive semi-definite matrix-valued function θ with measurable and a.e. finite entries such that $\theta^{\frac{1}{2}}(\cdot)\widehat{\Phi}(\cdot) \in L_2^{r\times 1}(\mathbb{R}^s)$,*
   (a) $\lim_{j\to-\infty} \widehat{\Phi}^*(A^{*j}\gamma)\theta(A^{*j}\gamma)\widehat{\Phi}(A^{*j}\gamma) = 1$ *a.e.* $\gamma \in \mathbb{R}^s$,
   (b) $H_0^*(\gamma + q)\theta(A^*\gamma)H_0(\gamma) + H_1^*(\gamma + q)H_1(\gamma) = \delta_{q,\mathbf{0}}\,\theta(\gamma)$ *for a.e.* $\gamma \in \sigma_\Phi$ *such that* $\gamma + q \in \sigma_\Phi$ *for any* $q \in A^{*-1}\mathbb{Z}^s/\mathbb{Z}^s$.

Proof. (1) $\leftrightarrow$ (2): We proceed by following the steps of the proof of the equivalence $(a) \leftrightarrow (b)$ of Theorem 1 for the case $\Phi = \Phi^d$, $\Psi = \Psi^d$ and $\Theta_{\mathrm{M}} = \Theta$. However, we observe that in the statement of part (b) of Theorem 1, X_Ψ is assumed to be a Bessel family, whereas in the statement of this theorem this hypothesis is removed. Recall that due to [52, Proposition 2.2] (or [50]), whenever $\Phi = \Phi^d$, then

$$\sum_{\lambda=\kappa(n)}^{\infty} \widehat{\Psi}^*(A^{*\lambda}(\gamma + n))\widehat{\Psi}(A^{*\lambda}\gamma) = \delta_{0,n}, \quad \text{a.e. } \gamma \in \mathbb{R}^s \tag{2.27}$$

if and only if X_Ψ is a Parseval frame for L_2, with $\kappa(n)$ as defined in Lemma 1. Therefore (1) is equivalent to (2.27), so, it suffices to prove that (2) is equivalent to (2.27). We do that by following exactly the same steps as we did in the proof of the equivalence $(a) \leftrightarrow (b)$ of Theorem 1.

(1) $\leftrightarrow$ (3): Take $\Psi = \Psi^d$, $\Phi = \Phi^d$ and $\Theta_{\mathrm{M}} = \Theta$. Then by (2.13) and (2.14) we derive that $\mu = \mu^d = \Theta^{1/2}$. The rest follow from the proof of equivalence $(a) \leftrightarrow (c)$ of Theorem 1.

(1) $\leftrightarrow$ (4): After noting again that $\Phi = \Phi^d$ and since the integer translates of the ϕ_i's from a Bessel sequence, it becomes apparent that this equivalence is a direct consequence of the proof of equivalence $(a) \leftrightarrow (d)$ of Theorem 1. □

3. Examples

We close with some examples illustrating applications of theorems 1 and 2. Throughout this section we use the dilation matrix $A = 2I_2$. All examples below are in two dimensions but one can extend these examples to three dimensions as well. More importantly, all constructs are compactly supported in space.

Example 1. We present a method for constructing Parseval frames of compactly supported wavelets on $[0,1]^2$ generated from a refinable function vector $\Phi = (\phi_1, \phi_2)^T \in L_2^{2\times 1}(\mathbb{R}^2)$ whose elements are supported on $[0,1]^2$ and satisfy conditions $(i)-(iv)$ of section 1. To do that we fulfill conditions (a) and (b) of item 4 of theorem 2 with $\theta = I_2$. With this selection of θ these two conditions imply that

(a) Φ must also satisfy $\widehat{\Phi}^*(0)\widehat{\Phi}(0) = |\widehat{\phi}_1(0)|^2 + |\widehat{\phi}_2(0)|^2 = 1$ and
(b) we need to define a $k \times 2$ wavelet mask H_1 such that

$$H_0^*(\gamma + q_i)H_0(\gamma) + H_1^*(\gamma + q_i)H_1(\gamma) = \delta_{i,0} I_2, \quad \gamma, \gamma + q_i \in \sigma_\Phi, \tag{3.1}$$

where $q_0 = (0,0), q_1 = \left(\frac{1}{2}, 0\right)$, $q_2 = \left(0, \frac{1}{2}\right)$ and $q_3 = \left(\frac{1}{2}, \frac{1}{2}\right)$.
The refinement mask of Φ has the form

$$H_0(\gamma) = \frac{1}{4}\Big(K + Le^{-2\pi i\gamma_1} + Me^{-2\pi i\gamma_2} + Ne^{-2\pi i(\gamma_1+\gamma_2)}\Big), \quad \gamma \in \mathbb{T}^2, \tag{3.2}$$

where $K = (K_{ij}), L = (L_{ij}), M = (M_{ij})$ and $N = (N_{ij})$ are 2×2 real (or complex) matrices. Let

$$X = 2\begin{pmatrix} K_{11} & L_{11} & M_{11} & N_{11} & K_{12} & L_{12} & M_{12} & N_{12} \\ K_{21} & L_{21} & M_{21} & N_{21} & K_{22} & L_{22} & M_{22} & N_{22} \end{pmatrix}$$

be a 2×8 matrix whose (real or complex) entries are defined in (3.2) and

$$W(\gamma) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ e^{-2\pi i \gamma_1} & 0 \\ e^{-2\pi i \gamma_2} & 0 \\ e^{-2\pi i (\gamma_1+\gamma_2)} & 0 \\ 0 & 1 \\ 0 & e^{-2\pi i \gamma_1} \\ 0 & e^{-2\pi i \gamma_2} \\ 0 & e^{-2\pi i (\gamma_1+\gamma_2)} \end{pmatrix},$$

where $\gamma = (\gamma_1, \gamma_2)$. Now from (3.2) we obtain

$$H_0(\gamma + q_i) = XW(\gamma + q_i), \ i = 0, ..., 3, \tag{3.3}$$

where the q_i's are defined in (3.1). Let $X = U\mathbf{\Sigma}V^*$ be the Singular Value Decomposition of the 2×8 matrix X, where

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} \Sigma_{2\times 2} & \mathbf{O}_{2\times 6} \end{array} \right)$$

and $\Sigma_{2\times 2} = \operatorname{diag}(\sigma_{11}, \sigma_{22})$ is a diagonal matrix whose diagonal entries are the positive square roots of the eigenvalues of the matrix $X\,X^*$, U is a 2×2 unitary matrix whose columns are eigenvectors of $X\,X^*$ and V is an 8×8 unitary matrix whose columns are eigenvectors of $X^*\,X$. From now on assume that both singular values of X are less than or equal to 1, i.e.

$$0 \leq \sigma_{22} \leq \sigma_{11} \leq 1.$$

Let $0 \leq s_0 \leq 2$ be an index such that

$$s_0 = \begin{cases} 0 & \text{whenever } \sigma_{11} = \sigma_{22} = 1 \\ 1, & \text{whenever } \sigma_{22} < 1 \text{ and } \sigma_{11} = 1 \\ 2 & \text{whenever } \sigma_{22} \leq \sigma_{11} < 1 \end{cases}. \tag{3.4}$$

Then we define an $(m + s_0) \times 2$ wavelet mask H_1 by

$$H_1(\gamma) = \left(\begin{array}{c|c} P_{s_0\times 2} & \mathbf{O}_{s_0\times 6} \\ \hline \mathbf{O}_{m\times 2} & Q_{m\times 6} \end{array} \right) \cdot V^* \cdot W(\gamma), \ (m \geq 6), \tag{3.5}$$

(see also [26]), where

(i) the $s_0 \times 2$ matrix P results from the 2×2 diagonal matrix $(I_2 - \Sigma^*\Sigma)^{\frac{1}{2}} = \operatorname{diag}(\sqrt{1-\sigma_{11}^2}, \sqrt{1-\sigma_{22}^2})$ by deleting its first $(2 - s_0)$ rows which correspond to zero diagonal elements and
(ii) for any $m \geq 6$, Q is any $m \times 6$ matrix with orthonormal columns.

By substituting (3.3) and (3.5) in the left hand side of (3.1) and by recalling that the block matrix-valued function $\{W(\gamma+q_i) : i = 0, ..., 3\}$ is unitary we obtain

$$\begin{aligned} & H_0^*(\gamma+q_i)H_0(\gamma) + H_1^*(\gamma+q_i)H_1(\gamma) = W^*(\gamma+q_i)\, X^*\, X\, W(\gamma) \\ + \quad & W^*(\gamma+q_i)\, V \left(\begin{array}{c|c} P^*_{2\times s_0} & \mathbf{O}_{2\times m} \\ \hline \mathbf{O}_{6\times s_0} & Q^*_{6\times m} \end{array}\right) \left(\begin{array}{c|c} P_{s_0\times 2} & \mathbf{O}_{s_0\times 6} \\ \hline \mathbf{O}_{m\times 2} & Q_{m\times 6} \end{array}\right) V^*\, W(\gamma) \\ = \quad & W^*(\gamma+q_i)\, V \left(\begin{array}{c|c} (\Sigma^*\,\Sigma)_{2\times 2} & \mathbf{O}_{2\times 6} \\ \hline \mathbf{O}_{6\times 2} & \mathbf{O}_{6\times 6} \end{array}\right) V^*\, W(\gamma) \\ + \quad & W^*(\gamma+q_i)\, V \left(\begin{array}{c|c} (P^*\,P)_{2\times 2} & \mathbf{O}_{2\times 6} \\ \hline \mathbf{O}_{6\times 2} & I_6 \end{array}\right) V^*\, W(\gamma) \\ = \quad & W^*(\gamma+q_i)\, V \left(\begin{array}{c|c} (\Sigma^*\,\Sigma + P^*\,P)_{2\times 2} & \mathbf{O}_{2\times 6} \\ \hline \mathbf{O}_{6\times 2} & I_6 \end{array}\right) V^*\, W(\gamma) \\ = \quad & W^*(\gamma+q_i)\, V \left(\begin{array}{c|c} I_2 & \mathbf{O}_{2\times 6} \\ \hline \mathbf{O}_{6\times 2} & I_6 \end{array}\right) V^*\, W(\gamma) \\ = \quad & W^*(\gamma+q_i)\, V\, I_8\, V^*\, W(\gamma) = W^*(\gamma+q_i)\, W(\gamma) = \delta_{i,0} I_2. \end{aligned} \tag{3.6}$$

The last equality in (3.6) follows from the fact that all columns collectively from all matrices $W(\gamma + q_i)$, $i = 0, ..., 3$ form a an orthonormal set. Therefore, condition (b) of item 4 of theorem 2 is satisfied for $\theta = I_2$, so, a class of affine Parseval frames of compactly supported wavelets on $[0, 1]^2$ has been constructed. Let us demonstrate a specific example.

We seek to construct a pair of functions

$$\phi_i(x, y) = (a_i x + b_i y + c_i)\chi_{[0,1)^2}(x, y),\ \ a_i, b_i, c_i \in \mathbb{R}\ , i = 1, 2$$

satisfying the refinability equation

$$\Phi(x, y) = K\,\Phi(2x, 2y) + L\,\Phi(2x-1, 2y) + M\,\Phi(2x, 2y-1) + N\,\Phi(2x-1, 2y-1)$$

for some 2×2 real matrices K, L, M and N. Then these functions ϕ_i must have the form

$$\left\{ \begin{array}{lcl} \phi_1(x, y) & = & (a_1 x + b_1 y + c_1)\chi_{[0,1)^2}(x, y) \\ \phi_2(x, y) & = & d(a_1 x + b_1 y + c_2)\chi_{[0,1)^2}(x, y) \end{array} \right. ,\ d \in \mathbb{R}\backslash\{0\},\ c_1 \neq dc_2.$$

If we take $d = 1$, $a_1 = b_1$ and assume that the corresponding jointly refinable functions ϕ_1, ϕ_2 form orthonormal pair and satisfy $\widehat{\phi}_1(0) = \frac{1}{\sqrt{2}}$ and $\widehat{\phi}_1(0) = -\frac{1}{\sqrt{2}}$, then we obtain two pairs of refinable function vectors of the form $(\phi_1, \phi_2)^T$ or $(\phi'_1, \phi'_2)^T = (-\phi_2, -\phi_1)^T$, where

$$\left\{ \begin{array}{lcl} \phi_1(x, y) & = & \sqrt{3}(x + y - 1 + \frac{1}{\sqrt{6}})\chi_{[0,1)^2}(x, y) \\ \phi_2(x, y) & = & \sqrt{3}(x + y - 1 - \frac{1}{\sqrt{6}})\chi_{[0,1)^2}(x, y) \end{array} \right. . \tag{3.7}$$

For the rest of this example we work with $\Phi = (\phi_1, \phi_2)^T$ only. Then the above condition $\widehat{\Phi}^*(0)\widehat{\Phi}(0) = 1$ is satisfied. Notice that ϕ_1 and ϕ_2 are not individually refinable but they are indeed jointly refinable. A routine space domain computation yields the following refinement mask for Φ

$$\begin{aligned} H_0(\gamma) = \ & \tfrac{1}{16} \begin{pmatrix} 3-\sqrt{6} & \sqrt{6}-1 \\ -1-\sqrt{6} & 3+\sqrt{6} \end{pmatrix} + \frac{1}{16}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\left(e^{-2\pi i\gamma_1} + e^{-2\pi i\gamma_2}\right) \\ & + \ \frac{1}{16}\begin{pmatrix} 3+\sqrt{6} & -1-\sqrt{6} \\ \sqrt{6}-1 & 3-\sqrt{6} \end{pmatrix} e^{-2\pi i(\gamma_1+\gamma_2)}. \end{aligned}$$

Also, the orthonormality of ϕ_i's implies that $G(\gamma) = I_2$ for a.e. $\gamma \in \mathbb{T}^2$ and so the conditions $(i)-(iv)$ of section 1 on Φ are satisfied. In this case, X has orthonormal rows. Using Mathematica, we estimate the singular values of X:

$$\sigma_{11} = \sigma_{22} = 1.$$

Therefore we may define an $m \times 2\,(m \geq 6)$ wavelet mask by (3.5). If we set $m = 6$ and $A = I_6$ in (3.5), we obtain (again using Mathematica) the wavelet mask H_1 expressed as the product:

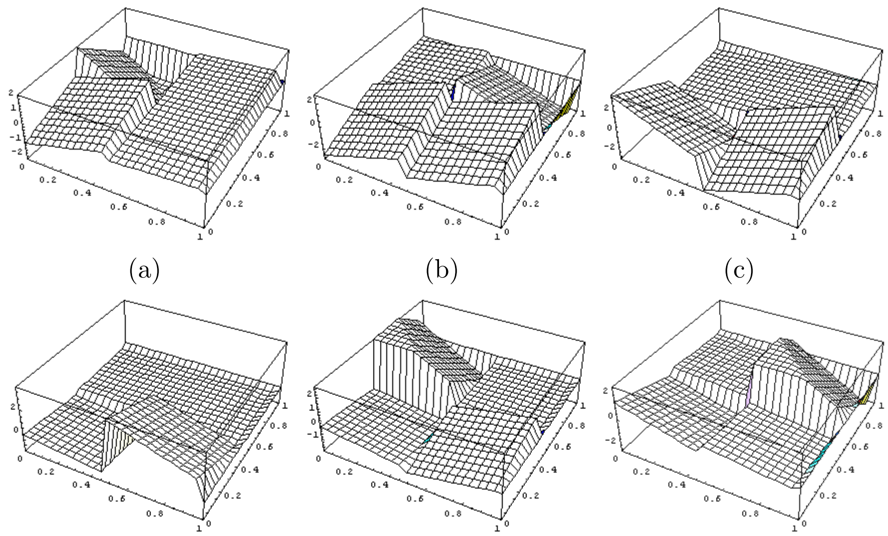
$$\begin{pmatrix} 0.354353 & 0.150611 & -0.875 & 0.237518 & 0.0793882 & -0.0331296 & -0.0331296 & -0.145647 \\ 0.322985 & 0.569315 & 0.125 & -0.591547 & 0.389891 & 0.106438 & 0.106438 & -0.177015 \\ -0.411331 & 0.609963 & -0.125 & 0.0729854 & -0.51536 & 0.286654 & 0.286654 & 0.088669 \\ -0.379964 & 0.191259 & -0.125 & -0.0979498 & 0.174137 & -0.852914 & 0.147086 & 0.120036 \\ -0.379964 & 0.191259 & -0.125 & -0.0979498 & 0.174137 & 0.147086 & -0.852914 & -0.120036 \\ -0.348597 & -0.227445 & -0.125 & -0.268885 & -0.136367 & 0.00751839 & 0.00751839 & -0.848597 \end{pmatrix} W(\gamma).$$

The six resulting wavelets are non separable, orthonormal and they are supported on $[0,1]^2$. They also have zero moments and are discontinuous on the lines $x = \frac{1}{2}$, $y = \frac{1}{2}$ as well as on the boundary of $[0,1]^2$ as shown in Fig. 1.

Example 2. This is a variation of the previous example. We modify the wavelet mask H_1 defined in (3.5) to generate a pair of affine dual frames for $L_2(\mathbb{R}^2)$ arising from the refinable function vector Φ of the previous example. Indeed, let us we extend the definition of the mask H_1 such that Q is an $m \times 6$ matrix with e.g. linearly independent (instead of orthonormal) columns. Define another mask H_1^d by

$$H_1^d(\gamma) = \left(\begin{array}{c|c} P_{s_0\times 2} & \mathbf{O}_{s_0\times 6} \\ \hline \mathbf{O}_{m\times 2} & R_{m\times 6} \end{array}\right) \cdot V^* \cdot W(\gamma),\ \ (m \geq 6),$$

where P, V, W are as in (3.5) and R is an $m \times 6$ real matrix such that $R\,Q = I_6$, which is the crucial condition we need to satisfy in order to construct a pair of dual affine frame wavelet masks. For example the rows of Q can form a frame and the rows of R form a dual of this frame,



(a) (b) (c)

(d) (e) (f)

FIGURE 1. Plots of the six orthonormal wavelets generated from the function vector Φ in (3.7).

both in $\mathbb{R}^6$. Then, by working as in (3.6) we verify that condition (ii) of item (d) of theorem 1 is satisfied with $\theta = I_2$, i.e.

$$H_0(\gamma + q_i)H_0(\gamma) + H_1^{d*}(\gamma + q_i)H_1(\gamma) = \delta_{i,0}I_2, \quad \gamma, \gamma + q_i \in \sigma_\Phi. \quad (3.8)$$

Thus, a pair (Ψ, Ψ^d) of affine dual frames for $L_2(\mathbb{R}^2)$ is generated from Φ.

In the previous example we modified the high-pass mask H_1 resulting in generating a pair of "dual" high pass masks. However, the core machinery of the frame wavelet construction remained the same as in example 1. In the next example we introduce in our construction a "dual" refinable function.

Example 3. We continue assuming that Φ is as in example 1. Now, consider a 2×2 measurable matrix valued function θ which is also unitary on σ_Φ. Hence, the vector-valued function $\Phi^d = (\phi_1^d, \phi_2^d)^T$ defined by

$$\widehat{\Phi}^d(\gamma) = \theta(\gamma)\widehat{\Phi}(\gamma), \gamma \in \sigma_\Phi$$

is square integrable on $\mathbb{R}^2$. The latter property implies

$$\Phi^d \in \overline{span}\{\tau_n\Phi : \ n \in \mathbb{Z}^2\}.$$

Moreover, the unitarity of θ and the invertibility of the Gram matrix G for a.e. γ in σ_Φ implies that the Gram matrix G^d in non singular for a.e. γ in σ_Φ as well. Observe that Φ^d is refinable with refinement mask

$$H_0^d(\gamma) = \theta(2\gamma)\, H_0(\gamma)\, \theta^*(\gamma),\ \gamma \in \mathbb{T}^2.$$

Also, $\widehat{\Phi}^{d*}(0)\theta(0)\widehat{\Phi}(0) = 1$, so condition (i) of item (d) of theorem 1 is satisfied and Φ^d satisfies all conditions $(i) - (iv)$ of section 1. In addition,

$$\|\widehat{\Phi}^{d*}(\cdot - k)\theta(\cdot)\widehat{\Phi}(\cdot)\|_{L_1} = \|\widehat{\Phi}^{d*}(\cdot - k)\widehat{\Phi}^d(\cdot)\|_{L_1} = \|\phi_1^d\|^2 + \|\phi_2^d\|^2 < \infty.$$

Let H_1^d be as in example 2. Define the dual wavelet mask

$$\widetilde{H}_1^d(\gamma) = H_1^d(\gamma)\, \theta^*(\gamma),\ \gamma \in \mathbb{T}^2.$$

Then

$$\begin{aligned} & H_0^{d*}(\gamma+q_i)\theta(2\gamma)H_0(\gamma) + \widetilde{H}_1^{d*}(\gamma+q_i)H_1(\gamma) = \theta(\gamma+q_i)(H_0^*(\gamma+q_i)H_0(\gamma) \\ + \quad & H_1^{d*}(\gamma+q_i)H_1(\gamma)) = \delta_{i,0}\theta(\gamma),\ \ \gamma, \gamma+q_i \in \sigma_\Phi, \end{aligned}$$

where the last equality was obtained from (3.8). Therefore condition (ii) of item (d) of theorem 1 is satisfied and so we obtain a pair of affine dual frames of $L_2(\mathbb{R}^2)$ generated from a pair (Φ, Φ^d) of refinable function vectors on $[0,1]^2$. If the entries of θ are trigonometric polynomials, then the resulting wavelets are compactly supported on $[0,1]^2$.

In the following example we modify the method introduced in Example 1 in order to obtain a Parseval frame of compactly supported wavelets on $[0,2]^2$.

Example 4. Let $\Phi = (\phi_1, \phi_2)^T \in L_2^{2\times 1}(\mathbb{R}^2)$ be a refinable function vector on $[0,2]^2$ satisfying conditions $(i) - (iv)$ of section 1. Condition (a) stated in example 1 must be satisfied by Φ as well. Since Φ is supported on $[0,2]^2$, the refinement mask of Φ has the form

$$\begin{aligned} H_0(\gamma) &= \frac{1}{4}\Big(A + Be^{-2\pi i\gamma_1} + Ce^{-2\pi i\gamma_2} + De^{-2\pi i(\gamma_1+\gamma_2)} + Ee^{-2\pi i(2\gamma_1+\gamma_2)} \\ &+ \quad Fe^{-2\pi i(\gamma_1+2\gamma_2)} + Ge^{-4\pi i\gamma_1} + Ke^{-4\pi i\gamma_2} + Le^{-4\pi i(\gamma_1+\gamma_2)}\Big), \quad (3.9) \end{aligned}$$

where $A = (A_{ij}), ..., L = (L_{ij})$ are 2×2 real matrices. Then, as in example 1,

$$H_0(\gamma+q_i) = X \cdot W(\gamma+q_i), i = 0, ..., 3,$$

where

$$X = \begin{pmatrix} A_{11} & ... & L_{11} & A_{12} & ... & L_{12} \\ A_{21} & ... & L_{21} & A_{22} & ... & L_{22} \end{pmatrix}$$

and

$$W(\gamma) = \left(\begin{array}{c|c} R_{9\times 1}(\gamma) & \mathbf{O}_{9\times 1} \\ \hline \mathbf{O}_{9\times 1} & R_{9\times 1}(\gamma) \end{array}\right) \qquad (3.10)$$

is a 18×2 block matrix valued function with

$$R^T(\gamma) = (1, e^{-2\pi i\gamma_1}, e^{-2\pi i\gamma_2}, e^{-2\pi i(\gamma_1+\gamma_2)}, e^{-2\pi i(2\gamma_1+\gamma_2)} e^{-2\pi i(\gamma_1+2\gamma_2)}, e^{-4\pi i\gamma_1}, e^{-4\pi i\gamma_2}, e^{-4\pi i(\gamma_1+\gamma_2)}).$$

This is where this example starts to differ from example 1. In this case, all columns collectively from all matrices $W(\gamma + q_i)$, $i = 0, ..., 3$ form a linearly independent set and not always an orthonormal set as we had in example 1 when we derived (3.6). This particular orthonormality plays a crucial role for the construction of a self dual wavelet mask in example 1. To overcome this difficulty, we introduce an invertible 18×18 weight matrix T such that

(i) $W_1(\gamma + q_i) = T\, W(\gamma + q_i)$ and
(ii) any pair of columns of any pair of the matrices $W_1(\gamma + q_i)$, $i = 0, ..., 3$ are orthogonal to each other and each column is a unit vector. T his is precisely the property we used to prove (3.6) in example 1.

This weight matrix T is given by

$$T = \left(\begin{array}{c|c} S_{9\times 9} & \mathbf{O}_{9\times 9} \\ \hline \mathbf{O}_{9\times 9} & S_{9\times 9} \end{array} \right),$$

where

$$S = \operatorname{diag}\Big(\frac{1}{4}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\Big) \tag{3.11}$$

is a 9×9 diagonal matrix. Then,

$$H_0(\gamma + q_i) = XW(\gamma + q_i) = \underbrace{X\, T^{-1}}_{X_1}\, \underbrace{T\, W(\gamma + q_i)}_{W_1(\gamma+q_i)} = X_1\, W_1(\gamma + q_i). \tag{3.12}$$

Now we can proceed as in example 1 performing SVD on X_1 being careful with the dimensions of the resulting matrices. Indeed, let $X_1 = U\,\Sigma\,V^*$ be the SVD of the 2×18 matrix X_1 defined in (3.12). By working as in (3.6), it is not hard to verify, that, if the two singular values of X_1 are less than or equal to 1, then the $(m + s_0) \times 2$ wavelet mask

$$H_1(\gamma) = \left(\begin{array}{c|c} P_{s_0\times 2} & \mathbf{O}_{s_0\times 16} \\ \hline \mathbf{O}_{m\times 2} & Q_{m\times 16} \end{array} \right) V^*\, W_1(\gamma),\ (m \geq 16) \tag{3.13}$$

satisfies (3.1). Here the index s_0 is defined in (3.4), the $s_0 \times 2$ matrix P depends on the singular values $\Sigma = \operatorname{diag}(\sigma_{11}, \sigma_{22})$ of the 2×18 matrix X_1 and Q is any $m \times 16$ matrix with orthonormal columns. More specifically, P results from the 2×2 diagonal matrix $(I_2 - \Sigma^*\Sigma)^{\frac{1}{2}} =$

diag($\sqrt{1-\sigma_{11}^2}, \sqrt{1-\sigma_{22}^2}$) by deleting its first $(2-s_0)$ rows which correspond to zero diagonal elements. Therefore the Oblique Extension Principle conditions of part 4 of theorem 2 are satisfied for $\theta = I_2$, so, an affine Parseval frame of compactly supported wavelets on $[0,2]^2$ is produced. Let us provide a specific example.

Take a compactly supported refinable function vector $\Phi = (\phi_1, \phi_2)^T$ on $[0,1]^2$ such that

$$\begin{cases} \phi_1(x,y) &= \frac{1}{\sqrt{2}}\big(\chi_{[0,\frac{1}{2})^2}(x,y) + \chi_{[\frac{1}{2},1)^2}(x,y)\big) \\ \phi_2(x,y) &= \frac{1}{\sqrt{2}}\big(\chi_{[0,\frac{1}{2})\times[\frac{1}{2},1)}(x,y) + \chi_{[\frac{1}{2},1)\times[0,\frac{1}{2})}(x,y)\big) \end{cases}.$$

Note that the functions ϕ_1 and ϕ_2 are orthogonal and they are not individually refinable but they are jointly refinable. The refinement mask of Φ is the following

$$H_0^{\Phi}(\gamma) = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\Big(1+e^{-2\pi i(\gamma_1+\gamma_2)}\Big)+\frac{1}{4}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\Big(e^{-2\pi i\gamma_1}+e^{-2\pi i\gamma_2}\Big). \tag{3.14}$$

The orthonormality of ϕ_1 and ϕ_2 implies that $G_\Phi(\gamma) = I_2$ a.e. on $\mathbb{T}^2$.

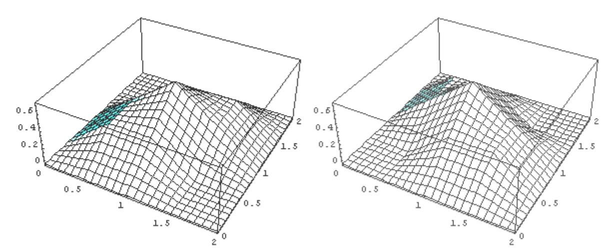
Let $\phi = \chi_{[0,1)^2}$. With the previously defined ϕ_1 and ϕ_2 we denote a new function vector $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2)^T$ such that

$$\begin{cases} \widetilde{\phi}_1 &= \phi_1 * \phi \\ \widetilde{\phi}_2 &= \phi_1 * \phi \end{cases}.$$

By working in the Fourier domain it is easy to see that $\widetilde{\Phi}$ is a refinable function vector on $[0,2]^2$ with mask

$$H_0^{\widetilde{\Phi}}(\gamma) = \frac{1}{16}\Big(1+e^{-2\pi i\gamma_1}+e^{-2\pi i\gamma_2}+e^{-2\pi i(\gamma_1+\gamma_2)}\Big)\, H_0^{\Phi}(\gamma). \tag{3.15}$$

The plots of $\widetilde{\phi}_1, \widetilde{\phi}_2$ are shown in Figure 4.



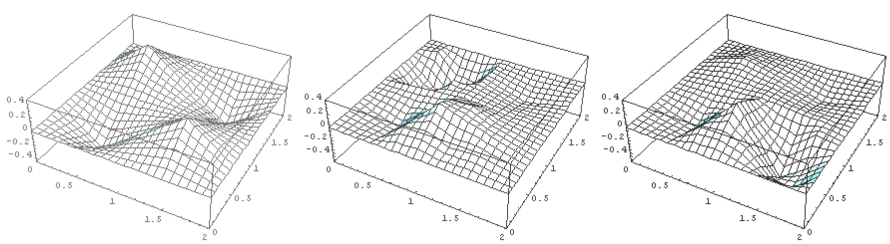
(a) (b)

FIGURE 2. Plots of $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$.

Also, $\widehat{\widetilde{\Phi}^*}(0)\widehat{\widetilde{\Phi}}(0) = 1$ and moreover $G_{\widetilde{\Phi}}$ is non singular for a.e. $\gamma \in \mathbb{T}^2$, otherwise $\widetilde{\phi}_2 \in \overline{\text{span}}\{\widetilde{\phi}_1(\cdot - n) : n \in \mathbb{Z}^2\}$ (or $\widetilde{\phi}_1 \in \overline{\text{span}}\{\widetilde{\phi}_2(\cdot - n) : n \in \mathbb{Z}^2\}$). Then, there exists a measurable $\mathbb{Z}^2$-periodic function β such that $\hat{\phi}_2\hat{\phi} = \beta\hat{\phi}_1\hat{\phi}$ a.e. But $\hat{\phi}$ vanishes only on a null set, hence, $\hat{\phi}_2 = \beta\hat{\phi}_1$ a.e. which contradicts the fact that $G_\Phi(\gamma)$ is non singular for a.e. γ. By combining (3.14) with (3.15) and with the above choice of S (see (3.11)) we compute the 2×18 matrix X_1 as in (3.12). We perform SVD of the matrix X_1 using Mathematica. The singular values of X_1 are

$$\sigma_{11} = 1,\ \sigma_{22} = 0.5.$$

Finally by taking $Q = I_{16}$ in (3.13), we obtain a 17×2 wavelet mask corresponding to a Parseval frame of 17 compactly supported continuous wavelets on $[0, 2]^2$ with zero moments. Below we give plots of three of the designed wavelets (Fig. 4).



(a) (b) (c)

FIGURE 3. Plots of three representative wavelets of the self-dual family Ψ produced from the refinable function vector $\widetilde{\Phi}$ in Example 4

Example 5. Consider the functions

$$\begin{cases} \phi_1(x,y) = \frac{1}{\sqrt{2}}\chi_{[0,1)^2}(x,y) \\ \phi_2(x,y) = \frac{1}{\sqrt{2}|Det(R)|}\chi_{R([0,1)^2)}(x,y) \end{cases},\ (x,y) \in \mathbb{R}^2,$$

where

$$R = \begin{pmatrix} 2k & 2l+1 \\ 2\mu+1 & 2\nu \end{pmatrix},\ k, l, \mu, \nu \in \mathbb{Z} - \{0\}.$$

First we note that $R(\mathbb{Z}^2) \subset \mathbb{Z}^2$ and

$$\widehat{\phi}_2(\gamma) = \widehat{\phi}_1(R^T\gamma),\ \gamma \in \mathbb{R}^2.$$

If R is an orthogonal matrix (whenever $k = \nu$ and $l + \mu = -1$), then R can be considered as a scaled rotation of $[0, 1]^2$. Obviously, ϕ_1 is refinable with mask

$$H_0(\gamma) = \frac{1}{4}\Big(1 + e^{-2\pi i\gamma_1}\Big)\Big(1 + e^{-2\pi i\gamma_2}\Big),\ \gamma \in \mathbb{T}^2$$

with which we associate the usual orthonormal wavelet mask (high-pass filter)

$$H_1(\gamma) = \begin{pmatrix} H_{11}(\gamma) \\ H_{12}(\gamma) \\ H_{13}(\gamma) \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \big(1 - e^{-2\pi i\gamma_1}\big)\big(1 + e^{-2\pi i\gamma_2}\big) \\ \big(1 + e^{-2\pi i\gamma_1}\big)\big(1 - e^{-2\pi i\gamma_2}\big) \\ \big(1 - e^{-2\pi i\gamma_1}\big)\big(1 - e^{-2\pi i\gamma_2}\big) \end{pmatrix}.$$

Then,

$$\overline{H_0(\gamma + q_j)}H_0(\gamma) + \sum_{i=1}^{3}\overline{H_{1i}(\gamma + q_j)}H_{1i}(\gamma) = \delta_{j,0},\ (j = 0, 1, 2, 3). \quad (3.16)$$

Moreover, ϕ_2 is refinable, because the entries of R are integers and ϕ_1 is refinable. In particular, we have

$$\widehat{\phi}_2(2\gamma) = H_0(R^T\gamma)\widehat{\phi}_2(\gamma).$$

Therefore, the function vector $\Phi = (\phi_1, \phi_2)^T$ is refinable with refinement mask

$$H_0^{\Phi}(\gamma) = \begin{pmatrix} H_0(\gamma) & 0 \\ 0 & H_0(R^T\gamma) \end{pmatrix}.$$

In addition, we have $\widehat{\Phi}^*(0)\widehat{\Phi}(0) = |\hat{\phi}_1(0)|^2 + |\hat{\phi}_2(0)|^2 = 1$. Also, the Gram matrix $G_\Phi(\gamma)$ is non singular for a.e. $\gamma \in \mathbb{T}^2$, otherwise $\phi_2 \in \overline{\text{span}}\{\phi_1(\cdot - n) : n \in \mathbb{Z}^2\}$. Indeed, ϕ_2 is compactly supported and $\{\phi_1(\cdot - n) : n \in \mathbb{Z}^2\}$ is an orthonormal set. Consequently, if $\phi_2 \in \overline{\text{span}}\{\phi_1(\cdot - n) : n \in \mathbb{Z}^2\}$, then $\phi_2 = \sum_{n\in\mathcal{I}} c_n\tau_n\phi_1$, where $\mathcal{I}$ is a finite subset of $\mathbb{Z}^2$. But the latter sum is a function with constant values on integer translates of $[0,1]^2$, hence it cannot be equal to ϕ_2. Therefore, $G_\Phi(\gamma)$ is non singular for a.e. $\gamma \in \mathbb{T}^2$. Once again we seek to construct a wavelet mask so that the Oblique Extension Principle conditions of part 4 of theorem 2 are satisfied. First, the continuity of $\hat{\phi}_1$ and $\hat{\phi}_2$ implies

$$\lim_{j\to-\infty}\widehat{\Phi}^*(2^j\gamma)\widehat{\Phi}(2^j\gamma) = |\hat{\phi}_1(0)|^2 + |\hat{\phi}_2(0)|^2 = 1\ .$$

Let us now define the 6×2 wavelet mask by

$$H_1^{\Phi}(\gamma) = \begin{pmatrix} H_1(\gamma) & \mathbf{0} \\ \mathbf{0} & H_1(R^T\gamma) \end{pmatrix}.$$

Since

$$R^T(\gamma + q_j) = R^T\gamma + R^Tq_j \text{ for any } q_j$$

and $R^T\left(\mathbb{Z}^2/2\mathbb{Z}^2\right) = \mathbb{Z}^2/2\mathbb{Z}^2$ with a bit of abuse of notation, because the rows and the columns of R add up to an odd number, we finally deduce, by taking also (3.16) into account that

$$H_0^{\Phi *}(\gamma + q_j) H_0^{\Phi}(\gamma) + H_1^{\Phi *}(\gamma + q_j) H_1^{\Phi}(\gamma) = \delta_{j,0}\, I_2.$$

Hence, both conditions of part 4 of theorem 2 are satisfied (with $\theta = I_2$), thus giving rise to an affine Parseval frame on $L_2(\mathbb{R}^2)$ with 6 compactly supported wavelets. Notice that three of these wavelets can be obtained by the action of the transform matrix R^{-1}.

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