Adaptive Lexicographic Optimization in Multi-Class M/GI/1 Queues

P.P. Bhattacharya, L. Georgiadis, P. Tsoucas

IBM Research Division

T.J. Watson Research Center

Yorktown Heights, NY 10598

I. Viniotis

Electrical and Computer Engineering Department

Center for Communications and Signal Processing

North Carolina State University

Raleigh, NC 27695

Abstract

We consider a multi-class $M/GI/1$ system, in which an average response time objective is associated with each class. The performance of each class is measured by the ratio of the average response time over the corresponding value of the objective. To achieve fairness in service allocation it is required to find a policy that lexicographically minimizes the vector of performance ratios arranged in non-increasing order. We provide such a policy that is adaptive, uses only knowledge of arrival and departure instants and is thus easy to implement. We also consider a variant of this policy which adapts faster to changes in the statistical parameters of the model. Both policies are analyzed via associated stochastic recursions using techniques of stochastic approximation.

Keywords: Stochastic Scheduling, Adaptive Control, Stochastic Approximation, Lexicographic optimization.

1 Introduction

We consider an $M/GI/1$ queue in which customers belonging to N classes arrive for service. Customers of different classes arrive at different rates and have different service requirements. Such models are used to analyze the performance of shared computer systems and telecommunication systems with heterogeneous traffic types.

To achieve efficient operation of such systems, a common approach is to provide a scheduling policy that minimizes the weighted sum of the average response times of the customer classes. In this case the optimal policy is a simple static priority rule [21],[3],[8]. In fact, with this linear cost function, Bernoulli feedback of customers into the system upon service completion can also be handled [26],[27]. See also [38] for the discounted cost case. More generally, these problems fall in the class of Multi-armed Bandit problems [39] for which simple index policies are known to be optimal.

In this paper we take a different approach. While the choice of cost function for a particular system is often ad hoc, it is more natural to associate an average response time objective with each class and consider its performance relative to the objective. Specifically, let g_i be the response time objective and let $\bar{R}_i^p$ denote the long-run average response time (assuming it exists) of class i customers under a scheduling policy p. Attention is restricted to the class Π of non-idling, non-preemptive, and non-anticipative policies; the last term means that scheduling decisions do not depend on future arrival and service times. We are interested in determining a policy in Π which lexicographically minimizes the vector of performance ratios

$$(\bar{R}_1^p/g_1, \dots, \bar{R}_N^p/g_N)^T, \tag{1}$$

arranged in non-increasing order (see Section 2 for the definition.) We will refer to this minimization as lexicographic.

Results on optimality crucially hinge on the possibility of characterizing the subset $\mathcal{A}$ of $\mathbb{R}_+^N$ that consists of the vectors of mean response times achievable by policies in Π. The set $\mathcal{A}$ is known to be the base of a polymatroid and is described in Section 2. The lexicographic minimization of vector (1) over the set $\mathcal{A}$ yields a unique point $\theta^* := (R_1^*/g_1, \ldots, R_N^*/g_N)^T$. Such a point has certain properties that capture fairness in resource allocation. These are described in remarks following Problem (P) of Section 2. Lexicographic minimization has been studied extensively in a deterministic context [15], [23].

The main contribution of this paper is two simple adaptive policies that (exactly and approximately, respectively) minimize (1) lexicographically. Three quantities are needed in order to specify our policies. Set $T_0 = 0$ and denote by T_n the end of the nth busy period, $n = 1, 2, \ldots$. For $i \in \mathcal{N} := \{1, \ldots, N\}$ denote by A_{ni} the number of customers of class i that have been served by time T_n, incremented by one; the additional unit is included for notational convenience. It is well known that $\{T_{n+1} - T_n\}_{n=0}^{\infty}$ are i.i.d. random variables whose distribution is invariant over policies in Π. For each $i \in \mathcal{N}$ the same holds true for $\{A_{n+1,i} - A_{ni}\}_{n=0}^{\infty}$. Finally, for $i \in \mathcal{N}$ and policy $u \in \Pi$ denote by $\eta_i^u(t)$ the number of customers of class i that are in the system at time $t \geq 0$, when policy u is employed. We can now describe the first of our policies. Let

$$\theta_{ni} := \frac{1}{A_{ni} g_i} \int_0^{T_n} \eta_i^{\pi}(t) dt, \ \ i \in \mathcal{N}, \tag{2}$$

denote the priority index of customers of class i at time T_n under a policy $\pi \in \Pi$ which, during the $(n+1)$st busy period, employs a fixed priority rule that gives priority to class j over class i if $\theta_{nj} > \theta_{ni}$. Ties can be resolved by assigning priorities according to an arbitrary but fixed rule. Thus, policy π gives priority to class j over class i if class j has worse current performance. Set $\theta_n := (\theta_{n1}, \ldots, \theta_{nN})^T$.

Our first goal is to prove that policy π is optimal. As will turn out, it suffices to show that

$$\lim_{n \to \infty} \theta_n = \theta^*, \quad \text{a.s.} \tag{3}$$

This is the object of Section 3. The analysis is facilitated by writing (2) in recursive form and applying techniques of stochastic approximation. However, the fact that the history, $n_i^{\pi}(t), \ 0 \leq t \leq T_n$, is equally weighted in (2), implies that π does not adequately adapt to an instantaneous change of the statistical parameters of the model, that occurs at a large time t. An informal discussion of this point based on the recursive form of (2) and its associated functional law of large numbers seems worthwhile.

Set $\tau := E\left[T_1 - T_0\right]$ and let λ_i be the rate of the Poisson arrival process of customers of class i. To derive a recursion for (2) set for $n = 1, 2, \ldots$, $i \in \mathcal{N}$,

$$J_{ni}(\theta_{n-1}) := \frac{\int_{T_{n-1}}^{T_n} \eta_i^\pi(t)dt}{\tau\lambda_i g_i}, \quad K_{ni} := \frac{A_{ni} - A_{n-1,i}}{\tau\lambda_i}, \quad a_{ni} := \frac{A_{ni}}{n\tau\lambda_i}, \tag{4}$$

$J_n(\cdot) := \left(J_{n1}(\cdot), \cdots, J_{nN}(\cdot)\right)^T$, $a_n := \left(a_{n1}, \cdots, a_{nN}\right)^T$, $K_n := \left(K_{n1}, \cdots, K_{nN}\right)^T$, with the convention that $a_{0i} = 1/\tau\lambda_i$. It is easy to check that θ_n and a_n satisfy for $i \in \mathcal{N}$ and $n = 0, 1, \ldots$,

$$\theta_{n+1,i} = \theta_{ni} + \frac{1}{n+1}\Big[J_{n+1,i}(\theta_n) - \theta_{ni}K_{n+1,i}\Big] + \frac{1}{n+1}\Big[J_{n+1,i}(\theta_n) - \theta_{ni}K_{n+1,i}\Big]\left[\frac{1}{a_{n+1,i}} - 1\right] \tag{5}$$

$$a_{n+1} = a_n + \frac{1}{n+1}\Big[K_{n+1} - 1_{\{n\geq 1\}}a_n\Big]. \tag{6}$$

Because of fluctuations in traffic rates and service requirements that arise in real systems it is important to consider the following situation. Suppose that, for n_0 large, at the end of the n_0th busy period there is an instantaneous change in the statistical parameters of the model such that the optimal point θ^* is displaced. This amounts to policy π using indices θ_n, $n \geq n_0$, generated by (5) and (6) with the initial value θ_{n_0} being near the old optimal point. It will be seen in Section 3 that convergence in (5) occurs for all $n_0 \geq 0$ and all initial vectors θ_{n_0}. However, the number of iterations necessary for convergence is an important figure of merit of policy π.

A functional law of large numbers for (5) is crucial in providing an estimate for this quantity, asymptotically as $n_0 \to \infty$. It states that for every $\epsilon > 0,\ T,\ \theta_0$,

$$\lim_{n_0\to\infty} P\left\{\max_{n_0\leq k\leq u(n_0,T)} ||\theta_k - \theta(t_k)|| > \epsilon\right\} = 0, \tag{7}$$

where the sequence $(t_k)_{k=n_0}^\infty$ is given by $t_{n_0} = 0,\ t_{k+1} = t_k + 1/(k+1),\ k = n_0, n_0+1\ldots$, $u(n_0,T) = \min\{k : k > n_0,\ t_k \geq T\}$, and given an initial condition $\theta_{n_0} = \theta_0,\ \theta(\cdot)$ is a deterministic function on $[0,\infty)$ into $I\!R_+^N$ such that $\theta(0)=\theta_{n_0}$ and $\theta(t)\to\theta^*$ as $t\to\infty$. For $\rho>0$ denote by τ_ρ the first time at which $\theta(t)$ enters to a ρ-ball of θ^*. By picking $T>\tau_\delta$, we see from (7) that for $n_0\to\infty$, with high probability the process θ_k will not visit a $\delta-$ball of θ^* before time $\tau_{\delta+\epsilon}$. Since $t_k\sim\ln(k/n_0)$, the number of steps needed for this visit is at least $k_{\delta_1}\sim n_0e^{\tau_{\delta+\epsilon}}$, asymptotically as $n_0\to\infty$. Therefore, the adaptivity of policy π, of which k_{δ_1} is a natural measure, deteriorates with time.

In stochastic approximation this situation is commonly remedied by replacing the $1/(n+1)$ factor in (5) and (6) by a small constant $\gamma>0$. This factor is known as the gain of the recursion. The second policy proposed in this paper, denoted by π^γ, acts as before but uses the indices of the modified recursions. A law of large numbers for the modified recursions shows that, in the situation of the previous paragraph, the time to enter a δ-ball of θ^* for the first time now becomes $\tau_{\delta+\epsilon}/\gamma$ as $\gamma\to 0$. However, the policy π^γ trades off convergence to θ^* for improved adaptability: Its performance indices have a stationary distribution that concentrates on θ^* as $\gamma\to 0$. The trade-off between speed of adaptation and proximity to the optimal is typical in stochastic approximation. Other choices of the gain in (5) and (6) lead to a variety of policies and the problem of choosing one for a particular application is of great practical significance. In [5] the issue of choosing the appropriate magnitude of a fixed gain is taken up. Ordinary differential equation (o.d.e.) limits and their associated diffusion limits which we do not discuss here are central in this regard too.

The above statements follow from the results of Sections 5 and 6. In Section 5 we establish a functional law of large numbers (also known as an o.d.e. limit) for our recursions. Since the vector field defined by the drift of the recursions is discontinuous, there arise considerable technical difficulties which seem to be inherent in applications of stochastic approximation to queueing systems. Our novel technique should be applicable to other situations as well. In Section 6 we prove that policy π^γ is asymptotically optimal as $\gamma\to 0$. Bounds are obtained for the difference between the optimal and the achieved mean response times.

This introduction concludes with a discussion of related previous work and some practical heuristics based on our policies. The problem of finding a policy p that satisfies $\bar{R}_i^p = g_i$ for every $i = 1, 2, \cdots, N$, was considered in [10]. Whenever this is possible, the authors showed that such a policy can be obtained by mixing N strict priority policies and the mixing proportions can be obtained by solving a linear program. An adaptive method for determining these proportions was proposed in [1]. A scheduling policy based on time-dependent priorities was provided in [13]. The problem of minimizing $\sum_{i=1}^{M} c_i \bar{R}_i$ subject to $\bar{R}_j \leq g_j$ for every $j = M+1, \cdots, M+N$, was considered in [36]. The authors proposed an algorithm which decomposes the set of classes into an ordered partition $\mathcal{N} = \cup_{i=1}^{k} P_i$. They prove the existence of an optimal policy which for $1 \leq i < j \leq N$, gives priority to classes in P_i over classes in P_j.

Compared to the work described above, the formulation we consider and the scheduling policy we describe have some advantages. When the performance objectives are specified for some classes independently of the others, it may be the case that there are no solutions to the problems considered in [10],[36]. In our formulation the policy proceeds to do the "best" that is possible under the circumstances. In order to implement the policies in [10],[36], knowledge of the arrival rates and the first and second moments of the service times of all classes is needed. The policy described in this paper only requires knowledge of arrival and departure instants. Moreover, from (2) it can be easily seen that θ_{n+1} can be computed from $\theta_n,\ A_n,\ A_{n+1}$ and $\int_{T_n}^{T_{n+1}} \eta_i^\pi(t)dt$. Since $\eta_i^\pi(\cdot)$ is a step function, the last integral can be computed as a summation. Therefore, the policy can be efficiently implemented on line and the memory requirements are minimal.

Another significant advantage of the present approach is its generality. It will be seen in the sequel that the optimality of our policy depends only on the fact that the region of average response times achievable by various policies in an $M/GI/1$ queue is the base of a polymatroid and explicit knowledge of the parameters of the region is not needed. It follows that our results apply to other systems with the same property. It was shown in [?] that these include systems obeying "strong" conservation laws. Such examples are multi-class Jackson networks with identical service rates for all classes at all nodes (see [37]) and multi-class $M/GI/c$ queues with identical service distributions for all classes (see [13]). The latter is of particular interest since its region of achievable average response times has not been calculated exactly. In [13], approximate methods were developed to deal with this problem. Thus, while none of the methods in [10], [36] can be used, our method is directly applicable since no knowledge of the region is required.

The necessity to alleviate some of the disadvantages associated with the policies in [10] and [36] motivated the formulation discussed in this paper. It was first presented in [17], where only the case of two classes was considered. Some of the results in [17] can be also shown by adapting the arguments in [32].

A disadvantage of policies that follow fixed priorities that are determined at the beginning of each busy period is that the variance of the response times may be high. To improve our policies in this regard it is natural to shorten the time between updates in (2). In general, it is plausible that they can be updated at any stopping time of the process of queue lengths without affecting their optimality property. Examples are fixed time instants and instants of completion of service.

A policy that seems to have low variance of response times is the one of time-dependent priorities (see [25], [12].) There, a vector $c := (c_1, \ldots, c_N)^T$ is specified and the priority index of a customer of class i is $c_i W$ where W is the time the customer has been in the system. The server is allocated (non-preemptively) to the customer with the highest index. Finding a simple adaptive rule for choosing c to achieve lexicographic minimization is an interesting problem. The response times resulting from the intuitive choice $c_i = 1/g_i$ converge to the optimal only as the utilization goes to 1 as can be seen from equation (3.48) in p.131 of [25]. However, the convergence is not uniform in the values of the goals $\{g_i\}$.

Nevertheless, a modification of our policies along these lines should further reduce variance: It will be seen in Remark 3.1 that if $\theta_{ni} = \theta_{nj}$ for some $i, j \in \mathcal{N}$, then no strict priority need be specified between classes i and j. This suggests that in practice a time-dependent priority policy can be used to allocate service between classes whose indices only differ by a small value from each other. This modification should result in a correspondingly small effect on the optimality of the algorithm and some reduction of variance.

In a forthcoming paper, we will focus on the optimality of policies with general update times and extensions of our results to multi-class $M/GI/1$ queues with Bernoulli feedback.

A few words on the notation used in this paper. For a set S, $|S|$ denotes its cardinality and 1_S denotes its indicator function. For two sets S_1 and S_2, $\bigotimes_{i=1}^{2} S_i$ denotes their cartesian product. For any vectors $x, y \in I\!R^N$, $\langle x, y\rangle$ denotes their inner product. A diagonal matrix A with $a_{11}, \cdots, a_{NN}$ as the diagonal elements is written as diag$\{a_{11}, \cdots, a_{NN}\}$. Finally, when $i > j$, the expressions $\sum_i^j$ and $\cup_i^j$ will be taken to represent 0 and $\emptyset$ respectively and $\inf\{x : x \in S\} = \infty$ when $S = \emptyset$.

2 Problem formulation

In this section the problem of lexicographic minimization is formulated. Necessary and sufficient optimality conditions are recalled.

We begin with some additional notation. For customers of class $i \in \mathcal{N}$ let $B_i(\cdot)$ denote the distribution of their service requirements. Denote the mean of this distribution by $1/\mu_i > 0$. Throughout, we will assume the usual stability condition: $\sum_{i=1}^N \rho_i < 1$, where $\rho_i := \lambda_i/\mu_i$, $i \in \mathcal{N}$.

We make the following assumption on the service times:

(A.1) The service requirements of every class have finite fourth moment.

For $i \in \mathcal{N}$ and policy $u \in \Pi$ denote by $R_i^u(k)$ the response time of the kth customer of class i to arrive in the system. Sample means of response times and queue lengths satisfy certain linear constraints that involve a set of non-negative constants $\{F(\cdot)\}$ specified as a set function on all subsets of $\mathcal{N}$; by convention $F(\emptyset) := 0$. The rest of the notation used in the next lemma was introduced in the previous section.

Lemma 2.1. *(a) For all $u \in \Pi$ we have*

$$\lim_{n\to\infty} \sum_{i\in\mathcal{N}} \rho_i \frac{1}{n} \sum_{k=1}^{n} R_i^u(k) = F(\mathcal{N}), \tag{8}$$

$$\liminf_{n\to\infty} \sum_{i\in S} \rho_i \frac{1}{n} \sum_{k=1}^{n} R_i^u(k) \geq F(S), \quad \text{for } S \subset \mathcal{N}. \tag{9}$$

(b) For any policy $u \in \Pi$ that gives priority to classes in set S over classes in set $\mathcal{N}\backslash S$, limits exist and equality obtains in (9).

For a proof see [18].

Denote now by $\bar{\Pi}$ the set of all policies u in Π for which the sample mean response times $1/n \sum_{k=1}^{n} R_i^u(k)$ converge a.s. as $n \to \infty$. An example in [18] shows that $\bar{\Pi}$ is a strict subset of Π. From Lemma 2.1 it follows that the set of mean response times achievable by policies in $\bar{\Pi}$ is the polytope

$$\mathcal{A} := \{R \in I\!R_+^N \ : \ \sum_{i\in\mathcal{N}} \rho_i R_i = F(\mathcal{N}),\ \sum_{i\in S} \rho_i R_i \geq F(S),\ S \subset \mathcal{N}\}. \tag{10}$$

Multi-class queueing networks whose mean response times are in a set of the form (10) are said to satisfy conservation laws. This was shown in [?] to imply that the set function $F(\cdot)$ is supermodular, i.e.,

$$F(S_1 \cup S_2) + F(S_1 \cap S_2) \geq F(S_1) + F(S_2), \quad S_1, S_2 \subset \mathcal{N}. \tag{11}$$

This property is used crucially in the sequel. Note that it can be established without explicit knowledge of the constants $F(\cdot)$. While explicit expressions are known for these in the case of $M/GI/1$ queues, (see, e.g., [16],) our results can be applied to systems for which these are not known. Certain difficulties encountered in [13] in the study of multi-class $M/GI/c$ queues with identical service distributions for all classes, are thus circumvented.

Relation (8) for $u \in \bar{\Pi}$ has been stated in [22] assuming only two moments on the distribution of service times. The proof, however, appears incomplete (see equation (11-91) on p.433.) In other previous work it has been shown (see [10],) that $\mathcal{A}$ is the set of achievable mean response times if one considers policies in $\bar{\Pi}$ whose actions during a busy period are independent of those taken in previous busy periods. Only finiteness of the second moment of the service times is required in this case.

We next present a lexicographic order that captures the notion of fairness in resource allocation [23], p89. For $i \in \mathcal{N}$, let $g_i > 0$ denote the average response time objective of the customers of class i. These are considered fixed throughout this paper. Define a map $\psi : I\!R^N \mapsto I\!R^N$ as

$$\psi(x) = \left(\frac{x_{i_1}}{g_{i_1}}, \frac{x_{i_2}}{g_{i_2}}, \cdots, \frac{x_{i_N}}{g_{i_N}} \right)^T ; \tag{12}$$

where $x_{i_1}/g_{i_1} \geq x_{i_2}/g_{i_2} \geq \cdots \geq x_{i_N}/g_{i_N}$.

For later use note that if the vector of indices at the end of a busy period is $\theta \in I\!R^N$, then throughout the next busy period policy π serves classes in a fixed priority corresponding to one of the permutations $\{i_1, \ldots, i_N\}$ of $\mathcal{N}$ induced by the map ψ. Denote this permutation by

$$\pi(\theta) := \{i_1^\pi(\theta), \ldots, i_N^\pi(\theta)\} . \tag{13}$$

The vector $\psi(x)$ is said to be lexicographically smaller [14] than the vector $\psi(y)$ (to be denoted $\psi(x) \preceq \psi(y)$) if the following condition holds: $\psi_1(x) \leq \psi_1(y)$ and $\psi_i(x) > \psi_i(y)$ for some $i = 2, \ldots, N$, implies $\psi_j(x) < \psi_j(y)$ for some $j < i$. We will write $\psi(x) \prec \psi(y)$ if $\psi(x) \neq \psi(y)$ and $\psi(x) \preceq \psi(y)$.

We can now state our optimization problem for policies in $\bar{\Pi}$. An extension to policies in Π will be given in Section 4.

Problem (P): Determine a policy $p^* \in \bar{\Pi}$ such that $\bar{R}^{p^*} = R^*$, where R^* is such that $\psi(R^*) \preceq \psi(R)$ for all $R \in \mathcal{A}$.

The existence of R^* follows merely because $\mathcal{A}$ is compact. Two properties of R^* and p^* are worth noting. They follow easily from the definition.

1. The vector R^* is also a minimal element of $\mathcal{A}$ in the usual min-max sense. It follows that if the objectives are achievable, then policy π^* achieves them, i.e.,

$$\max_{i \in \mathcal{N}} \{R_i^*/g_i\} = \min_{R \in \mathcal{A}} \max_{i \in \mathcal{N}} \{R_i/g_i\} \leq 1. \tag{14}$$

2. Policy p^* is fair in the following ways. First, it achieves a completely equitable allocation of the server if one is possible: Suppose that there is an $R \in \mathcal{A}$ such that $R_1/g_1 = \ldots = R_N/g_N$. (Such a point is unique because of the equality constraint in $\mathcal{A}$.) Then, $R = R^*$. Second, any policy $p \in \bar{\Pi}$ that reduces the ith largest performance ratio, i.e. $\psi_i(\bar{R}^p) < \psi_i(\bar{R}^{p^*})$ for $i = 2, \ldots, N$, necessarily worsens a performance ratio that is at least as large, i.e. $\psi_j(\bar{R}^P) > \psi(\bar{R}^{p^*})$ for some $j < i$.

We next recall from [14] a result which establishes uniqueness of R^* and gives necessary and sufficient optimality conditions for it. To state it, note that every $x \in I\!R^N$ can be uniquely represented by $M(x)$ real numbers $\{\nu_k(x)\}_{k=1}^{M(x)}$ and a partition $\{S_k(x)\}_{k=1}^{M(x)}$ of $\mathcal{N}$ such that

(i) $x_i/g_i =: \nu_j(x)$ for every $i \in S_j(x),\ \ j = 1, \cdots, M(x)$,

(ii) $\nu_1(x) > \nu_2(x) > \cdots > \nu_{M(x)}(x)$.

Therefore, the numbers $\{x_i/g_i\}_{i=1}^N$ take $M(x)$ distinct values, for $k = 1, 2, \cdots, M(x)$, $\nu_k(x)$ denotes the k^{th} largest value and $S_k(x)$ is the set containing the indices of the components which attain the k^{th} largest value.

Theorem 2.1. *The following conditions are necessary and sufficient for a vector $R^* \in I\!R^N$ to satisfy $\psi(R^*) \preceq \psi(R)$ for all $R \in \mathcal{A}$.*

$$\sum_{i \in S_j(R^*)} \rho_i R_i^* \;=\; F\left(\cup_{i=1}^{j} S_i(R^*)\right) - F\left(\cup_{i=1}^{j-1} S_i(R^*)\right) \ \text{ for every } \ j = 1, \cdots, M(R^*).$$

Furthermore, the conditions uniquely determine R^*.

For notational convenience, we will write $M^* := M(R^*)$ and for $i = 1, \ldots, M^*$, $\nu_i^* := \nu_i(R^*)$ and $S_i^* := S_i(R^*)$.

We conclude this section with the observation that the equality constraint in $\mathcal{A}$ can be relaxed without affecting the lexicographic minimum. Set

$$\mathcal{A}_0 = \{R \in I\!R^N : \quad \sum_{i \in S} \rho_i R_i \geq F(S) \text{ for every } S \subset \mathcal{N}\}. \tag{15}$$

The supermodularity of $F(\cdot)$ implies that for every $R \in \mathcal{A}_0$ there exists $R' \in \mathcal{A}$ such that $R'_i \leq R_i$ for all $i \in \mathcal{N}$. The following result is therefore immediate from Theorem 2.1.

Corollary 2.1. *$\psi(R^*) \prec \psi(R)$ for every $R \in \mathcal{A}_0 \setminus \mathcal{A}$.*

3 Optimality of policy π

The goal of this section is to prove the optimality of policy π defined in the Introduction.

It turns out that if the indices $\{\theta_n\}$ defined in (2) converge a.s. as $n \to \infty$, then policy π belongs to $\bar{\Pi}$. To see this note that for all $u \in \Pi$,

$$\int_0^{T_n} \eta_i^u(t)dt = \sum_{k=1}^{A_{ni}-1} R_i^u(k), \quad n = 1, 2, \ldots. \tag{16}$$

Therefore, the convergence of $\{\theta_n\}$ implies that for $i \in \mathcal{N}$, $(1/A_{ni}) \sum_{k=1}^{A_{ni}-1} R_i^u(k)$ converges a.s. as $n \to \infty$. By an argument similar to that used to establish partial reward limits from the known limits of renewal reward processes, see e.g. [40, Section 2.3], it follows that $(1/n) \sum_{k=1}^{n} R_i^u(k)$ converges a.s. as $n \to \infty$. Therefore, policy π solves Problem (P) of Section 2 if we prove

Theorem 3.1. *Under assumption (A.1) one has*

$$\lim_{n\to\infty} \theta_n = \theta^* := (R_1^*/g_1, \ldots, R_N^*/g_n)^T, \ a.s. \tag{17}$$

In the remaining of this section we develop a proof of this statement in several steps.

For convenience let us first restate recursion (5). For $i \in \mathcal{N}$ and $n = 0, 1, \ldots$,

$$\theta_{n+1,i} = \theta_{ni} + \frac{1}{n+1}\Big[J_{n+1,i}(\theta_n) - \theta_{ni}K_{n+1,i}\Big] + \frac{1}{n+1}\Big[J_{n+1,i}(\theta_n) - \theta_{ni}K_{n+1,i}\Big]\left[\frac{1}{a_{n+1,i}} - 1\right], \tag{18}$$

with some initial value θ_0. Assuming that all the processes involved are constructed on a suitable probability space $(\Omega, \mathcal{F}, P)$, for $n = 0, 1, \ldots,$ let $\mathcal{F}_n \subset \mathcal{F}$ be the σ-field generated by $\{(\theta_k, K_k),\ 0 \le k \le n\}$. Note that for each $n \ge 1$ the vector K_n is independent of the policy and that the vector $J_n(\cdot)$ depends on $\mathcal{F}_{n-1}$ only through the permutation $\pi(\theta_{n-1})$, defined in (13), that specifies the priority in which classes are being served during the nth busy period. It follows that the second term on the right in (18) is conditionally independent of $\mathcal{F}_n$ given θ_n.

Recursion (18) can thus be written in the stochastic approximation form

$$x_{n+1} = x_n + \frac{1}{n+1} f^{(1)}(x_n, y_{n+1}) + \frac{1}{(n+1)^{1+\delta}} f^{(2)}(x_n, y_{n+1}, w_{n+1}), \quad n = 0, 1, \cdots. \tag{19}$$

for some $\delta > 0$, and appropriate measurable functions $f^{(1)}(\cdot)$, $\ f^{(2)}(\cdot)$. The ith term in (19) identifies with the ith term in (18) for $i = 1, 2, 3$ and $y_{n+1} = (K_{n+1}, J_{n+1}(\theta_n))$, $w_n = a_n$. The triple (x_n, y_n, w_n) is $\mathcal{G}_n$-measurable, where $\{\mathcal{G}_n\}_{n=0}^{\infty}$ is a family of increasing σ-fields such that y_{n+1} is conditionally independent of $\mathcal{G}_n$ given x_n. As a consequence, the drift of (19) can be defined as

$$h(x_n) := E\left[f^{(1)}(x_n, y_{n+1}) | \mathcal{G}_n\right]. \tag{20}$$

Results on the convergence of recursions of the form (19) exist for a variety of different assumptions. Typically, under some conditions on $f^{(2)}$, the third term in (19) is of higher order and convergence is determined only from the properties of $f^{(1)}$. Due to some special features of our problem, application of existing results seems tedious. Instead we modify the general method of proof developed in [35], [20] and [33] . We first give a brief account of the method and indicate the difficulties and our modifications.

In order to show that (19) converges to a point x^*, it often suffices to provide a quadratic Liapunov-type function, i.e. a function

$$V(x) = \frac{1}{2} ||\Psi(x - x^*)||^2, \tag{21}$$

where Ψ is a $N \times N$ matrix satisfying the condition: for all $\epsilon > 0$,

$$\inf_{||x - x^*|| > \epsilon} \langle \Psi(x^* - x), \Psi h(x) \rangle > 0. \tag{22}$$

Thus, matrix Ψ transforms $\mathbb{R}^N$ so that the drift in (19) always has a component toward x^* uniformly for x outside each ϵ-ball of x^*.

A.s. convergence of $V(x_n)$ to a finite limit is obtained from the following result of Robbins and Siegmund [35].

Lemma 3.1. *On a probability space $(\Omega, \mathcal{F}, P)$ equipped with a sequence of σ-fields $\mathcal{F}_0 \subset \ldots \subset \mathcal{F}_n \subset \mathcal{F}$ let z_n, ξ_n and ζ_n be non-negative and $\mathcal{F}_n$-measurable random variables such that*

$$E\left[z_{n+1}|\mathcal{F}_n\right] \le z_n + \xi_n - \zeta_n, \quad n = 0, 1, \ldots. \tag{23}$$

Then, $\lim_{n\to\infty} z_n$ exists and is finite and $\sum_{k=0}^{\infty} \zeta_k < \infty$ a.s. on the event

$$\left\{\sum_{k=0}^{\infty} \xi_k < \infty\right\}. \tag{24}$$

This is applied with $z_n = V(x_n)$, $\zeta_n = 1/(n+1)\left\langle \Psi(x^* - x_n), \Psi h(x_n)\right\rangle$ and ξ_n is higher order terms. Finally, that $\lim_{n\to\infty} V(x_n) = 0$ follows from (22) by an argument that is presented at the end of this section.

Application of these results to our recursion (18) was carried out in a previous version of this work [6]. The factor $(1/a_{n+1,i} - 1)$ in the third term of (18) makes it complicated to obtain bounds on the $\mathcal{F}_n$-conditional expectation of this term. In what follows this difficulty is circumvented by a truncation technique.

We proceed by determining the drift of (18). For a fixed vector $\theta \in \mathbb{R}_+^N$ consider the fixed priority policy which serves classes in the order of the permutation $\pi(\theta)$ defined in (13); denote this policy also by $\pi(\theta)$. Let $\bar{R}(\theta)$ denote the vector of mean response times of policy $\pi(\theta)$ and set $\bar{J}_i(\theta_n) := E\left[J_{n+1,i}(\theta_n)|\mathcal{F}_n\right]$. From (16) and the point-wise regenerative theorem it follows that

$$\frac{\bar{R}_i(\theta)}{g_i} = \lim_{n\to\infty} \frac{1}{A_{ni} g_i} \sum_{k=0}^{n-1} \int_{T_k}^{T_{k+1}} \eta_i^{\pi(\theta)}(t)dt = \frac{E\left[\int_{T_0}^{T_1} \eta_i^{\pi(\theta)}(t)dt\right]}{\lambda_i \tau g_i} = \bar{J}_i(\theta), \tag{25}$$

and the drift in (18) is given by

$$E\left[J_{n+1,i}(\theta_n) - \theta_{ni} K_{n+1,i}|\mathcal{F}_n\right] = \bar{J}_i(\theta_n) - \theta_{ni}, \quad i \in \mathcal{N}. \tag{26}$$

The next result shows that (22) is satisfied for the matrix

$$\Psi := \mathrm{diag}\left\{(\rho_1 g_1)^{1/2}, \ldots, (\rho_N g_N)^{1/2}\right\}, \tag{27}$$

and hence that

$$V(\theta) := \frac{1}{2}\sum_{i=1}^{N} \rho_i g_i (\theta_i - \theta_i^*)^2, \tag{28}$$

is a Liapunov-type function for (18).

Lemma 3.2. *For all $\epsilon > 0$,*

$$\inf_{||\theta-\theta^*||>\epsilon} \sum_{i=1}^{N} \rho_i g_i (\theta_i^* - \theta_i)(\bar{J}_i(\theta) - \theta_i) > 0. \tag{29}$$

Proof. Rewriting the left hand side of (29) as

$$\sum_{i=1}^{N} \rho_i g_i (\theta_i^* - \theta_i)(\bar{J}_i(\theta) - \theta_i^*) + \sum_{i=1}^{N} \rho_i g_i (\theta_i - \theta_i^*)^2, \tag{30}$$

it will clearly suffice to show that

$$\sum_{i=1}^{N} \rho_i g_i \theta_i (\bar{J}_i(\theta) - \theta_i^*) \;\leq\; 0, \tag{31}$$

$$\sum_{i=1}^{N} \rho_i g_i \theta_i^* (\bar{J}_i(\theta) - \theta_i^*) \;\geq\; 0. \tag{32}$$

For (31), recalling the definition of $i_1, \cdots, i_N$ from (12), applied to θ, write for $k = 1, \ldots, N-1$,

$$\theta_{i_k} = \sum_{l=k}^{N-1} (\theta_{i_l} - \theta_{i_{l+1}}) + \theta_{i_N}. \tag{33}$$

Substituting in (31) and interchanging the order of summation yields the equivalent

$$\sum_{l=1}^{N-1} (\theta_{i_l} - \theta_{i_{l+1}}) \sum_{m=1}^{l} \rho_{i_m} g_{i_m} (\bar{J}_{i_m}(\theta) - \theta_{i_m}^*) + \theta_{i_N} \sum_{m=1}^{N} \rho_m g_m (\bar{J}_m(\theta) - \theta_m^*) \leq 0. \tag{34}$$

From Lemma 2.1 (b) and (25) note that for $k = 1, \ldots, N-1$,

$$\sum_{l=1}^{k} \rho_{i_l} g_{i_l} \bar{J}_{i_l}(\theta) = \sum_{l=1}^{k} \rho_{i_l} \bar{R}_{i_l}(\theta) = F(\{i_1, \ldots, i_k\}) \leq \sum_{l=1}^{k} \rho_{i_l} g_{i_l} \theta_{i_l}^*, \tag{35}$$

and equality holds in (35) for $k = N$. Since by definition $\theta_{i_l} \geq \theta_{i_{l+1}}$, for $l = 1, \cdots, N-1$, (34) follows.

For (32), recalling the definition of $\{\nu_i^*\}$, $\{S_i^*\}$, rewrite it as

$$\sum_{i=1}^{M^*} \nu_i^* \sum_{j \in S_i^*} \rho_j g_j (\bar{J}_j(\theta) - \theta_j^*) \leq 0, \tag{36}$$

and set $\nu_i^* = \sum_{l=i}^{M^*-1} (\nu_l^* - \nu_{l+1}^*) + \nu_{M^*}^*, \quad i = 1, \ldots, M^* - 1$. The proof concludes as before by noting that from Theorem 2.1 and for $k = 1, \ldots, M^* - 1$,

$$\sum_{i=1}^{k} \sum_{j \in S_i^*} \rho_j g_j \theta_j^* = F(\cup_{i=1}^k S_i) \leq \sum_{i=1}^{k} \sum_{j \in S_i^*} \rho_j g_j \bar{J}_j(\theta), \tag{37}$$

and equality holds in (37) for $k = M^*$. □

Remark 3.1. Note that if $\theta_{i_l} = \theta_{i_{l+1}}$ for some $1 \leq l \leq N-1$, (34) and Lemma 3.2 hold for policies that do not necessarily use a fixed priority between i_l and i_{l+1}. It can be seen that the remaining of the proof of Theorem 3.1 also goes through for this kind of policies and this suggests the heuristic presented in the Introduction.

We proceed by writing a recursion for (28). Using (18) we have for $n = 0, 1, \ldots,$

$$\begin{aligned} V(\theta_{n+1}) = V(\theta_n) \quad &- \quad \frac{1}{n+1} \sum_{i \in \mathcal{N}} \rho_i g_i (\theta_i^* - \theta_{ni}) \left[J_{n+1,i}(\theta_n) - \theta_{ni} K_{n+1,i} \right] \\ &+ \quad \frac{1}{n+1} \sum_{i \in \mathcal{N}} \rho_i g_i \left(\frac{1}{a_{n+1,i}} - 1 \right) (\theta_{ni} - \theta_i^*) \left[J_{n+1,i}(\theta_n) - \theta_{ni} K_{n+1,i} \right] \\ &+ \quad \frac{1}{(n+1)^2} \frac{1}{2} \sum_{i \in \mathcal{N}} \rho_i g_i \left(\frac{1}{a_{n+1,i}} \right)^2 \left[J_{n+1,i}(\theta_n) - \theta_{ni} K_{n+1,i} \right]^2 . \end{aligned} \tag{38}$$

Our goal is to apply Lemma 3.1 for $z_n = V(\theta_n)$. We must therefore turn our attention to the $\mathcal{F}_n$-conditional expectation of the third and fourth terms of (38). The following truncation scheme is crucial.

Let $\left\{ \{A_n^M\}_{n=1}^{\infty} \right\}_{M=1}^{\infty}$ be a collection of events such that:

(C.1) For $M = 1, 2, \ldots$, the sets A_n^M are decreasing in n and belong to $\mathcal{F}_n$, $n = 1, 2, \ldots$.

(C.2) The sequence $A^M := \cap_{n=0}^{\infty} A_n^M$ is increasing in M and

$$P\left(\cup_{M=1}^{\infty} A^M\right) = 1. \tag{39}$$

Set $V^M(n) := V(\theta_n) 1_{A_n^M}$, $n = 1, 2, \ldots$.

Lemma 3.3. *If* $\lim_{n\to\infty} V^M(n) = 0,\ a.s.\ M = 1, 2, \ldots,$ *then* $\lim_{n\to\infty} V(\theta_n) = 0,\ a.s.$

Proof. Note that

$$\{\lim_{n\to\infty} V^M(n) = 0\} = (A^M)^c \cup (A^M \cap \{\lim_{n\to\infty} V(\theta_n) = 0\}), \tag{40}$$

and therefore that

$$\lim_{M\to\infty} P\{\lim_{n\to\infty} V^M(n) = 0\} = \lim_{M\to\infty} \left(P\left((A^M)^c\right) + P\left(A^M \cap \{\lim_{n\to\infty} V(\theta_n) = 0\}\right)\right). \tag{41}$$

The left hand side equals 1 by hypothesis. From (39) we conclude that $P\{\lim_{n\to\infty} V(\theta_n) = 0\} = 1$. $\square$

To proceed we need to bound the quantity

$$\sigma_{n+1,i}^2(\theta_n) := E\left[(J_{n+1,i}(\theta_n) - \theta_{ni} K_{n+1,i})^2 \,|\mathcal{F}_n\right]. \tag{42}$$

To this end note that assumption (A.1) implies (see [19],)

$$E[(A_{1i} - A_{0i})^4] < \infty \qquad \text{for } i \in \mathcal{N} \quad \text{and} \quad E[(T_1 - T_0)^4] < \infty. \tag{43}$$

Since $K_{n+1,i}$ is independent of $\mathcal{F}_n$, (43) implies that

$$E\left[K_{n+1,i}^2 | \mathcal{F}_n\right] = E\left[K_{1i}^2\right] < \infty, \quad i \in \mathcal{N}. \tag{44}$$

The Cauchy-Schwartz inequality implies that

$$\begin{aligned}
\sup_{\theta \in I\!R_+^N} E\left[J_{n+1,i}^2(\theta_n) | \theta_n = \theta\right] &= \sup_{\theta \in I\!R_+^N} E\left[\left(\frac{\int_{T_0}^{T_1} \eta_i^{\pi(\theta)}(t) dt}{\lambda_i \tau g_i}\right)^2\right] \\
&\le \left(\frac{1}{\lambda_i g_i \tau}\right)^2 E\left[(T_1 - T_0)^2 (A_{1i} - A_{0i})^2\right] \\
&\le \left(\frac{1}{\lambda_i g_i \tau}\right)^2 E\left[(T_1 - T_0)^4\right]^{1/2} E\left[(A_{1i} - A_{0i})^4\right]^{1/2} < \infty.
\end{aligned} \tag{45}$$

Expanding (42) and from the Cauchy-Schwartz inequality we get

$$\sigma^2_{n+1,i}(\theta_n) \leq \left(\left(E\left[J^2_{n+1,i}(\theta_n)|\mathcal{F}_n \right] \right)^{1/2} + \theta_{ni} \left(E\left[K^2_{n+1,i} \right] \right)^{1/2} \right)^2 . \tag{46}$$

The following events will be used in the truncation scheme. For $M = 1, 2, \ldots,\ n = 1, 2, \ldots$ and some $\delta \in (1/2, 1)$, set

$$\begin{aligned}
D^M_{1n} &:= \{K_{ki} \leq M k^\delta,\ i \in \mathcal{N}, \quad k = 1, \ldots, n\}, & (47)\\
D^M_{2n} &:= \{1/a_{ki} \leq M,\ i \in \mathcal{N},\ k = 1, \ldots, n\}, & (48)\\
D^M_{3n} &:= \{|a_{ki} - 1| \leq M/k^{1-\delta},\ i \in \mathcal{N},\ k = 1, \ldots, n\}, & (49)\\
D^M_{4n} &:= \{||\theta_k - \theta^*|| \leq M,\ k = 1, \ldots, n\}. & (50)
\end{aligned}$$

These satisfy conditions (C.1) and (C.2) above. Only the verification of (39) is not immediate.

Lemma 3.4. *For $j = 1, \ldots, 4$, $P\left(\cup_{M=1}^\infty \cap_{n=1}^\infty D^M_{jn} \right) = 1$.*

Proof. For $j = 1$, note that since $\left(\cup_{i \in \mathcal{N}} \{K_{ni} > n^\delta,\ \text{infinitely often(i.o.)}\} \right)^c \subset \cup_{M=1}^\infty \cap_{n=1}^\infty D^M_{1n}$, it suffices to show that $P\{K_{ni} > n^\delta,\ \text{i.o.}\} = 0, \quad i \in \mathcal{N}$. From Chebychev inequality $P\{K_{ni} > n^\delta\} \leq E\left[K^2_{1i}\right]/n^{2\delta}$, $n = 1, 2, \ldots$, and the result follows from the fact that $E[K^2_{1i}] < \infty$ and the Borel-Cantelli lemma since $\delta > 1/2$.

Similarly for $j = 2$, recall that for $n = 1, 2, \ldots$,

$$a_{ni} = \frac{1}{\lambda_i \tau n} + \frac{1}{n} \sum_{l=1}^n K_{li} > 0. \tag{51}$$

It suffices to show that $P\{a_{ni} < 1 - \epsilon,\ \text{i.o.}\} = 0$ for $0 < \epsilon < 1$. But this follows from the strong law of large numbers.

For $j = 3$, (51) and the law of iterated logarithm [9] applied to $\{K_{ni}\}_{n=1}^\infty$ implies that for $\delta > 1/2$ and a.s.,

$$\frac{1}{\lambda_i \tau n} - \frac{1}{n^{1-\delta}} \leq a_{ni} - 1 \leq \frac{1}{\lambda_i \tau n} + \frac{1}{n^{1-\delta}}, \tag{52}$$

for all but finitely many values of n. This implies the result as in the cases above.

Finally for $j = 4$ note that from Lemma 2.1 (a) and (16) we have that

$$\lim_{n\to\infty} \sum_{i\in\mathcal{N}} \rho_i g_i \theta_{ni} = \lim_{n\to\infty} \sum_{i\in\mathcal{N}} \rho_i \frac{1}{A_{ni}} \sum_{k=1}^{A_{ni}-1} R_i^\pi(k) = F(\mathcal{N}).$$

This implies that $\limsup_{n\to\infty} ||\theta_n - \theta^*|| < \infty$ a.s. and the result follows. □

Proof of Theorem 3.1. By Lemma 3.3 it will suffice to show that $\lim_{n\to\infty} V^M(n) = 0$, $M = 1, 2, \ldots,$ for

$$V^M(n) := V(\theta_n) 1_{D_{1n}^M} 1_{B_{n-1}^M}, \quad n \geq 2 \ldots, \tag{53}$$

where $B_n^M := D_{2n}^M \cap D_{3n}^M \cap D_{4n}^M$. Taking conditional expectation in (38) and using the monotonicity in n of the sets $D_{jn}^M, \;\; j = 1, \cdots, 4$, we can write for $n \geq 2$,

$$\begin{aligned}
&E\left[V^M(n+1)|\mathcal{F}_n\right] \leq V^M(n) \qquad\qquad (54)\\
&-\frac{1}{n+1} \sum_{i\in\mathcal{N}} \rho_i g_i (\theta_i^* - \theta_{ni}) E\left[J_{n+1,i}(\theta_n) - \theta_{ni} K_{n+1,i} | \mathcal{F}_n\right] 1_{B_n^M} 1_{D_{1n}^M}\\
&+\frac{1}{n+1} \sum_{i\in\mathcal{N}} \rho_i g_i |\theta_{ni} - \theta_i^*| E\left[|J_{n+1,i}(\theta_n) - \theta_{ni} K_{n+1,i}| \, 1_{\{K_{n+1,i} > M(n+1)^\delta\}} \Big| \mathcal{F}_n\right] 1_{B_n^M} 1_{D_{1n}^M}\\
&+\frac{1}{n+1} \sum_{i\in\mathcal{N}} \rho_i g_i |\theta_{ni} - \theta_i^*| E\left[\left|\frac{1}{a_{n+1,i}} - 1\right| |J_{n+1,i}(\theta_n) - \theta_{ni} K_{n+1,i}| \, 1_{D_{1,n+1}^M} \Big| \mathcal{F}_n\right] 1_{B_n^M}\\
&+\frac{1}{(n+1)^2} \frac{1}{2} \sum_{i\in\mathcal{N}} \rho_i g_i E\left[\left(\frac{1}{a_{n+1,i}}\right)^2 (J_{n+1,i}(\theta_n) - \theta_{ni} K_{n+1,i})^2 \, 1_{D_{1,n+1}^M} \Big| \mathcal{F}_n\right] 1_{B_n^M}.
\end{aligned}$$

By $T_j^M(n)$ denote the ith term on the right hand side of (54) $j = 2, \ldots, 5$. Note that from Lemma 3.2, $T_2^M(n) \leq 0$. For the remaining terms the following bounds are obtained for $n \geq 2$, where C_M is a large enough deterministic constant that depends only on M.

(a) $T_3^M(n) \leq C_M / n^{1+\delta}$: From the Cauchy-Schwartz inequality and (42) we get

$$E\left[|J_{n+1,i}(\theta_n) - \theta_{ni} K_{n+1,i}| \, 1_{\{K_{n+1,i} > M(n+1)^\delta\}} \Big| \mathcal{F}_n\right] \leq \sigma_{n+1,i}(\theta_n) P\left(K_{n+1,i} > M(n+1)^\delta\right)^{1/2} \tag{55}$$

From (44)-(46) it follows that $\sigma_{n+1,i}(\theta_n)$ is uniformly bounded on D_{4n}^M. The desired bound now follows from (44) and the Chebychev inequality.

(b) $T_4^M(n) \leq C_M \left(1/n^2 + 1/n^{2-\delta}\right)$: Observe that $a_{n+1,i} \geq n/(n+1)a_{ni}$ and therefore,

$$\left|\frac{1}{a_{n+1,i}} - 1\right| \leq \frac{1}{a_{n+1,i}} \left(\frac{1}{n+1} + |a_{ni} - 1| + \frac{K_{n+1,i}}{n+1}\right) \leq \frac{2}{a_{ni}} \left(\frac{1}{n+1} + |a_{ni} - 1| + \frac{K_{n+1,i}}{n+1}\right). \tag{56}$$

The bound now follows from (47)-(50) and the uniform boundedness of $\sigma_{n+1,i}(\theta_n)$ on D_{4n}^M.

(c) $T_5^M(n) \leq C_M/n^2$: This bound follows from (56) and the uniform boundedness of $\sigma_{n+1,i}(\theta_n)$ on D_{4n}^M.

Setting $\tilde{T}^M(n) := T_3^M(n) + T_4^M(n) + T_5^M(n)$, the above bounds imply that $\sum_{n=0}^{\infty} \tilde{T}^M(n) < \infty$ a.s.. Lemma 3.1 applied for $z_n = V^M(n)$, $\zeta_n = -T_2^M(n)$, $\xi_n = \tilde{T}^M(n)$ yields that $\lim_{n\to\infty} V^M(n)$ exists and is finite and that $-\sum_{n=0}^{\infty} T_2^M(n) < \infty$ a.s.. To show that $P\left\{\lim_{n\to\infty} V^M(n) > 0\right\} = 0$, it suffices to show that $P\left\{\lim_{n\to\infty} V^M(n) > 1/m\right\} = 0, \quad m = 1, 2, \ldots$. Observe that on $\{\lim_{n\to\infty} V^M(n) > 1/m\}$, we necessarily have $1_{D_{1n}^M} = 1_{B_n^M} = 1$ for $n \geq 2$ and therefore, from Lemma 3.2, $-\sum_{n=0}^{\infty} T_2^M(n) = \infty$, which can only happen on a set of probability zero. $\square$

4 A generalization

In this section we present an extension of Problem (P) of Section 2 for all policies in Π. Recall that $R_i^p(n)$ denotes the response time of the n^{th} departing customer of class i and $R^p(n)$ is the associated vector in $\mathbb{R}^N$.

Problem (P'): Determine $p^* \in \Pi$ that satisfies

$$\psi\left(\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} R^{p^*}(k)\right)\right) \quad \preceq \quad \psi\left(\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} R^{p}(k)\right)\right), \text{ a.s., for every } p \in \Pi.$$

Our result is that policy π solves this problem as well.

Theorem 4.1. For all $p \in \Pi$,

$$\psi\left(\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}R^{\pi}(k)\right)\right) \;\preceq\; \psi\left(\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}R^{p}(k)\right)\right) \quad \text{a.s.} \tag{57}$$

Proof. Lemma 2.1 implies that for $p \in \Pi$,

$$\sum_{i=1}^{N}\rho_i \limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}R_i^{p}(k)\right) \geq \limsup_{n\to\infty}\sum_{i=1}^{N}\rho_i\left(\frac{1}{n}\sum_{k=1}^{n}R_i^{p}(k)\right) = F(\mathcal{N})$$

and for every $S \subset \mathcal{N}$, $S \neq \mathcal{N}$,

$$\sum_{i\in S}\rho_i \limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}R_i^{p}(k)\right) \geq \liminf_{n\to\infty}\sum_{i\in S}\rho_i\left(\frac{1}{n}\sum_{k=1}^{n}R_i^{p}(k)\right) \geq F(S).$$

Therefore, the vector $\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}R^{p}(k)\right)$ belongs to $\mathcal{A}_0$ defined in (15). However as shown in Section 3, $\bar{R}^{\pi} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}R^{\pi}(k) = R^{*}$ and (57) follows from Corollary 2.1. □

5 Finite time asymptotic behavior

As was seen in the Introduction, in order to assess the adaptivity of our policies π and π^{γ} it is necessary to study laws of large numbers for the indices used by these policies. A statement of this law for π was given in (7). The notation will be simpler if we work with the indices of π^{γ}. The modifications needed to obtain result (7) for the indices of π are straightforward and standard in stochastic approximation (see, e.g., [30].)

Let us first state the recursions for the indices of π^{γ}.

$$\theta_{ni}^{\gamma} = \theta_{n-1,i}^{\gamma} + \gamma\Big[J_{ni}(\theta_{n-1}^{\gamma}) - \theta_{n-1,i}^{\gamma}K_{ni}\Big] + \gamma\Big[J_{ni}(\theta_{n-1}^{\gamma}) - \theta_{n-1,i}^{\gamma}K_{ni}\Big]\left[\frac{1}{a_{ni}^{\gamma}} - 1\right] \tag{58}$$

$$a_{n}^{\gamma} = a_{n-1}^{\gamma} + \gamma\Big[K_n - 1_{\{n-1\geq 1\}}a_{n-1}^{\gamma}\Big], \quad n = 1, 2, \ldots, \tag{59}$$

with some initial conditions θ_0^{γ},a_0^{γ}.

It was seen in the Section 3 that convergence in (5) was determined primarily by its drift which satisfied (22). The third term in (5) is of higher order and has no effect on convergence.

The situation in Sections 5 and 6 is similar. The results only depend on the drift of (58) and the treatment of higher order terms is lengthy but inessential and can be done in a straightforward manner. For the sake of clarity we will make a simplification that leads to a recursion with the same drift but without the higher order terms. Henceforth assume that the arrival rates $\{\lambda_i\}$ are known and remain fixed. Then, it is natural to employ policy π' which in the $(n+1)$-st busy period serves classes in the order of permutation $\pi(P_n)$ (see (13),) where

$$P_i(T_n) := \frac{1}{\lambda_i g_i T_n} \int_0^{T_n} \eta_i^{\pi'}(t)dt, \quad i \in \mathcal{N}$$

or, equivalently, in the order of permutation $\pi(\theta_n)$ where

$$\theta_{ni} := P_i(T_n)\, \frac{T_n}{n\tau} = \frac{1}{\lambda_i g_i n\tau} \int_0^{T_n} \eta_i^{\pi'}(t)dt, \quad i \in \mathcal{N}. \tag{60}$$

Now $\{\theta_n\}$ satisfies for $n = 1, 2, \ldots$,

$$\theta_n \;=\; \theta_{n-1} + \frac{1}{n}\, \left[J_n(\theta_{n-1}) - \theta_{n-1}\right], \tag{61}$$

where

$$J_{ni}(\theta_{n-1}) = \frac{1}{\tau\lambda_i g_i} \int_{T_{n-1}}^{T_n} \eta_i^{\pi(\theta_{n-1})}(t)dt, \qquad i \in \mathcal{N}. \tag{62}$$

Replacing $1/n$ by γ, $0 < \gamma < 1$ in (61) we obtain

$$\theta_n^\gamma \;=\; \theta_{n-1}^\gamma + \gamma\left[J_n(\theta_{n-1}^\gamma) - \theta_{n-1}^\gamma\right]. \tag{63}$$

Observe that the conditional expectations of the second terms on the right hand side of (58) and (63) coincide.

The main result of this Section is a functional law of large numbers for (63). This is stated below as Theorem 5.2 where the deterministic limit $\theta(\cdot)$ is given by (65). Such limits have been studied extensively in in this context, [4], [29], and they also arise in a variety of other situations, [11]. There, the deterministic limit of a stochastic recursion is obtained as the unique integral curve of the vector field determined by the drift of the recursion and the initial condition. It is for this reason that none of the results in the literature seem to apply to our recursion. The term

$$E\left[J_{ni}(\theta^\gamma_{n-1})\Big|\theta^\gamma_{n-1}=\theta\right]=\frac{1}{\tau\lambda_i g_i}E\left[\int_{T_0}^{T_1}\eta_i^{\pi(\theta)}(t)dt\right]:=\bar{J}_i(\theta),\quad i\in\mathcal{N}, \tag{64}$$

in the drift of (58) is a discontinuous function of θ. To see this note that its points of discontinuity are, except in degenerate cases, the points of discontinuity of the permutation $\pi(\cdot)$ defined in (13). More importantly, the limit of recursion (63) can evolve on surfaces of discontinuity of $\bar{J}_i(\theta)$ as will be seen in Case 2 of Theorem 5.2 below. Thus, the limit of (63) cannot be characterized as the unique integral curve of the vector field defined by the drift and the initial condition. Indeed, it does not seem easy to guess what the deterministic limit is and whether it exists at all. Therein lies the contribution of this section.

To summarize our intentions, in Sections 5 and 6 we will assume that arrival rates are known and that the indices (60) are employed. In the remaining of this section we derive a functional law of large numbers for (63) and in Section 6 we study the long-term behavior of that recursion. The simplifications made allow us to present only the novel aspects of our results. The corresponding results for (58) can be obtained by straightforward but lengthy modifications. Finally, let us remark that one may be able to obtain the result of Section 3 from the o.d.e. limit of (58), provided that the estimates based on Lemma 5.4 below are sharpened. However, since one may only be interested in the optimality of policy π, the much shorter and direct proof of Section 3 seems worthwhile.

The limit

The limit of recursion (63) will be seen to be the solution of a 'piecewise linear' o.d.e.. Each piece is determined by the region of $\mathbb{R}_+^N$ in which the solution happens to be. To describe the limit, some additional notation is needed.

For a partition $\mathcal{U}=(S_i)_{i=1}^M$ of $\mathcal{N}$, a policy $p\in\bar{\Pi}$ is said to be of type $\mathcal{U}$ if, at every decision instant, customers of classes belonging to S_i are given priority over customers of classes belonging to S_j for $1\le i<j\le M$. Observe that for such a policy $p\in\bar{\Pi}$, the long-run average response

times $\{\bar{R}_i^p\}_{i=1}^N$ satisfy

$$\sum_{i\in\cup_{j=1}^k S_j} \rho_i \bar{R}_i^p = F\left(\cup_{j=1}^k S_j\right)$$

for every $k = 1, \cdots, M$.

A vector $\theta \in I\!R^N$ is said to be of type $\mathcal{U} = (S_i)_{i=1}^M$ if $i \in S_k$, $j \in S_l$ and $k < l$ implies that $\theta_i > \theta_j$. For a partition $\mathcal{U}$, let

$$D_{\mathcal{U}} := \left\{\theta \in I\!R_+^N \ : \ \theta \text{ is of type } \mathcal{U}\right\}.$$

The set of partitions of $\mathcal{N}$ with cardinality N will be denoted by $\mathcal{P}$.

For a partition $\mathcal{U} = (S_i)_{i=1}^M$ of $\mathcal{N}$ we will need to describe the set of average response time vectors achievable by a policy of type $\mathcal{U}$. To this end, set for $k = 1, \ldots, M$ and $S \subset S_k$,

$$\begin{aligned}
\bar{F}_k(S) &:= F\left(\cup_{i=1}^{k-1} S_i \cup S\right) - F\left(\cup_{i=1}^{k-1} S_i\right), \\
\mathcal{A}_{S_k} &:= \left\{x = (x_i)_{i\in S_k} \in I\!R_+^{|S_k|} \ : \ \sum_{S_k} \rho_i x_i = \bar{F}_k\left(S_k\right); \ \sum_{S'} \rho_i x_i \geq \bar{F}_k\left(S'\right), \ S' \subset S_k\right\}.
\end{aligned}$$

For $x \in I\!R_+^N$ and $S \subset \mathcal{N}$, let $x|_S$ denote the vector in $I\!R^{|S|}$ with coordinates in S.

The following result is a well known (see [15]) property of polymatroids.

Lemma 5.1 *For every $k = 1, \cdots, M$,*

(i) $\bar{F}_k(\cdot)$ is supermodular and

(ii) $\mathcal{A}_{S_k} = \left\{\bar{R}^p|_{S_k} \ : \ \text{policy } p \in \Pi_0 \text{ is of type } \left(\cup_{i=1}^{k-1} S_i, \ S_k, \ \cup_{i=k+1}^M S_i\right)\right\}$.

For $\theta \in I\!R_+^N$ with representation $\mathcal{U}(\theta) = \{S_i(\theta)\}_{i=1}^{M(\theta)}$, $\{\nu_i(\theta)\}_{i=1}^{M(\theta)}$ (see Section 2), let $\Theta^{(k)}(\theta) := \left(\Theta_i^{(k)}(\theta)\right)_{i\in S_k(\theta)}$ be such that $\psi(\Theta^{(k)}(\theta)) \preceq \psi(\Theta)$ for every $\Theta \in \mathcal{A}_{S_k(\theta)}$. From Lemma 5.1 and Theorem 2.1, we have that $\Theta^{(k)}(\theta)$ exists and is unique. Define the vector $\Theta(\theta) := (\Theta_1(\theta), \cdots, \Theta_N(\theta))^T$ by setting, for every $i \in \mathcal{N}$, $\Theta_i(\theta) := \Theta_i^{(k)}(\theta)$ whenever $i \in S_k(\theta)$.

We are now ready to describe the limit in probability of recursion (63) as $\gamma \to 0$ and over finite time intervals growing at rate $1/\gamma$. For an initial condition $\theta_0 \in I\!R_+^N$, let $\mathcal{U}(\theta_0) =$

$\{S_i(\theta_0)\}_{i=1}^{M(\theta_0)}$, $\{\nu_i(\theta_0)\}_{i=1}^{M(\theta_0)}$ and $\Theta(\theta_0)$ be as defined before. Let

$$\tau_{\mathcal{D}_{\mathcal{U}(\theta_0)}} := \inf \left\{ s \geq 0 \ : \ \theta(s) \notin D_{\mathcal{U}(\theta_0)} \right\}.$$

For times $0 \leq t < \tau_{\mathcal{D}_{\mathcal{U}(\theta_0)}}$, the limit is the solution of the o.d.e.

$$\dot{\theta}(t) = \Theta(\theta_0) - \theta(t), \ \theta(0) = \theta_0.$$

For times beyond $\tau_{\mathcal{D}_{\mathcal{U}(\theta_0)}}$, the limit is obtained by repeating the above procedure with $\theta\left(\tau_{\mathcal{D}_{\mathcal{U}(\theta_0)}}\right)$ as the initial condition. The function obtained in this way can be easily shown to be the unique solution of the integral equation

$$\theta(t) = \theta_0 + \int_0^t \left(\Theta\left(\theta\left(s\right)\right) - \theta\left(s\right)\right) ds. \tag{65}$$

Recall the definition of matrix A from (27) and of $\theta^* \in I\!R^N$ from (17).) We conclude the description of the limit by showing that the solution of (65) converges to θ^*as $t \to \infty$, for any initial condition. We first need the following result that can be derived by mimicking the proof of Lemma 3.2.

Lemma 5.2. *For every $\theta \in I\!R_+^N$,*

$$\langle \Psi\theta - \Psi\theta^*, \Psi\Theta(\theta) - \Psi\theta^* \rangle \leq 0.$$

and equality holds if and only if $\theta = \theta^*$.

Theorem 5.1. *The vector θ^* is a globally asymptotically stable point of (65).*

Proof. We show that $V(\theta) := 1/2||\Psi\theta - \Psi\theta^*||^2$ is a Liapunov function for (65). Indeed, whenever $\theta(t)$ is differentiable and different from θ^* we obtain, from Lemma 5.2, that

$$\frac{d}{dt} V\left(\theta(t)\right) = \langle \Psi\theta(t) - \Psi\theta^*, \Psi\Theta\left(\theta(t)\right) - \Psi\theta^* \rangle < 0.$$

□

Preliminary results

The results in this subsection are preparatory for the convergence proofs that follow. Recursion (63) is first written in an appropriate form and followed by three technical lemmas.

Recall from (64) the definition of $\bar{J}$. Also, let $J^\sigma := J_1(\theta_0)$ when θ_0 is of type $\sigma \in \mathcal{P}$, and $\bar{J}^\sigma := E\left[J^\sigma \mid \theta_0\right]$. Recursion (63) can be written as

$$\begin{aligned} \theta_n^\gamma &= \theta_0^\gamma + \gamma \sum_{k=0}^{n-1} J_{k+1}(\theta_k^\gamma) - \gamma \sum_{k=0}^{n-1} \theta_k^\gamma \\ &= \theta_0^\gamma + \gamma \sum_{k=0}^{n-1} \bar{J}(\theta_k^\gamma) - \gamma \sum_{k=0}^{n-1} \theta_k^\gamma + M_n^\gamma, \end{aligned} \tag{66}$$

where $M_n^\gamma := \gamma \sum_{k=0}^{n-1} (J_{k+1}(\theta_k^\gamma) - \bar{J}(\theta_k^\gamma))$, is a martingale with respect to the history $\mathcal{F}_n^\gamma$ of the process $\{\theta_n^\gamma\}_{n=0}^\infty$.

In the sequel, for a sequence $\{X_k\}_{k=0}^\infty$, we denote the right-continuous process $X_{\lfloor t/\gamma \rfloor}$ by $X(t)$, where $\lfloor \cdot \rfloor$ denotes the integer part of a non-negative number. Then, (66) can be written in the integral form

$$\theta^\gamma(t) = \theta^\gamma(0) - \int_0^{\gamma \lfloor t/\gamma \rfloor} \theta^\gamma(s) ds + \gamma \sum_{k=0}^{\lfloor t/\gamma \rfloor - 1} \bar{J}(\theta_k^\gamma) + M^\gamma(t) \tag{67}$$

We will need the following Lemmas.

Lemma 5.3. (Gronwall) *For* $\phi : [0, \infty) \to I\!R_+$ *and non-negative constants* $K_0,\ K_1,\ K_2$, *the integral inequality*

$$\phi(t) \leq (K_0 + K_1 t) + K_2 \int_0^t \phi(s) ds, \qquad t \geq 0,$$

implies that

$$\phi(t) \leq K_0 e^{K_2 t} + \frac{K_1}{K_2}\left(e^{K_2 t} - 1\right), \qquad t \geq 0.$$

For a proof see [31].

Lemma 5.4. (Lenglart) *Let* $(M_n, \mathcal{F}_n)_{n=0}^\infty$ *be an* $I\!R$*-valued, zero mean square integrable martingale with quadratic variation* $\{\langle M \rangle_n = \sum_{k=0}^{n-1} E\left[(M_{k+1} - M_k)^2 | \mathcal{F}_k\right]\}_{n=0}^\infty$. *For any* $\mathcal{F}_n$*-stopping* *time T, we have*

$$P\left\{\max_{0\le n\le T}|M_n|\ge\epsilon\right\}\le\frac{\delta}{\epsilon^2}+P\left\{\langle M\rangle_T\ge\delta\right\},$$

for all $\delta>0$, $\epsilon>0$.

For a proof see [24].

Corollary 5.1. *For all $\epsilon>0$,*

$$\lim_{\gamma\to 0}P\left\{\sup_{0\le s\le t}\|M^\gamma(s)\|\ge\epsilon\right\}=0.$$

Proof. Let $L=\max_{\sigma\in\mathcal{P}}E\|J^\sigma\|^2$. Notice that by assumption (A.1) and (45), $L<\infty$. From the definition of $M^\gamma(t)$ we have

$$\langle M^\gamma\rangle_t=\gamma^2\sum_{k=1}^{\lfloor t/\gamma\rfloor}E\|J_k(\theta^\gamma_{k-1})-\bar J(\theta^\gamma_{k-1})\|^2\le L\,\gamma\,t,$$

and the result follows from Lemma 5.4. □

The symbol $\Rightarrow$ denotes weak convergence.

Lemma 5.5. *If $\theta^\gamma(0)\Rightarrow\theta(0)$ as $\gamma\to 0$, then for $K>\max_{\sigma\in\mathcal{P}}\|\bar J^\sigma\|+\|\theta(0)\|$,*

$$\lim_{\gamma\to 0}P\left\{\sup_{0\le s\le t}\|\theta^\gamma(s)-\theta^\gamma(0)\|>K\left(e^t-1\right)\right\}=0.$$

Proof. From (67) we obtain

$$\theta^\gamma(s)-\theta^\gamma(0)=\gamma\sum_{k=1}^{\lfloor s/\gamma\rfloor}\bar J(\theta^\gamma_{k-1})-\int_0^s\theta^\gamma(u)du+\left(s-\lfloor s/\gamma\rfloor\gamma\right)\theta^\gamma(s)+M^\gamma(s).$$

By adding and subtracting appropriate terms involving $\theta(0)$ and $\theta^\gamma(0)$ and using the triangle inequality it follows that

$$\begin{aligned}(1-\gamma)\|\theta^\gamma(s)-\theta^\gamma(0)\| &\le s\left(\max_{\sigma\in\mathcal{P}}\|\bar J^\sigma\|+\|\theta(0)\|\right)+(s+\gamma)\|\theta^\gamma(0)-\theta(0)\|\\ &\quad+\gamma\|\theta(0)\|+\|M^\gamma(s)\|+\int_0^s\|\theta^\gamma(u)-\theta^\gamma(0)\|du,\end{aligned}$$

and by Lemma 5.3 (Gronwall),

$$\begin{aligned}\|\theta^\gamma(s)-\theta^\gamma(0)\| &\leq (1-\gamma)^{-1}\left(\sup_{0\leq s\leq t}\|M^\gamma(s)\|+\gamma\|\theta(0)\|+(t+\gamma)\|\theta^\gamma(0)-\theta(0)\|\right)e^{t/1-\gamma}\\ &\quad+\left(\max_{\sigma\in\mathcal{P}}\|\bar{J}^\sigma\|+\|\theta(0)\|\right)\left(e^{t/1-\gamma}-1\right).\end{aligned}$$

The result now follows from Corollary 5.1. □

Convergence

We intend to show the following result.

Theorem 5.2. *For every $\epsilon>0$, $t\geq 0$ and $\theta(0)\in I\!R_+^N$, if $\theta^\gamma(0)\Rightarrow\theta(0)$ as $\gamma\to 0$, then*

$$\lim_{\gamma\to 0}P\left\{\sup_{0\leq s\leq t}\|\theta^\gamma(s)-\theta(s)\|\geq\epsilon\right\}=0.$$

The essential difficulty in proving Theorem 5.2 is that the drift $h(\theta_{n-1}^\gamma)=\bar{J}(\theta_{n-1}^\gamma)-\theta_{n-1}^\gamma$ in (63) is a discontinuous function on $I\!R_+^N$. Suppose that $\theta(0)$ belongs to D_σ for some $\sigma\in\mathcal{P}$. Then, the finite time analysis of (63) can be carried out using standard results of stochastic approximation [29] and [33] only in the time interval $[0,\tau_{D_\sigma})$.

For clarity of presentation, we first give convergence proofs for two special cases of initial conditions (Theorems 5.3 and 5.4). Convergence for arbitrary initial conditions follows easily by combining these two special cases.

Case 1: We begin by considering the situation in which $\theta(0)$ is of type $\sigma\in\mathcal{P}$. In this case the limit is the solution of the differential equation

$$\dot{\theta}(t)=\bar{J}^\sigma-\theta(t),\qquad t\geq 0. \tag{68}$$

The next result establishes the convergence in the time interval $[0,\tau_{D_\sigma}]$. Our technique is closest to the one of [28]. For a subset D of $I\!R_+^N$ define the $\mathcal{F}_t^\gamma$-stopping time

$$\tau_D^\gamma:=\inf\left\{t\geq 0:\theta^\gamma(t)\notin D\right\}.$$

Theorem 5.3. *If $\theta^\gamma(0) \Rightarrow \theta(0)$ as $\gamma \to 0$, where $\theta(0)$ is of type $\sigma \in \mathcal{P}$, then for every $\epsilon > 0$ and $0 \le t < \infty$,*

$$\lim_{\gamma \to 0} P \left\{ \sup_{0 \le s \le \tau_{D_\sigma} \wedge t} \|\theta^\gamma(s) - \theta(s)\| \ge \epsilon \right\} = 0.$$

Proof. The main difficulty is that the drift $h(\theta)$ is discontinuous at τ_{D_σ}. The proof is carried out in two steps.

Step 1: We prove that for $0 \le t < \tau_{D_\sigma}$ and all $\epsilon > 0$,

$$\lim_{\gamma \to 0} P \left\{ \sup_{0 \le s \le t} \|\theta^\gamma(s) - \theta(s)\| \ge \epsilon \right\} = 0.$$

From (67) and the observation that for $s \le \tau^\gamma_{D_\sigma}$, $\bar{J}(\theta^\gamma_k) = \bar{J}^\sigma$, $\quad 0 \le k \le \lfloor s/\gamma \rfloor - 1$, we obtain

$$\begin{aligned} \theta^\gamma(s) - \theta(s) \quad &= \quad (\theta^\gamma(0) - \theta(0)) - \int_0^s (\theta^\gamma(u) - \theta(u))\, du + (s - \gamma \lfloor s/\gamma \rfloor)(\theta^\gamma(s) - \theta(s)) \\ &\quad + (s - \gamma \lfloor s/\gamma \rfloor)\, \theta(s) + M^\gamma(s) - (s - \gamma \lfloor s/\gamma \rfloor)\, \bar{J}^\sigma. \end{aligned}$$

By setting $U^\gamma(s) := \|\theta^\gamma(s) - \theta(s)\|$, we have that

$$\begin{aligned} (1-\gamma)\, U^\gamma(s \wedge \tau^\gamma_{D_\sigma}) \quad &\le \quad U^\gamma(0) + \int_0^s U^\gamma(u \wedge \tau^\gamma_{D_\sigma}) du \\ &\quad + \sup_{0 \le u \le t \wedge \tau^\gamma_{D_\sigma}} \|M^\gamma(u)\| + \gamma \left(\max_{0 \le s \le t} \|\theta(s)\| + \max_\sigma \|\bar{J}^\sigma\| \right), \end{aligned}$$

which after the application of Lemma 5.3 (Gronwall) gives

$$U^\gamma(s \wedge \tau^\gamma_{D_\sigma}) \le (1-\gamma)^{-1} \left(U^\gamma(0) + \sup_{0 \le u \le t \wedge \tau^\gamma_{D_\sigma}} \|M^\gamma(u)\| + \gamma \left(\max_{0 \le s \le t} \|\theta(s)\| + \max_\sigma \|\bar{J}^\sigma\| \right) \right) e^{t/(1-\gamma)}.$$

From Corollary 5.1 and the convergence of the initial condition we have that for all $\epsilon > 0$,

$$\lim_{\gamma \to 0} P \left\{ \sup_{0 \le s \le t} U^\gamma(s \wedge \tau^\gamma_{D_\sigma}) \ge \epsilon \right\} = 0. \tag{69}$$

It remains to remove the stopping time in the supremum of the event above. For this note that

$$P \left\{ \sup_{0 \le s \le t} U^\gamma(s) \ge \epsilon \right\} \le P \left\{ t \ge \tau^\gamma_{D_\sigma} \right\} + P \left\{ \sup_{0 \le s \le t} U^\gamma(s \wedge \tau^\gamma_{D_\sigma}) \ge \epsilon \right\}. \tag{70}$$

It therefore suffices to show that

$$\lim_{\gamma\to 0} P\left\{t \geq \tau^{\gamma}_{D_\sigma}\right\} = 0. \tag{71}$$

Since $t < \tau_{D_\sigma}$ there exists a $\zeta > 0$ such that

$$\inf_{s\leq t} \rho\left(\theta(s), D^c_\sigma\right) > \zeta,$$

where $\rho(x, D) = \inf\ \{\|x-y\| : y \in D\}$ denotes the distance of the set D from x. Now note that $\left\{t \geq \tau^{\gamma}_{D_\sigma}\right\} \subset \left\{\sup_{0\leq s\leq t} U^\gamma(s \wedge \tau^\gamma_{D_\sigma}) > \zeta\right\}$. Relation (71) follows as a consequence of (69).

Step 2: To complete the proof of the theorem, it suffices to show that if $\tau_{D_\sigma} \leq t < \infty$, then there exists a $t' \in [0, \tau_{D_\sigma})$ such that

$$\lim_{\gamma\to 0} P\left\{\sup_{t'\leq s\leq \tau_{D_\sigma}} \|\theta^\gamma(s) - \theta(s)\| \geq \epsilon\right\} = 0. \tag{72}$$

Observe that for all $t' \in [0, \tau_{D_\sigma})$

$$\sup_{t'\leq s\leq \tau_{D_\sigma}} \|\theta^\gamma(s)-\theta(s)\| \leq \sup_{t'\leq s\leq \tau_{D_\sigma}} \|\theta^\gamma(s)-\theta^\gamma(t')\|+\|\theta^\gamma(t')-\theta(t')\|+\sup_{t'\leq s\leq \tau_{D_\sigma}} \|\theta(t')-\theta(s)\|. \tag{73}$$

Pick $t' < \tau_{D_\sigma}$ large enough so that $\sup_{t'\leq s\leq \tau_{D_\sigma}} \|\theta(s)-\theta(t')\| < \epsilon/3$, and so that, for the constant K in Lemma 5.5, $K\left(e^{\tau_{D_\sigma}-t'} - 1\right) < \epsilon/3$. Note that from Step 1 above we have as a corollary that $\theta^\gamma(t') \Rightarrow \theta(t')$ as $\gamma \to 0$. Using Lemma 5.5, (72) follows from (73) and the proof is complete. $\square$

Case 2: We now consider the situation in which

$$\theta_i(0) = \theta_j(0), \quad i, j \in \mathcal{N}. \tag{74}$$

As mentioned before, the limit in this case is the solution of

$$\dot\theta(t) = \theta^* - \theta(t), \quad t \geq 0.$$

The difficulty here is that the drift in (63) is discontinuous at the initial condition $\theta(0)$, and furthermore, the trajectory of $\theta(t)$ may not leave the set where the drift is discontinuous. Assume that $\theta(0) \neq \theta^*$.

It will be convenient to transform the state space $I\!R_+^N$ by multiplying all vectors by the matrix Ψ. A transformed vector X will be denoted by $\tilde{X}$. Deviating slightly from this rule we set, $\tilde{J}(\theta) := \Psi \bar{J}(\theta)$.

Instead of considering the vector θ_k^γ we consider the projection and the vertical distance of $\tilde{\theta}_k^\gamma - \tilde{\theta}(0)$ from the line determined by the points $\tilde{\theta}(0)$ and $\tilde{\theta}^*$. They are defined as follows.

$$
\begin{aligned}
Q_k^\gamma &:= \left\langle \tilde{\theta}_k^\gamma - \tilde{\theta}(0), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle, \\
Z_k^\gamma &:= \tilde{\theta}_k^\gamma - \tilde{\theta}(0) - \mathbf{u} Q_k^\gamma,
\end{aligned}
$$

where

$$
\mathbf{u} := \frac{\tilde{\theta}^* - \tilde{\theta}(0)}{\|\tilde{\theta}^* - \tilde{\theta}(0)\|^2}.
$$

The basic idea of the proof is to show that $Z^\gamma(t)$ converges to zero (Corollary 5.3) and $Q^\gamma(t)$ converges to $\tilde{\theta}(s) - \tilde{\theta}(0)$ (Theorem 5.4) as $\gamma \to 0$, where in both cases convergence is in probability and uniformly over finite time intervals.

Observe that $\left\langle \tilde{J}(\theta) - \tilde{\theta}^*, \tilde{\theta}(0) \right\rangle = 0$ for $\theta(0)$ satisfying (74). Then from Lemma 3.2 we obtain, for all $\theta \in I\!R_+^N$,

$$
\begin{aligned}
\left\langle \tilde{\theta}^* - \tilde{\theta}(0), \tilde{J}(\theta) - \tilde{\theta}^* \right\rangle &\geq 0, & (75)\\
\left\langle \tilde{\theta} - \tilde{\theta}(0), \tilde{J}(\theta) - \tilde{\theta}^* \right\rangle &\leq 0. & (76)
\end{aligned}
$$

From (75), we also have

$$
\langle \tilde{\theta}^* - \tilde{\theta}(0), \tilde{J}(\theta) - \tilde{\theta}(0) \rangle \geq \|\tilde{\theta}^* - \tilde{\theta}(0)\|^2. \quad (77)
$$

Note that equality holds in (75) and (77) when θ is of type $\mathcal{U}(\theta^*)$.

Let

$$
\Delta_k(\theta_{k-1}^\gamma) := \Psi \left(J_k(\theta_{k-1}^\gamma) - \bar{J}(\theta_{k-1}^\gamma) \right)
$$

In terms of this notation the sequence $\{Q_k^\gamma\}_{k=0}^\infty$ can be written as

$$
Q_{k+1}^\gamma = (1-\gamma)^{k+1} Q_0^\gamma + \gamma \sum_{l=0}^{k} (1-\gamma)^l \left\langle \tilde{J}(\theta_{k-l}^\gamma) - \tilde{\theta}(0), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle
$$

$$
\begin{aligned}
& +\gamma \sum_{l=0}^{k}(1-\gamma)^l \left\langle \Delta_{k+1-l}(\theta^\gamma_{k-l}), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle \\
\geq\ & (1-\gamma)^{k+1} Q_0^\gamma + \left(1-(1-\gamma)^{k+1}\right) \|\tilde{\theta}^* - \tilde{\theta}(0)\|^2 \\
& +\gamma \sum_{l=0}^{k}(1-\gamma)^l \left\langle \Delta_{k+1-l}(\theta^\gamma_{k-l}), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle, \qquad (78)
\end{aligned}
$$

the inequality following from (77). When θ_l^γ is of type $\mathcal{U}(\theta^*)$ for every $l = 0, 1, \cdots, k$, then equality holds in (78).

Lemma 5.6. *For every $\epsilon > 0$,*

$$
\lim_{\gamma \to 0} P \left\{ \sup_{0 \le s \le t} \gamma \left| \sum_{l=0}^{\lfloor s/\gamma \rfloor - 1} (1-\gamma)^l \left\langle \Delta_{\lfloor s/\gamma \rfloor - l}(\theta^\gamma_{\lfloor s/\gamma \rfloor - l - 1}), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle \right| \geq \epsilon \right\} = 0.
$$

Proof. Set

$$
V^\gamma_{k+1} := \gamma \sum_{l=0}^{k} (1-\gamma)^l \left\langle \Delta_{k+1-l}(\theta^\gamma_{k-l}), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle.
$$

$\{V_k^\gamma\}_{k=0}^\infty$ satisfies the recursion

$$
V^\gamma_{k+1} - V^\gamma_k = \gamma \left\langle \Delta_{k+1}(\theta^\gamma_k), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle - \gamma V^\gamma_k,
$$

which in integral form can be written as

$$
V^\gamma(s) = \gamma \sum_{k=0}^{\lfloor s/\gamma \rfloor - 1} \left\langle \Delta_{k+1}(\theta^\gamma_k), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle - \int_0^s V^\gamma(u) du + (s - \gamma \lfloor s/\gamma \rfloor) V^\gamma(s).
$$

From the triangle inequality we have that for $0 \le s \le t$,

$$
(1-\gamma) |V^\gamma(s)| \le \sup_{0 \le s \le t} \left| \gamma \sum_{k=0}^{\lfloor s/\gamma \rfloor} \left\langle \Delta_{k+1}(\theta^\gamma_k), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle \right| + \int_0^s |V^\gamma(u)| \, du.
$$

From Lemma 5.3 (Gronwall),

$$
\sup_{0 \le s \le t} |V^\gamma(s)| \le (1-\gamma)^{-1} \sup_{0 \le s \le t} \left| \gamma \sum_{k=0}^{\lfloor s/\gamma \rfloor - 1} \left\langle \Delta_{k+1}(\theta^\gamma_k), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle \right| e^{t/1-\gamma}.
$$

Observe that $\gamma \sum_{k=0}^{\lfloor s/\gamma \rfloor -1} \left\langle \Delta_{k+1}(\theta_k^\gamma), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle$ is a $\mathcal{F}_s^\gamma$-martingale. The result follows now, by a similar argument as in Corollary 5.1. $\square$

Corollary 5.2. *If $\theta^\gamma(0) \Rightarrow \theta(0)$ as $\gamma \to 0$, then, for every $\epsilon > 0$,*

$$\lim_{\gamma \to 0} P\left\{ \inf_{0 \le s \le t} \left[Q^\gamma(s) - (1 - e^{-s}) \|\tilde{\theta}^* - \tilde{\theta}(0)\|^2 \right] \le -\epsilon \right\} = 0.$$

Proof. Since $\theta^\gamma(0) \Rightarrow \theta(0)$ as $\gamma \to 0$, we have that for all $\epsilon > 0$,

$$\lim_{\gamma \to 0} P\left\{ \sup_{0 \le s \le t} (1-\gamma)^{\lfloor s/\gamma \rfloor} Q_0^\gamma \ \ge \epsilon \right\} = 0.$$

Using in addition Lemma 5.6, we conclude from (78) that

$$\lim_{\gamma \to 0} P\left\{ \inf_{0 \le s \le t} \left(Q^\gamma(s) - (1 - (1-\gamma)^{\lfloor s/\gamma \rfloor}) \|\tilde{\theta}^* - \tilde{\theta}(0)\|^2 \right) \le -\epsilon \right\} = 0.$$

The result follows by observing that $\lim_{\gamma \to 0} (1-\gamma)^{\lfloor s/\gamma \rfloor} = e^{-s}$, uniformly in $s \in [0, t]$. $\square$

We now turn our attention to the sequence $\{Z_k^\gamma\}_{k=0}^\infty$. Let $T_C^\gamma := \inf\{s \ge 0 \ : \ \|Z^\gamma(s)\| > C\}$ for $C > 0$, and

$$\begin{aligned}
\Gamma_k(\theta_{k-1}^\gamma) &:= \Delta_k(\theta_{k-1}^\gamma) - \mathbf{u} \left\langle \Delta_k(\theta_{k-1}^\gamma), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle \\
H(\theta_{k-1}^\gamma) &:= \tilde{J}(\theta_{k-1}^\gamma) - \tilde{\theta}(0) - \mathbf{u} \left\langle \tilde{J}(\theta_{k-1}^\gamma) - \tilde{\theta}(0), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle.
\end{aligned}$$

We first show the following result.

Lemma 5.7. *If $\theta^\gamma(0) \Rightarrow \theta(0)$ as $\gamma \to 0$, then for every $\epsilon > 0$ and $C > 0$,*

$$\lim_{\gamma \to 0} P\left\{ \sup_{0 \le s \le t \wedge T_C^\gamma} \|Z^\gamma(s)\| \ge \epsilon \right\} = 0.$$

Proof. It can be easily seen that $\{Z_k^\gamma\}_{k=0}^\infty$ satisfies the following recursion.

$$Z_{k+1}^\gamma = (1-\gamma) Z_k^\gamma + \gamma H(\theta_k^\gamma) + \gamma \Gamma_{k+1}(\theta_k^\gamma). \tag{79}$$

From the definition of $H(\theta_k^\gamma)$ and $\Gamma_{k+1}(\theta_k^\gamma)$, we have the existence of finite constants U_H and U_Γ such that, $\max_k \|H(\theta_k^\gamma)\|^2 \leq U_H$ a.s., and $\max_k E\|\Gamma_{k+1}(\theta_k^\gamma)\|^2 \leq U_\Gamma$. From (79) we have

$$\begin{aligned}\|Z_{k+1}^\gamma\|^2 &\leq \|Z_k^\gamma\|^2 + \gamma^2 U_H + \gamma^2 \Gamma_{k+1}(\theta_k^\gamma) + 2\gamma^2 \langle H(\theta_k^\gamma), \Gamma_{k+1}(\theta_k^\gamma)\rangle \\ &\quad +2\gamma(1-\gamma)\langle Z_k^\gamma, \Gamma_{k+1}(\theta_k^\gamma)\rangle + 2\gamma(1-\gamma)\langle Z_l^\gamma, H(\theta_l^\gamma)\rangle,\end{aligned}$$

from which it follows that

$$\begin{aligned}\|Z^\gamma(s)\|^2 &\leq \|Z_0^\gamma\|^2 + \gamma s U_H + \gamma^2 \sum_{l=0}^{\lfloor s/\gamma\rfloor -1} \|\Gamma_{l+1}(\theta_l^\gamma)\|^2 \\ &\quad +2\gamma^2 \sum_{l=0}^{\lfloor s/\gamma\rfloor -1} \langle H(\theta_l^\gamma), \Gamma_{l+1}(\theta_l^\gamma)\rangle + 2\gamma(1-\gamma)\sum_{l=0}^{\lfloor s/\gamma\rfloor -1} \langle Z_l^\gamma, \Gamma_{l+1}(\theta_l^\gamma)\rangle \\ &\quad +2\gamma(1-\gamma)\sum_{l=0}^{\lfloor s/\gamma\rfloor -1} \langle Z_l^\gamma, H(\theta_l^\gamma)\rangle. \qquad (80)\end{aligned}$$

The Lemma will be proved if for every term, $T_k(s), k = 1,\ldots,6$, on the right hand side of (80), it is shown that for every $\epsilon > 0$,

$$\lim_{\gamma\to 0} P\left\{\sup_{0\leq s\leq t\wedge T_C^\gamma} |T_k(s)| \geq \epsilon\right\} = 0.$$

We consider each term separately.

a) $T_1(s) := \|Z_0^\gamma\|^2$: The assertion follows from the weak convergence of the initial condition.

b) $T_2(s) := \gamma s U_H$: Obvious.

c) $T_3(s) := \gamma^2 \sum_{l=0}^{\lfloor s/\gamma\rfloor -1} \|\Gamma_{l+1}(\theta_l^\gamma)\|^2$: Since the terms in the summation are non-negative,

$$\sup_{0\leq s\leq t} |T_3(s)| \leq \gamma^2 \sum_{l=0}^{\lfloor t/\gamma\rfloor -1} \|\Gamma_{l+1}(\theta_l^\gamma)\|^2.$$

Since $\gamma^2 E\left(\sum_{l=0}^{\lfloor t/\gamma\rfloor -1} \|\Gamma_{l+1}(\theta_l^\gamma)\|^2\right) \leq \gamma t U_\Gamma$, the result follows by Chebyshev's inequality.

d) $T_4(s) := \gamma^2 \sum_{l=0}^{\lfloor s/\gamma\rfloor -1} \langle H(\theta_l^\gamma), \Gamma_{l+1}(\theta_l^\gamma)\rangle$: Observe that the sum is a $\mathcal{F}_s^\gamma$-martingale. For the quadratic variation, $\langle T_4\rangle_\cdot$, of the sum we have that,

$$E\langle T_4\rangle_t \leq \gamma^4 \sum_{l=0}^{\lfloor t/\gamma\rfloor} E\langle H(\theta_l^\gamma), \Gamma_{l+1}(\theta_l^\gamma)\rangle^2 \leq \gamma^3 t U_H U_\Gamma.$$

The result now follows from Lemma 5.4, by applying Chebyshev's inequality to $\langle T_4 \rangle$..

e) $T_5(s) := \gamma(1-\gamma)\sum_{l=0}^{\lfloor s/\gamma \rfloor -1} \langle Z_l^\gamma, \Gamma_{l+1}(\theta_l^\gamma)\rangle$: Again, the sum is a $\mathcal{F}_s^\gamma$-martingale. Since $\|Z^\gamma(s)\| \leq C$ for $s < T_C^\gamma$, $\langle T_5 \rangle_{t \wedge T_C^\gamma} \to 0$ a.s. and the result follows from Lemma 5.4 (Lenglart).

f) $T_6(s) := 2\gamma(1-\gamma)\sum_{l=0}^{\lfloor s/\gamma \rfloor -1} \langle Z_l^\gamma, H(\theta_l^\gamma)\rangle$: Set

$$C(\theta_l^\gamma) := 1 - \frac{\left\langle \tilde{J}(\theta_l^\gamma) - \tilde{\theta}(0), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle}{\|\tilde{\theta}^* - \tilde{\theta}(0)\|^2}.$$

Observe that $U_C := \max_{\theta \in I\!R_+^N} C(\theta) < \infty$. From the definitions of Z_k^γ and $H(\theta_k^\gamma)$, it follows that

$$\begin{aligned}
\langle Z_l^\gamma, H(\theta_l^\gamma)\rangle &= \left\langle \tilde{\theta}_l^\gamma - \tilde{\theta}(0), H(\theta_l^\gamma) \right\rangle \\
&= C(\theta_l^\gamma) Q_l^\gamma + \left\langle \tilde{\theta}^\gamma(l) - \tilde{\theta}(0), \tilde{J}(\theta_l^\gamma) - \tilde{\theta}^* \right\rangle \\
&\leq C(\theta_l^\gamma) Q_l^\gamma
\end{aligned}$$

where the inequality follows from (76). From (77), we have that $C(\theta_l^\gamma) \leq 0$ for every l and consequently, from (78), we obtain

$$|T_6(s)| \leq 2U_C\, |Q_0^\gamma\,| + \, 2\gamma^2 U_C \sum_{l=0}^{\lfloor s/\gamma \rfloor -1} \left| \sum_{m=0}^{l-1} (1-\gamma)^m \left\langle \Delta_{l-m}(\theta_{l-m-1}^\gamma), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle \right|. \tag{81}$$

From the convergence of the initial condition, we have that the first term in (81) converges to 0 as $\gamma \to 0$. The second term is dominated by

$$2\, t\, U_C \sup_{0 \leq l \leq t} \gamma \left| \sum_{m=0}^{\lfloor l/\gamma \rfloor -1} (1-\gamma)^m \left\langle \Delta_{\lfloor l/\gamma \rfloor - m}(\theta_{\lfloor l/\gamma \rfloor - m-1}^\gamma), \tilde{\theta}^* - \tilde{\theta}(0) \right\rangle \right|$$

for which Lemma 5.6 applies. This completes the proof of Lemma 5.7. □

The stopping time T_C^γ can be removed from the supremum as in Step I in the proof of Theorem 5.2 (see (70)).

Corollary 5.3. *Under the same assumptions as in Lemma 5.7,*

$$\lim_{\gamma \to 0} P\left\{ \sup_{0 \leq s \leq t} \|Z^\gamma(s)\| \geq \epsilon \right\} = 0.$$

We will need the following simple geometric result. For any $\theta \in R^N$, let $Q(\theta)$ and $Z(\theta)$ denote, respectively, the projection and the vertical distance of $\theta - \theta(0)$ from the line joining θ^* and $\theta(0)$; i.e., $Q(\theta) := \langle \theta - \theta(0), \theta^* - \theta(0) \rangle$ and $Z(\theta) := \theta - \theta(0) - \mathbf{u}Q(\theta)$, with $\mathbf{u} = \|\theta^* - \theta(0)\|^{-2} (\theta^* - \theta(0))$. Recall that $\theta(0)$ satisfies (74).

Lemma 5.8. *For every $\epsilon_o > 0$, there exists $\delta > 0$ such that for every $\theta \in R^N$ with $Q(\theta) > \epsilon_o$ and $\|Z(\theta)\| < \delta$, $\theta_i^* > \theta_j^*$ implies $\theta_i > \theta_j$.*

Proof. We need to consider only the case where $\theta_i^* \neq \theta_j^*$ for some $i, j \in \mathcal{N}$. If we choose

$$\delta := \frac{\epsilon_o}{2} \quad \frac{\min\left\{|\theta_i^* - \theta_j^*| : \theta_i^* \neq \theta_j^*\right\}}{\|\theta^* - \theta(0)\|^2},$$

then $\theta_i^* > \theta_j^*$ implies

$$\begin{aligned} \theta_i - \theta_j &= Z_i(\theta) - Z_j(\theta) + (u_i - u_j)Q(\theta) \\ &\geq -\delta + \epsilon_o \left(\theta_i^* - \theta_j^*\right) \|\theta^* - \theta(0)\|^{-2} > 0. \end{aligned}$$

□

Recall that $\theta(\cdot)$ is the solution of the integral equation (65).

Theorem 5.4. *If $\theta^\gamma(0) \Rightarrow \theta(0)$ as $\gamma \to 0$ and $\theta(0)$ satisfies (74), then for every $\epsilon > 0$, $0 \leq t < \infty$,*

$$\lim_{\gamma \to 0} P\left\{\sup_{0 \leq s \leq t} \|\theta^\gamma(s) - \theta(s)\| \geq \epsilon\right\} = 0.$$

Proof. Assume first that $\theta(0) \neq \theta^*$. Set $Q(s) := (1 - e^{-s})\|\tilde{\theta}^* - \tilde{\theta}(0)\|^2$. Since $\tilde{\theta}^\gamma(s) - \tilde{\theta}(0) = \mathbf{u}Q^\gamma(s) + Z^\gamma(s)$, and $\tilde{\theta}(s) - \tilde{\theta}(0) = (1 - e^{-s})\left(\tilde{\theta}^* - \tilde{\theta}(0)\right)$, we have

$$\|\tilde{\theta}^\gamma(s) - \tilde{\theta}(s)\| \leq \|Z^\gamma(s)\| + \|\mathbf{u}\| \cdot \ |Q^\gamma(s) - Q(s)|$$

and, in view of Corollary 5.3, it suffices to show that for $\epsilon > 0$,

$$\lim_{\gamma \to 0} P\left\{\sup_{0 \leq s \leq t} |Q^\gamma(s) - Q(s)| \geq \epsilon\right\} = 0. \tag{82}$$

Observe that

$$\sup_{0\le u\le s}|Q^\gamma(u)-Q(u)|\le\sup_{0\le u\le s}|Q^\gamma(u)-Q^\gamma(0)|+|Q^\gamma(0)|+\sup_{0\le u\le s}|Q(u)|\,.$$

Fix $\epsilon>0$ and pick $s_o>0$ small enough such that (i) $\sup_{0\le u\le s_o}|Q(u)|<\epsilon/3$; and (ii)

$$\lim_{\gamma\to 0}P\left\{\sup_{0\le u\le s_o}|Q^\gamma(u)-Q^\gamma(0)|\ge\frac{\epsilon}{3}\right\}=0.$$

The final choice can be made by virtue of Lemma 5.5. From the convergence of the initial condition we have that

$$\lim_{\gamma\to 0}P\left\{\sup_{0\le s\le s_o}|Q^\gamma(s)-Q(s)|\ge\epsilon\right\}=0. \tag{83}$$

It remains to show that

$$\lim_{\gamma\to 0}P\left\{\sup_{s_o\le s\le t}|Q^\gamma(s)-Q(s)|\ge\epsilon\right\}=0. \tag{84}$$

Consider the events

$$\begin{aligned}\Omega_1^\gamma &:= \left\{\inf_{s_o\le s\le t}\left[Q^\gamma(s)-Q(s)\right]>-Q(s_o)/2\right\}\\ \Omega_2^\gamma &:= \left\{\sup_{s_o\le s\le t}\|Z^\gamma(s)\|<\delta\right\},\end{aligned}$$

where δ corresponds to the choice $\epsilon_o=Q(s_o)/2$ in Lemma 5.8. From Lemma 5.8, we have that for all sample paths in $\Omega^\gamma:=\Omega_1^\gamma\cap\Omega_2^\gamma$, $\theta^\gamma(s)$ is of type $\mathcal{U}(\theta^*)$ for every $s_o\le s\le t$. Consequently, for all these paths, an equality holds in (78) for $k=\lfloor s_o/\gamma\rfloor,\cdots,\lfloor t/\gamma\rfloor-1$ and we obtain, for $s_o\le s\le t$,

$$\begin{aligned}Q^\gamma(s) &= (1-\gamma)^{\lfloor s/\gamma\rfloor-\lfloor s_o/\gamma\rfloor}Q^\gamma(s_o)+\left[1-(1-\gamma)^{\lfloor s/\gamma\rfloor-\lfloor s_o/\gamma\rfloor}\right]\|\tilde\theta^*-\tilde\theta(0)\|^2\\ &\quad+\gamma\sum_{l=\lfloor s_o/\gamma\rfloor}^{\lfloor s/\gamma\rfloor-1}(1-\gamma)^l\left\langle\Delta_{\lfloor s/\gamma\rfloor-l}(\theta^\gamma_{\lfloor s/\gamma\rfloor-l-1}),\tilde\theta^*-\tilde\theta(0)\right\rangle.\end{aligned} \tag{85}$$

From Corollary 5.2 and Corollary 5.3, we have that $\lim_{\gamma\to 0}P(\Omega^\gamma)=1$. On the set Ω^γ, replace $Q^\gamma(s)$ with the right hand side of (85). From the choice of s_o and Lemma 5.6, we have that

$$\lim_{\gamma\to 0}P\left\{\sup_{s_o\le s\le t}|Q^\gamma(s)-Q(s)|\ge\epsilon\right\}=\lim_{\gamma\to 0}P\left\{\Omega^\gamma\cap\{\sup_{s_o\le s\le t}|Q^\gamma(s)-Q(s)|\ge\epsilon\}\right\}=0.$$

Finally, suppose that $\theta(0)=\theta^*$. With this initial condition, $\theta(t)\equiv\theta^*$. The proof in this case is derived by using the recursion

$$\theta_n^\gamma-\theta^*=(1-\gamma)\theta_{n-1}^\gamma+\gamma\left[\bar J(\theta_{n-1}^\gamma)-\theta^*\right]+\gamma\left[J_n(\theta_{n-1}^\gamma)-\bar J(\theta_{n-1}^\gamma)\right]$$

and the techniques of Lemma 5.7. $\square$

We provide now an outline of the proof of Theorem 5.2

Proof of Theorem 5.2. Let $\tau_0:=\tau_{D_{\mathcal{U}(\theta_0)}}$, and for $n\geq 1$, $\tau_n:=\inf\left\{s>\tau_{n-1}:\theta(s)\notin D_{\mathcal{U}(\theta_{\tau_{n-1}})}\right\}$. For the solution of (65) we have that $\tau_L\geq t$ for some $L\geq 0$. It is sufficient to show that

$$\lim_{\gamma\to 0}P\left\{\sup_{\tau_{n-1}\leq s\leq t\wedge\tau_n}\|\theta^\gamma(s)-\theta(s)\|\geq\epsilon\right\}=0,\qquad n=1,\cdots,L,$$

and this will follow by induction if it is shown that

$$\lim_{\gamma\to 0}P\left\{\sup_{0\leq s\leq t\wedge\tau_0}\|\theta^\gamma(s)-\theta(s)\|\geq\epsilon\right\}=0. \tag{86}$$

Recall that $x|_S$ denotes the vector in $\mathbb{R}^{|S|}$ with coordinates in $S\subset\mathcal{N}$. Using Lemma 5.1, the methodologies of the proof of Theorem 5.4 and step 1 of Theorem 5.3 it can be shown that for every $\epsilon>0$ and $t<\tau_0$,

$$\lim_{\gamma\to 0}P\left\{\sup_{0\leq s\leq t}\left\|\theta^\gamma(s)|_{S_k(\theta_0)}-\theta(s)|_{S_k(\theta_0)}\right\|\geq\epsilon\right\}=0,\qquad k=1,\cdots,M(\theta_0).$$

Using the same procedure as in the proof of step 2 of Theorem 5.3, we have that

$$\lim_{\gamma\to 0}P\left\{\sup_{0\leq s\leq t\wedge\tau_0}\left\|\theta^\gamma(s)|_{S_k(\theta_0)}-\theta(s)|_{S_k(\theta_0)}\right\|\geq\epsilon\right\}=0,\qquad k=1,\cdots,M(\theta_0),$$

which implies (86). $\square$

6 Long time asymptotic behavior of policy π^γ

In this section we establish the asymptotic optimality of policy π^γ as $\gamma\to 0$. Specifically we show that the long run average of the vector of response times

$$\left(g_1\frac{1}{n}\sum_{k=0}^{n-1}J_{k+1,1}(\theta_k^\gamma),\ldots,g_N\frac{1}{n}\sum_{k=0}^{n-1}J_{k+1,N}(\theta_k^\gamma)\right)^T$$

under policy π^γ converges a.s. to a finite limit as $n \to \infty$. The magnitude of the difference between this limit and the lexicographically minimum vector θ^* is bounded from above and the bound converges to zero as $\gamma \to 0$.

We first restate recursion (63) as

$$\begin{aligned} \theta_n^\gamma &= \theta_{n-1}^\gamma + \gamma \left[J_n(\theta_{n-1}^\gamma) - \theta_{n-1}^\gamma \right] & (87) \\ &= (1-\gamma)^n \theta_0 + \gamma \sum_{k=0}^{n-1} (1-\gamma)^{n-1-k} J_{k+1}(\theta_k^\gamma). & (88) \end{aligned}$$

The behavior of the sample mean response times is closely related to the long run behavior of $\{\theta_n^\gamma\}$. We begin by considering the existence of a stationary distribution of the $I\!R^N$-valued time-homogeneous Markov chain $\{\theta_n^\gamma\}_{n=0}^\infty$. The reader is referred to [34] for related definitions and results. The following result shows that for every $n = 1, 2, \cdots$, the conditional distribution of $J_n(\theta_{n-1}^\gamma)$ given θ_{n-1}^γ has a component that is absolutely continuous with respect to Lebesgue measure, $\ell(\cdot)$, restricted to an appropriate set. The assumption of Poisson arrivals is crucial.

Lemma 6.1. *There exists $G := \bigotimes_{i=1}^N [\alpha_i, \beta_i] \subset I\!R^N$ such that for every (Borel) measurable $D \subset G$, we have*

$$P\left\{ J_n(\theta_{n-1}^\gamma) \in D \mid \theta_{n-1}^\gamma = \theta \right\} \;\geq\; c\, \ell(D), \quad n = 1, 2, \cdots, \tag{89}$$

for some constant $c > 0$.

Proof. Observe that it suffices to show (89) for sets of the form $D = \bigotimes_{i=1}^N D_i$, where D_i is a measurable subset of $[\alpha_i, \beta_i]$ for every $i \in \mathcal{N}$. Indeed, then it is easy to show that (89) holds for finite disjoint unions of sets of the above form and, invoking the monotone class theorem, (89) would hold for all measurable subsets of G.

Since $B_i(0) < 1$ for every $i \in \mathcal{N}$, there exists $c_i > 0$ such that $B_i(c_i + \epsilon_i) - B_i(c_i^-) > 0$ for every $\epsilon_i > 0$; here $B_i(t^-) := \lim_{s \uparrow t} B_i(s)$. Set $\epsilon = \min_i (c_i/3)$, $\alpha_1 = (2c_1 + 2\epsilon)/(\tau\lambda_1 g_1)$, $\beta_1 = 3c_1/(\tau\lambda_1 g_1)$, and $\alpha_i = (c_i + \epsilon)/(\tau\lambda_i g_i)$, $\beta_i = (c_{i-1} + c_i)/(\tau\lambda_i g_i)$ for every $i = 2, \cdots, N$. Consider the event $\mathcal{E}$ in which, during the time interval $[T_{n-1}, T_n]$, exactly $N+1$ customers arrive: a customer of class 1 arrives at time T_{n-1}, yet another customer of class 1 arrives during its service time and after that, a customer of class i arrives during the service time of a customer of class $i-1$ for $i = 2, 3, \cdots, N$. Since the policy is non-idling and non-preemptive, on this event $\mathcal{E}$, the response times of the customers are independent of the policy and we can write

$$\begin{aligned} P\left\{J_n(\theta^\gamma_{n-1}) \in D \mid \theta^\gamma_{n-1} = \theta\right\} &\geq P\left\{(J_n(\theta^\gamma_{n-1}) \in D) \cap \mathcal{E} \mid \theta^\gamma_{n-1} = \theta\right\} \\ &= P\left\{(J_n(\theta^\gamma_{n-1}) \in D) \cap \mathcal{E}\right\}. \end{aligned} \tag{90}$$

For $D \subset I\!R$ and $x \in I\!R$, let $D - x$ denote the set $\{y - x : y \in D\}$. Also, let $d\mathbf{B}(\bar{s}_1, s_1, s_2, \cdots, s_N) := dB_1(\bar{s}_1)\prod_{i=1}^N dB_i(s_i)$. Then the right hand side in (90) can be written as

$$\int P\left\{(J_{n,i}(\theta^\gamma_{n-1}) \in D_i, i \in \mathcal{N}) \mid \mathcal{E}, \bar{s}_1, s_1, \cdots, s_N\right\} P\left\{\mathcal{E} \mid \bar{s}_1, s_1, \cdots, s_N\right\} d\mathbf{B}(\bar{s}_1, s_1, \cdots, s_N).$$

For convenience, set $\bar{D}_i := \{\tau\lambda_i g_i x : x \in D_i\}$ for $i \in \mathcal{N}$. From the memoryless nature of the arrivals and the independence of the arrivals and service times, it follows further that the above integral is equal to

$$\int \frac{\ell\left(\{\bar{D}_1 - \bar{s}_1 - s_1\} \cap [0, \bar{s}_1]\right)}{\bar{s}_1} \prod_{i=2}^N \frac{\ell\left(\{\bar{D}_i - s_i\} \cap [0, s_{i-1}]\right)}{s_{i-1}} P\left(\mathcal{E} | \bar{s}_1, s_1, \cdots, s_N\right) d\mathbf{B}(\bar{s}_1, s_1, \cdots, s_N). \tag{91}$$

Consider the integral only over the set

$$D_o = \{c_1 \leq \bar{s}_1 \leq c_1 + \epsilon \ \text{ and } \ c_i \leq s_i \leq c_i + \epsilon, \ i \in \mathcal{N}\}.$$

Observe that for the specific choices of $\{\alpha_i\}$ and $\{\beta_i\}$, we have that $\bar{D}_1 - \bar{s}_1 - s_1 \subset [0, \bar{s}_1]$ and $\bar{D}_i - s_i \subset [0, s_{i-1}]$ for every $i = 2, 3, \cdots, N$ whenever $(\bar{s}_1, s_1, s_2, \cdots, s_N) \in D_o$. Using this fact and the translation invariance of the Lebesgue measure, it follows that (91) is no smaller than

$$\ell(D)\left(\prod_{i=1}^N \tau\lambda_i g_i\right)\int_{D_o}\left(\bar{s}_1 \prod_{i=2}^N s_{i-1}\right)^{-1} P\left\{\mathcal{E} \mid \bar{s}_1, s_1, \cdots, s_N\right\} d\mathbf{B}(\bar{s}_1, s_1, \cdots, s_N).$$

From the choice of $\{c_i\}$, we have that $d\mathbf{B}$ measure of D_o is positive and the result follows by setting the coefficient of $\ell(D)$ above to $c > 0$. $\square$

Let $\ell 1_G(\cdot) := \ell(G\cap\cdot)$ with $G := \bigotimes_{i=1}^N[\alpha_i,\beta_i]$ as defined in Lemma 6.1. We will now show that the chain $\{\theta_n^\gamma\}$ is $\ell 1_G$-irreducible and will identify $\bar{G} \subset G$ such that every bounded measurable set $C \subset I\!R^N$ with $\ell 1_{\bar{G}}(C) > 0$ is a small set for the chain. Our approach is similar to that in example (f) (p. 5,12,15) of [34]. Let

$$
\begin{aligned}
V_n^\gamma &:= \gamma\sum_{k=0}^{n-1}(1-\gamma)^{n-1-k}J_{k+1}(\theta_k^\gamma),\\
G_n &:= \bigotimes_{i=1}^N\Big[\{1-(1-\gamma)^n\}\alpha_i+\epsilon_n, \{1-(1-\gamma)^n\}\beta_i-\epsilon_n\Big].
\end{aligned}
$$

with $\epsilon_n < 2^{-1}[1-(1-\gamma)^n](\beta_i-\alpha_i)$ and $\epsilon_n \to 0$ as $n\to\infty$. Fix $\theta \in I\!R^N$. From Lemma 6.1, it follows by an inductive argument (see Appendix), that there exists $c_n > 0$ (that is independent of D) such that

$$
P\left\{V_n^\gamma \in D \mid \theta_0^\gamma = \theta\right\} \;\geq\; c_n\,\ell(D), \quad \forall\; D \subset G_n, \quad n = 1,2,\cdots. \tag{92}
$$

Pick D such that $\ell 1_G(D) > 0$. Then from (88) and (92), we have

$$
\begin{aligned}
P\left\{\theta_n^\gamma \in D \mid \theta_0 = \theta\right\} &\geq P\left\{V_n^\gamma \in G_n \cap [D-(1-\gamma)^n\theta] \mid \theta_0^\gamma = \theta\right\}\\
&\geq c_n\,\ell\left(G_n\cap[D-(1-\gamma)^n\theta]\right).
\end{aligned} \tag{93}
$$

Since $\ell(G\cap D) > 0$ and $G_n \to G$, the quantity on the right is (strictly) positive for some n, sufficiently large and we have shown that

Lemma 6.2. *The Markov chain $\{\theta_n^\gamma\}$ is $\ell 1_G$-irreducible.*

Let $\bar{G} := \bigotimes_{i=1}^N[\alpha_i+\epsilon,\beta_i-\epsilon]$ with $0<\epsilon<(\beta_i-\alpha_i)/2$ for every $i \in \mathcal{N}$. Let $C \subset I\!R^N$ be a bounded measurable set. Since $G_n \to G$, it follows from the boundedness of C that there exists n_o (independent of θ) sufficiently large so that for every $n \geq n_o$,

$$
\left(D\cap\bar{G}\right) - (1-\gamma)^n\theta \subset\; G_n, \quad \forall\theta\in C. \tag{94}
$$

Applying (94) and the translation invariance of Lebesgue measure in (93), we now have, for every $D \subset I\!R^N$ and $\theta \in C$:

$$
P\left\{\theta_{n_o}^\gamma \in D \mid \theta_0 = \theta\right\} \;\geq\; P\left\{\theta_{n_o}^\gamma \in (D\cap\bar{G}) \mid \theta_0 = \theta\right\}
$$

$$\begin{aligned} &\geq \; c_{n_o}\,\ell\left(\left(D\cap\bar{G}\right)-(1-\gamma)^{n_o}\theta\right)\\ &= \; c_{n_o}\,\ell 1_{\bar{G}}\,(D)\,. \end{aligned}$$

This shows that

Lemma 6.3. *For the Markov chain $\{\theta_n^\gamma\}$, any bounded measurable set $C \subset I\!R^N$ such that $\ell 1_{\bar{G}}(C) > 0$ is a small set.*

We are now ready for our main result.

Theorem 6.1. *For every $0 < \gamma < 1$, the Markov chain $\{\theta_n^\gamma\}$ is Harris ergodic. The limiting r.v. θ_∞^γ has finite mean and $E||\theta_\infty^\gamma - \theta^*||^2 \leq c\gamma$, for some $c > 0$.*

Proof. We first establish positive Harris recurrence. With Ψ defined as in (27), let $V(\theta) := \frac{1}{2}||\Psi\theta - \Psi\theta^*||^2$. Using (87) we can write

$$\begin{aligned} V(\theta_n^\gamma) \;&=\; (1-\gamma)^2\,V(\theta_{n-1}^\gamma) \;+\; \gamma^2\frac{1}{2}\,||\Psi J_n(\theta_{n-1}^\gamma) - \Psi\theta^*||^2 \\ &\quad +\; \gamma(1-\gamma)\;\langle \Psi\theta_{n-1}^\gamma - \Psi\theta^*, \Psi J_n(\theta_{n-1}^\gamma) - \Psi\theta^*\rangle\,. \end{aligned} \tag{95}$$

Taking conditional expectation given θ_{n-1}^γ on both sides of (95) and using Lemma 3.2 we obtain

$$E\left[V(\theta_n^\gamma) - V(\theta_{n-1}^\gamma) \mid \theta_{n-1}^\gamma\right] \;\leq\; \gamma(\gamma-2)\,V(\theta_{n-1}^\gamma) \;+\; \gamma^2 c_1 \tag{96}$$

for some positive constant c_1. If we choose

$$C := \left\{\theta \in I\!R^N : ||\theta - \theta^*|| \leq M\right\},$$

a sufficiently large choice of M gives (see (96)): (a) $\sup_{\theta\in C^c} E\left[V(\theta_n^\gamma) - V(\theta_{n-1}^\gamma) \mid \theta_{n-1}^\gamma = \theta\right] < 0$, (b) $\sup_{\theta\in C} E\left[V(\theta_n^\gamma) \mid \theta_{n-1}^\gamma = \theta\right] < \infty$, and (c) $\ell 1_{\bar{G}}(C) > 0$. Positive recurrence now follows from Proposition 5.10 (page 77) in [34] and Lemmas 6.2 and 6.3.

Arguing as in Lemma 6.3, we can easily show that there exists $c_2 > 0$ and an integer n_o such that

$$P\left\{\theta_n^\gamma \in D \mid \theta_0^\gamma = \theta\right\} \;\geq\; c_2\;\ell 1_{\bar{G}}(D), \quad n = n_o,\; n_o+1,$$

for every $D \subset I\!R^N$ and $\theta \in C$, $C \subset I\!R^N$ being a bounded and measurable set with $\ell 1_{\bar{G}}(C) > 0$. From Problem 3.2 (p. 157) in [2], it follows that the chain $\{\theta_n^\gamma\}$ is aperiodic and therefore Harris ergodic.

We now show that $E\theta_\infty^\gamma < \infty$. Without loss of generality assume that $EV(\theta_0^\gamma) < \infty$. Taking expectation in (96) we obtain

$$EV(\theta_n^\gamma) \ \leq \ (1-\gamma)^2 EV(\theta_{n-1}^\gamma) \ + \ \gamma^2 c_1,$$

and iterating, this yields

$$EV(\theta_n^\gamma) \ \leq \ (1-\gamma)^{2n} EV(\theta_0^\gamma) \ + \ \gamma c_1/(2-\gamma). \tag{97}$$

Therefore, $\limsup_{n\to\infty} EV(\theta_n^\gamma) \ \leq \ \gamma c_1$. Since θ_n^γ converges weakly to θ_∞^γ as $n \to \infty$, from Theorem 6.3 in [7], we obtain

$$EV(\theta_\infty^\gamma) \ \leq \ \liminf_{n\to\infty} EV(\theta_n^\gamma) \ \leq \ \gamma c_1 \ < \ \infty. \tag{98}$$

From Jensen's inequality, it follows that $E\theta_\infty^\gamma < \infty$.

Finally, (98), implies that $E||\theta_\infty^\gamma - \theta^*||^2 \leq \frac{c_1}{\min_{i\in\mathcal{N}}(\rho_i g_i)}\gamma$. □

As an immediate consequence we have that the long run average of the response times under π^γ exists and that this limit approaches R^* as $\gamma \to 0$.

Corollary 6.1. *For $0 < \gamma < 1$,*

$$\theta^\gamma \ := \ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} J_{k+1}(\theta_k^\gamma)$$

exists a.s. and $||\theta^\gamma - \theta^*|| \leq \sqrt{c\gamma}$, where c is as in Theorem 6.1.

Proof. Some algebra (using (87)) shows that

$$\frac{1}{n}\sum_{k=0}^{n-1}\theta_k^\gamma \ = \ \frac{1}{n}\sum_{k=0}^{n-1} J_{k+1}(\theta_k^\gamma) \ + \ \frac{(\theta_0^\gamma - \theta_n^\gamma)}{n\gamma}.$$

From Theorem 6.1, we have $1/n\sum_{k=0}^{n-1}\theta_k^\gamma$ converges a.s. which implies θ_n^γ/n converges to 0 a.s.. Therefore θ^γ exists and is equal to $E\theta_\infty^\gamma$. The rest follows from Theorem 6.1 and Jensen's inequality. □

7 Appendix

In this appendix we present a proof of the inequality (92) in Section 6. Let

$$\begin{aligned} a_{ni} &:= \{1-(1-\gamma)^n\}\alpha_i + \epsilon_{ni} \\ b_{ni} &:= \{1-(1-\gamma)^n\}\beta_i - \epsilon_{ni} \\ G_n &= \bigotimes_{i=1}^{N} [a_{ni}, b_{ni}]\,. \end{aligned}$$

We have to show that for every $n = 1, 2, \cdots$, there exists $c_n > 0$ (independent of D) such that

$$P\{V_n^\gamma \in D \mid \theta_0 = \theta\} \;\geq\; c_n\, \ell(D), \quad D \subset G_n. \tag{99}$$

Let us first choose $\{\epsilon_n\}$ so that in addition to satisfying $\epsilon_n < [1-(1-\gamma)^n](\beta_i - \alpha_i)/2$ and $\epsilon_n \to 0$ as $n \to \infty$, we have

$$\epsilon_{n+1} = (1-\gamma)\left(\epsilon_n + \delta_n\right), \tag{100}$$

and $\delta_n > 0$ and $\delta_n \to 0$ as $n \to \infty$. That this is always possible can be seen, for example, from the choice

$$\begin{aligned} \epsilon_{ni} &:= \frac{e_i}{n}\ [1-(1-\gamma)^n]\,, \\ \delta_{ni} &:= \frac{\gamma e_i - (1-\gamma)\epsilon_{ni}}{(n+1)(1-\gamma)}, \qquad i \in \mathcal{N}, \end{aligned}$$

with $e_i = (\beta_i - \alpha_i)/4$.

Lemma 6.1 shows that (99) holds for $n = 1$. Assuming that (99) holds for n, we will show that it holds for $n+1$ as well. Consider first the case in which $D \subset G_{n+1}$ is such that $D \subset \otimes_{i=1}^{N}[c_i, d_i] \subset G_{n+1}$ with $0 < d_i - c_i = \gamma(\beta_i - \alpha_i)/2$ for every $i \in \mathcal{N}$. For $c \in I\!R$ and $D \subset I\!R$, let $D+c := \{\theta + c : \theta \in D\}$ and $D/c := \{\theta/c : \theta \in D\}$. Also, let $\bar\theta(v) := (1-\gamma)^n\theta + v$. Since

$$\begin{aligned} V_{n+1}^\gamma &= (1-\gamma)\, V_n \;+\; \gamma J_{n+1}(\theta_n^\gamma), \\ \theta_n^\gamma &= (1-\gamma)^n\, \theta_0 \;+\; V_n^\gamma, \end{aligned}$$

we can write

$$\begin{aligned} P\left\{V^\gamma_{n+1} \in D \mid \theta_0 = \theta\right\} &= \int P\left\{J_{n+1}(\bar\theta(v)) \in [D-(1-\gamma)v]/\gamma \,\Big|\, V^\gamma_n = v, \theta_0 = \theta\right\} \\ &\qquad\qquad dP\left\{V^\gamma_n \le v \,\Big|\, \theta_0 = \theta\right\} \\ &= \int P\left\{J_{n+1}(\bar\theta(v)) \in [D-(1-\gamma)v]/\gamma \,\Big|\, \theta^\gamma_n = \bar\theta(v)\right\} \\ &\qquad\qquad dP\left\{V^\gamma_n \le v \,\Big|\, \theta_0 = \theta\right\}. \end{aligned} \tag{101}$$

Assume first that there exists $D_o \subset I\!R^N$ such that

$$D_o \subset G_n; \quad [D-(1-\gamma)v]/\gamma \subset G = \otimes_{i=1}^N [\alpha_i, \beta_i], \quad \forall v \in D_o; \quad \text{and} \quad \ell(D_o) > 0. \tag{102}$$

Consider the integral in (101) only over the set $\{v \in D_o\}$. From Lemma 6.1, we have, using the translation invariance of Lebesgue measure, that the quantity on the right in (101) is no smaller than

$$c/\gamma^N \ \ell(D)\ P\left\{V^\gamma_n \in D_o \mid \theta_0 = \theta\right\}.$$

Setting $c_{n+1} := cc_n\ell(D_o)/\gamma^N > 0$, the induction hypothesis now gives (99) for the special choice of the set D. Since an arbitrary set $D \subset G_{n+1}$ can be written as a finite disjoint union of sets of this type the proof of (99) for the general case is straightforward.

It remains to show that $D_o \subset I\!R^N$ can be chosen so that (102) holds. Some algebra (using the definitions of $\{a_n\}$ and $\{b_n\}$) shows that

$$\begin{aligned} a_{n+1,i} &= (1-\gamma)a_{ni} + \gamma\alpha_i + (1-\gamma)\delta_{ni}, \\ b_{n+1,i} &= (1-\gamma)b_{ni} + \gamma\beta_i - (1-\gamma)\delta_{ni}, \end{aligned}$$

and this implies that (recall that $D \subset G_{n+1}$)

$$\inf_{\theta\in D}\left(\frac{\theta_i - \gamma\alpha_i}{1-\gamma}\right) \ge a_{ni} + \delta_{ni}; \quad \sup_{\theta\in D}\left(\frac{\theta_i - \gamma\beta_i}{1-\gamma}\right) \le b_{ni} - \delta_{ni}, \quad i \in \mathcal{N}. \tag{103}$$

From the choice of $\{c_i\}$ and $\{d_i\}$, we have further that

$$\sup_{\theta\in D}\left(\frac{\theta_i - \gamma\beta_i}{1-\gamma}\right) \le \frac{d_i - \gamma\beta_i}{1-\gamma} < \frac{c_i - \gamma\alpha_i}{1-\gamma} \le \inf_{\theta\in D}\left(\frac{\theta_i - \gamma\alpha_i}{1-\gamma}\right), \quad i \in \mathcal{N},$$

and

$$
\begin{aligned}
\inf_{\theta \in D} \left(\frac{\theta_i - \gamma \alpha_i}{1-\gamma} \right) \; - \; \sup_{\theta \in D} \left(\frac{\theta_i - \gamma \beta_i}{1-\gamma} \right) \;\; &> \;\; \frac{\gamma\left(\beta_i - \alpha_i\right) - (d_i - c_i)}{1-\gamma} \\
&= \;\; \frac{\gamma(\beta_i - \alpha_i)}{2(1-\gamma)}. \qquad (104)
\end{aligned}
$$

If we let

$$
\begin{aligned}
l_i \;\; &:= \;\; \max\left\{ a_{ni}, \; \sup_{\theta \in D} \left(\frac{\theta_i - \gamma \beta_i}{1-\gamma} \right) \right\}, \\
u_i \;\; &:= \;\; \min\left\{ b_{ni}, \; \inf_{\theta \in D} \left(\frac{\theta_i - \gamma \alpha_i}{1-\gamma} \right) \right\}, \qquad i \in \mathcal{N},
\end{aligned}
$$

then from (103)-(104), we have that

$$
u_i - l_i \; \geq \; \min\left(b_{ni} - a_{ni}, \delta_{ni}, \frac{\gamma(\beta_i - \alpha_i)}{2(1-\gamma)} \right) > 0
$$

for every $i \in \mathcal{N}$. Define $D_o := \otimes_{i=1}^{N} [l_i, u_i]$. It follows from the choices of $\{l_i\}$ and $\{u_i\}$ that (102) holds and the proof is complete.

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