Exploiting Wireless Channel State Information for Throughput Maximization*

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Abstract

We consider the problem of scheduling packets over channels with time-varying quality. This problem received a lot of attention lately in the context of devising methods for providing quality of service in wireless communications. Earlier work in this problem considered two cases. One case is that the arrival rate vector is in the throughput region and then policies that stabilize the system are pursued. The other case is that all packet queues are saturated and then policies that optimize an objective function of the channel throughputs are investigated. In this paper we address the case where no assumption on the arrival rates is made. We obtain a scheduling policy that maximizes the weighted sum of channel throughputs. Under the optimal policy, in the general case, the system may operate in a regime where some queues are stable while the rest become saturated. If stability for the whole system is at all possible, it is always achieved. The optimal policy is a combination of a criterion that gives priorities based on queue lengths and a strict priority rule. The scheduling mechanism switches between the two criteria based on thresholds on the queue lengths and is modulated by the availability of the channels. The analysis of the operation of the system involves the study of a vector process which in steady state has some of its components stable while the others are unstable. We adopted a novel model for time-varying channel availability that dispenses with the statistical assumptions and makes a rigorous description of system dynamics possible.

1 Introduction

The primary motivation of this work is to address the problem of scheduling transmissions of multiple data flows sharing the same wireless channel under general arrival rates. The relative delay tolerance of data applications, together with the bursty traffic characteristics, opens up the potential for scheduling transmissions so as to optimize throughput [3]. Given the above considerations, we examine a time-slotted parallel queue system with a single server. The condition of the associated channel of every queue varies with time between "on" and "off" states. In every time slot only one packet from a given queue can be transmitted, if the associated channel is in the "on" state and the queue is non empty. For such a system we design of scheduling policy that allocates the server to the queues in such a way that the weighted sum of channel throughputs is maximal.

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A related approach along these lines is proposed in [1], where the authors identify optimality properties for scheduling downlink transmissions to data users in CDMA networks. For arbitrary-topology networks, the problem of admission control and rate allocation to the users so that certain quality-of-service requirements are met, is investigated. A mathematical programming formulation is obtained for determining the optimal transmission schedule. The effect of wireless channels on the performance of transmission protocols such as TCP is examined through simulations in [2]. The authors conclude that channel state dependent scheduling can lead to significant improvement in channel utilization.

The problem of scheduling wireless channels with time-varying connectivity has been addressed in the past in several different contexts. In [15], optimal scheduling for a wireless system consisting of multiple queues and a single server is studied. The arrival processes to the queues are assumed i.i.d Bernoulli. The wireless channels can be in the "on" or in the "off" state according to i.i.d Bernoulli processes. The authors derived the system stability region; moreover, they showed that the policy that among the queues whose channel is "on" serves longest one, stabilizes the system whenever the arrival rates are within the stability region. In [16], Tassiulas considered a system that generalizes the one in [15] in the following aspects. First, a network with arbitrary topology is considered. Second, the topology is represented by a hidden Markov model instead of an independent and identically distributed (i.i.d) process. Third, anticipative scheduling policies are taken into consideration. Fourth, multiple link transmission rates are considered. In that context, after the characterization of the region of achievable throughputs, a transmission scheduling policy is proposed that achieves all throughput vectors achievable by any anticipative policy.

The problem of scheduling transmissions over a wireless channel with time-varying transmission rates is considered in [12], [3], [9] and [10]. The problem of providing a scheduling policy that stabilizes the system whenever the arrival rate vector lies within the stability region is dealt in [12] and in [10]. In [12], a finite set of channel states is assumed and every channel can be in one of these states. With each state there is an associated data rate representing the rate at which the queue is served if selected for transmission. The arrival processes to the queues are assumed mutually independent, ergodic, Markov chains with countable state space. Under these assumptions it was shown that the scheduling policy, called the *exponential rule*, makes the queues stable if there exists any policy that can do so. In [10], the authors consider the problem of power and server allocation in a multi-beam satellite downlink which transmits data to different ground locations over time-varying channels. The authors establish the stability region of the system and develop a power allocation policy, which stabilizes the system whenever the system is stabilizable and when the arrival and channel state processes are i.i.d.

In [9] and [3] the problem of developing a scheduling policy for efficient channel utilization is considered for the case that all the queues are infinite. In [9] the state of a channel is modeled by a stochastic process, which represents the level of performance of the given channel. A scheduling policy is provided which maximizes the average system performance given that a predetermined time-fraction assignment is achieved for all channels. In [3], the authors consider a base station serving data-users. The feasible rates of the users vary over time according to some stationary discrete-time stochastic process. A scheduling policy that exploits the variations in the channel conditions and maximizes the minimum throughput is developed.

The main contribution of this paper is the design and analysis of a scheduling policy for a wireless system with time-varying connectivity, for general arrival rates. This is an important situation that arises in practice, since the channel parameters and the arrival rates may not be known apriori, or may vary over time. In such a case, scheduling policies proposed before for maximizing throughput under various assumptions on the arrival rates may fail, and the system may have a rather erratic behavior. In the current work we consider the scheduling problem of maximizing the weighted sum of user throughputs. We provide a scheduling policy that is optimal under any arrival rates. In the most general case, under the optimal policy we propose, some queues will be stable while others will operate in saturation. Such a dynamic behavior makes the analysis of the system rather difficult. Instrumental in the analysis of our policy was the adoption of a "bounded burstiness" model for the variability of the channel inspired by "burstiness constrained" traffic models that have been used over the last several years in the analysis of communication networks [5], [4], [8].

The paper is organized as follows. In Section 2, the traffic and channel model is introduced. Specifically, the constraints on the arrival and channel availability processes are given. In Section 3 we provide the problem formulation and define the scheduling policy. In Section 4 the optimality proof of the proposed policy is given. Conclusions and suggestions for further work are discussed in Section 5.

1.1 Notations and Conventions

Before proceeding, we present some of the notations and conventions that we use throughout the paper. Sets of numbers are denoted by calligraphic capital letters. In particular we define $\mathcal{N} = \{1, ..., N\}$. A subset $\mathcal{S}$ of a set $\mathcal{D}$ is denoted by $\mathcal{S} \subseteq \mathcal{D}$ and a strict subset by $\mathcal{S} \subset \mathcal{D}$. In several places we will use sets as subscripts or arguments, say $F(\mathcal{S})$. To simplify notation and if there is no possibility for confusion, instead of $F(\{i_1, ..., i_k\})$ we write $F(i_1, ..., i_k)$. We write $\sum_{\mathcal{S}} y_i$ to denote $\sum_{i \in \mathcal{S}} y_i$. If $\mathcal{S} = \varnothing$, then we define $\sum_{\mathcal{S}} x_i = 0$. Also, $\cup_{i=k}^{l} \mathcal{D}_i = \varnothing$ if $k > l$. The cardinality of a set $\mathcal{S}$ is denoted by $|\mathcal{S}|$. If $\mathbf{X} = [x_{ij}]$ and $\mathbf{Y} = [y_{ij}]$ are matrices, then $\mathbf{X} \leq \mathbf{Y}$ ($\mathbf{X} < \mathbf{Y}$) means that $x_{ij} \leq y_{ij}$ ($x_{ij} < y_{ij}$) for all i and j. Finally, by $\mathbf{X}^T$ we denote the transpose of $\mathbf{X}$.

2 Traffic and Channel Model

We consider a system consisting of N channels. With each channel there is an associated queue holding packets that are to be transmitted over the given channel. Packets are of fixed size and time is divided in slots of unit length, equal to the transmission time of a packet. Slot $t \geq 1$ refers to the interval $(t-1, t]$. In the interval $(t-1, t]$ (slot t), $a_i(t)$ new packets join queue i to be transmitted over the corresponding channel. At the beginning of slot t, i.e., at time $t-1$, one packet among those already in one of the N queues may be chosen for transmission at slot t. The number of packets from queue i transmitted in slot t is $b_i(t)$ (therefore, $b_i(t)$ is either 0 or 1) and the number of packets in queue i at time $t \geq 0$ is $q_i(t)$. Hence the number of packets at queue i, $i \in \mathcal{N}$, evolves with time according to the equation

$$q_i(t) = (q_i(t-1) - b_i(t))^+ + a_i(t), \tag{1}$$

where $(x)^+ = \max\{0, x\}$.

Define $a_{\mathcal{S}}(t) = \sum_{\mathcal{S}} a_i(t)$ and $b_{\mathcal{S}}(t) = \sum_{\mathcal{S}} b_i(t)$. That is, $a_{\mathcal{S}}(t)$ is the number of arrivals in slot t to be transmitted over channel set $\mathcal{S}$, and $b_{\mathcal{S}}(t)$ is the number of packets transmitted over the channels in $\mathcal{S}$, in slot t. Since only one packet may be transmitted in one slot, we have

$$b_{\mathcal{S}}(t) = \begin{cases} 1, & \text{if } b_i(t) = 1 \text{ for one of the channels in } \mathcal{S} \\ 0, & \text{otherwise} \end{cases}.$$

At slot t, channel i may or may not be available for transmission of queue i packets. If the channel is available for transmission, we say that the channel is in the "on" state. We define for $\mathcal{S} \subseteq \mathcal{N}$, $\mathcal{S} \neq \varnothing$,

$$c_{\mathcal{S}}(t) = \begin{cases} 1, & \text{if at least one channel in } \mathcal{S} \text{ is "on" in slot } t \\ 0, & \text{otherwise} \end{cases},$$

and $c_{\varnothing}(t) \equiv 0$. For example, Figure 1 shows the channel availability for 3 channels during 15 time slots. According to the figure

- $c_{\{1\}}(t) = 1$ for $1 \le t \le 12$ and zero elsewhere.
- $c_{\{3\}}(t) = 1$ for $2 \le t \le 7$, $9 \le t \le 14$ and zero elsewhere.
- $c_{\{1,2\}}(t) = 1$ for $1 \le t \le 15$ and zero elsewhere.
- $c_{\{1,3\}}(t) = 1$ for $1 \le t \le 14$ and zero elsewhere.
- $c_{\{1,2,3\}}(t) = 1$ for $1 \le t \le 15$ and zero elsewhere.

Transmission over channel i may take place ($b_i(t) = 1$) only if the channel is in the "on" state and hence,

$$b_{\mathcal{S}}(t) \le c_{\mathcal{S}}(t). \tag{2}$$

If $x(t)$ is any of the quantities defined above, we denote

$$X(s,t) = \sum_{\tau=s+1}^{t} x(\tau).$$

We make the following assumptions regarding the traffic and channel availability processes.

Traffic Model.

$A_i(s,t)$ is $(\sigma_i^U, \sigma_i^L, \alpha_i)$-constrained, i.e., for any $t \ge s \ge 0$, it holds

$$\alpha_i(t-s) - \sigma_i^L \le A_i(s,t) \le \alpha_i(t-s) + \sigma_i^U, \tag{3}$$

where

$$\infty > \sigma_i^L \ge 0,\ \infty > \sigma_i^U \ge 0,\ \infty \ge \alpha_i \ge 0.$$

Parameter α_i is the packet arrival rate to queue i (i.e., $\alpha_i = \lim_{t\to\infty} \frac{A_i(0,t)}{t}$) for transmission over the corresponding channel. We allow for the possibility that $\alpha_i = \infty$, in order to include the case that some of the queues are infinite for $t \ge 1$. It follows from the definition that if $A_i(s,t)$ is $(\sigma_i^U, \sigma_i^L, \alpha_i)$-constrained for $i \in \mathcal{S}$, then $A_{\mathcal{S}}(s,t)$ is $\left(\sum_{\mathcal{S}} \sigma_i^U, \sum_{\mathcal{S}} \sigma_i^L, \sum_{\mathcal{S}} \alpha_i\right)$-constrained.

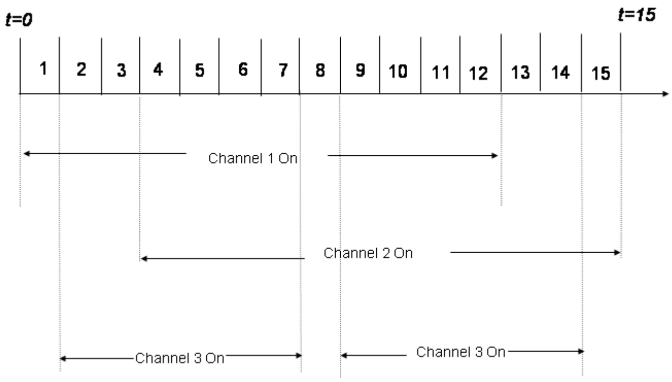

Figure 1: Channel Availability ("on" state) for 3 channels.

Channel Availability Model.

$C_{\mathcal{S}}(s,t)$, is $(\theta^U_{\mathcal{S}}, \theta^L_{\mathcal{S}}, F(\mathcal{S}))$-constrained, i.e., for any $t \geq s \geq 0$, it holds

$$F(\mathcal{S})(t-s) - \theta^L_{\mathcal{S}} \leq C_{\mathcal{S}}(s,t) \leq F(\mathcal{S})(t-s) + \theta^U_{\mathcal{S}}, \tag{4}$$

where

$$\infty > \theta^L_{\mathcal{S}} \geq 0, \ \ \infty > \theta^U_{\mathcal{S}} \geq 0.$$

We also use the convention $F(\varnothing) = \theta^L_{\varnothing} = \theta^U_{\varnothing} = 0$.

We refer to the inequalities in (3) and (4) as "burstiness constraints".

The definitions for the traffic model are standard, see e.g., [5], [6], [4], [8]. We elaborate on the Channel Availability Model. From (4) it follows that

$$\lim_{t\to\infty} \frac{C_{\mathcal{S}}(0,t)}{t} = F(\mathcal{S}), \tag{5}$$

that is, $F(\mathcal{S})$ is equal to the long-term fraction of time that at least one of the channels in $\mathcal{S}$ is in the "on" state. Also, from the definition of $c_{\mathcal{S}}(t)$ we have that for any subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{N}$, and for every t, it holds

$$c_{\mathcal{T}}(t) \leq c_{\mathcal{S}}(t), \ \text{if } \mathcal{T} \subseteq \mathcal{S},$$
$$c_{\mathcal{S}}(t) + c_{\mathcal{T}}(t) \geq c_{\mathcal{S}\cup\mathcal{T}}(t) + c_{\mathcal{S}\cap\mathcal{T}}(t),$$

and hence

$$C_{\mathcal{T}}(s,t) \leq C_{\mathcal{S}}(s,t), \ \text{if } \mathcal{T} \subseteq \mathcal{S}, \tag{6}$$
$$C_{\mathcal{S}}(s,t) + C_{\mathcal{T}}(s,t) \geq C_{\mathcal{S}\cup\mathcal{T}}(s,t) + C_{\mathcal{S}\cap\mathcal{T}}(s,t). \tag{7}$$

From (5), (6), (7) we conclude that $F(S)$ satisfies the following relations for any subsets $\mathcal{T}$ and $\mathcal{S}$ of $\mathcal{N}$.

$$F(\varnothing) = 0, \tag{8a}$$

$$F(\mathcal{T}) \leq F(\mathcal{S}),\ \mathcal{T} \subseteq \mathcal{S}, \tag{8b}$$

$$F(\mathcal{T}) + F(\mathcal{S}) \geq F(\mathcal{T} \cup \mathcal{S}) + F(\mathcal{S} \cap \mathcal{T}). \tag{8c}$$

The last property is known as the *submodularity* property.

As an example, suppose that the channel availability pattern in Figure 1, is repeated indefinitely, i.e., we have a periodic channel availability process. Consider the first channel, i.e., $\mathcal{S} = \{1\}$. It holds

$$c_{\{1\}}(t) = \begin{cases} 1, & \text{for } 1 \leq t \leq 12 \\ 0, & \text{for } 13 \leq t \leq 15 \end{cases}$$

and $c_{\{1\}}(t+15) = c_{\{1\}}(t)$, for every time-slot $t \geq 1$. Therefore we have

$$\left\lfloor \frac{t-s}{15} \right\rfloor 12 \leq C_{\{1\}}(s,t) \leq \left\lceil \frac{t-s}{15} \right\rceil 12, \text{ or}$$

$$\frac{12}{15}(t-s) - 12 \leq C_{\{1\}}(s,t) \leq \frac{12}{15}(t-s) + 12.$$

In conjunction with definition (4), the above inequality states that $C_{\{1\}}(s,t)$ is $(\theta^U_{\{1\}}, \theta^L_{\{1\}}, F(1))$-constrained, with $\theta^U_{\{1\}} = \theta^L_{\{1\}} = 12$ and $F(1) = 12/15$, i.e., $F(1)$ is equal to the long-term fraction of time that the first channel is on. Similarly we have that $C_{\mathcal{S}}(s,t)$ is $(\theta^U_{\mathcal{S}}, \theta^L_{\mathcal{S}}, F(\mathcal{S}))$-constrained and according to the figure

- For $\mathcal{S} = \{3\}$, $\theta^U_{\{3\}} = \theta^L_{\{3\}} = 12$ and $F(3) = 12/15$.
- For $\mathcal{S} = \{1,2\}$, $\theta^U_{\{1,2\}} = \theta^L_{\{1,2\}} = 0$ and $F(1,2) = 1$.
- For $\mathcal{S} = \{1,3\}$, $\theta^U_{\{1,3\}} = \theta^L_{\{1,3\}} = 14$ and $F(1,3) = 14/15$.
- For $\mathcal{S} = \{1,2,3\}$, $\theta^U_{\{1,2,3\}} = \theta^L_{\{1,2,3\}} = 0$ and $F(1,2,3) = 1$.

We close this section with a few comments on the adopted traffic and channel models. The assumption that the channel can be it two states only is applicable in networks with changing topology, e.g., Low-Earth-Orbit (LEO) satellite communications, metor-burst communication networks and networks with mobile users [16]. Furthermore the adopted burstiness-constrained model for the channel availability process is suitable for the representation of periodic connectivity processes arising in LEO satellite communications. While the “on-off” channel model is valid for several systems (see also [13], [14], [15]) it does not cover the case where several transmission rates are available depending on the channel state. We adopt this model here in order to simplify the situation and get a better insight into the problem at hand. Extension to multiple rates is an important open research topic. As will be seen, the adopted burstiness-constrained models make possible the complete description of system dynamics using mainly elementary (although not straightforward) techniques. Compared to introducing statistical assumptions for these models, there are both advantages and disadvantages. Note that the stationarity assumption is not needed in our model, although the existence of long-term averages is implied. On the other hand deterministic rather than stochastic bounds on process fluctuations are imposed.

3 Problem Formulation

Consider a scheduling policy π that at the beginning of slot t, i.e., at time $t-1$, decides which packet (if any) to transmit to one of the channels that are "on" at time t. Let

$$r_i^\pi = \lim_{t\to\infty}\inf \frac{B_i^\pi(0,t)}{t},$$

be the "throughput" of channel i under policy π.

Given costs w_i, $i \in \mathcal{N}$, $w_1 \geq w_2 \geq .... \geq w_N \geq 0$, our objective is to determine a policy such that the weighted sum of throughputs

$$\sum_{\mathcal{N}} w_i r_i^\pi,$$

is maximal.

Assume that the channel state at a given slot is known to the scheduler at the beginning of that slot and consider the following policy.

Scheduling Policy π^*.

> With queue i associate an index $I_i(q)$ of the form
>
> $$I_i(q) = \min\left(q, (N+1-i)\,T\right),$$
>
> where $T > 0$. At time t, consider the nonempty queues whose channel is "on". Among these queues, let i be the one with largest index $I_i(q_i(t))$ (if there are multiple such queues select one arbitrarily). Transmit a packet from queue i at slot $t+1$.

Our objective is to show that for T large enough, policy π^* maximizes the weighted sum of throughputs, irrespective of whether the overall system is stable or not. It is worth observing the following.

- Only the order of the costs w_i, $i \in \mathcal{N}$, not the actual values, determine policy π^*. This situation is similar to the well-known μc-rule in queueing theory.
- As will be seen, the traffic and channel model parameters determine how large T should be chosen. In other words, the policy depends on these parameters only through T. Again, the actual costs w_i do not have an effect on T. Although estimates of T can in principle be obtained through the analysis that follows, these will be too conservative. Moreover, in practice the traffic and channel parameters may not be known beforehand. Of course, one can pick very large values of T but this implies larger delays and slower convergence. Hence, development of adaptive schemes for determining T seems a more appropriate plausible way for choosing T. The development of such schemes is an important subject requiring further research work.
- Following the definition in [15], we call "Longest Connected Queue (LCQ) First", the scheduling policy which among the queues whose channels are "on", selects the one with the largest number of packets (if there are at least two such queues, pick one arbitrarily). Policy π^* has similarities with LCQ. In fact, when $q_i(t) \leq T$ for all $i \in \mathcal{N}$,

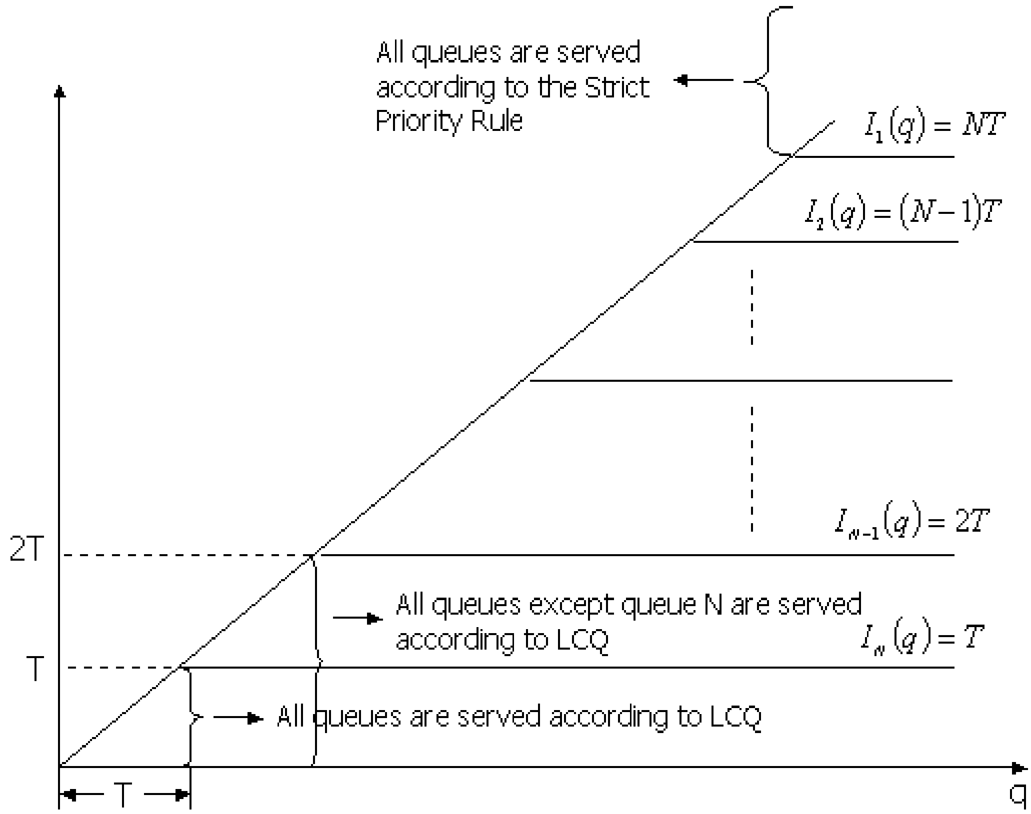

Figure 2: Operating diagram of policy π^*

the two policies are identical. On the other hand, when $q_i(t) \geq NT$ for all $i \in \mathcal{N}$, then $I_i(q_i(t)) = (N + 1 - i)T$ and hence π^* operates as a strict priority rule, giving higher priority to lower indexed queues. It is easy to see that this latter rule is optimal when all queues have always packets to transmit. In general, π^* operates as combination of the LCQ and the strict priority policy, and this enables it to provide the optimal throughput under any conditions on the arrival rates. The operating diagram of policy π^* is given in Figure 2. The comments seen in Figure 2 are true when all queue sizes belong to the intervals pointed by the brackets, e.g., the comment "all queues except queue N are served according to LCQ" is true when $q_i(t) \leq 2T$, for all $i \in \mathcal{N}$.

3.1 Achievable Throughput Space and Related Linear Optimization Problem

Consider that the system operates under an arbitrary scheduling policy π. From (2), the definitions of $B_{\mathcal{S}}(s,t)$, $C_{\mathcal{S}}(s,t)$ and (5), we have for any $\mathcal{S} \subseteq \mathcal{N}$,

$$\begin{aligned}
F(\mathcal{S}) &= \lim_{t\to\infty} \frac{C_{\mathcal{S}}(0,t)}{t} \\
&\geq \lim_{t\to\infty}\inf \frac{B^{\pi}_{\mathcal{S}}(0,t)}{t} \\
&= \lim_{t\to\infty}\inf \frac{\sum_{\mathcal{S}} B^{\pi}_i(0,t)}{t} \\
&\geq \sum_{\mathcal{S}} \lim_{t\to\infty}\inf \frac{B^{\pi}_i(0,t)}{t} \\
&= \sum_{\mathcal{S}} r^{\pi}_i. \qquad (9)
\end{aligned}$$

In addition, the fact that $A_i(0,t) \geq B^{\pi}_i(0,t)$ and (3) imply that for any $i \in \mathcal{N}$, it holds

$$0 \leq r^{\pi}_i \leq \alpha_i. \qquad (10)$$

From (9) and (10) we see that the maximum weighted sum of throughputs that can be achieved by any scheduling policy cannot exceed the value of the following optimization problem.

Linear Optimization Problem.

$$\max_{\{x_i\}_{i=1}^N} \sum_{i=1}^{N} w_i x_i,$$

subject to,

$$\begin{aligned}
\sum_{\mathcal{S}} x_i &\leq F(\mathcal{S}),\ \mathcal{S} \subseteq \mathcal{N}, && (11a)\\
x_i &\leq \alpha_i,\ i = 1,...,N, && (11b)\\
x_i &\geq 0,\ i = 1,...,N. && (11c)
\end{aligned}$$

and $F(\mathcal{S})$ satisfies (8a), (8b), (8c).

Let $\mathcal{N}_k = \{1,...,k\}$, and $\mathcal{N}_0 = \varnothing$. It can be shown that the solution to the previous optimization problem is given recursively by

$$x^*_k = \min\left\{\alpha_k, \min_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{F\left(k \cup \mathcal{D}\right) - \sum_{\mathcal{D}} x^*_i\right\}\right\}, \qquad (12)$$

for $k = 1,...,N$. The proof is given in the Appendix A.1.

Our objective in the next section is to show that scheduling policy π^* achieves the throughputs defined by (12) and therefore is optimal. Before proceeding with the details of the proof we discuss the problems encountered when one applies either the strict priority

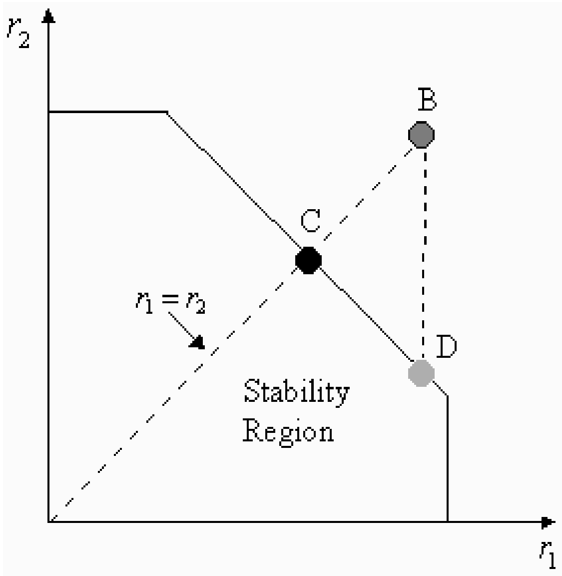

Figure 3: Throughput space for a symmetric system with two users.

rule or the LCQ policy to maximize the weighted sum of throughputs for all arrival rate vectors.

Regarding the strict priority rule, in [15] it is shown that there are cases where the arrival rates are within the throughput space of the system, and yet this rule renders the system unstable. That is, for at least one user the throughput is smaller than the arrival rate. However, as mentioned before the strict priority rule is optimal when all queues are infinite. On the other hand LCQ always stabilizes the system when the arrival rates are within the throughput space [15], and hence achieves the maximum weighted sum of throughputs (equal to weighted sum of arrival rates in this case). Next we give an example where the LCQ policy is suboptimal when the arrival rates are outside the throughput space.

Example 1. In Figure 3 we see the achievable throughput space (see equations (9) and (10)) of a symmetric system with two users. By symmetric we mean that the arrival as well as the channel constraints for the two users are identical. When the arrival rate vector is outside the stability region, e.g., point B, then the operating point under LCQ is C, which belongs to the boundary of the stability region and is such that both users receive equal throughputs (this follows from the symmetry of the system and the operation of LCQ). However, since $w_1 \geq w_2$ it can be shown easily that point D is the one that maximizes the weighted sum of throughputs. Hence D is the optimal point and LCQ, operating at point C, is suboptimal in this scenario.

From the above discussion it is evident that both LCQ and the strict priority rule while optimal for some cases, cannot maximize the weighted sum of throughputs for all arrival rates. Intuitively the proposed policy π^*, which switches between these two "extreme" policies according to system state, will be able to provide optimal throughputs regardless of any assumptions imposed on the arrival rates.

4 Optimality Proof

Since we deal only with policy π^* in this section, in order to simplify the notation we eliminate π^* from all related notations, e.g., we use r_i in place of $r_i^{\pi^*}$.

Before going into the details of the proof, we give an outline of the approach. In the general case, it can be shown that under π^*, a subset $\mathcal{U}$ of the queues will grow to infinity, while the rest of the queues will receive the maximum possible throughput, i.e., we have $r_i = \alpha_i,\ i \in \mathcal{N} - \mathcal{U}$. Call queues in $\mathcal{N} - \mathcal{U}$ "stable", and those in $\mathcal{U}$ "unstable". It can be proved that for any stable queue i, we have $r_i = x_i^*$. To determine the throughputs of the unstable queues we first show that for T large enough, each of the stable queues fluctuates in a certain range around kT, for some k, $0 \le k \le N$. This fact and the manner the indices are used to determine the scheduling decisions, implies that $r_k = x_k^*$ for all unstable queues. We mention that in the course of the proof, the fact that $r_i = x_i^*$, is established by starting from the smallest indices and moving to the largest, rather than by first proving the result for the stable and then for the unstable queues.

The following lemma will be useful in the sequel.

Lemma 1 *For any subsets $\mathcal{S}_1,\ \mathcal{S}_2$ of $\mathcal{N}$ and any t it holds*

$$\sum_{\mathcal{S}_1} q_i(t) - \sum_{\mathcal{S}_2} q_i(t) \le$$
$$\sum_{\mathcal{S}_1} q_i(t-1) - \sum_{\mathcal{S}_2} q_i(t-1) + \sum_{\mathcal{S}_1} \alpha_i + \sum_{\mathcal{S}_1} \sigma_i^U + F(\mathcal{S}_2) + \theta_{\mathcal{S}_2}^U.$$

Proof. This is immediate from the burstiness constraints on the arrival and channel availability processes and equation (1). ∎

We use the term "a set $\mathcal{G}$ has priority at time t over a set $\mathcal{Q}$", if given that at time t policy π^* chooses for transmission one of the packets in $\mathcal{G} \cup \mathcal{Q}$, this packet must belong to $\mathcal{G}$ provided that the queues in $\mathcal{G}$ have at least one packet and the associated channel is "on".

In order to simplify notation in the proofs, in the following we will use the symbol $\mathbb{O}$, to denote a finite nonnegative quantity that depends only on the parameters of the arrival and channel availability processes. In particular $\mathbb{O}$ depends *neither* on the time t *nor* on the policy parameter T. As will be clear from the proofs, in principle $\mathbb{O}$ can be explicitly computed, e.g., in Lemma 1, $\mathbb{O} = \sum_{\mathcal{S}_1} \alpha_i + \sum_{\mathcal{S}_1} \sigma_i^U + F(\mathcal{S}_2) + \theta_{\mathcal{S}_2}^U$.

Lemmas 2 and 3 below are used to determine the range around kT in which each of the stable queues fluctuates. To elucidate the meaning of Lemma 2, consider some set $\mathcal{L}$ of queues. Note that the average number of slots available for transmission of packets from the queues in a set $\mathcal{S} \subseteq \mathcal{N} - \mathcal{L}$ is at least $F(\mathcal{L} \cup \mathcal{S}) - F(\mathcal{L})$ (with equality when the queues in $\mathcal{L}$ have always packets to transmit and the set $\mathcal{L}$ has priority over set $\mathcal{S}$). Assume now that for any subset $\mathcal{S} \subseteq \mathcal{D} = \mathcal{N} - \mathcal{L}$ it holds,

$$\sum_{\mathcal{S}} \alpha_i \le F(\mathcal{L} \cup \mathcal{S}) - F(\mathcal{L}).$$

These inequalities state that the packet arrival rate to any subset $\mathcal{S}$ of $\mathcal{D}$ is smaller that the average number of slots available for transmission of packets from $\mathcal{S}$. It is intuitively plausible and it can be shown that these inequalities are sufficient conditions for the queues in $\mathcal{D}$ to be bounded under the LCQ policy. In our case, however, the situation is more complicated since π^* does not always operate as the LCQ policy, and in general we will have $\mathcal{D} \subseteq \mathcal{N} - \mathcal{L}$, i.e., there may be other queues in $\mathcal{N} - \mathcal{L}$ competing with the queues in $\mathcal{D}$, which may have priority at certain times. It turns out, however, that for the queues in $\mathcal{D}$ to remain bounded it is sufficient to ensure that whenever the queues in a subset of $\mathcal{D}$ are above a certain threshold, they have priority over the queues in $\mathcal{N} - \mathcal{L}$ and are served according to LCQ policy. This is made precise in the following lemma.

Lemma 2 *Suppose that there are queue sets $\mathcal{L} \subset \mathcal{N}$, $\mathcal{D} \subseteq \mathcal{N} - \mathcal{L}$, such that the following inequalities hold for any $\mathcal{S} \subseteq \mathcal{D}$.*

$$\sum_{\mathcal{S}} \alpha_i \leq F(\mathcal{L} \cup \mathcal{S}) - F(\mathcal{L}). \tag{13}$$

Suppose further that there are numbers $H(T) \geq 0$, $\Phi(T) > 0$, with $\lim_{T\to\infty} \Phi(T) = \infty$, such that when π^* operates with parameter T, the following hold.

a) The set $\mathcal{G}(t) = \{i \in \mathcal{D} : q_i(t) > H(T)\}$, has priority at time t over the queues in $\mathcal{N} - \mathcal{L}$.

b) If $\max_{i\in\mathcal{D}} \{q_i(t)\} \leq H(T) + \Phi(T)$, the queues in the set $\mathcal{G}(t)$ are served according to LCQ policy.

Then, there is a number $\mathbb{O}$ such that, if $\Phi(T) \geq \mathbb{O}$ and $\max_{i\in\mathcal{D}} \{q_i(0)\} \leq H(T)$, it holds

$$\max_{i\in\mathcal{D}} \{q_i(t)\} \leq H(T) + \mathbb{O}, \text{ for all } t \geq 0. \tag{14}$$

Proof. The proof is given in Appendix A.2. ■

The next lemma provides conditions under which it is known that the queue sizes of a set of queues do not fall below certain threshold after some time. We are essentially dealing with the inverse situation of Lemma 2. However, we need different arguments mainly because we can claim the truth of the lemma only after some time large enough to remove the effect of initial conditions.

Lemma 3 *Suppose that there are queue sets $\mathcal{L} \subset \mathcal{N}$, $\mathcal{D} \subseteq \mathcal{N} - \mathcal{L}$, such that the following inequalities hold for all $\mathcal{S} \subseteq \mathcal{D}$, $\mathcal{S} \neq \varnothing$.*

$$\sum_{\mathcal{S}} \alpha_i > F(\mathcal{L} \cup \mathcal{D}) - F(\mathcal{L} \cup \overline{\mathcal{S}}), \tag{15}$$

where $\overline{\mathcal{S}} = \mathcal{D} - \mathcal{S}$. Suppose further that there is a number $H(T) \geq 0$, such that when π^* operates with parameter T, the following hold.

a) The queues in $\mathcal{L}$ always have packets to transmit and have higher priority than the queues in $\mathcal{G}(t) = \{i \in \mathcal{D} : q_i(t) < H(T)\}$.

b) The queues in the set $\mathcal{G}(t)$ are served according to LCQ policy and have lower priority than the queues in $\mathcal{D} - \mathcal{G}(t)$.

Then there it a time τ_0 such that $\min_{i\in\mathcal{D}} \{q_i(t)\} \geq H(T) - \mathbb{O}$, for all $t \geq \tau_0$.

Proof. The proof is given in Appendix A.3. ■

Next we need to examine in more detail the structure of the optimal linear programming solution (12). According to (12), x_k^* may take values less than or equal to α_k. An index

Index	1	2	3	4	5	6	7	8	9	10
Condition	S	S	U	U	U	S	S	S	S	U

$\underbrace{1\;2}_{\mathcal{I}_1^s}\;\underbrace{3\;4\;5}_{\mathcal{I}_1^u}\;\underbrace{6\;7\;8\;9}_{\mathcal{I}_2^s}\;\underbrace{10}_{\mathcal{I}_2^u}$

Figure 4: Partitioning of the index set.

such that $x_k^* = \alpha_k$ is called "stable", while an index such that $x_k^* < \alpha_k$, "unstable". We therefore have that for a stable index k and for any set $\mathcal{D} \subseteq \mathcal{N}_{k-1}$,

$$\alpha_k \leq F(k \cup \mathcal{D}) - \sum_{\mathcal{D}} x_i^*. \tag{16}$$

Similarly, for an unstable index k and for any set $\mathcal{D}_k \subseteq \mathcal{N}_{k-1}$ such that

$$x_k^* = F(k \cup \mathcal{D}_k) - \sum_{\mathcal{D}_k} x_i^*,$$

it holds

$$F(k \cup \mathcal{D}_k) - \sum_{\mathcal{D}_k} x_i^* < \alpha_k. \tag{17}$$

The general structure of the vector $\{x_k^*\}_{k\in\mathcal{N}}$ is as follows. The set of indices is partitioned into index sets $\mathcal{I}_i^s$, $i = 1, ..., l_s$, $\mathcal{I}_i^u$, $i = 1, ..., l_u$ such that

- Indices in the set $\cup_{i=1}^{l_s} \mathcal{I}_i^s$ are stable. Indices in the set $\cup_{i=1}^{l_u} \mathcal{I}_i^u$ are unstable.
- Index set $\mathcal{I}_i^x$, $x \in \{s, u\}$ consists of successive integers.
- If $i > j$ then all indices in $\mathcal{I}_i^x$, $x \in \{s, u\}$ are larger than the indices in $\mathcal{I}_j^x$. Figure 4 shows an example of the partition of the index set for $N = 10$ channels. For convenience in the discussion we assume that for a given i, the indices in $\mathcal{I}_i^s$ are smaller than the indices in $\mathcal{I}_i^u$. Hence, for consistency, if index 1 is unstable, we define $\mathcal{I}_1^s = \varnothing$.

Denote by $u_1 < u_2 < ... < u_L$, $L = \left|\cup_{i=1}^{l_u} \mathcal{I}_i^u\right|$ the unstable indices. The following lemma describes some useful properties of stable and unstable indices, that are simple consequences of the definitions.

Lemma 4 *Consider the vector* $\{x_k^*\}_{k\in \mathcal{N}}$ *defined for* $k = 1, ..., N$, *by the recursion*

$$x_k^* = \min\left\{\alpha_k,\ \min_{\mathcal{D}\subseteq \mathcal{N}_{k-1}} \left\{F\left(k\cup\mathcal{D}\right) - \sum_{\mathcal{D}} x_i^*\right\}\right\}.$$

a) For any set $\mathcal{S} \subseteq \mathcal{I}_1^s$ *it holds,*

$$\sum_{\mathcal{S}} \alpha_i \leq F\left(\mathcal{S}\right).$$

b) For index $k \in \mathcal{I}_j^u$, $j = 1, ..., l_u$, *there is at least one index set* $\mathcal{D}_k$ *such that* $\mathcal{D}_k \subseteq \mathcal{N}_{k-1}$,

$$\sum_{k\cup\mathcal{D}_k} x_i^* = F\left(k\cup\mathcal{D}_k\right),$$

and for all $\mathcal{D} \subseteq \mathcal{N}_{k-1}$,

$$F\left(k\cup\mathcal{D}_k\right) - \sum_{\mathcal{D}_k} x_i^* \leq F\left(k\cup\mathcal{D}\right) - \sum_{\mathcal{D}} x_i^*.$$

c) If $k \in \mathcal{I}_j^u$ *is an unstable index, then for any set* $\mathcal{S} \subseteq \cup_{i=1}^{l_s}\mathcal{I}_i^s - \mathcal{D}_k$, *it holds*

$$\sum_{\mathcal{S}} \alpha_i \leq F\left(k\cup\mathcal{D}_k\cup\mathcal{S}\right) - F\left(k\cup\mathcal{D}_k\right).$$

Proof. a) Let $\mathcal{S} \subseteq \mathcal{I}_1^s$ and let k be the largest index in $\mathcal{S}$. Then, since $\mathcal{S} - \{k\} \in \mathcal{N}_{k-1}$, we have from (16) that

$$\alpha_k \leq F\left(k\cup(\mathcal{S}-k)\right) - \sum_{\mathcal{S}-k}\alpha_i$$

or,

$$\sum_{\mathcal{S}}\alpha_i \leq F\left(\mathcal{S}\right). \tag{18}$$

b) This follows directly from the definition of an unstable index.

c) Let l be the largest index in $\mathcal{S}$. If $l \in \cup_{i=1}^{j} I_i^s$, then $\mathcal{S} \subseteq \cup_{i=1}^{j}\mathcal{I}_i^s - \mathcal{D}_k \subseteq \mathcal{N}_{k-1}$. By part b),

$$F(k\cup\mathcal{D}_k) - \sum_{\mathcal{D}_k} x_j^* \leq F(k\cup\mathcal{D}_k\cup\mathcal{S}) - \sum_{\mathcal{D}_k\cup\mathcal{S}} x_j^*,$$

hence, taking into account that $\mathcal{S}\cap\mathcal{D}_k = \varnothing$ and $x_i^* = a_i$ for $i \in \mathcal{S}$, we have,

$$\sum_{\mathcal{S}} a_i^* \leq F(k\cup\mathcal{D}_k\cup\mathcal{S}) - F(k\cup\mathcal{D}_k).$$

Assume next that $l \in \cup_{i=j+1}^{l_s}\mathcal{I}_i^s$. Since $k\cup\mathcal{D}_k\cup(\mathcal{S}-l) \subseteq \mathcal{N}_{l-1}$, and $(k\cup\mathcal{D}_k)\cap(\mathcal{S}-l) = \varnothing$, we have from (16),

$$\begin{aligned}\alpha_l &\leq F(l\cup k\cup\mathcal{D}_k\cup(\mathcal{S}-l)) - \sum_{k\cup\mathcal{D}_k} x_i^* - \sum_{\mathcal{S}-l}\alpha_i \\ &= F(k\cup\mathcal{D}_k\cup\mathcal{S}) - F(k\cup\mathcal{D}_k) - \sum_{\mathcal{S}-l}\alpha_j,\end{aligned}$$

where we used the fact that by part b) of the lemma,

$$\sum_{k \cup \mathcal{D}_k} x_i^* = F(k \cup \mathcal{D}_k).$$

Therefore,

$$\sum_{\mathcal{S}} \alpha_j \leq F(k \cup \mathcal{D}_k \cup \mathcal{S}) - F(k \cup \mathcal{D}_k).$$

■

For an unstable index $k \in \mathcal{I}_j^u$ define by $\mathcal{P}_k$ the class of index sets that satisfy part b) of Lemma 4. In the next lemma, part a) essentially identifies stable indices in $\mathcal{I}_1^s$ whose corresponding queues, as will be be shown in conjunction with Lemma 3, stay above a given threshold after some time. Part b), is used to derive further lemmas that permit to extend this identification to indices in $\mathcal{I}_i^s$, $i \geq 2$.

Lemma 5 *a) For index u_1, there is a unique index set $\widehat{\mathcal{D}}_{u_1} \in \mathcal{P}_{u_1}$ such that for all $\overline{\mathcal{D}} \subset \widehat{\mathcal{D}}_{u_1}$ it holds*

$$x_{u_1}^* = F\left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right) - \sum_{\widehat{\mathcal{D}}_{u_1}} x_i^* < F\left(u_1 \cup \overline{\mathcal{D}}\right) - \sum_{\overline{\mathcal{D}}} x_i^*. \tag{19}$$

or, with $\mathcal{D} = \widehat{\mathcal{D}}_{u_1} - \overline{\mathcal{D}} \neq \varnothing$,

$$\sum_{\mathcal{D}} a_i > F\left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right) - F\left(u_1 \cup \overline{\mathcal{D}}\right). \tag{20}$$

b) For every index u_j, $j \geq 2$ there is an index set $\widetilde{\mathcal{D}}_{u_j} \in \mathcal{P}_{u_j}$ such that $u_1 \cup \widehat{\mathcal{D}}_{u_1} \subseteq \widetilde{\mathcal{D}}_{u_j}$.

Proof. a) Inequality (20) is immediate from (19) by observing that $\mathcal{D} \subseteq I_1^s$ and therefore, $x_i^* = a_i$, $i \in \mathcal{D}$. To prove (19), we will show that if $\mathcal{D}_1$, $\mathcal{D}_2$ belong to $\mathcal{P}_{u_1}$, then so does $\mathcal{D}_1 \cap \mathcal{D}_2$. This implies that the set

$$\widehat{\mathcal{D}}_{u_1} = \cap_{\mathcal{D} \in \mathcal{P}_{u_1}} \mathcal{D},$$

is the only one in $\mathcal{P}_{u_1}$ satisfying the required property.

According to the definition, it holds for any $\mathcal{D} \subseteq \mathcal{N}_{u_1 - 1}$

$$x_{u_1}^* = F\left(u_1 \cup \mathcal{D}_1\right) - \sum_{\mathcal{D}_1} x_i^* \leq F\left(u_1 \cup \mathcal{D}\right) - \sum_{\mathcal{D}} x_i^*, \tag{21}$$

$$x_{u_1}^* = F\left(u_1 \cup \mathcal{D}_2\right) - \sum_{\mathcal{D}_2} x_i^* \leq F\left(u_1 \cup \mathcal{D}\right) - \sum_{\mathcal{D}} x_i^*. \tag{22}$$

Replacing $\mathcal{D}$ by $\mathcal{D}_1 \cup \mathcal{D}_2$ in (21) and by $\mathcal{D}_1 \cap \mathcal{D}_2$ in (22), we have,

$$F\left(u_1 \cup \mathcal{D}_1\right) - \sum_{\mathcal{D}_1} x_i^* \leq F(u_1 \cup \mathcal{D}_1 \cup \mathcal{D}_2) - \sum_{\mathcal{D}_1 \cup \mathcal{D}_2} x_i^*,$$
$$F\left(u_1 \cup \mathcal{D}_2\right) - \sum_{\mathcal{D}_2} x_i^* \leq F(u_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2)) - \sum_{\mathcal{D}_1 \cap \mathcal{D}_2} x_i^*.$$

Adding these two inequalities and observing that

$$\sum_{\mathcal{D}_1} x_i^* + \sum_{\mathcal{D}_2} x_i^* = \sum_{\mathcal{D}_1\cup\mathcal{D}_2} x_i^* + \sum_{\mathcal{D}_1\cap\mathcal{D}_2} x_i^*,$$

we obtain

$$F(u_1 \cup \mathcal{D}_1) + F(u_1 \cup \mathcal{D}_2) \le F(u_1 \cup \mathcal{D}_1 \cup \mathcal{D}_2) + F(u_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2)),$$

which by the submodularity property implies that,

$$F(u_1 \cup \mathcal{D}_1) + F(u_1 \cup \mathcal{D}_2) = F(u_1 \cup \mathcal{D}_1 \cup \mathcal{D}_2) + F(u_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2)). \tag{23}$$

Replacing $\mathcal{D}$ by $\mathcal{D}_1 \cup \mathcal{D}_2$ both in (21) and (22) we have

$$F(u_1 \cup \mathcal{D}_1) - \sum_{\mathcal{D}_1} x_i^* + F(u_1 \cup \mathcal{D}_2) - \sum_{\mathcal{D}_2} x_i^* \le 2\left(F(u_1 \cup \mathcal{D}_1 \cup \mathcal{D}_2) - \sum_{\mathcal{D}_1\cup\mathcal{D}_2} x_i^*\right),$$

or,

$$F(u_1 \cup \mathcal{D}_1) + F(u_1 \cup \mathcal{D}_2) - \sum_{\mathcal{D}_1\cup\mathcal{D}_2} x_i^* - \sum_{\mathcal{D}_1\cap\mathcal{D}_2} x_i^* \le 2\left(F(u_1 \cup \mathcal{D}_1 \cup \mathcal{D}_2) - \sum_{\mathcal{D}_1\cup\mathcal{D}_2} x_i^*\right).$$

Using (23) we conclude

$$F(u_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2)) - \sum_{\mathcal{D}_1\cap\mathcal{D}_2} x_i^* \le F(u_1 \cup \mathcal{D}_1 \cup \mathcal{D}_2) - \sum_{\mathcal{D}_1\cup\mathcal{D}_2} x_i^*.$$

A similar argument shows that

$$F(u_1 \cup \mathcal{D}_1 \cup \mathcal{D}_2) - \sum_{\mathcal{D}_1\cup\mathcal{D}_2} x_i^* \le F(u_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2)) - \sum_{\mathcal{D}_1\cap\mathcal{D}_2} x_i^*,$$

and therefore,

$$F(u_1 \cup \mathcal{D}_1 \cup \mathcal{D}_2) - \sum_{\mathcal{D}_1\cup\mathcal{D}_2} x_i = F(u_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2)) - \sum_{\mathcal{D}_1\cap\mathcal{D}_2} x_i^*.$$

Finally,

$$\begin{aligned} 2x_{u_1}^* &= F(u_1 \cup \mathcal{D}_1) - \sum_{\mathcal{D}_1} x_i + F(u_1 \cup \mathcal{D}_2) - \sum_{\mathcal{D}_2} x_i^* \\ &= F(u_1 \cup \mathcal{D}_1 \cup \mathcal{D}_2) - \sum_{\mathcal{D}_1\cup\mathcal{D}_2} x_i^* + F(u_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2)) - \sum_{\mathcal{D}_1\cap\mathcal{D}_2} x_i^* \\ &= 2\left(F(u_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2)) - \sum_{\mathcal{D}_1\cap\mathcal{D}_2} x_i^*\right), \end{aligned}$$

where the second equality follows from (23). Therefore $\mathcal{D}_1 \cap \mathcal{D}_2 \in \mathcal{P}_{u_1}$.

b) Let $\mathcal{D} \in \mathcal{P}_{u_j}$, $j \geq 2$. Since $\mathcal{D} \cup u_1 \cup \widehat{\mathcal{D}}_{u_1} \subseteq \mathcal{N}_{u_j-1}$, it holds,

$$
\begin{aligned}
F\left(u_j \cup \mathcal{D}\right) - \sum_{\mathcal{D}} x_i^* &\leq F\left(u_j \cup \mathcal{D} \cup u_1 \cup \widehat{\mathcal{D}}_{u_1}\right) - \sum_{\mathcal{D} \cup u_1 \cup \widehat{\mathcal{D}}_{u_1}} x_i^* \qquad (24)\\
&= F\left(u_j \cup \mathcal{D} \cup u_1 \cup \widehat{\mathcal{D}}_{u_1}\right) - \sum_{u_1 \cup \widehat{\mathcal{D}}_{u_1}} x_i^* - \sum_{\mathcal{D}-(u_1 \cup \widehat{\mathcal{D}}_{u_1})} x_i^* \\
&= F\left(u_j \cup \mathcal{D} \cup u_1 \cup \widehat{\mathcal{D}}_{u_1}\right) - F(u_1 \cup \widehat{\mathcal{D}}_{u_1}) - \sum_{\mathcal{D}-(u_1 \cup \widehat{\mathcal{D}}_{u_1})} x_i^*.
\end{aligned}
$$

Taking into account that $\sum_{\mathcal{D}} x_i^* - \sum_{\mathcal{D}-(u_1 \cup \widehat{\mathcal{D}}_{u_1})} x_i^* = \sum_{\mathcal{D} \cap (u_1 \cup \widehat{\mathcal{D}}_{u_1})} x_i^*$, we have from the above inequality,

$$
\begin{aligned}
&\sum_{\mathcal{D} \cap (u_1 \cup \widehat{\mathcal{D}}_{u_1})} x_i^* \geq F(u_j \cup \mathcal{D}) + F(u_1 \cup \widehat{\mathcal{D}}_{u_1}) - F(u_j \cup \mathcal{D} \cup u_1 \cup \widehat{\mathcal{D}}_{u_1}) \\
&\geq F\left((u_j \cup \mathcal{D}) \cap \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)\right) \qquad (25)\\
&= F\left(\mathcal{D} \cap \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)\right), \qquad (26)
\end{aligned}
$$

where inequality (25) follows from the submodularity property and equality (26) from the fact that since $j > 1$, $u_j \cap \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right) = \varnothing$. Hence

$$
\sum_{\mathcal{D} \cap (u_1 \cup \widehat{\mathcal{D}}_{u_1})} x_i^* \geq F\left(\mathcal{D} \cap \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)\right). \qquad (27)
$$

Since $\{x_i^*\}_{i=1}^N$ is a feasible point of the linear optimization problem, it satisfies equation (11a), which for $\mathcal{S} = \mathcal{D} \cap (u_1 \cup \widehat{\mathcal{D}}_{u_1}) \subseteq \mathcal{N}$ gives

$$
\sum_{\mathcal{D} \cap (u_1 \cup \widehat{\mathcal{D}}_{u_1})} x_i^* \leq F\left(\mathcal{D} \cap \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)\right).
$$

Form the last inequality and (27) we obtain

$$
\sum_{\mathcal{D} \cap (u_1 \cup \widehat{\mathcal{D}}_{u_1})} x_i^* = F\left(\mathcal{D} \cap \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)\right).
$$

Observe that because of (24), this equality is true only if

$$
F\left(u_j \cup \mathcal{D}\right) - \sum_{\mathcal{D}} x_i^* = F\left(u_j \cup \mathcal{D} \cup u_1 \cup \widehat{\mathcal{D}}_{u_1}\right) - \sum_{\mathcal{D} \cup u_1 \cup \widehat{\mathcal{D}}_{u_1}} x_i^*,
$$

i.e., only if $\mathcal{D} \cup u_1 \cup \widehat{\mathcal{D}}_{u_1} \in \mathcal{P}_{u_j}$ and the lemma follows by setting $\widetilde{\mathcal{D}}_{u_j} = \mathcal{D} \cup u_1 \cup \widehat{\mathcal{D}}_{u_1}$. ■

Lemma 6 *The indices in the set $\mathcal{N} - \left(\widehat{\mathcal{D}}_{u_1} \cup u_1\right)$ satisfy the recursion*

$$
x_k^* = \min\left\{\alpha_k, \min_{\mathcal{D} \subseteq \mathcal{N}_{k-1} - \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)} \left\{F_1\left(k \cup \mathcal{D}\right) - \sum_{\mathcal{D}} x_i^*\right\}\right\},
$$

where

$$F_1\left(\mathcal{D}\right) = F\left(u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup\ \mathcal{D}\right) - F(u_1 \cup \widehat{\mathcal{D}}_{u_1}).$$

Proof. Consider an unstable index u_j, $j \geq 2$. According to part b) of Lemma 5 and part b) of lemma 4, we have for any $\mathcal{S} \subseteq \mathcal{N}_{u_j-1}$

$$x^*_{u_j} = F(u_j \cup \widetilde{\mathcal{D}}_{u_j}) - \sum_{\widetilde{\mathcal{D}}_{u_j}} x^*_i \leq F(u_j \cup \mathcal{S}) - \sum_{\mathcal{S}} x^*_i,$$

where $\widetilde{\mathcal{D}}_{u_j} = u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}_{u_j}$, $\mathcal{D}_{u_j} \subseteq \mathcal{N}_{u_j-1} - \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)$. Consider now any set $\mathcal{D} \subseteq \mathcal{N}_{u_j-1} - \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)$. Setting $\mathcal{S} = \mathcal{D} \cup \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)$ in the inequality above, we obtain,

$$x^*_{u_j} = F(u_j \cup u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}_{u_j}) - \sum_{u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}_{u_j}} x^*_i \leq F(u_j \cup u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}) - \sum_{u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}} x^*_i. \quad (28)$$

Since $\left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right) \cap \mathcal{D}_{u_j} = \varnothing$ we have

$$\sum_{u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}_{u_j}} x^*_i = \sum_{u_1 \cup \widehat{\mathcal{D}}_{u_1}} x^*_i + \sum_{\mathcal{D}_{u_j}} x^*_i = F(u_1 \cup \widehat{\mathcal{D}}_{u_1}) + \sum_{\mathcal{D}_{u_j}} x^*_i$$

Similarly,

$$\sum_{u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}} x^*_i = F(u_1 \cup \widehat{\mathcal{D}}_{u_1}) + \sum_{\mathcal{D}} x^*_i.$$

Replacing in (28) we obtain

$$x^*_{u_j} = F_1(u_j \cup \mathcal{D}_{u_j}) - \sum_{\mathcal{D}_{u_j}} x^*_i \leq F_1(u_j \cup \mathcal{D}) - \sum_{\mathcal{D}} x^*_i,$$

which together with the fact that $x^*_{u_j} < a_{u_j}$ implies the lemma for $k = u_j$.

Next, let $k \in \mathcal{I}^s_1 - \left(\widehat{\mathcal{D}}_{u_1} \cup u_1\right)$ and $\mathcal{D} \subseteq \mathcal{N}_{k-1} - \left(\widehat{\mathcal{D}}_{u_1} \cup u_1\right)$. Since $k \cup \mathcal{D} \subseteq \mathcal{I}^s_1 - \widehat{\mathcal{D}}_{u_1}$, we have by replacing in part c) of Lemma 4, $\mathcal{S}$ with $k \cup \mathcal{D}$ and k with u_1,

$$\sum_{k \cup \mathcal{D}} a_i \leq F(u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup k \cup \mathcal{D}) - F(u_1 \cup \widehat{\mathcal{D}}_{u_1}),$$

or,

$$a_k \leq F_1(\mathcal{D}) - \sum_{\mathcal{D}} x^*_i,$$

which implies the lemma for $k \in \mathcal{I}^s_1 - \left(\widehat{\mathcal{D}}_{u_1} \cup u_1\right)$.

Finally, assume that $k \in \cup_{j=2}^{l_s} \mathcal{I}^s_j$ and let $\mathcal{D} \subseteq \mathcal{N}_{k-1} - \left(\widehat{\mathcal{D}}_{u_1} \cup u_1\right)$. Since in this case

$$u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D} \subseteq \mathcal{N}_{k-1},$$

we have by the definition of x_k^* and taking into account that $\left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right) \cap \mathcal{D} = \varnothing$,

$$\begin{aligned}
x_k^* = a_k &\leq F(k \cup u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}) - \sum_{u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}} x_i^* \\
&= F(k \cup u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}) - \sum_{u_1 \cup \widehat{\mathcal{D}}_{u_1}} x_i^* - \sum_{\mathcal{D}} x_i^* \\
&= F(k \cup u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{D}) - F(u_1 \cup \widehat{\mathcal{D}}_{u_1}) - \sum_{\mathcal{D}} x_i^* \\
&= F_1(k \cup \mathcal{D}) - \sum_{\mathcal{D}} x_i^*,
\end{aligned}$$

which again implies the lemma. ■

Based on Lemma 6 we can now apply Lemma 5 with index set $\mathcal{N} - \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)$, to conclude the existence of a unique set $\widehat{\mathcal{D}}_{u_2} \subseteq \mathcal{N}_{u_2-1} - \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)$ such that for all $\mathcal{D} \subseteq \mathcal{N}_{u_2-1} - \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)$

$$x_{u_2}^* = F_1\left(u_2 \cup \widehat{\mathcal{D}}_{u_2}\right) - \sum_{\widehat{\mathcal{D}}_{u_2}} x_i^* \leq F_1\left(u_2 \cup \mathcal{D}\right) - \sum_{\mathcal{D}} x_i^*,$$

and for all $\overline{\mathcal{D}} \subset \widehat{\mathcal{D}}_{u_2}$,

$$x_{u_2}^* < F_1\left(u_2 \cup \overline{\mathcal{D}}\right) - \sum_{\overline{\mathcal{D}}} x_i^*.$$

In a similar manner, applying repeatedly Lemmas 6 and 5 we define the sets $\widehat{\mathcal{D}}_{u_j}$, $j = 1, ..., L$.

The next theorem provides the range within which the queues in $\mathcal{N}$ fluctuate.

Theorem 7 *Under policy π^*, if $q_i(0) = 0$, $i \in \mathcal{N}$, then for T large enough, the queues in $\cup_{j=1}^{l_u} \mathcal{I}_j^u$ tend to infinity. Moreover, there is a number $M = \mathbb{O}$ such that for $1 \leq j \leq L$ it holds*

$$\begin{aligned}
&\max_{i \in \widehat{\mathcal{D}}_{u_j}} \{q_i(t)\} \leq (N + 1 - u_j)T + M \textit{ for all } t \geq 0, \\
&\min_{i \in \widehat{\mathcal{D}}_{u_j}} \{q_i(t)\} \geq (N + 1 - u_j)T - M \textit{ for all } t \geq \tau_0,
\end{aligned}$$

and

$$\max_{i \in \cup_{j=1}^{l_s} \mathcal{I}_j^s - \cup_{j=1}^{L} \widehat{\mathcal{D}}_{u_j}} q_i(t) \leq M \quad \textit{for all } t.$$

It suffices to take $T > 2M$.

In the course of the proof of Theorem 7 we also prove the main result of this paper, that is,

Theorem 8 *The throughputs achieved by policy π^* for T large enough, satisfy the recursive equations*

$$x_k^* = \min\left\{\alpha_k, \min_{\mathcal{D}\subseteq\mathcal{N}_{k-1}}\left\{F\left(k\cup\mathcal{D}\right) - \sum_{\mathcal{D}} x_i^*\right\}\right\},$$

for $k = 1, ..., N$.

Proof of Theorem 7. From part a) of Lemma 4 we have that (13) holds in Lemma 2. Set now, $\mathcal{D} = \mathcal{I}_1^s$, $H\left(T\right) = H = (N + 1 - u_1)T$, $\Phi\left(T\right) = \Phi = T$, and $\mathcal{L} = \varnothing$. Conditions a), b), of the lemma 2 hold because of the definition of policy π^*. Indeed, for $i \in \mathcal{G}(t)$, $I_i(\ q_i(t)) > H$ while the queues in $\mathcal{N} - \mathcal{G}(t)$ have index at most H, so that condition a) is satisfied. Moreover, for $i \in \mathcal{G}(t)$, $I_i\left(q_i(t)\right) = q_i(t)$ whenever $q_i(t) \leq H + T$ and hence condition b) also holds.

Since $q_i(0) = 0 \leq H$, $i \in \mathcal{N}$, we conclude that for all $t \geq 0$,

$$\max_{i\in\mathcal{I}_1^s}\left\{q_i(t)\right\} \leq (N + 1 - u_1)T + \mathbb{O} \tag{29}$$

provided that $T \geq \mathbb{O}$.

Since $B_i(0, t) = A_i(0, t) - q_i(t)$, from (29) we conclude that for $i \in \mathcal{I}_1^s$,

$$r_i = \lim_{t\to\infty}\frac{B_i(0, t)}{t} = \lim_{t\to\infty}\frac{A_i(0, t)}{t} = \alpha_i = x_i^*. \tag{30}$$

Since $\widehat{\mathcal{D}}_{u_1} \subseteq \mathcal{I}_1^s$ we have from (17) and the definition of $\mathcal{I}_1^s$ that

$$F(u_1 \cup \widehat{\mathcal{D}}_{u_1}) - \sum_{\widehat{\mathcal{D}}_{u_1}}\alpha_i < \alpha_{u_1}. \tag{31}$$

Observing that

$$\begin{aligned}\sum_{u_1\cup\widehat{\mathcal{D}}_{u_1}} q_j(t) &= A_{u_1\cup\widehat{\mathcal{D}}_{u_1}}(0, t) - B_{u_1\cup\widehat{\mathcal{D}}_{u_1}}(0, t)\\ &\geq A_{u_1\cup\widehat{\mathcal{D}}_{u_1}}(0, t) - C_{u_1\cup\widehat{\mathcal{D}}_{u_1}}(0, t)\\ &\geq \left(\sum_{u_1\cup\widehat{\mathcal{D}}_{u_1}}\alpha_i - F\left(u_1\cup\widehat{\mathcal{D}}_{u_1}\right)\right)t - \mathbb{O},\end{aligned}$$

and taking into account (31) we conclude that

$$\lim_{t\to\infty}\sum_{u_1\cup\widehat{\mathcal{D}}_{u_1}} q_j(t) = \infty.$$

Since by (29) $q_i(t)$ is finite for $i \in \widehat{\mathcal{D}}_{u_1} \subseteq \mathcal{I}_1^s$, we conclude that $\lim_{t\to\infty} q_{u_1}(t) = \infty$.

Consider now Lemma 3 with $\mathcal{L} = u_1$, $\mathcal{D} = \widehat{\mathcal{D}}_{u_1}$ and $H(T) = H = (N + 1 - u_1)T$. Inequalities (15) hold since according to (20) in Lemma 5 for any nonempty subset $\mathcal{S} \subseteq \widehat{\mathcal{D}}_{u_1}$ we have

$$\sum_{\mathcal{S}}\alpha_i > F\left(u_1\cup\widehat{\mathcal{D}}_{u_1}\right) - F\left(u_1\cup\left(\widehat{\mathcal{D}}_{u_1} - \mathcal{S}\right)\right).$$

Notice also that since $\widehat{\mathcal{D}}_{u_1} \subseteq \mathcal{I}_1^s$, by the definition of the queue indices $I_i(q)$, the queues in $\widehat{\mathcal{D}}_{u_1}$ are served according to the LCQ policy when they are smaller than $H = (N+1-u_1)T$ and the queues in $\mathcal{G}(t) = \left\{i \in \widehat{\mathcal{D}}_{u_1} \ : q_i(t) < H\right\}$ have lower priority than the queues in $\widehat{\mathcal{D}}_{u_1} - \mathcal{G}(t)$, hence condition $b)$ of Lemma 3 also holds. Moreover, since $\lim_{t\to\infty} q_{u_1}(t) = \infty$, it holds that $I_{u_1}(q_{u_1}(t)) = H$, for t larger than or equal to some time t_0. Hence, for $t \geq t_0$ queue u_1 has priority over the queues in $\mathcal{S} \subseteq \widehat{\mathcal{D}}_{u_1}$ whenever $\max_{i\in\mathcal{S}} \{q_i(t)\} \leq H$. Therefore, condition a) holds as well. We conclude that there is some time $\tau_0^1 \geq t_0$ such that for $i \in \widehat{\mathcal{D}}_{u_1}$, and for all $t \geq \tau_0^1$, it holds

$$\min_{i\in\widehat{\mathcal{D}}_{u_1}} \{q_i(t)\} \geq (N+1-u_1)T - \mathbb{O}. \tag{32}$$

From Lemma 4, c) it follows that for any subset $\mathcal{S} \subseteq \mathcal{E}_1 = \cup_{j=1}^{l_s} \mathcal{I}_j^s - \widehat{\mathcal{D}}_{u_1}$ it holds

$$\sum_{\mathcal{S}} \alpha_i \leq F\left(u_1 \cup \widehat{\mathcal{D}}_{u_1} \cup \mathcal{S}\right) - F\left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right).$$

Applying now Lemma 2 with $\mathcal{D} = \mathcal{E}_1$, $H = (N+1-u_2)T$, $\Phi = T$ and $\mathcal{L} = u_1 \cup \widehat{\mathcal{D}}_{u_1}$, we conclude that for all $t \geq 0$,

$$\max_{i\in\mathcal{E}_1} \{q_i(t)\} \leq (N+1-u_2)T + \mathbb{O}, \tag{33}$$

provided that $T \geq \mathbb{O}$. Pick now T large enough so that

$$T - \mathbb{O} > \mathbb{O}. \tag{34}$$

Then since $u_2 > u_1$, it holds

$$(N+1-u_1)\,T - \mathbb{O} > (N+1-u_2)\,T + \mathbb{O}. \tag{35}$$

Inequalities (32), (33) and (35) and the fact that $q_{u_1}(t) \to \infty$, (i.e., $I_{u_1}(q_{u_1}(t)) = (N+1-u_1)\,T$, for $t \geq \tau_0^1$) imply that the queues in $u_1 \cup \widehat{\mathcal{D}}_{u_1}$ have higher priority over the rest of the queues for $t \geq \tau_0^1$ and that they are nonempty. Therefore, the queues in $u_1 \cup \widehat{\mathcal{D}}_{u_1}$ use all the available channel slots and we have

$$B_{u_1\cup\widehat{\mathcal{D}}_{u_1}}(\tau_0^1, t) = C_{u_1\cup\widehat{\mathcal{D}}_{u_1}}(\tau_0^1, t).$$

Hence

$$\lim_{t\to\infty} \frac{B_{u_1\cup\widehat{\mathcal{D}}_{u_1}}(0,t)}{t} = \lim_{t\to\infty} \frac{C_{u_1\cup\widehat{\mathcal{D}}_{u_1}}(0,t)}{t}.$$

Taking into account (30) and the fact that

$$\lim_{t\to\infty} \frac{C_{u_1\cup\widehat{\mathcal{D}}_{u_1}}(0,t)}{t} = F\left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right),$$

we conclude that $\lim_{t\to\infty}(B_{u_1}(0,t)/t)$ exists and

$$r_{u_1} = \lim_{t\to\infty} \frac{B_{u_1}(0,t)}{t} = F\left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right) - \sum_{\widehat{\mathcal{D}}_{u_1}} \alpha_i,$$

that is, $r_{u_1} = x_{u_1}^*$.

For the indices in the set $\mathcal{N} - \left(u_1 \cup \widehat{\mathcal{D}}_{u_1}\right)$ and based on the definition of $\widehat{\mathcal{D}}_{u_j}$, we can use similar arguments to verify the rest of the claims. ■

5 Conclusions

In this paper we considered the problem of scheduling transmissions to multiple users over a wireless channel with time-varying connectivity. We presented a scheduling policy that maximizes the weighted sum of channel throughputs in a general setting where no assumptions on the arrival rates are imposed. Instrumental in the analysis was the adoption of a burstiness-constrained model for the description of the wireless channel. This model makes the rigorous description of the system dynamics possible, without relying on statistical assumptions.

The proposed optimal scheduling policy is fairly simple and the only parameter that needs to be determined is T. The analysis presented in this work applies to "on-off" channels models. A subject of further study is the extension of the analysis to include multi-rate channels and more general optimization functions. In addition the consideration of packet delays is a practical matter that needs to be addressed. Another issue for further study is the development of an adaptive control mechanism for determining the value of policy parameter T according to observed system performance. Finally a general topic, where intense research is devoted lately in the area of wireless communications, is that of exploring the interaction of scheduling policies with higher layer protocols. In this respect, an interesting subject of future work is to assess the interaction of the proposed policy with the congestion control mechanism of the TCP/IP protocol.

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A Appendix

A.1 Proof of the Solution to the Linear Optimization Problem (equation (12))

In this appendix we prove formula (12). Specifically, we prove the following theorem, where we denote $\min\{x, y\} = x \bigvee y$.

Theorem 9 *Let $w_1 \geq w_2 \geq ... \geq w_N \geq 0$. The point*

$$x_k^* = \alpha_k \bigvee \bigvee_{\mathcal{D} \subseteq \mathcal{N}_{k-1}} \left\{ F(k \cup \mathcal{D}) - \sum_{\mathcal{D}} x_i^* \right\},\ k \in \mathcal{N}, \tag{36}$$

maximizes $\sum_{i=1}^N w_i x_i$ subject to the constraints

$$\sum_{\mathcal{S}} x_i \leq F(\mathcal{S}),\ \mathcal{S} \subseteq \mathcal{N}, \tag{37}$$

$$x_i \leq \alpha_i,\ i = 1, ..., N, \tag{38}$$

$$x_i \geq 0,\ i = 1, ..., N. \tag{39}$$

We first need some auxiliary lemmas.

Lemma 10 *The point x^* is a feasible point, i.e., satisfies (37), (38), (39).*

Proof. Let $\mathcal{D} \subseteq \mathcal{N}$ and let n be the largest coordinate index in D. Then,

$$\sum_{\mathcal{D}} x_i^* = x_n^* + \sum_{\mathcal{D}-n} x_i^* \leq \bigvee_{\mathcal{G}\subseteq\mathcal{N}_{n-1}} \left\{ F\left(n \cup \mathcal{G}\right) - \sum_{\mathcal{G}} x_i^* \right\} + \sum_{\mathcal{D}-n} x_i^*$$
$$\leq F\left(n \cup \{\mathcal{D}-n\}\right) - \sum_{\mathcal{D}-n} x_i^* + \sum_{\mathcal{D}-n} x_i^* = F(\mathcal{D}).$$

In the second inequality above we used the fact that $\mathcal{D} - n \subseteq \mathcal{N}_{n-1}$. On the other hand, by definition we have $x_k^* \leq a_k$. Hence equations (37) and (38) are satisfied. To show (39), notice that since $\sum_{\mathcal{D}} x_i^* \leq F(\mathcal{D})$, we have

$$\begin{aligned} x_k^* &= \alpha_k \bigvee \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F\left(k \cup \mathcal{D}\right) - \sum_{\mathcal{D}} x_i^* \right\} \\ &\geq \alpha_k \bigvee \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F\left(k \cup \mathcal{D}\right) - F\left(\mathcal{D}\right) \right\} \geq 0, \end{aligned}$$

where the last inequality follows from (8b) and from the fact that $\alpha_k \geq 0$. ■

Lemma 11 *For every subset $\mathcal{G} \subseteq \mathcal{N} - \mathcal{N}_k$ it holds*

$$\bigvee_{\mathcal{D}\subseteq\mathcal{N}_k} \left\{ F\left(\mathcal{G} \cup \mathcal{D}\right) + \sum_{\mathcal{N}_k-\mathcal{D}} x_i^* \right\} \geq \bigvee_{\mathcal{D}\subseteq\mathcal{N}_k} \left\{ F\left(\mathcal{G} \cup \mathcal{D}\right) + \sum_{\mathcal{N}_k-\mathcal{D}} \alpha_i \right\}.$$

Proof. For $k = 1$ the lemma states that for $\mathcal{G} \subseteq \mathcal{N} - \mathcal{N}_1$,

$$\left(F\left(\mathcal{G}\right) + x_1^*\right) \bigvee F(\mathcal{G} \cup 1) \geq \left\{F\left(\mathcal{G}\right) + \alpha_1\right\} \bigvee F(\mathcal{G} \cup 1).$$

Indeed, since $x_1^* = a_1 \bigvee F\left(1\right)$ we have

$$F\left(\mathcal{G}\right) + x_1^* = \left\{F\left(\mathcal{G}\right) + \alpha_1\right\} \bigvee \left\{F(\mathcal{G}) + F\left(1\right)\right\} \geq \left\{F\left(\mathcal{G}\right) + \alpha_1\right\} \bigvee F(\mathcal{G} \cup 1),$$

where the last inequality follows from the fact that $F(\mathcal{G}) + F\left(1\right) \geq F(\mathcal{G} \cup 1)$ and $\mathcal{G} \cap 1 = \emptyset$.

Assume that it is true for $k - 1$. Let $\mathcal{G} \subseteq \mathcal{N} - \mathcal{N}_k$. Then by splitting the sets in $\mathcal{N}_k$ in those that contain k and those that do not, we have

$$\bigvee_{\mathcal{D}\subseteq\mathcal{N}_k} \left\{ F\left(\mathcal{G} \cup \mathcal{D}\right) + \sum_{\mathcal{N}_k-\mathcal{D}} x_i^* \right\} =$$
$$\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F\left(\mathcal{G} \cup k \cup \mathcal{D}\right) + \sum_{\mathcal{N}_{k-1}-\mathcal{D}} x_i^* \right\} \bigvee \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F\left(\mathcal{G} \cup \mathcal{D}\right) + x_k^* + \sum_{\mathcal{N}_{k-1}-\mathcal{D}} x_i^* \right\}.$$

Now, by the induction hypothesis, and because $\mathcal{G} \cup k \subseteq \mathcal{N} - \mathcal{N}_{k-1}$, it holds,

$$\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F\left(\mathcal{G} \cup k \cup \mathcal{D}\right) + \sum_{\mathcal{N}_{k-1}-\mathcal{D}} x_i^* \right\} \geq \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F\left(\mathcal{G} \cup k \cup \mathcal{D}\right) + \sum_{\mathcal{N}_{k-1}-\mathcal{D}} \alpha_i \right\}.$$

Next,

$$F(\mathcal{G}\cup\mathcal{D})+x_k^*+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*=$$

$$F(\mathcal{G}\cup\mathcal{D})+a_k\bigvee\left\{\bigvee_{\mathcal{Q}\subseteq\mathcal{N}_{k-1}}\left\{F(k\cup\mathcal{Q})-\sum_{\mathcal{Q}}x_i^*\right\}\right\}+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*$$

$$=\left(a_k+F(\mathcal{G}\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*\right)\bigvee$$

$$\bigvee_{\mathcal{Q}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup\mathcal{D})+F(k\cup\mathcal{Q})-\sum_{\mathcal{Q}}x_i^*+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*\right\}.$$

Using (8c), we can write for $\mathcal{Q}$, $\mathcal{D}$, subsets of $\mathcal{N}_{k-1}$

$$F(\mathcal{G}\cup\mathcal{D})+F(k\cup\mathcal{Q})-\sum_{\mathcal{Q}}x_i^*+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*$$

$$\geq F(\mathcal{G}\cup k\cup\mathcal{Q}\cup\mathcal{D})+F(\mathcal{Q}\cap\mathcal{D})-\sum_{\mathcal{Q}\cap D}x_i^*+\sum_{\mathcal{N}_{k-1}-(\mathcal{D}\cup\mathcal{Q})}x_i^*$$

$$\geq F(\mathcal{G}\cup k\cup\mathcal{Q}\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-(\mathcal{D}\cup\mathcal{Q})}x_i^*,$$

where the last inequality follows from the fact that $F(\mathcal{Q}\cap\mathcal{D})\geq\sum_{\mathcal{Q}\cap\mathcal{D}}x_i^*$.

Hence,

$$F(\mathcal{G}\cup\mathcal{D})+x_k^*+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*$$

$$\geq\left(\alpha_k+F(\mathcal{G}\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*\right)\bigvee\bigvee_{\mathcal{Q}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup k\cup\mathcal{Q}\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-(\mathcal{D}\cup\mathcal{Q})}x_i^*\right\},$$

and

$$\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup\mathcal{D})+x_k^*+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*\right\}\geq\left(a_k+\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*\right\}\right)\bigvee$$

$$\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}}\bigvee_{\mathcal{Q}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup k\cup\mathcal{Q}\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-(\mathcal{D}\cup\mathcal{Q})}x_i^*\right\}$$

$$=\left(\alpha_k+\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*\right\}\right)\bigvee\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup k\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*\right\}$$

$$\geq\left(\alpha_k+\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}\alpha_i\right\}\right)\bigvee\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup k\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}\alpha_i\right\}$$

$$=\bigvee_{\mathcal{D}\subseteq\mathcal{N}_k}\left\{F(\mathcal{G}\cup\mathcal{D})+\sum_{\mathcal{N}_k-\mathcal{D}}\alpha_i\right\},$$

where we used the fact that

$$\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}}\bigvee_{\mathcal{Q}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup k\cup\mathcal{Q}\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-(\mathcal{D}\cup\mathcal{Q})}x_i^*\right\}=\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}}\left\{F(\mathcal{G}\cup k\cup\mathcal{D})+\sum_{\mathcal{N}_{k-1}-\mathcal{D}}x_i^*\right\},$$

because if $\mathcal{D}\subseteq\mathcal{N}_{k-1}$ and $\mathcal{Q}\subseteq\mathcal{N}_{k-1}$ then $\mathcal{Q}\cup\mathcal{D}\subseteq\mathcal{N}_{k-1}$. In the last equality we used as before the argument for splitting the sets in $\mathcal{N}_k$ in those that contain k and those that do not and the last inequality follows from the induction hypothesis. From the above we conclude that

$$\bigvee_{D\subseteq\mathcal{N}_k}\left\{F(G\cup D)+\sum_{\mathcal{N}_k-D}x_i^*\right\}\geq\bigvee_{D\subseteq\mathcal{N}_{k-1}}\left\{F(G\cup k\cup D)+\sum_{\mathcal{N}_{k-1}-D}\alpha_i\right\}\bigvee$$

$$\bigvee_{D\subseteq\mathcal{N}_k}\left\{F(G\cup D)+\sum_{\mathcal{N}_k-D}\alpha_i\right\}=\bigvee_{D\subseteq\mathcal{N}_k}\left\{F(G\cup D)+\sum_{\mathcal{N}_k-D}\alpha_i\right\}.$$

■

Lemma 12 *If y is a point satisfying (37), (38), (39), it holds*

$$\sum_{i=1}^k y_i\leq\bigvee_{\mathcal{D}\subseteq\mathcal{N}_k}\left\{\sum_{\mathcal{D}}\alpha_i+F(\mathcal{N}_k-\mathcal{D})\right\}=\sum_{i=1}^k x_i^*.$$

Proof. We have for any set $\mathcal{D}\subseteq\mathcal{N}_k$

$$\sum_{i=1}^k y_i=\sum_{i\in\mathcal{D}}y_i+\sum_{i\in\mathcal{N}_k-\mathcal{D}}y_i\leq\sum_{\mathcal{D}}\alpha_i+F(\mathcal{N}_k-\mathcal{D}).$$

Hence

$$\sum_{i=1}^{k} y_i \le \bigvee_{\mathcal{D}\subseteq\mathcal{N}_k} \left\{ \sum_{\mathcal{D}} \alpha_i + F(\mathcal{N}_k - \mathcal{D}) \right\}.$$

We will use induction to show

$$\sum_{i=1}^{k} x_i^* = \bigvee_{\mathcal{D}\subseteq\mathcal{N}_k} \left\{ \sum_{\mathcal{D}} \alpha_i + F(\mathcal{N}_k - \mathcal{D}) \right\}.$$

For $k = 1$ this follows from the definition (36). Assume now that it holds for $k-1$. Then, we have

$$\begin{aligned}
\sum_{i=1}^{k} x_i^* &= x_k^* + \sum_{i=1}^{k-1} x_i^* \\
&= \left\{ \alpha_k \vee \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F(k \cup \mathcal{D}) - \sum_{\mathcal{D}} x_i^* \right\} \right\} + \sum_{i=1}^{k-1} x_i^* \\
&= \left\{ \sum_{i=1}^{k-1} x_i^* + \alpha_k \right\} \vee \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F(k \cup \mathcal{D}) + \sum_{\mathcal{N}_{k-1}-\mathcal{D}} x_i^* \right\} \\
&= \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ \sum_{D} \alpha_i + \alpha_k + F(\mathcal{N}_{k-1} - \mathcal{D}) \right\} \vee \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F(k \cup \mathcal{D}) + \sum_{\mathcal{N}_{k-1}-\mathcal{D}} x_i^* \right\},
\end{aligned}$$

where the last equality follows from the induction hypothesis. Using now Lemma 11 we have

$$\sum_{i=1}^{k} x_i^* \ge \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ \sum_{\mathcal{D}} \alpha_i + \alpha_k + F(\mathcal{N}_{k-1} - \mathcal{D}) \right\} \vee \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F(k \cup \mathcal{D}) + \sum_{\mathcal{N}_{k-1}-\mathcal{D}} \alpha_i \right\}.$$

Where we used the fact that

$$\bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ F(k \cup \mathcal{D}) + \sum_{\mathcal{N}_{k-1}-\mathcal{D}} \alpha_i \right\} = \bigvee_{\mathcal{D}\subseteq\mathcal{N}_{k-1}} \left\{ \sum_{\mathcal{D}} \alpha_i + F(\mathcal{N}_k - \mathcal{D}) \right\},$$

and once again the argument of splitting the sets in $\mathcal{N}_k$ in those that contain k and those that do not. Since x^* is feasible, equality must hold. ■

Theorem 9 follows now easily.

Proof of Theorem 9 . Write

$$\sum_{i=1}^{N} w_i x_i^* = \sum_{i=1}^{N-1} \left((w_i - w_{i+1}) \sum_{k=1}^{i} x_i^* \right) + w_N \sum_{k=1}^{N} x_i^*.$$

Using the fact that $c_i \ge c_{i+1}$, and Lemma 12 we have for any feasible point y

$$\sum_{i=1}^{N} w_i x_i^* \ge \sum_{i=1}^{N-1} \left((w_i - w_{i+1}) \sum_{k=1}^{i} y_i \right) + w_N \sum_{k=1}^{N} y_i = \sum_{i=1}^{N} w_i y_i.$$

■

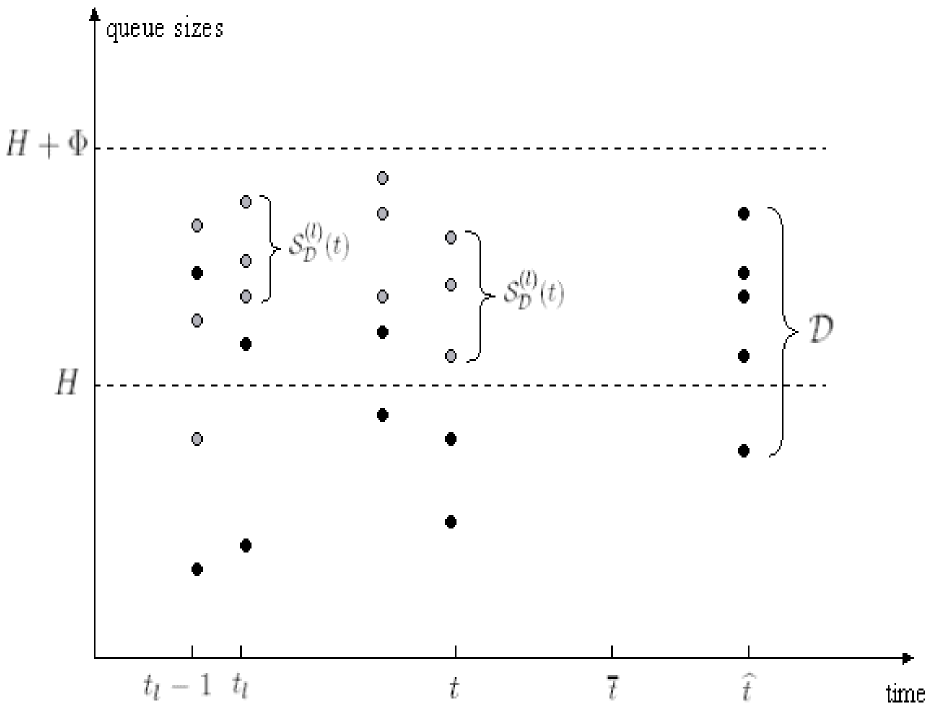

Figure 5: Evolution with time of the queue sizes in the set $\mathcal{D}$. Time $\widehat{t}$ is the largest time for which the maximum queue size in $\mathcal{D}$ is less than $H+\Phi$. During the time interval $[t_l, t]$ the index set corresponding to the l largest queues among the queues in $\mathcal{D}$ remains the same and above H. At time $t_l - 1$ either this set changes or it drops bellow H.

A.2 Proof of Lemma 2

In this appendix we give the proof of Lemma 2.

Proof. We first need some notation. For $\mathcal{D} \subseteq \mathcal{N}$, denote by $q_{\mathcal{D}}^{(l)}(t)$ the l^{th} maximum of $\{q_i(t)\}_{i\in\mathcal{D}}$. Hence $q_{\mathcal{D}}^{(1)}(t) = \max_{i\in\mathcal{D}}\{q_i(t)\}$ and $q_{\mathcal{D}}^{(|\mathcal{D}|)}(t) = \min_{i\in\mathcal{D}}\{q_i(t)\}$. Let $\pi_l^{\mathcal{D}}(t)$, $l \in \{1, ..., |\mathcal{D}|\}$ be a permutation of the indices in $\mathcal{D}$ such that $q_{\pi_l^{\mathcal{D}}(t)}(t) = q_{\mathcal{D}}^{(l)}(t)$. Also, let $\mathcal{S}_{\mathcal{D}}^{(l)}(t) = \cup_{j=1}^{l}\pi_j^{\mathcal{D}}(t)$ (i.e., the set containing the l largest queues among the queues in $\mathcal{D}$) and $\overline{\mathcal{S}}_{\mathcal{D}}^{(l)}(t) = \mathcal{D} - \mathcal{S}_{\mathcal{D}}^{(l)}(t)$.

In the following H and Φ stand for $H(T)$ and $\Phi(T)$ respectively. Let $D = |\mathcal{D}|$. Define also,

$$y_l(t) = \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t)} (q_i(t) - H)^+ .$$

We observe that for $2 \leq l \leq D-1$, it holds

$$y_{l-1}(t) + y_{l+1}(t) = 2y_l(t) + \left(q_{\mathcal{D}}^{(l+1)}(t) - H\right)^+ - \left(q_{\mathcal{D}}^{(l)}(t) - H\right)^+, \tag{40}$$

and

$$y_2(t) = 2y_1(t) + \left(q_{\mathcal{D}}^{(2)}(t) - H\right)^+ - \left(q_{\mathcal{D}}^{(1)}(t) - H\right)^+, \tag{41}$$

$$y_{D-1}(t) = y_D(t) - \left(q_{\mathcal{D}}^{(D)}(t) - H\right)^+. \tag{42}$$

Figure 5 will be useful in the following. Let $\widehat{t} \le \infty$ be the largest time for which $\Phi + H \ge q_{\mathcal{D}}^{(1)}(t)$ for all $t \le \widehat{t}$. Consider a time $t \le \widehat{t}$ such that $q_{\mathcal{D}}^{(l)}(t) > H$ and let $t_l - 1$ be the largest time before t, such that either $\mathcal{S}_{\mathcal{D}}^{(l)}(t) \ne \mathcal{S}_{\mathcal{D}}^{(l)}(t_l - 1)$ or $q_{\mathcal{D}}^{(l)}(t_l - 1) \le H$ (for $l = D$ only the second situation makes sense). Time t_l is well defined since by assumption $\max_{i \in \mathcal{D}} \{q_i(0)\} \le H$. In the time interval $[t_l, t]$, the set $\mathcal{S}_{\mathcal{D}}^{(l)}(t)$ remains the same, i.e., $\mathcal{S}_{\mathcal{D}}^{(l)}(\tau) = \mathcal{S}_{\mathcal{D}}^{(l)}(t_l)$, $t_l \le \tau \le t$, and all the queues in this set are bigger than H, i.e., nonempty. Because of assumptions a) and b) and because $q_{\mathcal{D}}^{(1)}(t) \le \Phi + H$, this set of queues has priority over the queues in $\mathcal{N} - \mathcal{L}$ and since its queues are nonempty, they use all the available slots in $[t_l, t]$. Since these slots are at least $C_{\mathcal{L} \cup \mathcal{S}_{\mathcal{D}}^{(l)}(t_l)}(t_l, t) - C_{\mathcal{L}}(t_l, t)$, we have $B_{\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)}(t_l, t) \ge C_{\mathcal{L} \cup \mathcal{S}_{\mathcal{D}}^{(l)}(t_l)}(t_l, t) - C_{\mathcal{L}}(t_l, t)$. Setting $\overline{F}(\mathcal{S}) = F(\mathcal{L} \cup \mathcal{S}) - F(\mathcal{L})$, we conclude

$$\begin{aligned}
\sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)} q_i(t) &= \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)} q_i(t_l) + A_{\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)}(t_l, t) - B_{\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)}(t_l, t) \\
&\le \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)} q_i(t_l) + \left(\sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)} \alpha_i - \overline{F}\left(\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)\right) \right)(t - t_l) + \mathbb{O} \\
&\le \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)} q_i(t_l) + \mathbb{O}.
\end{aligned} \tag{43}$$

In the first inequality above we used the burstiness constraints on the arrival and channel availability processes. In the second inequality we used (13). Because of the way t_l is defined, we have that for $i \in \mathcal{S}_{\mathcal{D}}^{(l)}(t_l)$ it holds $q_i(t) > H$ and $q_i(t_l) > H$. Therefore by subtracting lH from both sides of inequality (43) we obtain

$$\sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)} (q_i(t) - H)^+ \le \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t_l)} (q_i(t_l) - H)^+ + \mathbb{O}. \tag{44}$$

Hence,

$$y_l(t) \le y_l(t_l) + \mathbb{O}. \tag{45}$$

Let $2 \le l \le D - 1$. We will show now that

$$y_l(t_l) \le \frac{1}{2} y_{l-1}(t_l) + \frac{1}{2} y_{l+1}(t_l) + \mathbb{O}. \tag{46}$$

Note that by definition $q_{\mathcal{D}}^{(l)}(t_l) > H$ and therefore from equation (40) it holds

$$y_l(t_l) = \frac{1}{2} y_{l-1}(t_l) + \frac{1}{2} y_{l+1}(t_l) + \frac{1}{2}\left(q_{\mathcal{D}}^{(l)}(t_l) - H\right) - \frac{1}{2}\left(q_{\mathcal{D}}^{(l+1)}(t_l) - H\right)^+. \tag{47}$$

We consider two cases.

Case 1. $q_{\mathcal{D}}^{(l)}(t_l-1) \le H$. Then, there must be an index $i_0 \in \mathcal{S}_{\mathcal{D}}^{(l)}(t_l)$ such that $q_{i_0}(t_l-1) \le H$ and $q_{\mathcal{D}}^{(l)}(t_l) \le q_{i_0}(t_l)$. Using Lemma 1 with $\mathcal{S}_1 = \{i_0\}$, $\mathcal{S}_2 = \varnothing$, we conclude that

$$q_{i_0}(t_l) \le q_{i_0}(t_l-1) + \mathbb{O} \le H + \mathbb{O}.$$

Hence,

$$q_{\mathcal{D}}^{(l)}(t_l) - H \le \mathbb{O}. \tag{48}$$

From (48) and (47) (taking also into account that $\left(q_{\mathcal{D}}^{(l+1)}(t_l) - H\right)^+ \ge 0$) follows (46).

Case 2. $\mathcal{S}_{\mathcal{D}}^{(l)}(t_l) \ne \mathcal{S}_{\mathcal{D}}^{(l)}(t_l-1)$ and $q_{\mathcal{D}}^{(l)}(t_l-1) > H$. Then, there must be indices $i_0 \in \mathcal{S}_{\mathcal{D}}^{(l)}(t_l)$, $j_0 \in \overline{\mathcal{S}}_{\mathcal{D}}^{(l)}(t_l)$, such that $q_{i_0}(t_l-1) \le q_{j_0}(t_l-1)$ and $q_{\mathcal{D}}^{(l)}(t_l) - q_{\mathcal{D}}^{(l+1)}(t_l) \le q_{i_0}(t_l) - q_{j_0}(t_l)$. By using Lemma 1 with $\mathcal{S}_1 = \{i_0\}$, $\mathcal{S}_2 = \{j_0\}$, we conclude that

$$q_{i_0}(t_l) - q_{j_0}(t_l) \le q_{i_0}(t_l-1) - q_{j_0}(t_l-1) + \mathbb{O} \le \mathbb{O}.$$

Hence,

$$q_{\mathcal{D}}^{(l)}(t_l) - q_{\mathcal{D}}^{(l+1)}(t_l) \le \mathbb{O}. \tag{49}$$

Since $\left(q_{\mathcal{D}}^{(l+1)}(t_l) - H\right)^+ \ge q_{\mathcal{D}}^{(l+1)}(t_l) - H$, from (47) we have

$$y_l(t_l) \le \frac{1}{2}y_{l-1}(t_l) + \frac{1}{2}y_{l+1}(t_l) + \frac{1}{2}\left(q_{\mathcal{D}}^{(l)}(t_l) - q_{\mathcal{D}}^{(l+1)}(t_l)\right).$$

which in conjunction with (49) shows (46).

Combining now (46) and (45) we get

$$y_l(t) \le \frac{1}{2}y_{l-1}(t_l) + \frac{1}{2}y_{l+1}(t_l) + \mathbb{O}. \tag{50}$$

Similarly, we have with an analogous definition of t_1 and t_D

$$y_1(t) \le \frac{1}{2}y_2(t_1) + \mathbb{O}, \tag{51}$$

$$y_D(t) \le y_{D-1}(t_D) + \mathbb{O}. \tag{52}$$

If $q_{\mathcal{D}}^{(l)}(t) \le H$, then we have from (40), (41) and (42) that (50) - (52) still hold with $t_l = t$.

Fix now a time $\overline{t} \le \widehat{t}$ and define

$$\overline{y}_l(\overline{t}) = \max_{t \le \overline{t}} y_l(t) < \infty. \tag{53}$$

From (50), it follows for $2 \le l \le D-1$, that for any t, $0 \le t \le \overline{t}$,

$$y_l(t) \le \frac{1}{2}\overline{y}_{l-1}(\overline{t}) + \frac{1}{2}\overline{y}_{l+1}(\overline{t}) + \mathbb{O}. \tag{54}$$

Similarly, from (51) and (52) it follows that for any t, $0 \le t \le \overline{t}$,

$$y_1(t) \le \frac{1}{2}\overline{y}_2(\overline{t}) + \mathbb{O}, \tag{55}$$

$$y_D(t) \le \overline{y}_{D-1}(\overline{t}) + \mathbb{O}. \tag{56}$$

Therefore we have for $2 \leq l \leq D-1$,

$$\overline{y}_l\left(\overline{t}\right) \leq \frac{1}{2}\overline{y}_{l-1}\left(\overline{t}\right) + \frac{1}{2}\overline{y}_{l+1}\left(\overline{t}\right) + \mathbb{O}, \tag{57}$$

and

$$\overline{y}_1\left(\overline{t}\right) \leq \frac{1}{2}\overline{y}_2\left(\overline{t}\right) + \mathbb{O}, \tag{58}$$

$$\overline{y}_D\left(\overline{t}\right) \leq \overline{y}_{D-1}\left(\overline{t}\right) + \mathbb{O}. \tag{59}$$

The above inequalities can be written in matrix form as:

$$\left(\mathbf{I}-\mathbf{B}\right)\mathbf{Y} \leq \overline{\mathbb{O}}, \tag{60}$$

where $\mathbf{Y} = \left[\overline{y}_1(\overline{t})...\overline{y}_D\left(\overline{t}\right)\right]^T$, and $\mathbf{I}$ is the unity matrix, $\overline{\mathbb{O}}$ is a matrix whose elements are of type $\mathbb{O}$ and

$$\mathbf{B} = \left[\begin{array}{ccccc} 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]. \tag{61}$$

Since the row sums of $\mathbf{B}$ are all less than or equal to 1 and the sum of the first raw is 1/2, i.e., less than 1, it follows from the Perron-Frobenius Theorem [11], that the eigenvalues of $\mathbf{B}$ are all smaller than 1 in absolute value. Therefore, the matrix $\left(\mathbf{I}-\mathbf{B}\right)^{-1}$ has nonnegative elements. Hence, we can multiply (60) with $\left(\mathbf{I}-\mathbf{B}\right)^{-1}$ to get $\mathbf{Y} \leq \left(\mathbf{I}-\mathbf{B}\right)^{-1}\mathbb{O}$, or

$$\overline{y}_l(\overline{t}) \leq \mathbb{O}, \;\; l = 1, ..., D. \tag{62}$$

Therefore, for $0 \leq t \leq \overline{t} \leq \widehat{t}$, we have

$$\max_{i \in \mathcal{D}}\left\{q_i\left(t\right)\right\} - H = q_{\mathcal{D}}^{(1)}(t) - H \leq \left(q_{\mathcal{D}}^{(1)}(t) - H\right)^+ = y_1\left(t\right) \leq \overline{y}_l(\overline{t}) \leq \mathbb{O}$$

or,

$$\max_{i \in \mathcal{D}}\left\{q_i\left(t\right)\right\} \leq H + \mathbb{O}, \, 0 \leq t \leq \overline{t} \leq \widehat{t}. \tag{63}$$

We will show next that we can pick $\Phi \geq \mathbb{O}$ so that (63) holds for all $t \geq 0$. Indeed if $\widehat{t} < \infty$, from (63) and Lemma 1 and for any choice of Φ, we have

$$\max_{i \in \mathcal{D}}\left\{q_i\left(\widehat{t}+1\right)\right\} \leq H + \mathbb{O}. \tag{64}$$

Pick $\Phi \geq \mathbb{O}$, where $\mathbb{O}$ is as computed in (64). For this choice of Φ, if $\widehat{t}$ were finite, we would have by the definition of $\widehat{t}$,

$$\max_{i \in \mathcal{D}}\left\{q_i\left(\widehat{t}+1\right)\right\} > H + \Phi,$$

which contradicts (64). ■

A.3 Proof of Lemma 3

In this Appendix the proof of Lemma 3 is given.

Proof. In the following H stands for $H(T)$. We redefine $q_{\mathcal{D}}^{(l)}(t)$ and $y_l(t)$, first defined in the proof of Lemma 2, as follows. For $\mathcal{D} \subseteq \mathcal{N}$, denote by $q_{\mathcal{D}}^{(l)}(t)$ the l^{th} *minimum* of $\{q_i(t)\}_{i\in\mathcal{D}}$. Hence $q_{\mathcal{D}}^{(1)}(t) = \min_{i\in\mathcal{D}}\{q_i(t)\}$ and $q_{\mathcal{D}}^{(|\mathcal{D}|)}(t) = \max_{i\in\mathcal{D}}\{q_i(t)\}$. The definitions of $\pi_l^{\mathcal{P}}(t)$, $\mathcal{S}_{\mathcal{D}}^{(l)}(t)$, $\overline{\mathcal{S}}_{\mathcal{D}}^{(l)}(t)$ remain the same. Also, define

$$y_l(t) = \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t)} (H - q_i(t))^+ . \tag{65}$$

In the following we will need to define several time instances. For a schematic representation of these times refer to Figure 6.

Let $t > 0$. If $q_{\mathcal{D}}^{(l)}(t) \geq H$, then define $t_l^{(1)} = t$. Else (i.e., if $q_{\mathcal{D}}^{(l)}(t) < H$) let $t_l^{(1)} - 1$ be the largest time before t (if it exists) such that $\mathcal{S}_{\mathcal{D}}^{(l)}(t) \neq \mathcal{S}_{\mathcal{D}}^{(l)}(t_l^{(1)} - 1)$, or $q_{\mathcal{D}}^{(l)}(t_l^{(1)} - 1) \geq H$; if no such $t_l^{(1)}$ exists, define $t_l^{(1)} = 0$.

If $t_l^{(1)} = 0$, then for τ in the interval $[0, t]$ the set $\mathcal{S}_{\mathcal{D}}^{(l)}(\tau)$ remains the same and $q_{\mathcal{D}}^{(l)}(\tau) < H$, $0 \leq \tau \leq t$. Moreover, according to assumptions a) and b) of the lemma, the queues in $\mathcal{L} \cup \overline{\mathcal{S}}_{\mathcal{D}}^{(l)}(t)$ have priority over the queues in $\mathcal{S}_{\mathcal{D}}^{(l)}(t)$. Notice also that by definition, whenever some of the queues in $\mathcal{S}_{\mathcal{D}}^{(l)}(t)$ are nonempty, the queues in $\overline{\mathcal{S}}_{\mathcal{D}}^{(l)}(t)$ are nonempty as well. Therefore, if one of the channels in $\mathcal{L} \cup \overline{\mathcal{S}}_{\mathcal{D}}^{(l)}(t)$ is "on" at time t, this slot cannot be used for transmission of packets of queues in $\mathcal{S}_{\mathcal{D}}^{(l)}(t)$. We conclude that

$$\begin{aligned} B_{\mathcal{S}_{\mathcal{D}}^{(l)}(t)}(0,t) &\leq C_{\mathcal{L}\cup\mathcal{D}}(0,t) - C_{\mathcal{L}\cup\overline{\mathcal{S}}_{\mathcal{D}}^{(l)}(t)}(0,t) \\ &\leq \left(F(\mathcal{L}\cup\mathcal{D}) - F(\mathcal{L}\cup\overline{\mathcal{S}}_{\mathcal{D}}^{(l)}(t)\right) t + \mathbb{O}. \end{aligned}$$

Hence, by setting $\overline{F}(\mathcal{S}) = F(\mathcal{L}\cup\mathcal{D}) - F(\mathcal{L}\cup\overline{\mathcal{S}})$, we have,

$$\begin{aligned} \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t)} q_i(t) &= \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t)} q_i(0) + A_{\mathcal{S}_{\mathcal{D}}^{(l)}(t)}(0,t) - B_{\mathcal{S}_{\mathcal{D}}^{(l)}(t)}(0,t) \\ &\geq \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t)} q_i(0) + \left(\sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t)} \alpha_i - \overline{F}\left(\mathcal{S}_{\mathcal{D}}^{(l)}(t)\right)\right) t - \mathbb{O} \\ &\geq \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t)} q_i(0) - \mathbb{O}. \end{aligned} \tag{66}$$

where the last inequality follows from (15). Since $q_{\mathcal{D}}^{(l)}(\tau) < H$, $0 \leq \tau \leq t$, subtracting lH from both sides of (66) and taking into account the definition (65) we have for $2 \leq l \leq D-1$,

$$y_l(t) \leq y_l(0) + \mathbb{O} \leq \frac{1}{2} y_{l-1}(0) + \frac{1}{2} y_{l+1}(0) + y_l(0) + \mathbb{O}, \tag{67}$$

and

$$y_1(t) \leq y_1(0) + \mathbb{O} \leq \frac{1}{2} y_2(0) + y_1(0) + \mathbb{O}, \tag{68}$$

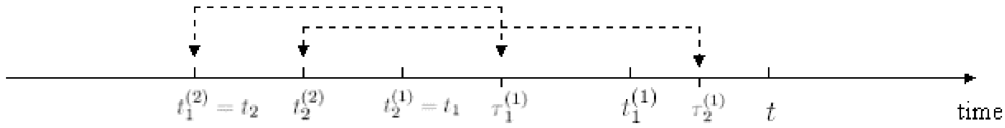

Figure 6: Schematic representation of the times $t_l^{(j)}$, $\tau_l^{(j)}$ and t_j, for $D = 2$.

$$y_D(t) \leq y_D(0) + \mathbb{O} \leq y_{D-1}(0) + y_D(0) + \mathbb{O}. \tag{69}$$

The inclusion of additional terms in the second inequalities in (67)-(69) is made in order to treat the various cases in a uniform fashion, see inequalities (73)-(75) below. Assume next that $t_l^{(1)} > 0$. Following an analogous reasoning as in the proof of Lemma 2 we have that for $2 \leq l \leq D-1$,

$$y_l(t) \leq \frac{1}{2} y_{l-1}(t_l^{(1)}) + \frac{1}{2} y_{l+1}(t_l^{(1)}) + \mathbb{O}, \tag{70}$$

and

$$y_1(t) \leq \frac{1}{2} y_2\left(t_1^{(1)}\right) + \mathbb{O}, \tag{71}$$

$$y_D(t) \leq y_{D-1}(t_D^{(1)}) + \mathbb{O}. \tag{72}$$

Combining (67)-(72) (i.e., the cases $t_l^{(1)} > 0$ and $t_l^{(1)} = 0$) we have for $2 \leq l \leq D-1$

$$y_l(t) \leq \frac{1}{2} y_{l-1}(t_l^{(1)}) + \frac{1}{2} y_{l+1}(t_l^{(1)}) + \mathbf{1}_{\left\{t_l^{(1)}=0\right\}} y_l(0) + \mathbb{O}, \tag{73}$$

and

$$y_1(t) \leq \frac{1}{2} y_2\left(t_1^{(1)}\right) + \mathbf{1}_{\left\{t_1^{(1)}=0\right\}} y_1(0) + \mathbb{O}, \tag{74}$$

$$y_D(t) \leq y_{D-1}(t_D^{(1)}) + \mathbf{1}_{\left\{t_D^{(1)}=0\right\}} y_D(0) + \mathbb{O}, \tag{75}$$

where $\mathbf{1}_{\left\{t_l^{(1)}=0\right\}} = 1$ if $t_l^{(1)} = 0$ and zero otherwise.

If $z(t)$, $t \geq 0$, is a real function defined on the nonnegative integers, define

$$\overline{z}[a, b] = \max_{\tau \in [a,b]} z(\tau). \tag{76}$$

Let $t_1 = \min_{l \in \{1, \ldots D\}} \left\{t_l^{(1)}\right\}$. Using definition (76) and (73)-(75), yield: for $2 \leq l \leq D-1$,

$$y_l(t) \leq \frac{1}{2} \overline{y}_{l-1}[t_1, t] + \frac{1}{2} \overline{y}_{l+1}[t_1, t] + \mathbf{1}_{\{t_1=0\}} y_l(0) + \mathbb{O},$$

and

$$y_1(t) \leq \frac{1}{2} \overline{y}_2[t_1, t] + \mathbf{1}_{\{t_1=0\}} y_1(0) + \mathbb{O}.$$

$$y_D(t) \leq \overline{y}_{D-1}[t_1, t] + \mathbf{1}_{\{t_1=0\}} y_D(0) + \mathbb{O}.$$

The above inequalities can be written in matrix form as

$$\mathbf{Y}(t) \leq \mathbf{B}\overline{\mathbf{Y}}^{(1)} + \mathbf{1}_{\{t_1=0\}}\mathbf{Y}(0) + \overline{\mathbb{O}}, \tag{77}$$

where $\mathbf{B}$ is defined by (61) in the proof of lemma 2, $\mathbf{Y}(t) = [y_1(t), ..., y_D(t)]^T$ and $\overline{\mathbf{Y}}^{(1)} = [\overline{y}_1[t_1,t] \ \overline{y}_2[t_1,t] \ ... \ \overline{y}_D[t_1,t]]^T$. Let $\tau_l^{(1)} \in [t_1,t]$, $l = 1,..,D$, be the time where $\overline{y}_l[t_1,t]$ is achieved, (i.e., $y_l(\tau_l^{(1)}) = \overline{y}_l[t_1,t]$). For $l = 1,...,D$, the time $t_l^{(2)}$, is defined for the set $\mathcal{S}_{\mathcal{D}}^{(l)}(\tau_l^{(1)})$ exactly as the time $t_l^{(1)}$ were defined for the sets $\mathcal{S}_{\mathcal{D}}^{(l)}(t)$. Generally, provided that $t_l^{(j)}$are defined, define recursively:

- $t_j = \min_{l\in\{1,...D\}}\left\{t_l^{(j)}\right\}$.
- $\tau_l^{(j)} \in [t_j,t]$, $l = 1,..,D$, the time where $\overline{y}_l[t_j,t]$ is achieved.
- If $q_{\mathcal{D}}^{(l)}(\tau_l^{(j)}) \geq H$, then define $t_l^{(j+1)} = \tau_l^{(j)}$. Else, let $t_l^{(j+1)} - 1$, be the largest time (if one exists) before $\tau_l^{(j)}$ such that $\mathcal{S}_{\mathcal{D}}^{(l)}(\tau_l^{(j)}) \neq \mathcal{S}_{\mathcal{D}}^{(l)}(t_l^{(j+1)} - 1)$, or $q_{\mathcal{D}}^{(l)}(t_l^{(j+1)} - 1) \geq H$. If no such time exists, set $t_l^{(j+1)} = 0$.

Arguing as above we have,

$$\overline{\mathbf{Y}}^{(j)} \leq \mathbf{B}\overline{\mathbf{Y}}^{(j+1)} + \mathbf{1}_{\{t_{j+1}=0\}}\mathbf{Y}(0) + \overline{\mathbb{O}}, \tag{78}$$

where $\overline{\mathbf{Y}}^{(j)} = [\overline{y}_1[t_j,t] \ \overline{y}_2[t_j,t] \ ... \ \overline{y}_D[t_j,t]]^T$.

Let $k(t)+1$ be the smallest integer such that either $t_{k(t)+1} = 0$ or $0 < t_{k(t)} \leq t_{k(t)+1}$.

We consider two cases depending on whether $t_{k(t)+1}$ is larger than zero, or zero.

case 1. $0 < t_{k(t)} \leq t_{k(t)+1}$. In this case we have $\left[t_{k(t)+1}, t\right] \subseteq \left[t_{k(t)}, t\right]$ which implies that $\overline{\mathbf{Y}}^{(k(t))} \geq \overline{\mathbf{Y}}^{(k(t)+1)}$. Therefore by using inequality (78) we conclude that

$$\overline{\mathbf{Y}}^{(k(t)+1)} \leq \mathbf{B}\overline{\mathbf{Y}}^{(k(t)+1)} + \overline{\mathbb{O}}.$$

From the last inequality and the Perron-Frobenius theorem we have that

$$\overline{\mathbf{Y}}^{(k(t)+1)} \leq (\mathbf{I} - \mathbf{B})^{-1}\overline{\mathbb{O}} = \overline{\mathbb{O}} \tag{79}$$

Observing that $\mathbf{1}_{\{t_{j+1}=0\}} = 0$ for $1 \leq j \leq k(t)+1$, we have from (78) that

$$\mathbf{Y}(t) \leq \mathbf{B}^{k(t)+1}\overline{\mathbf{Y}}^{(k(t)+1)} + \left(\mathbf{I} + \mathbf{B} + ... + \mathbf{B}^{k(t)}\right)\overline{\mathbb{O}}. \tag{80}$$

From the Perron-Frobenius theorem we also have that for any k,

$$\mathbf{B}^k \leq \left(\mathbf{I} + \mathbf{B} + ... + \mathbf{B}^k\right) \leq \sum_{i=0}^{\infty}\mathbf{B}^i = (\mathbf{I} - \mathbf{B})^{-1}. \tag{81}$$

Inequalities (79), (80), (81) implies,

$$\mathbf{Y}(t) \leq (\mathbf{I} - \mathbf{B})^{-1}\overline{\mathbb{O}} + (\mathbf{I} - \mathbf{B})^{-1}\overline{\mathbb{O}} = \overline{\mathbb{O}}$$

Hence
$$\mathbb{O} \geq y_1(t) = (H - q_{\mathcal{D}}^{(1)}(t))^+ \geq H - q_{\mathcal{D}}^{(1)}(t) = H - \min_{i\in\mathcal{D}} \{q_i(t)\}$$
and the lemma follows.

case 2. $t_{k(t)+1} = 0$. Observing that $\mathbf{1}_{\{t_j=0\}} = 0$ for $1 \leq j \leq k(t)$, we have from (78) and by using the Perron-Frobenius theorem,

$$\mathbf{Y}(t) \leq \mathbf{B}^{k(t)+1}\overline{\mathbf{Y}}^{(k(t)+1)} + \mathbf{B}^{k(t)}\mathbf{Y}(0) + (\mathbf{I}-\mathbf{B})^{-1}\overline{\mathbb{O}}.$$

By the definition of $y_l(t)$ it holds for any t,

$$y_l(t) \leq |\mathcal{D}|\, H. \tag{82}$$

Therefore, we can write

$$\mathbf{Y}(t) \leq \mathbf{B}^{k(t)+1}\mathbf{Z} + \mathbf{B}^{k(t)}\mathbf{Z} + (\mathbf{I}-\mathbf{B})^{-1}\overline{\mathbb{O}}, \tag{83}$$

where $\mathbf{Z}$ is a column matrix such that all it's entries are equal to $|\mathcal{D}|\, H$.

We claim that
$$k(t) \geq \gamma t - 1, \tag{84}$$
where
$$\gamma = \frac{\delta}{|\mathcal{D}|\, H + \mathbb{O}},$$
and
$$\delta = \min_{\substack{\mathcal{S}\subseteq\mathcal{D}\\ \mathcal{S}\neq\varnothing}} \left\{ \sum_{\mathcal{S}} \alpha_i - \overline{F}(\mathcal{S}) \right\} > 0.$$
This can be proved as follows. Define

$$\begin{aligned} &I_i(t) = t_{i-1} - t_i, \;\; 2 \leq i \leq k(t)+1 \\ &I_1(t) = t - t_1. \end{aligned}$$

By the definition of $k(t)$, we have $I_i(t) > 0$ for $1 \leq i \leq k(t)+1$. Moreover,

$$\sum_{i=1}^{k(t)+1} I_i(t) = t. \tag{85}$$

From the definition of times t_j, there exists a time $t' \geq t_{j-1}$, and an index l such that $q_{\mathcal{D}}^{(l)}(\tau) < H$, $t_j \leq \tau \leq t'$ and

$$\begin{aligned} \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t')} q_i(t') &\geq \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t')} q_i(t_j) + \left(\sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t')} \alpha_i - \overline{F}\left(\mathcal{S}_{\mathcal{D}}^{(l)}(t')\right) \right) (t' - t_j) - \mathbb{O} \\ &\geq \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t')} q_i(t_j) + \delta\left(t' - t_j\right) - \mathbb{O} \geq \sum_{\mathcal{S}_{\mathcal{D}}^{(l)}(t')} q_i(t_j) + \delta I_j(t) - \mathbb{O}. \end{aligned} \tag{86}$$

Subtracting lH from both sides of (86) we have

$$0 \leq y_l(t') \leq y_l(t_j) - \delta I_j(t) + \mathbb{O},$$

or, taking also into account (82),

$$I_j(t) \leq \frac{|\mathcal{D}|\, H + \mathbb{O}}{\delta}.$$

The last inequality and (85) imply (84).

From the Perron-Frobenius theorem we have that $\lim_{k\to\infty} \mathbf{B}^k = \mathbf{0}$ (where $\mathbf{0}$ is a matrix whose elements are all zero). Therefore, we can pick τ_0 large enough so that for $t > \tau_0$ it holds,

$$\mathbf{B}^{\lfloor \gamma t \rfloor}\mathbf{Z} + \mathbf{B}^{\lfloor \gamma t \rfloor - 1}\mathbf{Z} \leq \mathbb{O}.$$

With this choice of τ_0, the lemma follows from (84) and (83). ∎