Optimal Overload Response in Sensor Networks

Leonidas Georgiadis, *Senior Member, IEEE,* and Leandros Tassiulas, *Senior Member, IEEE*

***Abstract*— A single commodity network that models the information flow in an arbitrary topology sensor field that collects and forwards information to a backbone through certain designated gateway nodes is considered. Resilient operation in overload stress situations caused by unpredictable traffic or topology variations is considered. A fluid model is adopted where *superflows* model traffic forwarding and backlog formations at the network level. Quantitative performance metrics of the overload including *throughput*, *lexicographic minimization*, *most balanced allocation* and *amount of lost traffic* due to buffer overflow are considered to capture the information loss process due to overflow in the network. Optimal superflows with respect to these metrics are characterized and a distributed asynchronous algorithm that computes such superflows is given. The characterization of the optimal superflow amounts to obtaining a structural decomposition of the network in a sequence of disjoint subregions with decreasing overload such that traffic flows only from regions of higher overload to regions of lower overload.**

***Index Terms*— Network Control, Overload Response, Sensor Networks, Distributed Algorithms, Lexicographic Optimization, Most Balanced Allocation, Lost Traffic Minimization, Throughput Maximization.**

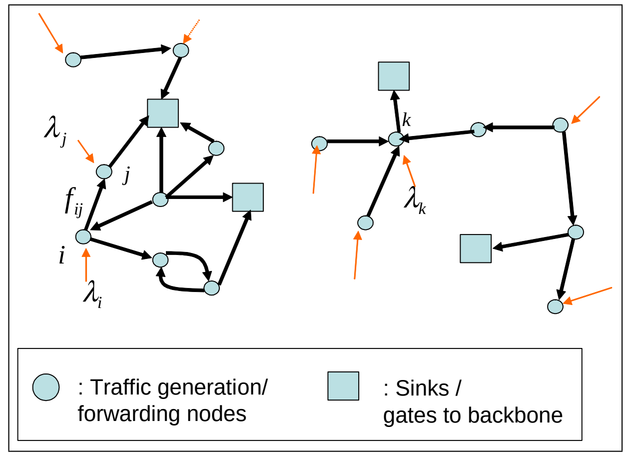

Fig. 1. The information collection network is depicted. Information might be generated at any node and is destined to any of the gateway nodes (squares) through which it is forwarded to the backbone. Multihop transmission might be required through intermediate nodes in which case routing and flow control will take place.

I. Introduction

Unpredictability in traffic load variations, link capacity fluctuations, topology modifications, node failures or various types of intentional misbehavior may lead a network to overload conditions. A smooth and balanced system response in those stressful situations is essential for effective crisis management in the network. This is more of an issue in wireless ad-hoc and sensor networks where due to the nature of the system and the likely scenarios of operation, anomalous behavior of that type is more likely to occur.

We consider a network consisting of an arbitrary spatial arrangement of nodes. Information may be generated at any node in the network and needs to be forwarded to a collection of hub (sink) nodes, see Figure 1. The spatial distribution of traffic generation intensity is specified by the vector of traffic generation rates at each node. In overload condition this vector may lie outside the feasibility region of the system, that is there may be no feasible flow to transfer the information to the sinks, given the capacity of the system. This may occur for instance in a sensor network where traffic generation is event driven and activity scenarios may light up different portions of the system creating spatially localized, temporary overloads. In that case inevitably traffic backlogs will occur in the nodes. The distribution of the backlog build-up is an indication of the behavior of the system.

L. Georgiadis is with Aristotle University of Thessaloniki, Dept. of Electrical and Computer Engineering, Thessaloniki 54124, Greece. (e-mail: leonid@auth.gr)

L. Tassiulas is with the University of Thessaly, Computer Engineering and Telecommunications Dept., Volos 38221, Greece. (e-mail: leandros@uth.gr)

In this work we study the operation of the system in overload. A fluid model is considered where the information flow induced by the routing policy is represented by superflows. A superflow is a generalized notion of flow, where the aggregate incoming flow in a node may exceed the outgoing, i.e. flow conservation at the nodes need not necessarily hold. Superflows permit us to model more accurately overload situations. Specifically, a superflow is a vector with a nonnegative element for each network link representing the information forwarding rate at that link. The difference of incoming minus the outgoing flow from a node is the backlog buildup rate at the node. We call this difference “node overload”. The vector of node overloads under a certain routing policy is the quantitative performance objective that represents the overload response of the network to the routing policy.

We show that in the space of node overload vectors there is one that is lexicographically minimal and we characterize it. The overload corresponding to this vector also maximizes the information rate that reaches the sinks. Furthermore we show that this vector is the unique solution for a wide class of optimization problems where the optimization objective function is the sum of any nondecreasing convex function of node overloads. We call that vector “most balanced” overload vector and any superflow that induces the most balanced overload vector, “most balanced” superflow. Finally we present a distributed adaptive superflow reallocation policy converging to a most balanced superflow.

The paper is organized as follows. In Section II we present some related work. In Section III we present the system mode. The performance objectives under consideration are discussed in Section IV. Properties and characterization of the optimal solution are provided in Section V. In Section VI we propose a distributed adaptive superflow reallocation policy converging to a most balanced superflow. The case where optimization of a weighted overload vector is desirable is discussed in Section VII. We conclude with discussions and directions for further work in Section VIII.

II. Preliminaries and Some Related Work

There are two viewpoints in studying information flow in networks like the one we described above, the microscopic and the macroscopic. At the microscopic level we keep track of the dynamic evolution of the system at the packet level modeling the instantaneous information backlog dynamics through appropriate stochastic queueing networks; the associated routing and flow control algorithms are viewed operating at the packet level. At the macroscopic level, under the assumption that the stochastic dynamic flows at the links and nodes of the network have long term averages, we focus on average flows; hence we have a fluid model of the system and we study different routing policies through the properties of their induced fluid flows.

At the microscopic modeling regime let $a_i(t)$ be the total amount of information generated at node $i \in K$ (K is the set of nodes where traffic is generated) in the time interval $[0, t]$ and $a_{ij}(t)$ the total amount of information transferred to node j from node i through link (i, j) in the same time interval. If $q_i(t)$ is the information backlog at node i at time t then,

$$q_i(t) = q_i(0) + a_i(t) + \sum_{j \in N_{in}(i)} a_{ji}(t) - \sum_{j \in N_{out}(i)} a_{ij}(t),\ i \in K, \tag{1}$$

where $N_{in}(i)$ and $N_{out}(i)$ are respectively the set of incoming and outgoing neighbors of node i.

Stability of the network means bounded backlogs over time, i.e.,

$$\sup E[q_i(t)] < \infty,\ i \in K.$$

Assuming that the long term averages of the stochastic flows $a_i(t),\ a_{ij}(t)$ exist,

$$\lim_{t\to\infty} a_i(t)/t = \lambda_i, \quad \text{a.s.,}$$
$$\lim_{t\to\infty} a_{ij}(t)/t = f_{ij}, \quad \text{a.s.,}$$

the stability of the network implies from (1) that,

$$\lambda_i = \sum_{j \in N_{out}(i)} f_{ij} - \sum_{j \in N_{in}(i)} f_{ji},\ i \in K. \tag{2}$$

while the link capacity constraint implies that

$$0 \le f_{ij} \le c_{ij},\ i \in K,\ j \in N_{out}(i). \tag{3}$$

Equations (2) and (3) are called flow conservation and link capacity constraints respectively and are necessary conditions for stability. While the arrival rate vector $\boldsymbol{\lambda} = (\lambda_1, .., \lambda_K)$ is the average spatial statistical profile of the exogenous traffic and is not affected by the network control policy, the vector of flows $\boldsymbol{f} = (f_{ij} : i \in K,\ j \in N_{out}(i))$ is the result of the routing policy and we may say that it characterizes the routing policy as far as its long term behavior is concerned. A

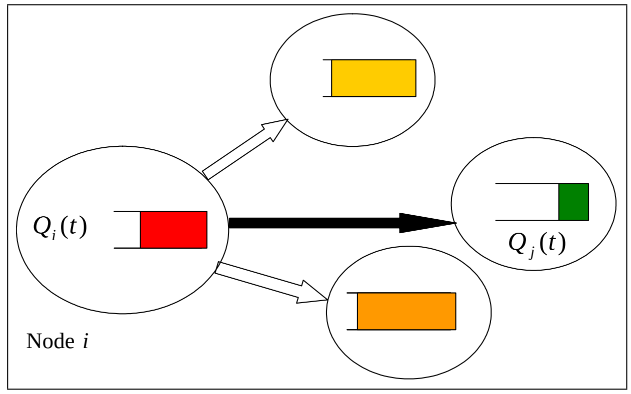

Fig. 2. A graphical depiction of the operation of the Adaptive Back Pressure policy at a certain node is given. A node transmits packet with full capacity to a downstream neighbor that has lower backlog than its own while it idles a link directed to a node with backlog higher than its own.

necessary condition for the feasibility of an arrival rate vector $\boldsymbol{\lambda}$ is that there is a flow vector $\boldsymbol{f}$ satisfying (2) and (3). Let $\mathbf{F}_{\boldsymbol{\lambda}}$ be the collection of flow vectors satisfying (2) and (3) for arrival rate vector $\boldsymbol{\lambda}$. Then each possible routing policy that guarantees stable operation of the network corresponds to a flow vector in $\mathbf{F}_{\boldsymbol{\lambda}}$.

The behavior of the system in the stability regime has been studied extensively in the past. A dynamic routing and flow control policy, the Adaptive Back Pressure (ABP) policy has been proposed and analyzed, showing that it achieves maximum throughput in the network. ABP is a distributed control policy where node i controls the transmissions of its outgoing links based only on its own backlog as well as its outgoing neighbors, without any knowledge of the network topology or its statistics. In the context of the network under consideration ABP operates as follows at each node i, see Figure 2.

- At time t node i compares its backlog $q_i(t)$ with the backlog of its one hop downstream neighbor $j \in N_{out}(i)$.
- If $q_j(t) \ge q_i(t)$ then link (i, j) idles and no packet is transmitted (flow control is performed).
- If $q_j(t) < q_i(t)$ then link (i, j) transmits full speed a packet from i to j.

The above policy was proposed initially in [19], [22], in the context of a multihop radio network and in combination with a max weight radio access control policy. It was shown that the combined routing-scheduling scheme achieves maximum throughput and stabilizes the network if it is possible to do so. More specifically it was shown that under the statistical assumption of i.i.d. arrivals and given that the arrival rate vector is such that $\mathbf{F}_{\boldsymbol{\lambda}}$ is nonempty, the ABP policy achieves stability of the network. In other words its fluid flow profile behaves as one of the feasible flows of $\mathbf{F}_{\boldsymbol{\lambda}}$. A similar policy has been considered later in [20] in the context of a multiclass service network and its dynamic behavior was analyzed extending the results in [22] for deterministic (σ, ρ) traffic profiles, i.e. profiles that comply to the output streams of a (σ, ρ) regulator (see [4] for more details on (σ, ρ) regulated traffic). In [21] the policy was studied under general Markov modulated statistics and batch processing to account for synchronization deficiencies of different servers due to unequal service times. A generalization of the policy, incorporating power control for time-varying channels was presented in [14]. A policy similar to ABP was proposed and studied in [1], [2], in the context of adversarial queueing theory. That is, its performance was analyzed under arrival traffic patterns that might be the worst possible within a certain family of arrival patterns, for instance all possible arrival patterns at the output of a (σ, ρ) regulator. It was shown that the policy achieves maximum throughput in that context as well. Finally the ABP policy has been considered in the context of general service systems in manufacturing and transportation problems in [5], [6], and its maximum throughput properties were verified.

In all the works discussed above the system was studied in its stability region, i.e. when the load did not exceed capacity. In the following we study the system in the overload region resorting directly to the fluid model. The behavior of the system in overload has been considered recently by several researchers [23], [13], [7], [18], proposing flow control at the edge in combination with backlog balancing inside the network, to achieve desirable throughput. In the current paper we study the network overload build-up and we characterize the behavior of the policies from that perspective. Results of this paper were presented in partial form in [9].

Finally, we note that the model and the problem considered in this paper can address, in a limiting sense, the continuous assignment problem of Hajek in [11].

III. Definitions, Model and Assumptions

The topology of the network is represented by a directed graph $G = (N, L)$. The set N consists of network nodes that may generate and forward traffic, and a node d that represents collectively all gateway nodes to the infrastructure network. The set of links L includes a link between any two nodes that may communicate directly. It also includes a link to node d for any network node that may communicate directly with a gateway node. Link $l \in L$ has capacity c_l. Given a set of nodes $S \subset N$, let $L_{in}(S)$ be the set of links that start at some node out of S, i.e. in $S^c = N - S$, and end at some node in S. With $N_{in}(S)$ we denote the set of nodes in S^c that are starting points of the links in $L_{in}(S)$. For simplicity of exposition define $L_{out}(S) = L_{in}(S^c)$ and $N_{out}(S)$ the set of nodes in S^c that are ending points of the links in $L_{out}(S)$. Without loss of generality we assume that $L_{out}(\{d\}) = \varnothing$. The set $L_{in}(S)$ will also be referred to as the set of "incoming links" of the cut (S, S^c). Similarly, the set $L_{out}(S)$ will be referred to as the set of "outgoing links" of the cut (S, S^c). For simplicity, if S consists of a single node i, $S = \{i\}$, we write simply $L_{in}(i)$ and similarly for the other notations. As usual, summation over indices of an empty set is defined as zero, i.e., $\sum_{i\in X} a_i = 0$ if $X = \varnothing$.

Denote by $K = N - \{d\}$. Information is generated at node $i \in K$ at rate $\lambda_i \geq 0$ and is destined to sink node d. A "superflow" $\boldsymbol{f} = \{f_l\}_{l\in L}$, is any nonnegative vector with one element for each link that satisfies the following constraints.

$$\lambda_i + \sum_{l\in L_{in}(i)} f_l - \sum_{l\in L_{out}(i)} f_l \geq 0, \; i \in K \tag{4a}$$

$$0 \leq f_l \leq c_l, \; l \in L. \tag{4b}$$

The inequality in (4a) may be strict since we allow for the possibility of overload at a node. In case equality holds for every node in K, the superflow reduces to the standard "flow" definition. The quantity

$$q_i = \lambda_i + \sum_{l\in L_{in}(i)} f_l - \sum_{l\in L_{out}(i)} f_l,$$

is the rate at which traffic is accumulated at node $i \in K$ for the specific superflow vector. We refer to q_i as the "overload" at node i under superflow $\boldsymbol{f}$. We extend the definition of overload to sink node d by defining $q_d = 0$. We denote by $\mathbf{F}_{\boldsymbol{\lambda}}$ the set of superflows satisfying (4), and by $\mathbf{Q}_{\boldsymbol{\lambda}}$ the set of overload vectors induced by superflows in $\mathbf{F}_{\boldsymbol{\lambda}}$.

The throughput $T_{\boldsymbol{f}}$ of a superflow is defined as the sum of flow intensities at the links terminating at the sink node, i.e.,

$$T_{\boldsymbol{f}} = \sum_{l\in L_{in}(d)} f_l. \tag{5}$$

The traffic load Λ of the network is the sum of exogenous arrivals intensities,

$$\Lambda = \sum_{i\in K} \lambda_i.$$

It can be derived from the definition of q_i that for any subset $S \subseteq K$ it holds,

$$\sum_{i\in S} q_i = \sum_{i\in S} \lambda_i + \sum_{l\in L_{in}(S)} f_l - \sum_{l\in L_{out}(S)} f_l \geq 0. \tag{6}$$

From the above equation for $S = K$ and the assumption that $L_{in}(S) = L_{out}(d) = \varnothing$, we get,

$$T_{\boldsymbol{f}} = \Lambda - \sum_{i\in K} q_i. \tag{7}$$

If the superflow is a flow, then $q_i = 0$ for all $i \in K$ and $\Lambda = T_{\boldsymbol{f}}$, i.e., the throughput equals the traffic load.

IV. Performance Objectives

The performance of the policy in overload mode is quantified through the overload vector $\boldsymbol{q}$ of the corresponding superflow. Several physically important properties of a policy correspond to certain mathematical properties of the overload vector. From (7) we see that in order to maximize the network throughput it is enough to minimize the aggregate overload

$$\min_{\boldsymbol{q}\in\mathbf{Q}_{\boldsymbol{\lambda}}} \sum_{i\in K} q_i. \tag{8}$$

Also observe that if all the buffers at the nodes are equal, the time to buffer overflow of node i is $(q_i)^{-1}$, and the time to first buffer overflow in the network is, $\min_{i\in K}\{q_i{}^{-1}\} = (\max_{i\in K} q_i)^{-1}$. Hence overload vectors that are solutions to the following problem,

$$\min_{\boldsymbol{q}\in\mathbf{Q}_{\boldsymbol{\lambda}}} \max_{i\in K} q_i, \tag{9}$$

maximize the time to first buffer overflow in the network. A stronger criterion than (9) is lexicographic minimization. This optimization, also known as min-max optimization [3], is based on the following order relation between vectors. Given a vector $\boldsymbol{v}=(v_1,...,v_n)$, let $\overline{v}_i$, $i=1,...,n$ be the ith maximal coordinate of $\boldsymbol{v}$. We say that vector $\boldsymbol{v}$ is *lexicographically smaller* than vector $\boldsymbol{u}$, denoted by $\boldsymbol{v}\prec\boldsymbol{u}$, if either $\overline{v}_1 < \overline{u}_1$, or for some i, $1 \le i < n$, $\overline{v}_j = \overline{u}_j$ for $1 \le j \le i$ and $\overline{v}_{i+1} < \overline{u}_{i+1}$. If in addition we allow for the possibility that $\overline{v}_i = \overline{u}_i$, for all $i = 1, 2, ..n$, we denote $\boldsymbol{v}\preccurlyeq\boldsymbol{u}$. Note that if for two vector $\boldsymbol{v}$, $\boldsymbol{u}$ we have $\boldsymbol{v} \preccurlyeq \boldsymbol{u}$ then by definition $\max_i v_i \le \max_i u_i$.

According to the previous discussion, attempting to both maximize throughput and minimize the time to first buffer overflow amounts to solving simultaneously problems (8) and (9). In general solving each of these problems separately, does not guarantee a solution to the other one. However, as will be shown in the sequel, lexicographic minimization of node overloads does provide optimal solution to both of these problems.

In fact, it turns out the an even stronger property than lexicographic minimization holds for the network under consideration. To this end, we introduce the following partial ordering. We say that vector $\boldsymbol{v}$ is *more balanced* than vector $\boldsymbol{u}$, denoted by $\boldsymbol{v}\vdash\boldsymbol{u}$, if the following inequalities hold

$$\sum_{l=1}^{i} \overline{v}_l \le \sum_{l=1}^{i} \overline{u}_l,\ i=1,...,n. \tag{10}$$

Note that if for the overload vectors $\boldsymbol{q}^1$, $\boldsymbol{q}^2$ we have that $\boldsymbol{q}^1 \vdash \boldsymbol{q}^2$ then it follows that $\boldsymbol{q}^1\preccurlyeq \boldsymbol{q}^2$ and furthermore, because of (7), the throughput under $\boldsymbol{q}^1$ is larger than under $\boldsymbol{q}^2$. Moreover, in case the node buffers are finite and of equal size and the buffers of all nodes are empty at time $t=0$, a most balanced overload vector *minimizes at any time* t, *the amount of lost traffic* due to buffer overflow. To see this, assume that the node buffers are of size $A \ge 0$. Then, under overload vector $\boldsymbol{q}$, node i will start loosing traffic after time $t_{q_i} = A/q_i$. Therefore, defining

$$i_t \triangleq \max_{1\le i\le n} \left\{ \overline{q}_i \ge \frac{A}{t} \right\}, \tag{11}$$

the amount of traffic lost due to buffer overflow is (define $\overline{q}_i\left(t - A/\overline{q}_i\right) = 0$ if $\overline{q}_i = 0$),

$$F_{\boldsymbol{q}}^{lost}(t) \triangleq \sum_{i=1}^{i_t} \overline{q}_i \left(t - \frac{A}{\overline{q}_i} \right) = t \sum_{i=1}^{i_t} \overline{q}_i - i_t A. \tag{12}$$

Let now $\boldsymbol{q}^*$ be a most balanced overload vector and again $i_t^* \triangleq \max_{1\le i\le n}\{\overline{q}_i^* \ge A/t\}$. We will show that $F_{\boldsymbol{q}}^{lost}(t) \ge F_{\boldsymbol{q}^*}^{lost}(t)$ for any $t \ge 0$. Indeed, consider two cases

a) $i_t \ge i_t^*$. Then, since for $i \le i_t$ it holds $\overline{q}_i \ge A/t$, we have,

$$t \sum_{i=i_t^*+1}^{i_t} \overline{q}_i \ge \left(i_t - i_t^*\right) A. \tag{13}$$

Therefore,

$$\begin{aligned} F_{\boldsymbol{q}}^{lost}(t) &= t\sum_{i=1}^{i_t^*} \overline{q}_i - i_t^* A + t \sum_{i=i_t^*+1}^{i_t} \overline{q}_i - \left(i_t - i_t^*\right) A \\ &\ge t\sum_{i=1}^{i_t^*} \overline{q}_i^* - i_t^* A \quad \text{by (10) and (13)} \\ &= F_{\boldsymbol{q}^*}^{lost}(t). \end{aligned}$$

b) $i_t < i_t^*$. Then, since for $i > i_t$ it holds $\overline{q}_i < A/t$, we have,

$$t \sum_{i=i_t+1}^{i_t^*} \overline{q}_i < \left(i_t^* - i_t\right) A. \tag{14}$$

Therefore,

$$\begin{aligned} F_{\boldsymbol{q}}^{lost}(t) &= t\sum_{i=1}^{i_t^*} \overline{q}_i - i_t^* A - t \sum_{i=i_t+1}^{i_t^*} \overline{q}_i + \left(i_t^* - i_t\right) A \\ &> t\sum_{i=1}^{i_t^*} \overline{q}_i^* - i_t^* A \quad \text{by (10) and (14)} \\ &= F_{\boldsymbol{q}^*}^{lost}(t). \end{aligned}$$

Hence a "most balanced" overload vector according to relation $\vdash$ is a very desirable property. A potential complication is that relation $\vdash$ is a partial ordering and not any two overload vectors are comparable with respect to that ordering, unlike the throughput or the lexicographic criterion that are total orderings. While it is certain than an optimal throughput overload vector exist and the same holds for the lexicographically optimal solution ([8]), this is not always the case for a most balanced overload vector. It is shown in the following that for the network under consideration a lexicographically minimal vector is also most balanced.

V. Properties and Characterization of Most Balanced Superflows

Consider a superflow, SABP, characterized by the following inequalities that hold for any link $l=(i,j) \in L$ (for simplicity of the notation we define $f_{ij} = f_{(i,j)}$).

SABP superflow.

$$\text{If } q_i < q_j, \text{ then } f_{ij} = 0, \tag{15a}$$

$$\text{if } q_i > q_j, \text{ then } f_{ij} = c_{ij}. \tag{15b}$$

A superflow satisfying inequalities (15) is called "Superflow of Adaptive Back Pressure policy". The reason for this terminology is that, as will be seen in Section VI, a SABP superflow can be obtained as limit of superflows induced by a distributed adaptive flow update policy that is similar to the ABP policy described in the introduction. The SABP superflow can also be thought of as the equilibrium point of "selfish routing" in cases where the only information available to users is the backlog change rate at the node where they are located, as well as at the outgoing neighbors of that node. Hence the node (or agents located at the node) directs its traffic only to nodes with smaller overloads in the hope that this way the traffic will encounter smaller congestion. Problems related to selfish routing have been the subject of several studies, see [15] and the references therein.

The main result of the paper is summarized in the following theorem.

Theorem 1: A superflow induces a most balanced overload vector if and only if it is SABP. The most balanced overload vector is unique - however there may be more than one superflows inducing the most balanced overload vector.

The proof of the theorem is outlined in the following. For more details the reader is referred to the Appendix.

We start with the following lemma which shows that a superflow inducing a lexicographically minimal node overload vector is SABP.

Lemma 2: Let $\boldsymbol{f}^*$ be a superflow inducing a lexicographically minimal vector. Then $\boldsymbol{f}^*$ is SABP.

To proceed we need some further properties that are satisfied by the overload vector induced by a superflow. Because of (6) and (4b) we have for any subset of nodes $S \subseteq K$,

$$\sum_{i \in S} q_i = \sum_{i \in S} \lambda_i + \sum_{l \in L_{in}(S)} f_l - \sum_{l \in L_{out}(S)} f_l \geq \left(\sum_{i \in S} \lambda_i - \sum_{l \in L_{out}(S)} c_l \right)^+ . \tag{16}$$

Let us define for any $S \subseteq K, \ S \neq \varnothing$,

$$B(S) \triangleq \left(\sum_{i \in S} \lambda_i - \sum_{l \in L_{out}(S)} c_l \right)^+ . \tag{17}$$

For $S = \varnothing$ we use the convention $B(S) = 0$. The following lemma provides a lower bound on the maximum overload values on subsets of K. For a set X, $|X|$ denotes the number of its elements. If $S = \varnothing$, we use the convention $B(S)/|S| = 0$.

Lemma 3: Under any superflow $\boldsymbol{f}$ inducing overload vector $\boldsymbol{q}$, for any $S \subseteq K$,

$$\max_{i \in S} q_i \geq B(S)/|S|, \tag{18}$$

and

$$\max_{i \in K} q_i \geq \max_{S \subseteq K} B(S)/|S|. \tag{19}$$

Let

$$\widehat{R}_1 = \max_{S \subseteq K} \frac{B(S)}{|S|},$$
$$\mathcal{S}_1 = \left\{ S : \frac{B(S)}{|S|} = \widehat{R}_1, \ S \subseteq K, \ S \neq \varnothing \right\},$$
$$\widehat{S}_1 = \cup_{S \in \mathcal{S}_1} S.$$

The next lemma shows the basic property of SABP superflows related to min-max optimization.

Lemma 4: Let $\boldsymbol{f}$ be a SABP superflow inducing overload vector $\boldsymbol{q}$. Then (19) is achieved with equality by $\boldsymbol{q}$ and $\widehat{S}_1$ is the set of nodes with maximal overloads under $\boldsymbol{f}$, i.e.,

$$q_i = \max_{j \in K} \{q_j\} = \widehat{R}_1, \ i \in \widehat{S}_1.$$

Hence, $\widehat{S}_1$ is the set of nodes with maximal overload under a lexicographically minimal superflow.

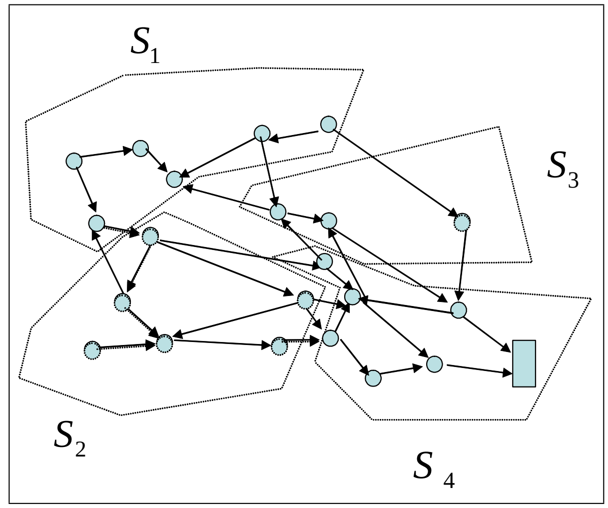

Fig. 3. A partitioning of the network to regions of nodes with the same level of traffic load intensity is depicted. Region $\widehat{S}_1$ includes all nodes with maximum backlog build-up rate under the SABP policy. It corresponds to a "hot spot" of the network. Region $\widehat{S}_2$ includes all nodes with second maximum backlog build-up rate and so on. Region $\widehat{S}_4$ includes all nodes that are not overloaded and the backlog build-up rate there is zero.

Consider now a SABP superflow. If $\widehat{S}_1 = K$ then according to the previous discussion any SABP superflow has overload vector $\boldsymbol{q}^*$ such that,

$$q_i^* = \frac{B(K)}{|K|}, \ i \in K.$$

Assume next that $\widehat{S}_1 \subset K$. Since $\max_{i \in \widehat{S}_1^c} q_i < q$, we conclude from the definition of SABP superflow that,

$$f_{ij} = 0, \text{ for all } (i,j) \in L_{in}\left(\widehat{S}_1\right), \tag{20}$$

$$f_{ij} = c_{ij}, \text{ for all } (i,j) \in L_{out}\left(\widehat{S}_1\right). \tag{21}$$

Consider the reduced network where the subgraph that consists of the nodes in $\widehat{S}_1$ is removed and for each link in $L_{out}\left(\widehat{S}_1\right)$ we put an exogenous arrival source to the node in $\widehat{S}_1^c$ where the link terminates with intensity equal to the link capacity. We can apply the same argument to the reduced graph in order to determine the set $\widehat{S}_2$ of nodes on which the second largest overload for any SABP superflow is achieved. In this manner we end-up getting node sets $\widehat{S}_1, \widehat{S}_2, ..., \widehat{S}_L$, where on set $\widehat{S}_l$ the lth maximal overload for any SABP superflow is achieved, see Figure 3 for a graphical representation. Hence any SABP superflow determines uniquely the node overloads. Since by Lemma 2 any most balanced superflow is also a SABP superflow, we conclude.

Lemma 5: A superflow induces a lexicographically minimal overload vector if and only if it is SABP. The lexicographically minimal overload vector is unique.

While the lexicographically minimal overload vector is unique, there may be multiple superflows that achieve this overload vector. An example is shown in Figure 4.

The next lemma shows that a lexicographically minimal overload vector is also most balanced.

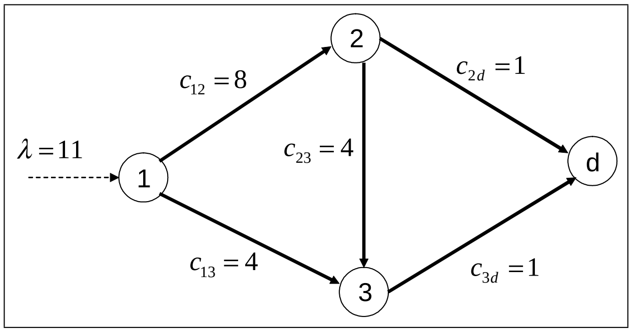

Fig. 4. For the network in the figure, the lexicographically optimal overload vector is $q_1 = q_2 = q_3 = 3$. Two superflows that achieve this vector are: 1) $f_{12} = f_{13} = 4,\ f_{23} = 0,\ f_{2d} = f_{3d} = 1$, and 2) $f_{12} = 8,\ f_{23} = 4,\ f_{13} = 0,\ f_{2d} = f_{3d} = 1$.

Lemma 6: Let $\boldsymbol{q}^*$ be the overload vector induced by a SABP superflow. Then

$$\boldsymbol{q}^* \vdash \boldsymbol{q} \text{ for all } \boldsymbol{q} \in \mathbf{Q}_{\boldsymbol{\lambda}}. \tag{22}$$

Combining Lemmas 5 and 6 we obtain Theorem 1.

The next Theorem shows that (22) is equivalent to minimizing the sum of any convex nondecreasing function of node overloads.

Theorem 7: For a vector of real numbers $\boldsymbol{q} = (q_i)_{i=1}^n$, it holds

$$\sum_{l=1}^{i} \overline{q}_l^* \leq \sum_{l=1}^{i} \overline{q}_l, \text{ for all } i = 1, ..., n,$$

if and only if

$$\sum_{i=1}^{n} g(q_i^*) \leq \sum_{i=1}^{n} g(q_i), \tag{23}$$

for any convex nondecreasing function $g(q)$.

We note that property (23) is the defining property of "most balanced" assignment in [11].

VI. Distributed Asynchronous SABP Computation Policies

In this section we present an asynchronous distributed method for computing SABP superflows that relies on the following local adjustment of a flow, done on a per link basis.

Link Flow Update Rule

For link (i, j) do the following flow update:

- If $q_i > q_j$ and $f_{ij} < c_{ij}$ then increase f_{ij} until either $f_{ij} = c_{ij}$ or $q_i = q_j$
- If $q_i < q_j$ and $f_{ij} > 0$ then decrease f_{ij} until either $f_{ij} = 0$ or $q_i = q_j$

If the above iteration is performed infinitely often by each link, without any need for coordination of successive iterations among links we have convergence to an SABP superflow. More specifically, let $t_n,\ n = 0, 1, ...$ be a sequence of flow adjustment times, $t_n > t_{n-1},\ n = 1, 2, ...$ and $l_n = (i_n, j_n),\ n = 0, 1, ...$ be a sequence of links such that the origin node of link l_n performs the flow adjustment operation described above at time t_n on link l_n. Assume that the update operation on each link is performed infinitely often, i.e. for any time T and for any link $l = (i, j)$ there is an update instant $t_k > T$ at which node i performs the update on link l. We have the following theorem

Theorem 8: Let $\boldsymbol{f}^n$ be the superflow vector at time t_n and $\boldsymbol{q}^n$ the associated overload vector. If $\boldsymbol{q}^*$ is the unique overload vector associated with all SABP superflows, then

$$\lim_{n\to\infty} \boldsymbol{q}^n = \boldsymbol{q}^*,$$
$$\lim_{n\to\infty} \boldsymbol{f}^n = \boldsymbol{f}^*.$$

where $\boldsymbol{f}^*$ is some SABP superflow.

As discussed in Section V, while $\boldsymbol{q}^*$ is unique, $\boldsymbol{f}^*$is not. The particular SABP superflow to which convergence is obtained in Theorem 8 depends on the sequence $l_n,\ n = 0, ..$ according to which link flows are adjusted.

VII. Balancing a Weighted Overload Vector

Several of the results of the previous sections can be extended to the case where it is of interest to determine a most balanced weighted overload vector in the sense of achieving lexicographic minimization of the vector $\boldsymbol{p} = (q_i/a_i)_{i\in N}$ for given constants $a_i > 0,\ i \in N$. This is of interest in situations where the buffer sizes of network nodes may differ.

In this section we assume that for a given weight vector $\boldsymbol{a} = (a_i)_{i\in N}$, the performance vector of interest is $\boldsymbol{p} = (q_i/a_i)_{i\in N}$. For this performance vector, Theorem 1 still holds, with "most balanced" replaced by "lexicographically minimal". The only change necessary for the proofs to work is to replace for any set S of nodes, $|S|$ with $M(S) = \sum_{i\in S} a_i$. More specifically, we define an SABP$^{\boldsymbol{a}}$ superflow as follows

SABP$^{\boldsymbol{a}}$ superflow.

$$\text{If } \frac{q_i}{a_i} < \frac{q_j}{a_j}, \text{ then } f_{ij} = 0,$$
$$\text{if } \frac{q_i}{a_i} > \frac{q_j}{a_j}, \text{ then } f_{ij} = c_{ij}.$$

We then have the following theorem.

Theorem 9: For a given weight vector $\boldsymbol{a} = (a_i)_{i\in N}$, a superflow induces a lexicographically minimal vector $\boldsymbol{p} = (q_i/a_i)_{i\in N}$ if and only if it is SABP$^{\boldsymbol{a}}$. The lexicographically minimal weighted overload vector is unique - however there may be more than one superflows inducing this vector.

Also, it holds true the partitioning of the network nodes into sets $\widehat{S}_1^{\boldsymbol{a}}, \widehat{S}_2^{\boldsymbol{a}}, ..., \widehat{S}_L^{\boldsymbol{a}}$, where on set $\widehat{S}_m^{\boldsymbol{a}}$ the mth maximal of $\boldsymbol{p}^* = (q_i^*/a_i)_{i\in N}$ for any SABP$^{\boldsymbol{a}}$ superflow $\boldsymbol{f}^*$ is achieved, and for any $M,\ 1 \leq M \leq L$,

$$f_{ij}^* = 0, \text{ for all } (i, j) \in L_{in}\left(\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{a}}\right), \tag{25}$$

$$f_{ij}^* = c_{ij}, \text{ for all } (i, j) \in L_{out}\left(\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{a}}\right). \tag{26}$$

For general $\boldsymbol{a}$, a lexicographically minimal overload vector $\boldsymbol{p}^* = (q_i^*/a_i)_{i\in N}$ is not most balanced in the sense of (10). However, two main properties discussed in Section IV, namely *throughput maximization* and *lost traffic minimization* still hold.

Regarding throughput minimization, observe first that if $\boldsymbol{p}^* = \mathbf{0}$ then the throughput is $\sum_{i\in K} \lambda_i$ and therefore maximal. If on the other hand $\boldsymbol{p}^* \neq \mathbf{0}$, then $L > 1$ and $\widehat{S}_L^{\boldsymbol{a}} = \{i \in N : q_i^* = 0\}$. From (25), (26), (6) we have,

$$\sum_{i \in N} q_i^* = \sum_{i \in \cup_{m=1}^{L-1} \widehat{S}_m^{\boldsymbol{a}}} q_i^* = \sum_{i \in \cup_{m=1}^{L-1} \widehat{S}_m^{\boldsymbol{a}}} \lambda_i - \sum_{l \in L_{out}\left(\cup_{m=1}^{L-1} \widehat{S}_m^{\boldsymbol{a}}\right)} c_{ij}.$$

Consider now any other overload vector $\boldsymbol{q}$. We have,

$$\begin{aligned} \sum_{i \in N} q_i &\geq \sum_{i \in \cup_{m=1}^{L-1} \widehat{S}_m^{\boldsymbol{a}}} q_i \\ &\geq \sum_{i \in \cup_{m=1}^{L-1} \widehat{S}_m^{\boldsymbol{a}}} \lambda_i - \sum_{l \in L_{out}\left(\cup_{m=1}^{L-1} \widehat{S}_m^{\boldsymbol{a}}\right)} c_{ij} \quad \text{by (6)} \\ &= \sum_{i \in N} q_i^*, \end{aligned}$$

which by (7) implies throughput maximization.

To address the issue of lost traffic minimization, let $A_i > 0$ be the buffer size of node i. Consider an SABP$^{\boldsymbol{A}}$ superflow and let $\boldsymbol{p}^* = (q_i^*/A_i)_{i \in N}$ be the induced performance vector. Let γ_m by the mth maximal of $\boldsymbol{p}^* = (q_i^*/A_i)_{i \in N}$. For an overload vector $\boldsymbol{q}$, let $F_{M,\boldsymbol{q}}^{lost}(t)$ be the amount of lost traffic due to buffer overflow up to time t, by nodes in the set $\cup_{m=1}^{M} \widehat{S}_m^{\boldsymbol{A}}$. We then have the following theorem.

Theorem 10: Under an SABP$^{\boldsymbol{A}}$ superflow, if all buffers are empty at time 0, then for any $M = 1, 2, .., L-1$, for any t in the interval $[1/\gamma_M, 1/\gamma_{M+1})$, it holds,

$$F_{M,\boldsymbol{q}}^{lost}(t) \geq F_{\boldsymbol{q}^*}^{lost}(t).$$

Since by definition $F_{M,\boldsymbol{q}}^{lost}(t) \leq F_{\boldsymbol{q}}^{lost}(t)$ for all $t \geq 0$, $F_{\boldsymbol{q}^*}^{lost}(t) = 0$ for $0 \leq t < 1/\gamma_1$, and $\gamma_L = 0$, we conclude,

Corollary 11: Under an SABP$^{\boldsymbol{A}}$ superflow, if all buffers are empty at time 0, then for any $t \geq 0$,

$$F_{\boldsymbol{q}}^{lost}(t) \geq F_{\boldsymbol{q}^*}^{lost}(t).$$

VIII. Discussion and Conclusions

A problem related to the one discussed in this paper has been studied by Sasaki and Hajek in [16], following earlier related studies [17], [12]. They consider a single commodity flow network with a feasible arrival traffic load vector and some initial backlog distribution at the nodes. They provide various algorithm for computing a fluid dynamic routing policy that evacuates the network in an optimal manner. More specifically under the optimal policy the amount of remaining traffic in the network (aggregate backlog) is minimized sample path-wise, i.e. at every time instance. A direct consequence is that both the network evacuation time is minimized as well as the total delay; the latter being the time integral of the remaining backlog till evacuation. The optimal routing policy is obtained by flow relaxation resulting in a quasi-static routing policy that is expressed as a sequence of feasible flows that remain fixed for certain time intervals. That study was for traffic load within the feasible region. The same question may be phrased in the more general case of traffic load not being necessarily within the feasibility region. A more general problem that combines our approach in this paper with delay consideration will be to minimize losses and under the condition of minimum losses to minimize delay. A combination of an SABP flow routing, with flow relaxation in the resulting feasible component of the network will achieve that conditional multiobjective optimization.

Polynomial time algorithms for determining a SABP superflow exist, but they do not lead to distributed adaptive policies which are of main concern when ad-hoc and sensor networks are considered.

There is an apparent similarity between SABP flows in the fluid model studied in this paper and the adaptive back pressure policy that performs packet level forwarding in a stochastic dynamic model of the system. We conjecture that the fluid flow limit of the stochastic dynamic model of the network operated under the adaptive back pressure policy is an SABP flow when the arrival rate vector is outside the stability region. When the arrival rate vector is in the feasible region the conjecture is true. From the global stability property of ABP it follows that the fluid flow limit of the network in the feasible load case will be a feasible flow for the particular traffic load vector. By definition of SABP every feasible flow is SABP.

In this work we concentrated on the routing and forwarding aspects of information transmission. A topic of further investigation is to consider cross-layer issues where wireless node interactions, rate adaptations and power control are also taken into account. In this case, it turns out that lexicographic minimization of node overloads does not guarantee throughput maximization and a different formulation is required. A step towards this direction has been taken in our recent work [10].

References

[1] W. Aiello, E. Kushilevitz, R. Ostrovsky, and A. Rosen, “Adaptive packet routing for bursty adversarial traffic,” in *Proc. ACM Symposium on Theory of Computing*, 1998.

[2] B. Awerbuch, P. Berenbrink, A. Brinkmann, and C. Scheideler, “Simple routing strategies for adversarial systems,” in *Proc. IEEE Symposium on Foundations of Computer Science*, 2001.

[3] D. Bertsekas and R. Gallager, *Data Networks*. Englewood Cliffs, New Jersey 07632: Prentice Hall, 1992.

[4] R. Cruz, “A calculus for network delay, part i: Network elements in isolation,” *IEEE Transactions on Information Theory*, vol. 37, pp. 114–131, 1991.

[5] G. Dai and W. Lin, “Maximum pressure policies in stochastic processing networks,” *Operations Research*, to appear.

[6] G. Dai, W. L. R. Moorthy, and C. P. Teo, “Berth allocation planning optimization in container terminals,” preprint.

[7] A. Eryilmaz and R. Srikant, “Fair resource allocation in wireless networks using queue-length based scheduling and congestion control,” in *IEEE Infocom*, Maimi, Florida, USA, March 13-17 2005.

[8] L. Georgiadis, P. Georgatsos, K. Floros, and S. Sartzetakis, “Lexicographically optimal balanced networks,” *IEEE/ACM Transactions on Networking*, vol. 10, no. 6, pp. 818–829, December 2002.

[9] L. Georgiadis and L. Tassiulas, “Most balanced overload response in sensor networks,” in *International Symposium on Information Theory*, Adelaide, Australia, 2005.

[10] ——, “Robust network response to overload traffic fluctuations,” in *44th Allerton Conference on Communication, Control and Computing*, Illinois, USA, September 28-30 2005.

[11] B. Hajek, “Performance of global load balancing by local adjustment,” *IEEE Transactions on Information Theory*, vol. 36, no. 6, pp. 1398–1414, November 1990.

[12] B. Hajek and R. G. Ogier, “Optimal dynamic routing in communication networks with continuous traffic,” *Networks*, vol. 14, pp. 457–487, 1984.

[13] M. J. Neely, E. Modiano, and C.-P. Li, "Fairness and optimal stochastic control for heterogenous networks," in *IEEE Infocom 2005*, Miami, Florida, USA, March 13-17 2005.

[14] M. J. Neely, E. Modiano, and C. E. Rohrs, "Dynamic power allocation and routing for time varying wireless networks," *IEEE Journal on Selected Areas in Communications, Special Issue on Wireless Ad-hoc Networks*, vol. 23, no. 1, pp. 89–103, January 2005.

[15] T. Roughgarden and E. Tardos, "How bad is selfish routing?" *ACM*, vol. 49(2), pp. 236–259, 2002.

[16] G. Sasaki and B. Hajek, "Optimal dynamic routing in single commodity networks by iterative methods," *IEEE Transactions on Communications*, vol. com-35, no. 11, pp. 1199–1206, November 1987.

[17] A. Segall, "The modeling of adaptive routing in data communication networks," *IEEE Transactions on Communications*, vol. COM-25, pp. 85–95, January 1977.

[18] A. L. Stolyar, "Maximizing queueing network utility subject to stability: Greedy primal-dual algorithm," *Queueing Systems*, to appear.

[19] L. Tassiulas, "Dynamic link activation scheduling in multihop radio networks with fixed of changing topology," Ph.D. dissertation, University of Maryland, College Park, 1991.

[20] ——, "Adaptive back-pressure congestion control based on local information," *IEEE Transactions on Automatic Control*, vol. 40, no. 2, pp. 236–250, February 1995.

[21] L. Tassiulas and P. Bhattacharya, "Allocation of interdendent resources for maximum throughput," *Stochastic Models*, vol. 16, no. 1, 2000.

[22] L. Tassiulas and A. Ephremides, "Stability properties of constrained queueing systems and scheduling policies for maximum throughtput in multihop radio networks," *IEEE Transactions on Automatic Control*, vol. 37, no. 12, pp. 1936–1949, December 1992.

[23] V. Tsibonis, L. Georgiadis, and L. Tassiulas, "Exploiting wireless channel state information for throughput maximization," *IEEE Transactions on Information Theory*, vol. 50, no. 11, pp. 2566–2582, November 2004.

APPENDIX

A. Proofs of Lemmas and Theorems

1) ***Lemma 2*** *: Let $\boldsymbol{f}^*$ be a superflow inducing a lexicographically minimal vector. Then $\boldsymbol{f}^*$ is SABP.*

Proof: Let $\boldsymbol{q}^*$ be the overload vector under superflow $\boldsymbol{f}^*$. Let for some link (i,j), $q_i^* < q_j^*$, but $f_{ij}^* > 0$. Reduce f_{ij}^* by $\min\left\{f_{ij}^*, \left(q_j^* - q_i^*\right)/2\right\}$, i.e., create a new superflow $\boldsymbol{f}$ such that $f_l = f_l^*$ for all links different than (i,j),and for link (i,j),

$$f_{ij} = \max\left\{0, f_{ij}^* - \left(q_j^* - q_i^*\right)/2\right\}.$$

This either makes $f_{ij} = 0$ or equalizes q_i and q_j, where $\boldsymbol{q}$ is the overload under $\boldsymbol{f}$. In either case, $\boldsymbol{q} \prec \boldsymbol{q}^*$, a contradiction. Similarly, if for $j \neq d$ it holds $q_i^* > q_j^*$, but $f_{ij}^* < c_{ij}$, create a new superflow $\boldsymbol{f}$ such that $f_l = f_l^*$ for all links different than (i,j),and for link (i,j),

$$f_{ij} = \min\left\{c_{ij},\ f_{ij}^* + \left(q_i^* - q_j^*\right)/2\right\}.$$

This either makes $f_{ij} = c_{ij}$ or equalizes q_i and q_j. In either case, $\boldsymbol{q} \prec \boldsymbol{q}^*$, a contradiction. If $j = d$, the same argument can be applied by defining

$$f_{ij} = \min\left\{c_{ij},\ f_{ij}^* + q_i^*\right\}.$$

■

2) ***Lemma 3.*** *: Under any superflow $\boldsymbol{f}$ inducing overload vector $\boldsymbol{q}$, for any $S \subseteq K$,*

$$\max_{i \in S} q_i \geq B(S)/\left|S\right|, \tag{27}$$

and

$$\max_{i \in K} q_i \geq \max_{S \subseteq K} B(S)/\left|S\right|. \tag{28}$$

Proof: If $\max_{i \in S} q_i < B(S)/\left|S\right|$ then $\sum_{i \in S} q_i < B(S)$, which contradicts (16). Hence

$$\max_{i \in S} q_i < \frac{B(S)}{|S|}.$$

Inequality (28) follows by observing that $\max_{i \in K} q_i = \max_{S \subseteq K} \max_{i \in S} q_i$. ■

3) ***Lemma 4****: Let $\boldsymbol{f}$ be a SABP superflow inducing overload vector $\boldsymbol{q}$. Then (19) is achieved with equality and $\widehat{S}_1$ is the set of nodes with maximal overloads under $\boldsymbol{f}$, i.e.,*

$$q_i = \max_{j \in K}\left\{q_j\right\} = \widehat{R}_1,\ i \in \widehat{S}_1.$$

Hence, $\widehat{S}_1$ is the set of nodes with maximal overload under a lexicographically minimal superflow.

We first need the following lemmas.

Lemma 12: If for two subsets of K, S_1 and S_2, it holds,

$$\sum_{i \in S_1} \lambda_i \geq \sum_{l \in L_{out}(S_1)} c_l, \tag{29}$$

$$\sum_{i \in S_2} \lambda_i \geq \sum_{l \in L_{out}(S_2)} c_l, \tag{30}$$

then

$$B(S_1) + B(S_2) \leq B(S_1 \cup S_2) + B(S_1 \cap S_2). \tag{31}$$

Proof: Notice that in general it holds,

$$\sum_{l \in L_{out}(S_1)} c_l + \sum_{l \in L_{out}(S_2)} c_l \geq \sum_{l \in L_{out}(S_1 \cup S_2)} c_l + \sum_{l \in L_{out}(S_1 \cap S_2)} c_l.$$

This, (17), (29) and (30) imply that

$$\begin{aligned} B(S_1) + B(S_2) &= \sum_{i \in S_1} \lambda_i + \sum_{i \in S_2} \lambda_i - \\ &\quad \sum_{l \in L_{out}(S_1)} c_l - \sum_{l \in L_{out}(S_2)} c_l \\ &\leq \sum_{i \in S_1 \cup S_2} \lambda_i + \sum_{i \in S_1 \cap S_2} \lambda_i - \\ &\quad \sum_{l \in L_{out}(S_1 \cup S_2)} c_l - \sum_{l \in L_{out}(S_1 \cap S_2)} c_l \\ &\leq B(S_1 \cup S_2) + B(S_1 \cap S_2). \end{aligned}$$

■

Lemma 13: If for two subsets of K, S_1 and S_2, it holds,

$$\frac{B(S_1)}{|S_1|} = \frac{B(S_1)}{|S_2|} = \widehat{R}_1, \tag{32}$$

then

$$\frac{B(S_1 \cup S_2)}{|S_1 \cup S_2|} = \widehat{R}_1. \tag{33}$$

Moreover, if $S_1 \cap S_2 \neq \varnothing$ then also,

$$\frac{B(S_1 \cap S_2)}{|S_1 \cap S_2|} = \widehat{R}_1. \tag{34}$$

Proof: By the definition of $\widehat{R}_1$ we have for any nonempty subsets A_1 and A_2 of K,

$$\frac{B(A_1)}{|A_1|} \leq \widehat{R}_1,\ \frac{B(A_2)}{|A_2|} \leq \widehat{R}_1,$$

which implies that

$$\frac{B(A_1)+B(A_2)}{|A_1|+|A_2|} \le \widehat{R}_1, \tag{35}$$

with equality if and only if

$$\frac{B(A_1)}{|A_1|} = \frac{B(A_2)}{|A_2|} = \widehat{R}_1. \tag{36}$$

If $\widehat{R}_1 = 0$ then the definition of $\widehat{R}_1$ implies that $B(S) = 0$ for any subset of K and the lemma follows. Assume next that $\widehat{R}_1 > 0$, hence $B(S_1) > 0$ and $B(S_2) > 0$. By Lemma 12 we have

$$B(S_1) + B(S_2) \le B(S_1 \cup S_2) + B(S_1 \cap S_2).$$

Taking also into account (32) and the fact that $|S_1| + |S_2| = |S_1 \cup S_2| + |S_1 \cap S_2|$, we have,

$$\begin{aligned} \widehat{R}_1 &= \frac{B(S_1)+B(S_2)}{|S_1|+|S_2|} \\ &\le \frac{B(S_1 \cup S_2)+B(S_1 \cap S_2)}{|S_1 \cup S_2|+|S_1 \cap S_2|}. \end{aligned} \tag{37}$$

If $S_1 \cap S_2 = \varnothing$ then $B(S_1 \cap S_2) = 0$, hence the definition of $\widehat{R}_1$ and (37) imply (33). If $S_1 \cap S_2 \neq \varnothing$, (37) and (35) imply that,

$$\frac{B(S_1 \cup S_2)+B(S_1 \cap S_2)}{|S_1 \cup S_2|+|S_1 \cap S_2|} = \widehat{R}_1,$$

and by (36), equations (33) and (34) hold. ■

Proof: (**of Lemma 4**). Let S_1 be the set of all indices i such that $q_i = \max_{j \in K}\{q_j\} \triangleq q$. If $q = 0$ then the assertion follows directly from Lemma 3. Assume next that $q > 0$. Because of (4) and (15) we have,

$$\begin{aligned} q\,|S_1| &= \sum_{i \in S_1} q_i \\ &= \sum_{i \in S_1} \lambda_i - \sum_{l \in L_{out}(S_1)} c_l \\ &= B(S_1). \end{aligned}$$

Hence $q = B(S_1)/|S_1| \le \widehat{R}_1$. Also, Lemma 13 implies that $\widehat{R}_1 = B(\widehat{S}_1)/\left|\widehat{S}_1\right|$. Taking also into account (28) we conclude,

$$q = \frac{B(S_1)}{|S_1|} = \frac{B(\widehat{S}_1)}{\left|\widehat{S}_1\right|} = \widehat{R}_1. \tag{38}$$

By the definition of $\widehat{S}_1$ we have,

$$S_1 \subseteq \widehat{S}_1.$$

Now, from the definition of q we have

$$\begin{aligned} \sum_{i \in \widehat{S}_1} q_i &\le \left|\widehat{S}_1\right| q \\ &= B(\widehat{S}_1), \text{ by (38)}. \end{aligned}$$

This and (16) imply that

$$\sum_{i \in \widehat{S}_1} q_i = B(\widehat{S}_1),$$

which in turn implies that $q_i = q$, $i \in \widehat{S}_1$. Hence

$$\widehat{S}_1 \subseteq S_1,$$

and the lemma follows. Since according to Lemma 2 any lexicographically minimal superflow is SABP, we conclude that the same holds for any lexicographically minimal superflow. ■

*4) **Lemma 6**: Let $\boldsymbol{q}^*$ be the overload vector induced by a SABP superflow. Then*

$$\boldsymbol{q}^* \vdash \boldsymbol{q} \text{ for all } \boldsymbol{q} \in \mathbf{Q}_{\boldsymbol{\lambda}}. \tag{39}$$

We first need the following lemma.

Lemma 14: a) If $\widehat{R}_n > 0$ for $n = 1,..,m \le L$, then the following equalities hold,

$$\sum_{n=1}^{l} \widehat{R}_n \left|\widehat{S}_n\right| = B\left(\cup_{n=1}^{l} \widehat{S}_n\right) > 0,\ l = 1,2,...m.$$

b) The following inequalities hold for any overload vector $\boldsymbol{q}$.

$$\sum_{i \in \cup_{n=1}^{l} \widehat{S}_n} q_i \ge \sum_{n=1}^{l} \widehat{R}_n \left|\widehat{S}_n\right|, l = 1,2,...,L.$$

Proof: a) We use induction. For $l = 1$ the equality holds by the definition of $\widehat{R}_1$. Assume now that $m > 1$ and that equality holds for $l < m$. Since by definition we then have $\widehat{R}_n > 0,\ n = 1,..,l$, we conclude,

$$\begin{aligned} \sum_{n=1}^{l} \widehat{R}_n \left|\widehat{S}_n\right| &= B\left(\cup_{n=1}^{l} \widehat{S}_n\right) \\ &= \sum_{i \in \cup_{n=1}^{l} \widehat{S}_n} \lambda_i - \sum_{l \in L_{out}(\cup_{n=1}^{l} \widehat{S}_n)} c_l > 0. \end{aligned} \tag{40}$$

By the definition of $\widehat{R}_{l+1}$ we have

$$\begin{aligned} \widehat{R}_{l+1}\left|\widehat{S}_{l+1}\right| &= \sum_{i \in \widehat{S}_{l+1}} \lambda_i + \sum_{l \in L_{out}(\cup_{n=1}^{l} \widehat{S}_n) \cap L_{in}(\widehat{S}_{l+1})} c_l \\ &\quad - \sum_{l \in L_{out}(\widehat{S}_{l+1}) - L_{in}(\cup_{n=1}^{l} \widehat{S}_n)} c_l. \end{aligned} \tag{41}$$

From (40) and (41) we get

$$\begin{aligned} \sum_{n=1}^{l+1} \widehat{R}_n \left|\widehat{S}_n\right| &= \sum_{i \in \cup_{n=1}^{l+1} \widehat{S}_n} \lambda_i - \sum_{l \in L_{out}(\cup_{n=1}^{l+1} \widehat{S}_n)} c_l \\ &= B\left(\cup_{n=1}^{l+1} \widehat{S}_n\right) > 0. \end{aligned}$$

b) If $\widehat{R}_l > 0$ for some $l = 1,2,...,L$, we have

$$\begin{aligned} \sum_{i \in \cup_{n=1}^{l} \widehat{S}_n} q_i &\ge B\left(\cup_{n=1}^{l} \widehat{S}_n\right) \text{ by (16)} \\ &= \sum_{n=1}^{l} \widehat{R}_n \left|\widehat{S}_n\right| \text{ by part a)}. \end{aligned}$$

If $\widehat{R}_L = 0$ and $L = 1$ the inequality is trivial. If $\widehat{R}_L = 0$ and $L > 1$ then by definition $\widehat{R}_l > 0$, $1 \leq l \leq L-1$, and

$$\begin{aligned}\sum_{i \in \cup_{n=1}^{L} \widehat{S}_n} q_i &\geq \sum_{i \in \cup_{n=1}^{L-1} \widehat{S}_n} q_i \\ &\geq \sum_{n=1}^{L-1} \widehat{R}_n \left| \widehat{S}_n \right| \text{ by part a)} \\ &= \sum_{n=1}^{L} \widehat{R}_n \left| \widehat{S}_n \right| \text{ since } \widehat{R}_L = 0.\end{aligned}$$

■

Proof: **(of Lemma 6)** By definition we have

$$\sum_{i \in \cup_{n=1}^{l} \widehat{S}_n} \overline{q}_i^* = \sum_{n=1}^{l} \widehat{R}_n \left| \widehat{S}_n \right|, \; l = 1, ..., L. \tag{42}$$

This and part b) of Lemma 14 imply that

$$\sum_{i \in \cup_{n=1}^{l} \widehat{S}_n} \overline{q}_i \geq \sum_{i \in \cup_{n=1}^{l} \widehat{S}_n} \overline{q}_i^*, \; l = 1, ..., L. \tag{43}$$

Let

$$\pi_{\widehat{S}_n}(i) : \left[\left(\sum_{k=1}^{n-1} \left| \widehat{S}_k \right| + 1 \right), \sum_{k=1}^{n} \left| \widehat{S}_k \right| \right] \to \widehat{S}_n,$$

represent an ordering of the indices in $\widehat{S}_n$ such that,

$$q_{\pi_{\widehat{S}_n}(i)} \geq q_{\pi_{\widehat{S}_n}(i+1)}.$$

Consider the permutation $\pi(i)$ of indices in K consisting of the concatenation of $\pi_{\widehat{S}_n}(i)$, $n = 1, .., L$. That is, if

$$\sum_{k=1}^{n-1} \left| \widehat{S}_k \right| < i \leq \sum_{k=1}^{n} \left| \widehat{S}_k \right|,$$

then,

$$\pi(i) = \pi_{\widehat{S}_n}(i).$$

We will show that

$$\sum_{l=1}^{i} q_{\pi(l)} \geq \sum_{l=1}^{i} \overline{q}_l^*, \quad i = 1, ..., |K|. \tag{44}$$

The lemma will then follow since $q_d = q_d^* = 0$ and by the definition of $\overline{\boldsymbol{q}}$ we have for any permutation $\pi(i)$ of the indices in K,

$$\sum_{l=1}^{i} \overline{q}_l \geq \sum_{l=1}^{i} q_{\pi(l)}, \; i = 1, ..., |K|. \tag{45}$$

We show (44) using induction. For $i = 1$, (44) holds since by definition

$$\begin{aligned}q_{\pi(1)} &= \max_{i \in \left| \widehat{S}_1 \right|} q_i \\ &\geq \max_{i \in \left| \widehat{S}_1 \right|} \frac{B\left(\widehat{S}_1\right)}{\left| \widehat{S}_1 \right|} \text{ by (27)} \\ &= \overline{q}_1^*.\end{aligned}$$

Assume now that (44) holds for $1, 2, ..., i$, but fails to holds hold for $i+1$. That is,

$$\sum_{l=1}^{k} q_{\pi(l)} \geq \sum_{l=1}^{k} \overline{q}_l^*, \; k = 1, ..., i, \tag{46}$$

but

$$\sum_{l=1}^{i+1} q_{\pi(l)} < \sum_{l=1}^{i+1} \overline{q}_l^*. \tag{47}$$

We will then have

$$q_{\pi(i+1)} < \overline{q}_{i+1}^*. \tag{48}$$

Because of (43), it must hold for some m, $1 \leq m \leq L$,

$$\sum_{k=1}^{m-1} \left| \widehat{S}_k \right| \leq i < i+1 < \sum_{k=1}^{m} \left| \widehat{S}_k \right|.$$

But since $\overline{q}_i^* = \widehat{R}_m$ for $i \in \widehat{S}_m$, (48) and the definition of $\pi(i)$ implies that

$$q_{\pi(k)} < \widehat{R}_m, \; k = i+1, ..., \sum_{k=1}^{m} \left| \widehat{S}_k \right|,$$

which together with (47) implies that

$$\sum_{i \in \cup_{n=1}^{m} \widehat{S}_n} q_i < \sum_{n=1}^{m} \widehat{R}_n \left| \widehat{S}_n \right|,$$

which contradicts (43). ■

5) ***Theorem*** *7: For a vector of real numbers $q = (q_i)_{i=1}^n$, it holds*

$$\sum_{l=1}^{i} \overline{q}_l^* \leq \sum_{l=1}^{i} \overline{q}_l, \text{ for all } i = 1, ..., n, \tag{49}$$

if and only if

$$\sum_{i=1}^{n} g\left(q_i^*\right) \leq \sum_{i=1}^{n} g\left(q_i\right), \tag{50}$$

for any convex nondecreasing function $g(q)$.

Proof: For the only if part, assume that (49) holds and let

$$\overline{q}_i = \overline{q}_i^* + \varepsilon_i, \; i = 1, .., n.$$

Because of (49) we have,

$$\sum_{l=1}^{i} \varepsilon_l \geq 0, \; i = 1, ..., n. \tag{51}$$

Since $g(x)$ is convex, it holds for any x and $x + \varepsilon$ in the domain of definition of $g(x)$,

$$g(x + \varepsilon) \geq g(x) + \varepsilon g'(x), \tag{52}$$

where $g'(x)$ is the left derivative of $g(x)$ at x. Since $g(x)$ is convex, $g'(x)$ is nondecreasing and hence,

$$g'\left(\overline{q}_i^*\right) \geq g'\left(\overline{q}_{i+1}^*\right), \; i = 2, ..., n. \tag{53}$$

Moreover, since $g(x)$ is nondecreasing,

$$g'(x) \geq 0. \tag{54}$$

Based on the above and defining for convenience $g'\left(\overline{q}^*_{n+1}\right) = 0$, we can write,

$$\begin{aligned}\sum_{i=1}^{n} g\left(q_i\right) &= \sum_{i=1}^{n} g\left(\overline{q}_i\right)\\ &= \sum_{i=1}^{n} g\left(\overline{q}^*_i + \varepsilon_i\right)\\ &\geq \sum_{i=1}^{n} g\left(\overline{q}^*_i\right) + \sum_{i=1}^{n} \varepsilon_i g'\left(\overline{q}^*_i\right) \text{ by (52)}\\ &= \sum_{i=1}^{n} g\left(\overline{q}^*_i\right) + \sum_{i=1}^{n}\left(\sum_{l=1}^{i}\varepsilon_l\right)\left(g'\left(\overline{q}^*_i\right) - g'\left(\overline{q}^*_{i+1}\right)\right)\\ &\geq \sum_{i=1}^{n} g\left(\overline{q}^*_i\right) \text{ by (51), (53) and (54).}\\ &= \sum_{i=1}^{n} g\left(q^*_i\right).\end{aligned}$$

For the if part, define for an i, $1 \leq i \leq n$, the nondecreasing convex function,

$$g_i(x) = \max\left\{x, \overline{q}_i\right\}. \tag{55}$$

Applying (50) to $g_i(x)$ we have

$$\sum_{l=1}^{n} g_i\left(\overline{q}^*_l\right) \leq \sum_{l=1}^{n} g_i\left(\overline{q}_l\right). \tag{56}$$

By the definition of $g_i(x)$,

$$g_i\left(\overline{q}_l\right) = \left\{\begin{array}{ll}\overline{q}_l & l \leq i\\ \overline{q}_i & l \geq i+1\end{array}\right. . \tag{57}$$

Combining the above, we arrive at

$$\begin{aligned}\sum_{l=1}^{i}\overline{q}^*_l + (n-i)\,\overline{q}_i &= \sum_{l=1}^{i}\overline{q}^*_l + \sum_{l=i+1}^{n}\overline{q}_i\\ &\leq \sum_{l=1}^{i} g_i(\overline{q}^*_l) + \sum_{l=i+1}^{n} g_i(\overline{q}^*_l) \text{ by (55)}\\ &\leq \sum_{l=1}^{n} g_i\left(\overline{q}_l\right) \text{ by (56)}\\ &= \sum_{l=1}^{i} g_i(\overline{q}_l) + \sum_{l=i+1}^{n} g_i(\overline{q}_l)\\ &= \sum_{l=1}^{i}\overline{q}_l + (n-i)\,\overline{q}_i \text{ by (57),}\end{aligned}$$

from which we conclude that for any i, $1 \leq i \leq n$,

$$\sum_{l=1}^{i}\overline{q}^*_l \leq \sum_{l=1}^{i}\overline{q}_l.$$

■

*6) **Theorem 8**: Let $\boldsymbol{f}^n$ be the superflow vector at time t_n and $\boldsymbol{q}^n$ the associated overload vector. If $\boldsymbol{q}^*$ is the unique overload vector associated with all SABP superflows, then*

$$\lim_{n\to\infty}\boldsymbol{q}^n = \boldsymbol{q}^*,$$
$$\lim_{n\to\infty}\boldsymbol{f}^n = \boldsymbol{f}^*.$$

where $\boldsymbol{f}^*$ is some SABP superflow.

We first need the following lemmas.

Lemma 15: Let $l_n = (i_n, j_n)$. It holds,

$$\left(q^n_{i_n}\right)^2 + \left(q^n_{j_n}\right)^2 \geq \left(q^{n+1}_{i_n}\right)^2 + \left(q^{n+1}_{j_n}\right)^2. \tag{58}$$

The inequality is strict if and only if one of the following two conditions holds: either a) $q^n_{i_n} > q^n_{j_n}$ and $f^n_{l_n} < c_{l_n}$ or b) $q^n_{i_n} < q^n_{j_n}$ and $f^n_{l_n} > 0$.

Proof: If $j_n = d$, then according to the Link Flow Update Rule described in Section VI it holds,

$$q^{n+1}_{i_n} = q^n_{i_n} - \beta,\ q^{n+1}_{j_n} = q_d = 0, \tag{59}$$

where

$$0 \leq \beta \leq q^n_{i_n}, \tag{60}$$

with $\beta > 0$ if and only if $q^n_{i_n} > 0$ and $f^n_{l_n} < c_{l_n}$.

If $j \neq d$ then,

$$q^{n+1}_{i_n} = q^n_{i_n} - \beta,\ q^{n+1}_{j_n} = q^n_{j_n} + \beta, \tag{61}$$

where

$$\left(q^n_{i_n} - q^n_{j_n}\right)/2 \geq \beta \geq 0 \text{ if } q^n_{i_n} \geq q^n_{j_n}, \tag{62a}$$
$$\left(q^n_{i_n} - q^n_{j_n}\right)/2 \leq \beta \leq 0 \text{ if } q^n_{i_n} < q^n_{j_n}, \tag{62b}$$

with $\beta \neq 0$ if and only if the conditions stated in the lemma hold.

From (59), (60) we conclude that, if $j_n = d$,

$$\left(q^{n+1}_{i_n}\right)^2 + \left(q^{n+1}_{j_n}\right)^2 = \left(q^n_{i_n} - \beta\right)^2 \leq \left(q^n_{i_n}\right)^2 + \left(q^n_{j_n}\right)^2.$$

with strict inequality holding iff $\beta > 0$, that is, only if and only if $q_{n,i_n} > 0$ and $f^n_{l_n} < c_{l_n}$.

From (61), (62a) we conclude that if $j_n \neq d$, then,

$$\left(q^{n+1}_{i_n}\right)^2 + \left(q^{n+1}_{j_n}\right)^2 = \left(q^n_{i_n} - \beta\right)^2 + \left(q^n_{j_n} + \beta\right)^2. \tag{63}$$

It is easy to see by simple calculations that under (62), it holds,

$$\left(q^n_{i_n} - \beta\right)^2 + \left(q^n_{j_n} + \beta\right)^2 \leq \left(q^n_{i_n}\right)^2 + \left(q^n_{j_n}\right)^2, \tag{64}$$

with strict inequality holding iff $\beta \neq 0$, that is, if and only if the conditions stated in the lemma hold. ■

Observing that at each link flow adjustment time t_n only the overloads of nodes i_n and j_n may be updated, we conclude from Lemma 15,

Corollary 16: Let $l_n = (i_n, j_n)$. It holds,

$$\sum_{i\in N}\left(q^n_i\right)^2 \geq \sum_{i\in N}\left(q^{n+1}_i\right)^2. \tag{65}$$

The inequality is strict if and only if one of the following two conditions holds: either a) $q^n_{i_n} > q^n_{j_n}$ and $f^n_{l_n} < c_{l_n}$ or b) $q^n_{i_n} < q^n_{j_n}$ and $f^n_{l_n} > 0$.

Lemma 17: It holds for any update time t_n, and any link $l = (i, j)$,

$$\left|f^{n+1}_{ij} - f^n_{ij}\right| \leq \left|q^{n+1}_i - q^n_i\right|.$$

Proof: If at time t^n the flow of link l is not updated, then $f_{ij}^{n+1} = f_{ij}^n$ and the inequality holds trivially. If on the other hand the link is updated, then since this is the only update that is taking place, it holds

$$q_i^{n+1} = q_i^n - \left(f_{ij}^{n+1} - f_{ij}^n\right),$$

that is,

$$\left|f_{ij}^{n+1} - f_{ij}^n\right| = \left|q_i^{n+1} - q_i^n\right|.$$

■

Proof: **(of Theorem 8)** Corollary 16 implies that $\sum_{i\in N}\left(q_i^n\right)^2$ converges at $n \to \infty$. Since the superflows $\boldsymbol{f}^n$ and the overload vectors $\boldsymbol{q}^n$, $n = 1, 2, ..$, take values in a compact set, for any subsequence of $(\boldsymbol{f}^n, \boldsymbol{q}^n)$ there is a convergent subsequence $(\boldsymbol{f}^{n_k}, \boldsymbol{q}^{n_k})$ so that

$$\lim_{k\to\infty} \boldsymbol{q}^{n_k} = \widetilde{\boldsymbol{q}}, \tag{66}$$

$$\lim_{k\to\infty} \boldsymbol{f}^{n_k} = \widetilde{\boldsymbol{f}}. \tag{67}$$

Moreover, by (65),

$$\sum_{i\in N}\left(q_i^{n_k}\right)^2 \ge \sum_{i\in N}\widetilde{q}_i^2 \text{ for all } k. \tag{68}$$

Clearly, $\widetilde{\boldsymbol{f}}$ is a superflow with overload vector $\widetilde{\boldsymbol{q}}$. Moreover, $\widetilde{\boldsymbol{f}}$ is SABP and hence $\widetilde{\boldsymbol{q}} = \boldsymbol{q}^*$. To see this note that if $\widetilde{\boldsymbol{f}}$ is not a SABP then we can employ the Link Flow Update Rule in Section VI, with $\widetilde{\boldsymbol{f}}$ as the initial superflow, and for a link $l = (i, j)$ for which either a) $\widetilde{q}_i > \widetilde{q}_j$ and $\widetilde{f}_{ij} < c_{ij}$ or b) $q_{i_n}^n < q_{j_n}^n$ and $\widetilde{f}_{ij} > 0$. According to Corollary 16 for the resulting new overload $\boldsymbol{q}'$ we have

$$\sum_{i\in K}\widetilde{q}_i^2 > \sum_{i\in N}\left(q_i'\right)^2. \tag{69}$$

Since the flow on link l is updated infinitely often, we can pick k large enough and an update instant for link l, larger than t_{n_k} so that for the resulting overload $\boldsymbol{q}'$ *after* the link update, (69) holds. Since $\sum_{i\in N}\left(q_i^{n_k}\right)^2$ is monotonically decreasing, this means that we can find k large enough so that $\sum_{i\in N}\left(q_i^{n_k}\right)^2$ is reduced below $\sum_{i\in K}\widetilde{q}_i^2$, a contradiction to (68).

Resorting to the general result that a sequence converges to α iff any subsequence contains a further subsequence converging to α, we conclude that (66) holds. Since $\lim_{n\to\infty}\left|q_i^{n+1} - q_i^n\right| = 0$ for any node i by (66), we conclude from Lemma 17 that $\lim_{n\to\infty}\left|f_l^{n+1} - f_l^n\right| = 0$ for any link l and hence (67) holds. ■

7) ***Theorem 10****: Under an SABP*$^{\boldsymbol{A}}$ *superflow, if all buffers are empty at time* 0, *then for any* $M = 1, 2, .., L-1$, *for any* t *in the interval* $[1/\gamma_M, 1/\gamma_{M+1})$, *it holds,*

$$F_{M,\boldsymbol{q}}^{lost}(t) \ge F_{\boldsymbol{q}^*}^{lost}(t).$$

Proof: For the lost traffic for $t \in [1/\gamma_M, 1/\gamma_{M+1})$ we have,

$$\begin{aligned} F_{\boldsymbol{q}^*}^{lost}(t) &= \sum_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}} q_i^*\left(t - \frac{A_i}{q_i^*}\right) \\ &= t\sum_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}} q_i^* - \sum_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}} A_i. \end{aligned}$$

Due to (25), (26), (6), it holds,

$$\sum_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}} q_i^* \le \sum_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}} q_i$$

and hence

$$F_{\boldsymbol{q}^*}^{lost}(t) \le t\sum_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}} q_i - \sum_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}} A_i.$$

Let $\boldsymbol{q}^M = \left(q_i^M\right)_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}} = \left(q_i\right)_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}}$ and let $\overline{A}_i$ be the buffer size of the node with the ith maximal element of $\boldsymbol{q}^M$. Define also,

$$i_t^M \triangleq \max_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}}\left\{\overline{q}_i^M \ge \frac{\overline{A}_i}{t}\right\}.$$

Then,

$$\begin{aligned} F_{M,\boldsymbol{q}}^{lost}(t) &= t\sum_{i=1}^{i_t^M}\overline{q}_i^M - \sum_{i=1}^{i_t^M}\overline{A}_i \\ &= \left(t\sum_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}} q_i - \sum_{i\in\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}} A_i\right) \\ &\quad - \left(t\sum_{i=i_t+1}^{\left|\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}\right|}\overline{q}_i^M - \sum_{i=i_t+1}^{\left|\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}\right|}\overline{A}_i\right) \\ &\ge F_{\boldsymbol{q}^*}^{lost}(t) - \left(t\sum_{i=i_t+1}^{\left|\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}\right|}\overline{q}_i^M - \sum_{i=i_t+1}^{\left|\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}\right|}\overline{A}_i\right). \end{aligned} \tag{70}$$

By definition of i_t^M, for $i > i_t^M$ we have $t < \frac{\overline{A}_i}{\overline{q}_i^M}$ and hence,

$$t \le \frac{\sum_{i=i_t+1}^{\left|\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}\right|}\overline{A}_i}{\sum_{i=i_t+1}^{\left|\cup_{m=1}^M \widehat{S}_m^{\boldsymbol{A}}\right|}\overline{q}_i^M}. \tag{71}$$

Form (70) and (71) the theorem follows. ■

L eonidas Georgiadis (S'76, M'78, SM'96) received the Diploma degree in electrical engineering from Aristotle University, Thessaloniki, Greece, in 1979, and his M.S. and Ph.D. degrees both in electrical engineering from the University of Connecticut, in 1981 and 1986 respectively. From 1981 to 1983 he was with the Greek army.

From 1986 to 1987 he was Research Assistant Professor at the University of Virginia, Charlottesville. In 1987 he joined IBM T. J. Watson Research Center, Yorktown Heights as a Research Staff Member. Since October 1995, he has been with the Telecommunications Department of Aristotle University, Thessaloniki, Greece.

His interests are in the area of wireless networks, high speed networks, distributed systems, routing, scheduling, congestion control, modeling and performance analysis.

In 1992 he received the IBM Outstanding Innovation Award for his work on Goal-Oriented Workload Management for Multi-class systems.

L eandros Tassiulas (S'89, M'91, SM'05) was born in 1965, in Katerini, Greece. He obtained the Diploma in Electrical Engineering from Aristotle University of Thessaloniki, Greece in 1987, and the M.S. and Ph.D. degrees in Electrical Engineering from the University of Maryland, College Park in 1989 and 1991 respectively. He is Professor in the Dept. of Computer and Telecommunications Engineering, University of Thessaly, Greece and Research Professor in the Dept. of Electrical and Computer Eng. and the Institute for Systems Research, University of Maryland College Park since 2001. He has held positions as Assistant Professor at Polytechnic University New York (1991-95), Assistant and Associate Professor University of Maryland College Park (1995-2001) and Professor University of Ioannina Greece (1999-2001). His research interests are in the field of computer and communication networks with emphasis on architectures and protocols of wireless systems, sensor networks, high-speed internet, satellite communications and stochastic network optimization. Dr. Tassiulas received a National Science Foundation (NSF) Research Initiation Award in 1992, an NSF CAREER Award in 1995 an Office of Naval Research, Young Investigator Award in 1997 a Bodosaki Foundation award in 1999 and the INFOCOM '94 best paper award.