SPECTRUM OF THE LAPLACIAN AND RIESZ TRANSFORM ON LOCALLY SYMMETRIC SPACES

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Abstract. We assume that the discrete part of the spectrum of the Laplacian on a noncompact locally symmetric space is non empty and we prove that the Riesz transform is bounded on $L^p$ for all $p$ in an interval around 2.

1. Introduction and statement of the results

Let $M$ be a complete, non-compact, connected Riemannian manifold. Let us denote by $dx$ the Riemannian measure and by $\nabla$ the gradient. We shall also denote by $L^p(M, dx)$, $p \geq 1$. If $|.|$ is the length in the tangent space then one can define the (positive) Laplace-Beltrami operator $\Delta$, as well as its square root $\Delta^{1/2}$, as self adjoint and positive operators on $L^2$ by the formula

$$ (\Delta f, f) = \|\nabla f\|_2^2 = \|\Delta^{1/2} f\|_2^2, \quad f \in C^\infty_0(M). $$

Hence the Riesz transform $\nabla \Delta^{-1/2}$ is bounded on $L^2$. The basic issue to ask, which was raised in [31], is for which complete noncompact Riemannian manifold, and for which $p \in (1, \infty)$, the Riesz transform is bounded on $L^p$ i.e. there exists a constant $c_p > 0$ such that

$$ \|\nabla f\|_p \leq c_p \|\Delta^{1/2} f\|_p, \quad f \in C^\infty_0(M). \quad (1.1) $$


In the present work we study the Riesz transform on a noncompact locally symmetric space. To state our results we need to recall few basic concepts about symmetric spaces. These standard facts can be found in [17].

Let $G$ be a noncompact and connected semi-simple Lie group with finite center. We denote by $K$ a compact maximal subgroup of $G$ and we consider the symmetric space $X = G/K$.

Let us denote by $\mathfrak{g}$ (resp. $\mathfrak{k}$) the Lie algebra of $G$ (resp. $K$) and let $\mathfrak{p}$ the subspace of $\mathfrak{g}$ which is orthogonal to $\mathfrak{k}$ with respect to the Killing form. We recall that the restriction of the Killing form on $\mathfrak{p}$ is positive and it defines a Riemannian structure on $X$. We shall denote by $\Delta$ the Laplacian and by $d_X(.,.)$ the Riemannian distance.

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Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We denote by $\mathfrak{a}^*$ the dual of $\mathfrak{a}$. For $\alpha \in \mathfrak{a}^*$ we set $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [H, x] = \alpha (H) \text{ for every } H \in \mathfrak{a}\}$. If $\alpha$ and $\mathfrak{g}_\alpha$ are non-zero we say that $\alpha$ is a restricted root. We denote by $m_\alpha = \dim \mathfrak{g}_\alpha$ the multiplicity of the root $\alpha$.

Let $\Sigma^+ \subset \mathfrak{a}^*$ be a choice of positive roots. A fundamental quantity is

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$ 

Since the Killing form on $\mathfrak{a}$ is positive, it induces an inner product on $\mathfrak{a}^*$, and so $\|\rho\|$ is well defined. It is well known that the spectrum of $\Delta$ on $L^2 (X)$ is equal to $[\|\rho\|^2, \infty)$.

Let $\Gamma$ be a discrete and torsion free subgroup of $G$. We denote by $M$ the locally symmetric space $\Gamma \backslash G / K$. Since $\Gamma$ is torsion free, $M$ equipped with the projection of the Riemannian structure of $X$, becomes a complete Riemannian manifold with negative Ricci curvature. We shall denote also by $\Delta$ the Laplacian, by $d(.,.)$ and $dx$ the Riemannian distance and measure of $M$.

We recall that the $L^2$-spectrum of $\Delta$ on a noncompact locally symmetric space is in general unknown. In the present work we shall assume that it is equal to

$$(1.2) \quad \{\lambda_0, ..., \lambda_m\} \cup [\|\rho\|^2, \infty),$$

where the eigenvalues $0 \leq \lambda_0 < \cdots < \lambda_m$ are of finite multiplicity. This is the case if $M$ is the quotient of the hyperbolic space $\mathbb{H}^{n+1}$ by a geometrically finite Kleinian group $\Gamma$, i.e. when $M = \Gamma \backslash \mathbb{H}^{n+1} = \Gamma \backslash SO(n+1,1) / SO(n+1)$, [18]. Note that in this case $\|\rho\| = n/2$.

In order to estimate the bottom of the spectrum $\lambda_0$ we need the critical exponent $\delta (\Gamma)$ of the group $\Gamma$ which is defined as follows. Let $x_0$ be a fixed point of $X$ and for $R > 0$ we denote by $n_R$ the cardinal of the set $\{\gamma \in \Gamma : d_X (x_0, \gamma x_0) < R\}$. Then

$$\delta (\Gamma) = \limsup_{R \to \infty} \frac{\log n_R}{R}.$$ 

We always have that $0 \leq \delta (\Gamma) \leq 2 \|\rho\|$, [32, p.33].

Let $\mathfrak{a}^+$ be a positive Weyl chamber associated with a choice of a set of positive roots $\Sigma^+ \subset \mathfrak{a}^*$. We set

$$\rho_{\min} = \min_{H \in \mathfrak{a}^*, \|H\| = 1} \rho (H).$$

In [32, Theorem 3.8] Weber, following Leuzinger [19], proved that the point spectrum of $\Delta$ is empty when $0 \leq \delta (\Gamma) \leq \rho_{\min}$. So, the point spectrum appears in the case when $\delta (\Gamma) > \rho_{\min}$. Further, if $\delta (\Gamma) < \|\rho\| + \rho_{\min}$, then $\lambda_0 > 0$, while if $\delta (\Gamma) \geq \|\rho\| + \rho_{\min}$ then $\lambda_0$ may be equal to 0. This is for example the case when $\text{vol} (M) < \infty$, since the constants belong in $L^2$.

Note that in the case of a Kleinian group $M = \Gamma \backslash \mathbb{H}^{n+1}$, if $\delta (\Gamma) > n/2$, then $\lambda_0 = \delta (\Gamma) (n - \delta (\Gamma))$. Thus $\lambda_0 = 0$ if $\delta (\Gamma) = n$.

Before stating our result on the $L^p$-boundedness of the Riesz transform, let us make clear that its proof depends on the properties of the $L^2$-eigenfunctions associated to the point spectrum. In fact we shall show in Theorem 1 below that they belong also in $L^p (M)$ for $p$ in some interval $(r_1, r_2)$ around 2. This fact is a generalization of some results in [14], (see also [30]) and it is inspired from the work [24] of N. Lohoué.
Theorem 1. If \( \delta(\Gamma) > \rho_{\text{min}} \), then every \( L^2 \)-eigenfunction \( u_j \) with eigenvalue \( \lambda_j \), \( j \leq m \), belongs in \( L^p \) for all \( p \in (r_1, r_2) \), where
\[
  r_1 = 2 \left\{ \left( 1 - \left( \frac{\lambda_m}{\| \rho \|^2} \right)^2 \right)^{1/2} + 1 \right\}^{-1},
\]
and
\[
  r_2 = 2 \left\{ \left( 1 - \left( \frac{\lambda_m}{\| \rho \|^2} \right)^2 \right)^{1/2} + 1 \right\}.
\]

Let \( r'_2 \) be the conjugate of \( r_2 \). Note that \( r_1 \leq r'_2 \).

Theorem 2. (i) If \( 0 \leq \delta(\Gamma) \leq \rho_{\text{min}} \), then for all \( p \in (1, \infty) \), there is a constant \( c_p > 0 \) such that
\[
  \| \nabla f \|_p \leq c_p \left\| \Delta^{1/2} f \right\|_p, \quad f \in C^\infty_0 (M).
\]

(ii) If \( \delta(\Gamma) > \rho_{\text{min}} \) and \( \lambda_0 \neq 0 \), then (1.3) is valid for all \( p \in (r'_2, r_2) \).

(iii) If \( \delta(\Gamma) \geq \| \rho \| + \rho_{\text{min}} \) and \( \lambda_0 = 0 \), then (1.3) is valid for all \( p \in (r'_2, r_2) \) and for all \( f \in C^\infty_0 (M) \) such that
\[
  \int_M u'_0 (x) f (x) \, dx = 0,
\]
where \( u'_0, j \leq k_0 \), are the \( L^2 \)-harmonic functions.

For the proof of Theorem 2 we use Theorem 1 and the following local version of (1.3): for all \( p \in (1, \infty) \), there are positive constants \( c_1 \) and \( c_2 \) depending on \( p \) such that
\[
  \| \nabla f \|_p \leq c_1 \left\| \Delta^{1/2} f \right\|_p + c_2 \| f \|_p, \quad f \in C^\infty_0 (M).
\]

Inequality (1.4) has been proved by Lohoué in [21] for complete manifolds with bounded geometry and extended by Bakry, [10, Theorem 4.1, p.160], in the case when the injectivity radius is not bounded below.

Next we deal with the case when \( M = \Gamma \backslash \mathbb{H}^{n+1} \). In this case the point spectrum is non empty if \( \delta(\Gamma) > n/2 \). Using the results of [14], one can replace in Theorem 2 the interval \( (r'_2, r_2) \) by \( (r_1, r'_1) \) which is bigger.

Theorem 3. If \( M = \Gamma \backslash \mathbb{H}^{n+1} \), where \( \Gamma \) is geometrically finite Kleinian group with \( \delta(\Gamma) > n/2 \), then claims (ii) and (iii) of Theorem 2 are valid for all \( p \in (r_1, r'_1) \).

The \( L^p \)-boundedness of the Riesz transform is extensively studied in various geometric settings, as Riemannian manifolds of polynomial volume growth [6, 7, 8, 10, 9, 12, 11, 20, 27, 28], or exponential volume growth [10, 21, 6], Lie groups [1, 22, 23], Cartan-Hadamard manifolds and symmetric spaces of noncompact type [21, 4], discrete groups [2, 16] or graphs [29]. For an extended list of references see [6]. See also [25, 26] for the related problem of multipliers on locally symmetric spaces and Kleinian groups.

Finally, let us say a few words about claims (ii) and (iii) of Theorem 2 where it is proved that the Riesz transform is bounded on \( L^p \) not for all \( p \), but only for \( p \) in an interval \( (r'_2, r_2) \) around 2. This is due to the fact, we proved in Theorem 1 above, that the \( L^2 \)-eigenfunctions associated to the discrete spectrum belong in \( L^p \) only for \( p \in (r'_2, r_2) \). The phenomenon where the Riesz transform is bounded on \( L^p \) for certain values of \( p \) has been observed first by H.-Q. Li [20] and Coulhon and Duong.
[11]. In [20, Theorem 1] Li proves that on a certain class of conical manifolds, the Riesz transform is bounded on $L^p$ if and only if $p \in (1, p_0)$ for some $p_0 > 2$. Later in [6], it is proved that on manifolds satisfying the doubling volume property and the Poincaré inequality, the Riesz transform is bounded on $L^p$ for $p \in (2, p_0)$ for some $p_0 > 2$, if and only if the heat operator $e^{-t\Delta}$ satisfies

$$\sup_{t>0} \sqrt{t} \|\nabla e^{-t\Delta}\|_{p \to p} < \infty,$$

for all $p$ in the same region. As it is observed in [13] the class of conical manifolds treated by Li satisfy the doubling volume property and the Poincaré inequality, and further, (1.5) is satisfied precisely in the region where Li proves the $L^p$-boundedness of the Riesz transform. Finally, it is worth mentioning that in the case of non-compact symmetric spaces we treat here, the approach of [6] does not apply, since these manifolds don’t satisfy the doubling volume property.

Throughout this article the different constants will always be denoted by the same letter $c$. When their dependence or independence is significant, it will be clearly stated.

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2. Proof of Theorem 1

Let $T$ be an operator on $L^2(M)$ and let us denote by $\text{sp}(T)$ its spectrum. Let us also recall that we made the assumption that

$$\text{sp}(\Delta) = \{\lambda_0, \ldots, \lambda_m\} \cup [\|\rho\|^2, \infty),$$

and that the eigenvalues $\lambda_0, \ldots, \lambda_m$ are of finite multiplicity. Thus the eigenspace

$$E_j = \{f \in L^2(M) : \Delta f = \lambda_j f\}, \quad j \leq m,$$

has finite dimension. So, $E_j$ is spanned by a finite number of eigenfunctions $u_j^k$, $k \leq k_j$. For simplicity we write $u_j$ instead of $u_j^k$.

Let us recall that a point $\tilde{x} \in M$ is identified with the trajectory $\{\gamma x; \gamma \in \Gamma\}$ of the point $x \in X$. Let us denote by $P_t = e^{-t\Delta}, t > 0$, the heat semigroup on $M$ and by $p_t(\tilde{x}, \tilde{y})$ its kernel, i.e. the heat kernel on $M$.

For the proof of Theorem 1 we need the following lemma.

**Lemma 1.** Let $x_0$ be a fixed point in $X$. If $p > 2$, then for

$$\beta > \|\rho\| (p - 2),$$

there exists $c > 0$ such that

$$p_1(\tilde{x}, \tilde{x})^{(p-2)/2} \leq c e^{\beta d(\tilde{x}, \tilde{x}_0)}, \quad \text{for all } \tilde{x} \in M.$$

Proof. Let

$$P(s; x, y) = \sum_{\gamma \in \Gamma} e^{-s d(\gamma x, y)} \quad s > \delta(\Gamma), \quad x, y \in X,$$

be the Poincaré series of $\Gamma$. As it is shown in [32, p.46], for $\eta > 0$, there exists a positive constant $c(\eta)$ such that

$$p_1(\tilde{x}, \tilde{x}) \leq c(\eta) P(\delta(\Gamma) + \eta; x, x), \quad x \in X.$$

Further, in [32, p.36], it is proved that for $s > 2\|\rho\|$ and $x_0 \in X$ fixed, there exists a positive constant $c(s, x_0)$ such that

$$P(s; x, x) \leq c(s, x_0) e^{s d(\tilde{x}, \tilde{x}_0)}, \quad \text{for all } x \in X.$$
Let us now choose $\eta$ in (2.4) such that

$$\delta(\Gamma) + \eta = 2(\|\rho\| + \varepsilon), \quad \varepsilon > 0.$$  

Using (2.5) we get

$$p_1(\tilde{x}, \tilde{x})^{(p-2)/2} \leq ce^{(\|\rho\|+\varepsilon)d(\tilde{x}, \tilde{x}_0)(p-2)} \leq ce^{\beta d(\tilde{x}, \tilde{x}_0)},$$

provided that $\beta > \|\rho\| (p-2)$.  

Proof of Theorem 1. Let $u_j$ be an $L^2$-eigenfunction with eigenvalue $\lambda_j$. We have to show that $\|u_j\|_p < \infty$ for all $p \in (r_1, r_2)$. Let us treat first the case $p > 2$. We have

$$P_t u_j(\tilde{x}) = e^{-\lambda_j t} u_j(\tilde{x}) = \int_M p_t(\tilde{x}, \tilde{y}) u_j(\tilde{y}) d\tilde{y}.$$ 

Taking $t = 1/2$, and using the semigroup property of $p_t(\tilde{x}, \tilde{y})$, it follows that

$$|u_j(\tilde{x})| \leq e^{\lambda_j/2} \int_M p_{1/2}(\tilde{x}, \tilde{y}) |u_j(\tilde{y})| d\tilde{y} \leq e^{\lambda_j/2} \left( \int_M p_{1/2}(\tilde{x}, \tilde{y})^2 d\tilde{y} \right)^{1/2} \left( \int_M |u_j(\tilde{y})|^2 d\tilde{y} \right)^{1/2} = e^{\lambda_j/2} p_1(\tilde{x}, \tilde{x})^{1/2} \|u_j\|_2.$$  

We write

$$\|u_j\|^p_p = \int_M |u_j(\tilde{x})|^{p-2} |u_j(\tilde{x})|^2 d\tilde{x} = \int_M |u_j(\tilde{x})|^{p-2} e^{-\beta d(\tilde{x}, \tilde{x}_0)} e^{\beta d(\tilde{x}, \tilde{x}_0)} |u_j(\tilde{x})|^2 d\tilde{x},$$

where $\tilde{x}_0$ is a fixed point in $M$ and $\beta$ is given by (2.2). Combining (2.6) and (2.7) we get that

$$\|u_j\|^p_p \leq e^{(p-2)\lambda_j/2} \|u_j\|^{(p-2)}_2 \times \int_M p_1(\tilde{x}, \tilde{x})^{(p-2)/2} e^{-\beta d(\tilde{x}, \tilde{x}_0)} e^{\beta d(\tilde{x}, \tilde{x}_0)} |u_j(\tilde{x})|^2 d\tilde{x}.$$ 

Using (2.3) it follows that if $\beta$ is as above, then

$$\|u_j\|^p_p \leq ce^{(p-2)\lambda_j/2} \|u_j\|^{(p-2)}_2 \int_M e^{\beta d(\tilde{x}, \tilde{x}_0)} |u_j(\tilde{x})|^2 d\tilde{x}.$$  

Combining (2.9) with Agmon’s $L^2$-weighted estimate [3, p.55]

$$\int_M |u_j(\tilde{x})|^2 e^{2(1-\varepsilon)d(\tilde{x}, \tilde{x}_0)} d\tilde{x} \leq c,$$  

for all $\varepsilon > 0$, it follows that $\|u_j\|_p < \infty$ provided that

$$2(1-\varepsilon) \left( \frac{\|\rho\|^2 - \lambda_j}{\|\rho\|^2} \right)^{1/2} \geq \beta > \|\rho\| (p-2),$$

i.e. when

$$p < 2 \left( 1 - \frac{\lambda_j}{\|\rho\|^2} \right)^{1/2} + 2.$$
The case \( p \in (r_1, 2) \) is a particular case of a more general result of M. Taylor [30, p.783-4]. So, we shall only give the tools we need for its proof. First, let us recall that \( M \) has exponential volume growth; for \( R > 0 \), the volume \( V(x, R) \) of the ball \( B(x, R) \) satisfies

\[
V(x, R) \sim R^{2{a-1}} e^{2\|\rho\|R},
\]

(2.11) where \( a = \text{rank } X = \dim a \), [32, p.33].

Using (2.11) and the fact that eigenfunctions \( u_j \) with eigenvalue \( \lambda_j \) have exponential decay

\[
\int_{B(x_0, R)} |u_j(x)|^2 dx \leq ce^{-2R(\|\rho\|^2-\lambda_j)}^{1/2},
\]

Taylor proves in [30, p.783-4], that \( u_j \in L^p \) for all \( p \in (r_1, 2) \).

Let us now present the \( L^p \)-properties of the eigenfunctions \( u_j \) in some interesting particular cases.

**Proposition 1.** (i). If \( \text{vol}(M) < \infty \), then \( u_j \in L^p(M) \) for all \( p \in [1, 2] \).

(ii). If \( \text{dim } M \geq 3 \) and \( M \) has bounded geometry, then every \( L^2 \)-eigenfunction belongs in \( L^p \) for all \( p > 2 \).

**Proof.** (i). Just note that for all \( p \in [1, 2) \), we have that \( L^2(M) \subset L^p(M) \).

(ii). In [32, Theorem 3], Weber proved that in this case every \( L^2 \)-eigenfunction \( u_j \) is bounded. So, for \( p > 2 \), we have that

\[
\int_M |u_j(x)|^p dx \leq \|u_j\|_\infty^{p-2} \|u_j\|_2^2 < \infty.
\]

\( \square \)

**Remark 1.** In the particular case of Kleinian groups \( M = \Gamma \backslash \mathbb{H}^{n+1} \), Davies, Simon and Taylor have obtained in [14] the following results.

(i). If \( \text{vol}(M) = \infty \), then every \( L^2 \)-eigenfunction belongs in \( L^p \) for all \( p \in (r_1, r_2) \), where

\[
r_1 = 2 \left\{ 1 + \left(1 - \left(\frac{\lambda_m}{\|\rho\|^2}\right)\right)^{1/2} \right\}^{-1} \quad \text{and} \quad r_2 = r_1',
\]

(2.12) cf. [14, Propotisions 12 and 18]. Note that

\[
r_1' = 2 \left\{ 1 - \left(1 - \left(\frac{\lambda_m}{\|\rho\|^2}\right)\right)^{1/2} \right\}^{-1},
\]

and that \( \|\rho\| = n/2 \).

(ii). If \( \text{vol}(M) < \infty \), then \( r_1 = 1 \) and

(iii). If \( \text{dim } M \geq 3 \), and \( M \) has bounded geometry, then \( r_2 = \infty \).

**Remark 2.** In the case when \( M = \Gamma \backslash \mathbb{H}^{n+1} \) and \( M \) contains a cusp of rank \( r \), then [15, Theorem 5.4]

\[
p_1(\tilde{x}, \tilde{x}) \leq ce^{-\delta(\Gamma)(n-\delta(\Gamma))} e^{rd(x,x_0)}.
\]

Bearing in mind that \( \|\rho\| = n/2 \) and arguing as in the proof of Theorem 1, we get that \( u_j \in L^p \), provided that

\[
p < \frac{4}{r} \left(\frac{n^2}{4} - \lambda_j\right)^{1/2} + 2.
\]
3. Proof of Theorems 2 and 3

As it shown in Theorem 1, all $L^2$-eigenfunctions $u_j$, $j = 0, 1, \ldots, m$, belong also in $L^p$, $p \in (r'_2, r_2)$. Let us denote by $L^p_m$ the span in $L^p$ of $u_j$, $j = 0, 1, \ldots, m$. Since, by our assumption, all the eigenvalues have finite multiplicity, $L^p_m$ is finite dimensional. It follows that

$$L^p = L^p_m \oplus (L^p_m)^\perp,$$

where

$$(L^p_m)^\perp = \{ f \in L^p : \langle f, u_j \rangle = 0, \ 0 \leq j \leq m \},$$

is the complement of $L^p_m$ in $L^p$.

Let us denote by $\pi_m$ the projection of $L^p$ on $L^p_m$:

$$\pi_m (f) = \sum_{0 \leq j \leq m} \langle f, u_j \rangle u_j, \ f \in L^p.$$

An operator $T$ on $L^p$ is then written as

$$Tf = T\pi_m (f) + T (I - \pi_m) (f).$$

We shall use the above decomposition to prove the following lemma.

**Lemma 2.** If $\lambda_0 \neq 0$, then $\Delta^{-1/2}$ is bounded on $L^p$ for all $p \in (r'_2, r_2)$.

**Proof.** We shall first show that $\Delta^{-1/2}\pi_m$ is bounded on $L^p$. Since $\lambda_j \neq 0$ for all $j \leq m$, we have that

$$\Delta^{-1/2}\pi_m (f) = \Delta^{-1/2} \left( \sum_{0 \leq j \leq m} \langle f, u_j \rangle u_j \right)$$

$$= \sum_{0 \leq j \leq m} \langle f, u_j \rangle \Delta^{-1/2} u_j$$

$$= \sum_{0 \leq j \leq m} \langle f, u_j \rangle \lambda_j^{-1/2} u_j. \quad (3.1)$$

By Theorem 1, $u_j \in L^p$ for all $p \in (r_2, r'_2)$. Thus if $f \in L^p$ and $q$ is the conjugate of $p$, by (3.1) we get that

$$\left\| \Delta^{-1/2}\pi_m (f) \right\|_p \leq \sum_{0 \leq j \leq m} |\langle f, u_j \rangle| \lambda_j^{-1/2} \|u_j\|_p$$

$$\leq \sum_{0 \leq j \leq m} \lambda_j^{-1/2} \|f\|_p \|u_j\|_q \|u_j\|_p$$

$$\leq c \|f\|_p. \quad (3.2)$$

It remains to show that $\Delta^{-1/2} (I - \pi_m)$ is also bounded on $L^p$. For that we use the following Laplace transform formula:

$$\Delta^{-1/2} (I - \pi_m) f = c \int_{0}^{\infty} e^{-t\Delta} (I - \pi_m) f \frac{dt}{\sqrt{t}}$$

$$= c \int_{0}^{\infty} P_t (I - \pi_m) f \frac{dt}{\sqrt{t}}. \quad (3.3)$$

Next, we shall prove that there is a positive constant $c(p)$ such that
If we denote by $P$ the operator $P(t)$, then

$$\|P(t)(I - \pi_m)\|_{p \to p} \leq e^{-tc(p)},$$

for all $p \in (r', r_2)$.

Combining (3.3) and (3.4) we obtain that

$$\left\|\Delta^{-1/2}(I - \pi_m)f\right\|_p \leq c \int_0^\infty \left\|P(t)(I - \pi_m)f\right\|_p \frac{dt}{\sqrt{t}} \leq c \int_0^\infty e^{-tc(p)} \left\|f\right\|_p \frac{dt}{\sqrt{t}} \leq c \left\|f\right\|_p.$$  (3.5)

Thus, to complete the proof of the lemma, it remains to prove (3.4). For that we write

$$P_t = P_t\pi_m + P_t(I - \pi_m).$$

It is easy to see that $P_t$ leaves invariant both $L^p_m$ and $(L^p_m)^\perp$. This implies that $P_t\pi_m$ is an operator on $L^p_m$ and $P_t(I - \pi_m)$ on $(L^p_m)^\perp$. Clearly, the $L^2$-spectrum of $P_t\pi_m$ is equal to $\{e^{-t\lambda_0}, \ldots, e^{-t\lambda_m}\}$. This, combined with the fact the spectrum of $P_t$ is equal to

$$\{e^{-t\lambda_0}, \ldots, e^{-t\lambda_m}\} \cup \left[e^{-t\left\|\rho\right\|^2}, \infty\right),$$

implies that the $L^2$-spectrum of $P_t(I - \pi_m)$ is equal to $\left[e^{-t\left\|\rho\right\|^2}, \infty\right)$.

This gives that

$$\|P_t(I - \pi_m)\|_{2 \to 2} \leq e^{-t\left\|\rho\right\|^2}.$$  (3.6)

Also, since $P_t$ is a contraction on $L^p$ for all $p \geq 1$, it follows that

$$\|P_t(I - \pi_m)f\|_{r_2} \leq \|P_t\|_{r_2 \to r_2} \|(I - \pi_m)f\|_{r_2} \leq \|(I - \pi_m)f\|_{r_2} \leq c \left\|f\right\|_{r_2}.$$  (3.7)

By interpolation and duality we have that

$$\|P_t(I - \pi_m)\|_{p \to p} \leq e^{-tc(p)},$$

for all $p \in (r', r_2)$ and the proof of (3.4) is complete.  \qed

**Lemma 3.** If $\lambda_0 = 0$, then $\Delta^{-1/2}$ is bounded on $(L^p_0)^\perp$ for all $p \in (r_2, r')$.

**Proof.** Let us denote by $u_{0,j}$, $j \leq k$, the $L^2$-eigenfunctions with eigenvalue $0$, i.e. the $L^2$-harmonic functions. Let $L^p_0$ be the span in $L^p$ of $u_{0,j}$, $j \leq k$. If for example $\text{vol}(M) < \infty$, then $1 \in L^p_0$ for all $p \geq 1$.

Let us now assume that for some $j \leq k$, $u_{0,j}$ satisfies

$$\left\|u_{0,j}\right\|_2 \leq c \left\|\Delta^{1/2}u_{0,j}\right\|_2 = c \left\|\lambda_0^{1/2}u_{0,j}\right\|_2 = 0.$$

It follows that $u_{0,j} = 0$, and consequently $\Delta^{-1/2}$ is not bounded on $L^p_0$.

Since $\dim L^p_0 < \infty$, we have that

$$L^p = L^p_0 \oplus (L^p_0)^\perp$$

where

$$(L^p_0)^\perp = \left\{f \in L^p : \langle f, u_{0,j} \rangle = 0, \ j \leq k\right\}.$$
Proceeding as in the previous case when \( \lambda_0 \neq 0 \), one can see that the \( L^2 \)-spectrum of \( P_t \) on \( (L^p_0)^\perp \) is equal to \( \{ e^{-t\lambda_1}, \ldots, e^{-t\lambda_m} \} \cup \left[ e^{-t\|\rho\|^2}, \infty \right) \).

But \( \lambda_1 \neq 0 \) and the same arguments as in Lemma 2, allow us to prove that \( \Delta^{-1/2} \) is bounded on \( (L^p_0)^\perp \).

\( \square \)

End of proof of Theorems 2 and 3. To prove claim (i) of Theorem 2 we recall that in this case the \( L^2 \)-spectrum of \( \Delta \) is equal to \( \left[ \|\rho\|^2, \infty \right) \). This implies that

\[
\|P_t\|_{2\to 2} \leq e^{-t\|\rho\|^2}.
\]

Also, \( \|P_t\|_{1\to 1} \leq 1 \) and by interpolation and duality, we have that

\[
\|P_t\|_{p\to p} \leq e^{-tc(p)},
\]

for all \( p \in (1, \infty) \). Arguing as in (3.5) we get that \( \|\Delta^{-1/2}\|_{p\to p} \leq c_p < \infty \) for all \( p \in (1, \infty) \).

Claims (ii) and (iii) of Theorem 2 follow from Lemma 2 and Lemma 3 while for the proof of Theorem 3, instead of Theorem 1 we use the results of [14] we presented in the Remark 1. \( \square \)

REFERENCES


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