An Adaptive Framework for Addressing Fairness Issues in Wireless Networks

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Abstract

We present a framework for designing optimal policies addressing resource allocation problems in wireless networks. We consider a general utility function optimization objective. Specific choices of the utility functions lead to policies that satisfy several well-known fairness criteria, e.g., *max-min* and *proportional-fair.* Traditional approaches for solving these nonlinear optimization problems in an off-line manner, lead to nonadaptive policies that usually rely on system parameters, which may not be known a priory. Within our framework the development of such policies is based on the adaptive employment of policies that solve linear optimization problems. In several situations the development of policies for these linear problems is fairly simple and depends minimally on system parameters. Subsequently we apply this method to three specific wireless resource allocation problems. In particular we consider wireless fading channel systems and provide optimal policies for a) uplink optimal power allocation for constant bit rate connections, b) uplink optimal average throughput allocation, and c) downlink optimal scheduling with limited transmission rate capabilities over multiple fading channels.

Keywords: Wireless Networks, Adaptive Policies, Resource Allocation, Fairness, Stochastic Approximation.

1 Introduction

A main design issue in wireless networks is the development of mechanisms for sharing limited and possibly time-varying system resources (e.g., bandwidth, base station buffer space, base station power), by a number of users that may have limited energy resources. The users may have conflicting performance requirements (throughput, delay, power consumption) and hence the question arises, how to allocate system resources in a manner that achieves a compromise between overall system performance and individual user satisfaction. In this work we provide a framework for designing adaptive scheduling policies for

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various wireless resource allocation problems. In particular the ability of several policies to allocate the wireless resources according to prespecified fairness criteria can be examined within this framework. For the framework to be applicable, the system must satisfy certain assumptions. These assumptions are natural for a variety of practical systems.

It turns out that in certain systems, optimal policies can be derived when the optimization objective is a linear combination of user performance requirements. These policies are often simple and depend minimally on system parameters. However, the linear optimization criterion, while very important from the overall system perspective, may lead to overallocation of resources to certain users and underallocation to some others. Hence, optimization objectives other than simple linear combinations have been proposed in the literature in order to address this problem. Well-known among them, is the *proportional-fair,* [8], [9], and the *max-min* allocation, [2]. These objectives are special or limiting cases of utility function optimization. In the latter optimization, with each user there is an associated utility function, usually nonlinear, expressing the user satisfaction (or dissatisfaction) for receiving certain performance and the objective is to optimize the sum of user utilities, [8], [9], [6]. The design of policies for these criteria requires new approaches. A direct off-line approach for policy design based on associated nonlinear optimization problems usually results in policies that depend on system parameters, which may not be known a priori, or are difficult to compute. In a recent work, [12], a framework for opportunistic scheduling is developed, which addresses the utility optimization problem. The policies provided in the latter work are stationary, and depend on parameters that require the a priori knowledge of system statistics.

In this paper we present a systematic approach for designing policies appropriate for addressing the fairness issue, based on the knowledge of policies that solve the linear objective criterion. More specifically, it turns out that under certain assumptions that several systems of interest satisfy, policies achieving general utility function optimization may be obtained by employing in an adaptive fashion known policies for linear objective optimization. By "adaptive" we mean that the coefficients in the linear objective optimization are updated at regular intervals according to observed system performance and system state; next, until the next update interval, a policy that solves the linear optimization problem is employed and so on (see Section 2 for the details). Hence in a sense, for this class of systems, obtaining a policy that solves the linear optimization problem provides immediately a policy for solving more general fairness-related objectives. Oftentimes, the latter policy is as simple as its linear counterpart and depends minimally on system parameters.

The approach outlined above, is based on stochastic approximation. Stochastic approximation methods have been used to address problems in wireless communication, e.g. in [5] where the objective is to maximize the minimum of user throughputs. The method presented in this paper can address more general utility optimization problems. It has been employed successfully to the solution of specific control problems related to scheduling jobs in multi-class queueing systems, [4], and recently to downlink scheduling in wireless communications [19], for the system considered in [5]. However, the essential features of the approach are hidden by the technical details of the specific problem addressed. An objective of this paper is to reveal the essential simplicity of the method and to describe it in a systematic fashion so that it can be easily identified whether other problems fall within the same framework. In addition, based on the proposed methodology, we also provide new, adaptive policies for a number of problems related to wireless communication.

For the system in [5], results similar to ours were obtained independently in [11]. The method in [11] is different than ours and is based on the study of the associated system ordinary differential equation, an approach that has also been used in [3].

The rest of the paper is organized as follows. In Section 2 we present the basic framework based on which adaptive optimal policies for wireless resource allocation problems can be designed. In Section 3, we apply the framework developed in Section 2 to design optimal scheduling policies for two resource allocation problems arising in uplink scheduling over wireless fading channels when successive decoding is available. In Section 4, the same framework is applied to a resource allocation problem arising in downlink scheduling over multiple fading channels. In Section 5, we discuss approaches to relax several of the technical assumptions made in Section 2. Finally, in Section 6 the conclusions of this work are given.

2 Motivation and Basic Approach

In the following we denote the space of N-dimensional real vectors by $\Re^N$. Vectors are denoted with boldface letters, e.g., $\mathbf{x} = (x_i)_{i=1}^N$. The inner product of two vectors $\mathbf{x}$, $\mathbf{y} \in \Re^N$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$, i.e., $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^N x_i y_i$. For $x \in \Re$, we define $(x)^+ = \max\{0, x\}$.

Consider N users that need access to a common resource. Assume that the system time is slotted and that slot t, $t \geq 1$, refers to the time interval $[t-1, t)$. An allocation policy u decides how to allocate the resource to the users at the beginning of each slot. There may be constraints on policy u, e.g., the user transmission rates or user powers may be limited. We denote by U the set of all admissible policies. Let $x_i^u(t)$ be the performance measure of interest to user i, at slot t, under a resource allocation policy $u \in U$. We assume that this measure is the time average of quantities occurring during each time slot, i.e.,

$$x_i^u(t) = \frac{1}{t} \sum_{\tau=1}^{t} X_i^u(\tau), \tag{1}$$

where $X_i^u(\tau)$ is the quantity occurred in slot τ. For example, $x_i^u(t)$ may be the average throughput of user i up to time t (in bps), in which case $X_i^u(t)$ is the number of bits transmitted at slot t. Alternatively, $x_i^u(t)$ may represent the average power consumed by user i up to time t, and $X_i^u(t)$ the power consumed by user i at slot t.

With user i there is an associated utility function $f_i(x)$ which can be interpreted as the user "satisfaction" (or "dissatisfaction") for having performance measure equal to x. A general optimization cost related to fairness is then to minimize or maximize an overall function of the form,

$$F^u(t) = \sum_{i=1}^{N} f_i\left(x_i^u(t)\right), \tag{2}$$

as t goes to infinity. For example, if the performance objectives are user throughputs and $f_i(x)$ is interpreted as the user satisfaction for receiving throughput x, then the objective is to maximize (2). If on the other hand the performance measure is the consumed average user power and $f_i(x)$ is interpreted as the cost for consuming average power x, then the objective is to minimize (2). For the sake of definiteness we will describe below the steps and assumptions involved in the maximization problem. At the end of the section we will describe the modifications needed to address the minimization problem.

The utility optimization involved in the maximization of (2)-for large t-constitutes a general framework for providing fairness guarantees, special cases of which are well-known fairness criteria. In particular, if *max-min* allocation of the system resource is desired [2], then such an allocation can be approximated by using a family of utility functions of the form $f_i^{(m)}(x) = c - g(x)^m$,where c is constant and $g(\cdot)$ is a differentiable, decreasing, convex and positive function, e.g., $f_i^{(m)}(x) = 1 - 1/x^m$. By adopting these functions as reward functions, it can be shown that the allocation of performance measures converges to the *max-min* allocation as m tends to $+\infty$, [6]. If it is considered appropriate to apply the *proportional-fairness* criterion, then the logarithmic functions can be chosen, i.e., $f_i(x) = \ln(x)$, [8], [9]. Another useful criterion is *harmonic-mean-fairness*, where $f_i(x) = -1/x.$ The choice $f_i(x) = x^{1-\alpha}/(1-a)$ leads to *max-min* fairness when $\alpha \to \infty$, *proportional-fairness* when $\alpha \to 1$ and *harmonic-fairness* when $a = 2$, [15], [1].

For the maximization problem the total long-term reward obtained by a policy π is defined as

$$F^u = \limsup_{t\to\infty} \sum_{i=1}^{N} f_i\left(x_i^u(t)\right), \tag{3}$$

and we are interested in finding an admissible policy with maximal F^u. We assume that $\{f_i(\cdot)\}_{i=1}^N$ are concave, nondecreasing, twice continuously differentiable real functions. We are taking lim sup in (3) since it may not be known a priori whether the limit exists under an arbitrary admissible scheduling policy. Formally we have the following problem.

> **Problem (P)**: Determine a policy u^* such that $F^{u^*} \geq F^u$,
> for all policies in U (admissible policies).

The constraints imposed on admissible policies, as well as the physical system constraints, imply that the user performance measures lay in a certain region $\mathcal{A}$ of the N-dimensional space called "achievable region". More specifically, it can often be shown that for any policy $u \in U$, it holds

Achievable Region:

$$\lim_{t\to\infty} \left(\inf \left\{ |\mathbf{x}^u(t) - \mathbf{x}| : \mathbf{x} \in \mathcal{A} \right\}\right) = 0. \tag{4}$$

Assume now that $\mathcal{A}$ is compact and consider the following problem.

Problem (D):

$$\text{maximize} \sum_{i=1}^{N} f_i(x_i),$$
$$\text{subject to } \mathbf{x} \in \mathcal{A}.$$

Since $\mathcal{A}$ is compact and the objective function continuous, there is an optimal solution $\mathbf{x}^*$ to the problem. It can also be shown that under the same conditions on $\mathcal{A}$ and on the optimization objective function, (4) implies that for any $u \in U$ (see Appendix A.1),

$$F^u = \limsup_{t\to\infty} \sum_{i=1}^{N} f_i(x_i^u(t)) \le \sum_{i=1}^{N} f_i(x_i^*). \tag{5}$$

Therefore, if we are able to find a policy u^* such that

$$\liminf_{t\to\infty} \sum_{i=1}^{N} f_i\left(x_i^{u^*}(t)\right) \ge \sum_{i=1}^{N} f_i(x_i^*), \tag{6}$$

we will have from (5) and (6) that

$$\lim_{t\to\infty} \sum_{i=1}^{N} f_i\left(x_i^{u^*}(t)\right) = F^{u^*} = \sum_{i=1}^{N} f_i(x_i^*). \tag{7}$$

Hence, from (5) and (7) we conclude that u^* is a policy solving Problem (P).

As we saw above, the crucial point is to find a policy that satisfies (6). We will provide such a policy whose design in based on stochastic approximation techniques. There are several theorems related to stochastic approximation, with varying generality and technical complexities. We describe below a fairly simplified version of a Theorem appearing in [4] which is sufficient for our discussion and avoids several of the complicating technical issues that may arise in specific applications. In Section 5 we indicate cases where the more general version is needed.

Theorem 1 *In $\Re^N$ consider a stochastic sequence $\{\mathbf{y}(t)\}_{t=1}^{\infty}$, which satisfies the recursion,*

$$\mathbf{y}(t+1) = \mathbf{y}(t) + \frac{1}{t+1}\mathbf{G}(t+1), \quad t \geq 1. \tag{8}$$

Assume that the following conditions hold:

a) There exists a compact set $\mathcal{A} \subset \Re^N$ such that $\{\mathbf{y}(t)\}_{t=1}^{\infty}$ converges to $\mathcal{A}$ a.s., i.e.,

$$\lim_{t\to\infty} \left(\inf\left\{|\mathbf{y}(t) - \mathbf{y}| : \mathbf{y} \in \mathcal{A}\right\}\right) = 0.$$

b) $\mathbf{G}(t)$ is bounded, i.e., $|\mathbf{G}(t)| \leq B$.

c) There exists a twice continuously differentiable function $V : \Re^N \to \Re$ such that

$$\left\langle \nabla V\left(\mathbf{y}(t)\right), E\left[\mathbf{G}(t+1)\,|\mathbf{y}(\tau),\ \tau \leq t\right]\right\rangle < -V\left(\mathbf{y}(t)\right). \tag{9}$$

Then,

$$\lim_{t\to\infty} \left(V\left(\mathbf{y}(t)\right)\right)^{+} = 0.$$

Theorem 1 is related to Problem (P) as follows. We rewrite (1) (we omit the dependence on u for simplicity).

$$\begin{aligned} x_i\left(t+1\right) &= \frac{1}{t+1}\sum_{\tau=1}^{t+1} X_i\left(\tau\right) = \frac{1}{t+1}\sum_{\tau=1}^{t} X_i\left(\tau\right) + \frac{1}{t+1} X_i\left(t+1\right) \\ &= x_i\left(t\right) + \frac{1}{t+1}\left[X_i\left(t+1\right) - x_i\left(t\right)\right]. \end{aligned} \tag{10}$$

We identify $\mathbf{x}(t)$ with vector $\mathbf{y}(t)$ in Theorem 1, hence $\mathbf{G}(t+1) = \mathbf{X}\left(t+1\right) - \mathbf{x}\left(t\right)$. We also set

$$V(\mathbf{x}) = \sum_{i=1}^{N} f_i(x_i^*) - \sum_{i=1}^{N} f_i(x_i). \tag{11}$$

With these identifications, we have

$$\begin{aligned} &\left\langle \nabla V\left(\mathbf{y}(t)\right), E\left[\mathbf{G}(t+1)\,|\mathbf{y}(\tau),\ \tau \leq t\right]\right\rangle = \\ &\qquad \sum_{i=1}^{N} f_i'(x_i(t))x_i(t) - \sum_{i=1}^{N} f_i'(x_i(t)) E\left[X_i(t+1)\,|\mathbf{x}(\tau),\ \tau \leq t\right]. \end{aligned} \tag{12}$$

The (random) vector $\mathbf{X}(t+1)$ depends on the allocation policy. Let us now impose the following condition on the selected policy u^*.

Condition (C): Policy u^* is such that the policy-induced vector

$$\left(E\left[X_i(t+1)\,|\mathbf{x}(\tau),\ \tau \leq t\right]\right)_{i=1}^{N},$$

is a solution to the following linear optimization problem

$$\begin{gathered}\text{maximize } \sum_{i=1}^{N} c_i x_i \\ \text{subject to } \mathbf{x} \in \mathcal{A},\end{gathered}$$

where $c_i = f_i'(x_i(t)),\ i = 1, 2, ..., N.$

Under condition (C), since $\mathbf{x}^*$ (the solution to Problem (D)) belongs to $\mathcal{A}$, we will have,

$$\sum_{i=1}^{N} f_i'(x_i(t)) E\left[X_i(t+1)\left|\mathbf{x}(\tau),\ \tau \leq t\right.\right] \geq \sum_{i=1}^{N} f_i'(x_i(t)) x_i^*,$$

and hence,

$$\sum_{i=1}^{N} f_i'(x_i(t)) x_i(t) - \sum_{i=1}^{N} f_i'(x_i(t)) E\left[X_i(t+1)\left|\mathbf{x}(\tau),\ \tau \leq t\right.\right] \leq \sum_{i=1}^{N} f_i'(x_i(t)) \left(x_i(t) - x_i^*\right). \quad (13)$$

Using now the fact that $f_i(x)$ is concave, we have

$$f_i'(x_i(t)) \left(x_i(t) - x_i^*\right) \leq f_i(x_i(t)) - f_i\left(x_i^*\right),$$

and taking also into account (12), (13) and (11), we conclude that

$$\begin{aligned}\left\langle \nabla V\left(\mathbf{x}(t)\right), E\left[\mathbf{G}(t+1)\left|\mathbf{y}(\tau),\ \tau \leq t\right.\right]\right\rangle &\leq -\left(\sum_{i=1}^{N} f_i(x_i^*) - \sum_{i=1}^{N} f_i(x_i(t))\right) \\ &= -V\left(\mathbf{x}(t)\right).\end{aligned}$$

Assuming that $\left|\mathbf{X}\left(t+1\right) - \mathbf{x}\left(t\right)\right| \leq B$ and using Theorem 1 we have,

$$\lim_{t\to\infty} \left(\sum_{i=1}^{N} f_i(x_i^*) - \sum_{i=1}^{N} f_i(x_i\left(t\right))\right)^+ = 0.$$

Hence,

$$\begin{aligned}\sum_{i=1}^{N} f_i(x_i^*) - \liminf_{t\to\infty} \sum_{i=1}^{N} f_i(x_i(t)) &= \limsup_{t\to\infty} \left(\sum_{i=1}^{N} f_i(x_i^*) - \sum_{i=1}^{N} f_i(x_i(t))\right) \\ &\leq \lim_{t\to\infty} \left(\sum_{i=1}^{N} f_i(x_i^*) - \sum_{i=1}^{N} f_i(x_i\left(t\right))\right)^+ = 0,\end{aligned}$$

and (6) holds.

From the discussion above we see that the design of a policy that solves Problem (P) may follow the steps below.

POLICY DESIGN STEPS

Assumptions on $\{f_i(\cdot)\}_{i=1}^{N}$: concave, nondecreasing, twice continuously differentiable real functions.

- *Step 1.* Determine the system achievable region.
- *Step 2.* Ensure that $|\mathbf{X}(t+1) - \mathbf{x}(t)| \leq B$.
- *Step 3.* Find a policy that satisfies Condition (C), i.e., solves the linear optimization problem. Employ this policy at time t using as constant coefficients for the linear optimization problem the derivatives $f_i'(x_i(t))$.

In several applications, the optimization in Step 3 is relatively easy to solve. Also, it often turns out that the policy that solves the linear optimization problem relies minimally on system parameters and statistics and thus is fairly simple. We will see examples of these properties in the next Sections.

If minimization instead of maximization is sought, then the following modifications are needed in the algorithmic steps above.

1. The functions $\{f_i(\cdot)\}_{i=1}^{N}$ need to be convex instead of concave.
2. The policy objective function is defined as $F^u = \liminf_{t\to\infty} F^u(t)$.
3. The linear optimization problem involves minimization instead of maximization.

3 Uplink Scheduling in Wireless Fading Channels with Successive Decoding

In this section we develop optimal policies for two resource allocation problems arising in uplink scheduling over fading channels when successive decoding (or successive interference cancellation) is employed. In both of these problems the scheduling decisions are based on the knowledge of the channel fading state in a slot-by-slot fashion and on the "history" of system evolution. In the first problem we present an optimal scheduling policy when the performance measure of a user is its average power consumption and in the second problem an optimal policy is developed when the performance measure is the user throughput.

3.1 Optimal Power Control

In this section we apply the framework developed above to the design of an optimal adaptive resource allocation policy for an uplink (user to base station) wireless fading channel with a single base station, where successive decoding is available. We are interested in policies that keep track of the channel fading states for the various users in a slot-by-slot fashion. Based on this channel state information the policies determine the user transmission powers as well as the decoding order at the base station, in such a way that the Signal to Interference Ratio (SIR) of each user remains above a prespecified threshold and the average user powers satisfy certain fairness criteria. A constant user SIR is associated with constant transmission rate. Hence, in this case scheduling aims at providing to each user a constant bit rate connection over all slots, while at the same time minimizing an objective function of user average power consumption. An information theoretic study of this system has been provided in [7].

3.1.1 System Model

We consider N users that need to access a single base station through a wireless fading channel. The system is time-slotted and slot t refers to the time interval $[t-1,t)$. We are interested in the uplink control of user transmissions in such a way that the users consume power in a fair manner (see Figure 1). Let $h_i\left(t\right)$ denote the fading of user i at slot t. That is, if user i transmits with power $P_i(t)$ at slot t, the received power at the base station is $h_i(t)P_i(t)$. It is assumed that the base station has knowledge (through measurements) of the vector of users fades $\mathbf{h}(t)=\left(h_1\left(t\right),...,h_N\left(t\right)\right)$ at time $t-1$ and informs the users about the powers they should use for transmission at slot t (the fading remains constant in the interval $[t-1,t)$). To avoid technical difficulties we assume that the fading process $\mathbf{h}(t)$, $t=1,2,...$ consists of i.i.d. random variables taking values on discrete set H and denote $\overline{\Pr}\left(\mathbf{h}\right)=\Pr\left(\mathbf{h}(t)=\mathbf{h}\right)$.

Regarding coding we assume that successive decoding is available at the base station. Successive decoding is a technique, which successively subtracts off the decoded signal from the composite signal. Therefore when successive decoding is applied, users that are decoded later experience reduced interference since the interference from previously decoded users is subtracted off. For details on successive decoding, see e.g., [16], [20]. The order of decoding at any given slot, is determined by $\pi\left(i\right),\, i=1,...,N$, which is a permutation of $\{1,2,...,N\}$ such that user $\pi\left(i\right)$ is decoded after users $\pi\left(i+1\right),...,\pi\left(N\right),\, i\leq N-1$. Therefore user $\pi\left(N\right)$ is decoded first (i.e., receives the lowest priority) while user $\pi\left(1\right)$ is decoded last (receives the highest priority).

Assume that the decoding order $\pi\left(i\right),\, i=1,...,N$, is employed at a given slot t. The

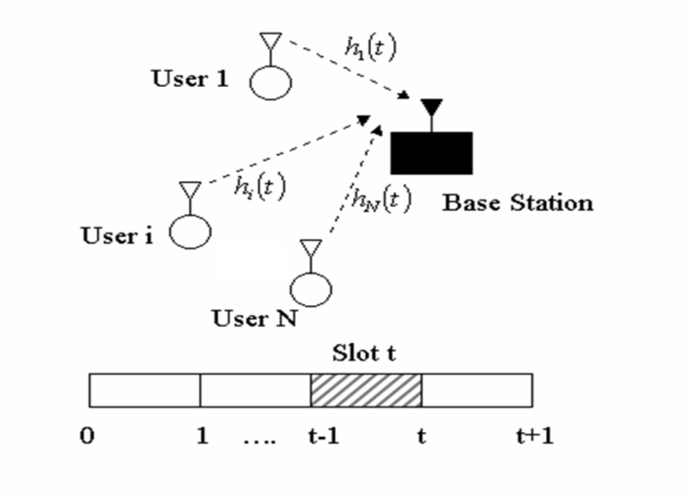

Figure 1: System Model

SIR of user $\pi(i)$ at slot t, which is denoted as $\gamma_{\pi(i)}(t)$, is given by

$$\gamma_{\pi(i)}(t) = \frac{h_{\pi(i)}(t) P_{\pi(i)}(t)}{\sigma^2 + \sum_{j<i} h_{\pi(j)}(t) P_{\pi(j)}(t)}, \ i = 1, ..., N,$$

where $\sum_{j<1} f(j) = 0$ and σ^2 denotes the variance of the background noise which is assumed to be White Gaussian.

With each user i there is an associated target SIR a_i. At every slot the SIR experienced by user i must be no less than its target SIR, i.e., it is required that

$$\gamma_i(t) \geq a_i, \text{ for every } i = 1, ..., N \text{ and every } t \geq 1. \tag{14}$$

3.1.2 Problem Formulation

We are interested in controlling the uplink transmissions, i.e., in deciding the decoding permutation and performing power control in every slot, in such a way that the user SIR constraints defined in (14) are met while at the same time an objective function of the user average transmission powers is minimized.

First we proceed with the description of the set of admissible policies U. A policy u decides the decoding permutation and performs power control in every slot. In order to make its scheduling decisions in a given slot t, a policy can use the history of the system up to time $t-1$, as well as the current state of the channel, i.e., the vector $\mathbf{h}(t)$.

Let $P_i^u(t)$ denote the power allocated to user i at slot t by a policy u. The average power consumption for user i up to slot t, under a policy u, is given by

$$p_i^u(t) = \frac{1}{t}\sum_{\tau=1}^{t} P_i^u(\tau). \tag{15}$$

Following the notation of Section 2 the total cost of scheduling policy u is defined as

$$F^u = \liminf_{t\to\infty} \sum_{i=1}^{N} f_i\left(p_i^u(t)\right), \tag{16}$$

where $\{f_i(\cdot)\}_{i=1}^N$ are convex, nondecreasing, twice continuously differentiable functions defined on $[0,\infty)$. We seek an admissible policy u^* such that the total cost associated with this policy is minimal. In particular we are interested in the problem.

Problem (P1): Determine a policy u^* such that $F^{u^*} \le F^u$, for all $u \in U$.

The design of such a policy that solves Problem (P1) follows the algorithmic steps presented in Section 2.

Step 1: *Determine the Achievable Region.*

If a decoding permutation $\pi(i)$ is given and the channel is at state $\mathbf{h}$, then in order to satisfy the SIR requirements of the users, the user powers must belong to the region,

$$F(\pi, \mathbf{h}) = \left\{ \mathbf{P} : \frac{h_{\pi(i)}P_{\pi(i)}}{\sigma^2 + \sum_{j<i} h_{\pi(j)}P_{\pi(j)}} \ge a_{\pi(i)},\ i = 1, ..., N \right\}.$$

It is easily seen that the solution $\mathbf{P}(\pi, \mathbf{h})$ to the system of equations

$$\frac{h_{\pi(i)}P_{\pi(i)}}{\sigma^2 + \sum_{j<i} h_{\pi(j)}P_{\pi(j)}} = a_{\pi(i)},\ i = 1, ..., N,$$

always exists, belongs to $F(\pi, \mathbf{h})$ and is minimal among the vectors in $F(\pi, \mathbf{h})$, i.e.,

$$P_i(\pi, \mathbf{h}) \le P_i, \text{ for } i = 1, ...N,\ \mathbf{P} \in F(\pi, \mathbf{h}).$$

Therefore, it is sufficient to restrict attention to policies which choose power vector $\mathbf{P}(\pi, \mathbf{h})$ when the fading state is $\mathbf{h}$ and permutation π is selected. Hence a scheduling policy in U must choose at any fading state $\mathbf{h}$, the decoding order π, which in turn determines $\mathbf{P}(\pi, \mathbf{h})$. Let u be such a policy and let $t(\pi, \mathbf{h})$ be the amount of time up to time t that policy u employs decoding order π when the fading state is $\mathbf{h}$. Let also,

$$t(\mathbf{h}) = \sum_{\pi} t(\pi, \mathbf{h}), \tag{17}$$

be the amount of time up to time t that the system is in state $\mathbf{h}$.

Then, the average power of user i under policy u at time t is

$$\begin{aligned} p_i^u(t) &= \frac{\sum_{\tau=1}^{t} P_i^u(\tau)}{t} = \frac{\sum_{\mathbf{h}}\sum_{\pi} P_i(\pi,\mathbf{h})t(\pi,\mathbf{h})}{t} \\ &= \sum_{\mathbf{h}} \frac{t(\mathbf{h})}{t} \sum_{\pi} P_i(\pi,\mathbf{h}) \frac{t(\pi,\mathbf{h})}{t(\mathbf{h})}. \end{aligned} \tag{18}$$

Since the fading states are assumed independent, it holds

$$\lim_{t\to\infty} \frac{t(\mathbf{h})}{t} = \overline{\Pr}(\mathbf{h}). \tag{19}$$

Let us denote by $\mathcal{C}(\mathbf{h})$ the convex hull of the power vectors $\mathbf{P}(\pi,\mathbf{h})$, where π ranges over all permutations of the set $\{1,2....,N\}$. We observe from (17) that at any time t, it holds

$$\sum_{\pi} \mathbf{P}(\pi,\mathbf{h}) \frac{t(\pi,\mathbf{h})}{t(\mathbf{h})} \in \mathcal{C}(\mathbf{h}). \tag{20}$$

From (18), (19) and (20) it can be easily seen (see Appendix A.2) that (4) is satisfied for the region that is a convex combination of $\mathcal{C}(\mathbf{h})$, i.e.,

$$\mathcal{P} = \left\{ \mathbf{p} : \mathbf{p} = \sum_{\mathbf{h}} \mathbf{p}(\mathbf{h}) \overline{\Pr}(\mathbf{h}),\ \mathbf{p}(\mathbf{h}) \in \mathcal{C}(\mathbf{h}) \right\}. \tag{21}$$

Step 2: *Ensure that* $|\mathbf{X}(t+1) - \mathbf{x}(t)| \leq B$.

For the system under consideration we observe that

$$|\mathbf{X}(t+1)| \leq \max_{\pi,\mathbf{h}} \{|\mathbf{P}(\pi,\mathbf{h})|\}$$

which implies that $\mathbf{x}(t)$ is also bounded.

Step 3: *Find a policy that satisfies Condition (C).*

We need to find a policy u^* such that at time t the vector

$$\left(\mathrm{E}\left[P_i^{u^*}(t+1) \middle| \mathbf{p}^{u^*}(\tau),\ \tau \leq t \right] \right)_{i=1}^{N},$$

is a solution to the following linear optimization problem

$$\begin{gathered} \text{minimize } \sum_{i=1}^{N} f_i'\left(p_i^{u^*}(t)\right) p_i \\ \text{subject to } \mathbf{p} \in \mathcal{P}. \end{gathered}$$

We will show that the minimization above is achieved by the following simple policy.

Policy u^* : At time t, select the permutation π^* so that

$$\frac{f'_{\pi^*(1)}\left(p_{\pi^*(1)}\left(t\right)\right)}{h_{\pi^*(1)}\left(t+1\right)} \geq .. \geq \frac{f'_{\pi^*(N)}\left(p_{\pi^*(N)}\left(t\right)\right)}{h_{\pi^*(N)}\left(t+1\right)}. \tag{22}$$

To show this, we proceed as follows. Observe that conditioned on $\mathbf{p}^{u^*}(t)$ and $\mathbf{h}(t+1)$, $P_i^{u^*}(t+1)$ is independent of $\mathbf{p}^{u^*}(\tau)$, $\tau < t$. Taking also into account that $\mathbf{h}(t)$ is an i.i.d. process and hence $\mathbf{h}(t+1)$ independent of $\mathbf{p}^{u^*}(t)$, we have

$$\begin{aligned}
& \mathrm{E}\left[P_i^{u^*}(t+1)\left|\mathbf{p}^{u^*}(\tau),\ \tau \leq t\right.\right] \\
= & \sum_{\mathbf{h}} \mathrm{E}\left[P_i^{u^*}(t+1)\left|\mathbf{p}^{u^*}(t), \mathbf{h}(t+1) = \mathbf{h}\right.\right] \Pr\left(\mathbf{h}(t+1) = \mathbf{h} \middle| \mathbf{p}^{u^*}(t)\right) \\
= & \sum_{\mathbf{h}} \mathrm{E}\left[P_i^{u^*}(t+1)\left|\mathbf{p}^{u^*}(t), \mathbf{h}(t+1) = \mathbf{h}\right.\right] \overline{\Pr}(\mathbf{h}) \\
= & \sum_{\mathbf{h}} P_i(\pi^*, \mathbf{h}) \overline{\Pr}(\mathbf{h}).
\end{aligned}$$

Denoting for simplicity $f'_i\left(p_i^{u^*}(t)\right) = f'_i$, we have

$$\sum_{i=1}^{N} f'_i \mathrm{E}\left[P_i^{u^*}(t+1)\left|\mathbf{p}^{u^*}(\tau),\ \tau \leq t\right.\right] = \sum_{\mathbf{h}} \left(\sum_{i=1}^{N} f'_i P_i(\pi^*, \mathbf{h})\right) \overline{\Pr}(\mathbf{h}).$$

Consider now any point $\mathbf{p} \in \mathcal{P}$. From (21) we have

$$\mathbf{p} = \sum_{\mathbf{h}} \mathbf{p}(\mathbf{h}) \overline{\Pr}(\mathbf{h}),\ \mathbf{p}(\mathbf{h}) \in \mathcal{C}(\mathbf{h}).$$

Hence

$$\sum_{i=1}^{N} f'_i p_i = \sum_{\mathbf{h}} \left(\sum_{i=1}^{N} f'_i p_i(\mathbf{h})\right) \overline{\Pr}(\mathbf{h}),\ \mathbf{p}(\mathbf{h}) \in \mathcal{C}(\mathbf{h}).$$

Using an approach similar to the one used in [10] (or based on a result in [7, Page 2819]) it can be shown that the sum $\sum_{i=1}^{N} f'_i p_i(\mathbf{h})$, for $\mathbf{p}(\mathbf{h}) \in \mathcal{C}(\mathbf{h})$ is minimized when permutation π^* is employed. Therefore,

$$\sum_{i=1}^{N} f'_i p_i \geq \sum_{\mathbf{h}} \left(\sum_{i=1}^{N} f'_i P_i(\pi^*, \mathbf{h})\right) \overline{\Pr}(\mathbf{h}),$$

i.e., policy u^* indeed satisfies Condition (C). Therefore policy u^* solves (P1).

It is worth noting that the proposed optimal policy u^*, does not require any information about the channel statistics. In particular, u^* is a fairly simple adaptive policy that in each time slot decides a decoding permutation by using only a) the channel fading state in that slot, and b) the observed average power consumption up to that slot. Then, once the decoding permutation has been defined for slot t, the transmission power schedule for that slot is easily computed (i.e., the vector $\mathbf{P}(\pi^*, \mathbf{h}(t))$).

In [7, Algorithm 4.1] an algorithm for minimizing the maximum average user power is presented. This latter algorithm is off-line, relies heavily on the knowledge of channel statistics (the complete pdf is required for the fading of each channel) and involves complicated computations (integration over the channel state space).

3.2 Optimal Throughput Policies

Here we consider the same channel model as in Section 3.1. However, we do not impose any SIR constraints, i.e., it is not required that the user bit rate be larger than a given threshold at all times. Instead, the performance of interest is the long-term user throughput. Hence the policies in Section 3.1 are geared towards supporting Constant Bit Rate (CBR) connections while the policies in the current section are appropriate for "best effort" traffic. The study of the capacity region of this system, as well as of some related linear optimization problems has been done in [18].

3.2.1 System Model

The channel model is the same as in Section 3.1. We also assume that there is an instantaneous constraint on the power levels at which users can transmit. Specifically we assume that there is a peak power constraint for each user transmission in any given slot, i.e.,

$$P_i(t) \leq \widehat{P}_i, \text{ for all } i = 1, ..., N,\ t \geq 1. \tag{23}$$

We assume again that successive decoding is available at the base station and that the rate achieved by a user in slot t is related to its SIR through Shannon's formula for the information theoretic capacity. In particular, if $R_i(t)$ is the rate for user i at slot t, we have that

$$R_i(t) = \frac{1}{2}\log\left(1 + \gamma_i(t)\right). \tag{24}$$

3.2.2 Problem Formulation

A scheduling policy u consists of two components. In particular, a policy defines the decoding permutation as well as power levels at which user transmissions take place in any given slot. The scheduler (located at the base station) uses the channel state information $\mathbf{h}(t)$, as well as the "history" of the system evolution up to time slot $t-1$, in order to make a scheduling decision regarding slot t and informs the users accordingly.

The "throughput" of user i at time t, under policy u is

$$r_i^u(t) = \frac{1}{t} \sum_{\tau=1}^{t} R_i^u(\tau). \tag{25}$$

Following the notation of Section 2, the total reward of scheduling policy u is

$$F^u = \limsup_{t \to \infty} \sum_{i=1}^{N} f_i(r_i^u(t)), \tag{26}$$

where $\{f_i(\cdot)\}_{i=1}^N$ are concave, nondecreasing, twice continuously differentiable functions defined on $[0, \infty)$.

We are interested in the problem

Problem (P2): Determine a policy u^* such that $F^{u^*} \geq F^u$, for all $u \in U$.

The design of an optimal policy (i.e., a policy that solves (P2)) follows the algorithmic steps presented in Section 2.

Step 1: *Determine the Achievable Region.*

Let $\overline{\mathbf{h}}$ be a random vector with distribution the joint distribution of the fading states, i.e., $\overline{\Pr}(\mathbf{h})$. Using a special case of a result obtained in [18, page 2808] we have that the achievable region of the system, is the set of rates defined by

$$\mathcal{R} = \bigcup_{\mathbf{P}(\mathbf{h})} \mathcal{R}(\mathbf{P}(\mathbf{h})), \tag{27}$$

where

$$\mathcal{R}(\mathbf{P}(\mathbf{h})) = \left\{ \mathbf{r} : \sum_{i \in \mathcal{S}} r_i \leq \mathrm{E}_{\overline{\mathbf{h}}} \left[\frac{1}{2} \log \left(1 + \frac{1}{\sigma^2} \sum_{i \in \mathcal{S}} \overline{h}_i P_i(\overline{\mathbf{h}}) \right) \right], \ \forall \mathcal{S} \subseteq \{1, ..., N\} \right\},$$

and the union ranges over all vectors $\mathbf{P}(\mathbf{h})$ such that

$$0 \leq P_i(\mathbf{h}) \leq \widehat{P}_i, \ i = 1, ..., N.$$

Since $\log(x)$ is increasing, it follows that $\mathcal{R}(\mathbf{P}(\mathbf{h})) \subseteq \mathcal{R}\left(\widehat{\mathbf{P}}\right)$, where $\widehat{\mathbf{P}} = \left(\widehat{P}_i\right)_{i=1}^N$, i.e.,

$$\mathcal{R} = \left\{ \mathbf{r} : \sum_{i \in \mathcal{S}} r_i \leq \mathrm{E}_{\overline{\mathbf{h}}} \left[\frac{1}{2} \log \left(1 + \frac{1}{\sigma^2} \sum_{i \in \mathcal{S}} \overline{h}_i \widehat{P}_i \right) \right], \ \forall \mathcal{S} \subseteq \{1, ..., N\} \right\}. \tag{28}$$

This region corresponds to the policy that allocates at each time slot the power vector $\widehat{\mathbf{P}}$, i.e., the maximum allowable power to each user, independent of the fading state [18, page

2799]. Therefore, we may restrict attention to the set of policies U such that for every $u \in U$ it holds

$$P_i^u(t) = \widehat{P}_i, \text{ for every } i = 1, .., N, \ t \geq 1. \tag{29}$$

Hence, in essence a scheduling policy $u \in U$, defines only the decoding order in any given slot by taking into account the channel state information in that slot as well as the “history” of the system up to that slot.

Step 2: *Ensure that* $|\mathbf{X}(t+1) - \mathbf{x}(t)| \leq B$.

For the system under consideration we have that under any policy u,

$$\mathbf{X}(t+1) = \mathbf{R}^u(t+1), \ \mathbf{x}(t) = \mathbf{r}^u(t).$$

The facts that the fading process and the transmit powers are bounded, imply through (24) that $\mathbf{R}^u(t)$ is bounded as well. This in turn implies through (25) that $\mathbf{r}^u(t)$ is also bounded and hence the same holds for $\mathbf{R}^u(t+1) - \mathbf{r}^u(t)$.

Step 3: *Find a policy that satisfies Condition (C).*

We will show that the following policy is optimal.

Policy u^*: At any slot $t+1$ the decoding permutation π^* satisfies

$$f'_{\pi^*(1)}\left(r_{\pi^*(1)}(t)\right) \geq f'_{\pi^*(2)}\left(r_{\pi^*(2)}(t)\right) ... \geq f'_{\pi^*(N)}\left(r_{\pi^*(N)}(t)\right).$$

For any given state $\mathbf{h}$ consider the set

$$\mathcal{R}(\mathbf{h}) = \left\{\mathbf{r} : \sum_{i \in \mathcal{S}} r_i \leq \frac{1}{2}\log\left(1 + \frac{1}{\sigma^2}\sum_{i \in \mathcal{S}} h_i \widehat{P}_i\right), \text{ for all } \mathcal{S} \subset \{1, ..., M\}\right\}.$$

Denoting for simplicity $f'_i(r_i^{u^*}(t)) = f'_i$ and using an approach similar to [10], or [18], it is easy to prove that if policy u^* is employed, then for any point $\mathbf{r}(\mathbf{h}) \in \mathcal{R}(\mathbf{h})$ it holds

$$\sum_{i=1}^{N} f'_i \mathrm{E}\left[R_i^{u^*}(t+1) \middle| \mathbf{r}^{u^*}(\tau), \ \tau \leq t, \, \mathbf{h}(t+1) = \mathbf{h}\ \right] \geq \sum_{i=1}^{N} f'_i r_i(\mathbf{h}).$$

Consider now an arbitrary point $\mathbf{r} \in \mathcal{R}$. According to [18], $\mathbf{r}$ can be written as

$$r_i = \sum_{\mathbf{h}} r_i(\mathbf{h}) \overline{\Pr}(\mathbf{h}),$$

with $\mathbf{r}(\mathbf{h}) \in \mathcal{R}(\mathbf{h})$.

Hence under policy u^*, as in Section 3.1.2 it holds,

$$\begin{aligned}
&\sum_{i=1}^{N} f_i' \mathrm{E}\left[R_i^{u^*}(t+1)\,|\mathbf{r}(\tau),\ \tau \leq t\ \right] \\
=\ & \sum_{i=1}^{N} f_i' \sum_{\mathbf{h}} \mathrm{E}\left[R_i^{u^*}(t+1)\,|\mathbf{r}(t),\mathbf{h}(t+1)=\mathbf{h}\ \right] \overline{\Pr}(\mathbf{h}) \\
\geq\ & \sum_{i=1}^{N} \sum_{\mathbf{h}} f_i' r_i(\mathbf{h}) \overline{\Pr}(\mathbf{h}) = \sum_{i=1}^{N} f_i' \sum_{\mathbf{h}} r_i(\mathbf{h}) \overline{\Pr}(\mathbf{h}) = \sum_{i=1}^{N} f_i' r_i.
\end{aligned}$$

From the last inequality it is concluded that policy u^* is such that the vector

$$\left(\mathrm{E}\left[R_i^{u^*}(t+1)\,\middle|\,\mathbf{r}^{u^*}(\tau),\ \tau \leq t\ \right]\right)_{i=1}^{N},$$

is a solution to the linear optimization problem over the achievable region $\mathcal{R}$. Hence condition (C) is satisfied and the proposed policy u^* solves Problem (P2).

4 Downlink Scheduling over Multiple Wireless Fading Channels

In this section we deal with downlink scheduling (base station to users) over wireless fading channels under the assumption that transmission to multiple users from the base station can take place simultaneously as long as there are sufficient resources available. Simultaneous transmission to multiple users can be achieved by employing spread spectrum techniques that utilize multiple channels, e.g., via orthogonal codes. In such an environment, a policy scheduling user transmissions so as to maximize the sum of the users throughputs, while ensuring that throughput allocations are proportional to prespecified weights, was presented in [13]. This is essentially a problem of maximizing the minimum of user weighted throughputs. We adopt the system model of [13] and we use the framework presented in Section 2 to derive the achievable region and an optimal policy associated with the maximization of the objective function given in (2) when the performance measure of interest is the user throughputs. As stated in Section 2, appropriate choice of user objective functions permits the approximate solution to the *max-min* optimization problem which is stricter than the optimization considered in [13].

We now give the description of the system model. We consider N users that receive information from the base station through multiple wireless fading channels. We assume that the transmission rate to user i in any given slot t, $R_i(t)$, can take any value from the set $S^i = \left\{0, R_i^1, ..., R_i^{M_i}\right\}$, i.e., $R_i(t) \in S^i$. The power consumption for user i at slot t, $P_i(t)$, is linearly related to its transmission rate and in particular $P_i(t) = g_i(t) R_i(t)$.

Here, $g_i(t)$ is the per bit power consumption for transmission to user i at slot t, and is varying with time due to fading. The channel state in slot t is described by the vector $\mathbf{g}(t) = (g_1(t), ..., g_N(t))$. The assumptions for the process $\mathbf{g}(t)$, $t = 1, 2, ...$ are the same as the assumptions used for the fading process in Sections 3.1 and 3.2. The main system resource to be shared is the total transmission power P at the base station and it is required that

$$\sum_{i=1}^{N} P_i(t) = \sum_{i=1}^{N} g_i(t) R_i(t) \leq P, \text{ for all } t. \tag{30}$$

Hence, an admissible scheduling policy for this system decides in every slot the rates at which transmission to users take place (note that a user may not be scheduled in a slot if its assigned rate is zero), subject to the total power constraint given in (30).

The throughput of user i up to time t, the associated reward functions and the total reward of an admissible policy u are the same as in Section 3.2.2. We are interested in the problem

Problem (P3): Determine a policy u^* such that $F^{u^*} \geq F^u$, for all $u \in U$.

Let the channel be in a given state $\mathbf{g}$. Then, in order to satisfy the system resource constraint (30), the user rates must belong to the region

$$\mathcal{R}^F(\mathbf{g}) = \left\{ \mathbf{R} \in S^1 \times \cdots \times S^N \colon \sum_{i=1}^{N} g_i R_i \leq P \right\}.$$

Consider a scheduling policy u and define by $t(\mathbf{R}, \mathbf{g})$ the amount of time up to time t that policy u selects the transmission rate vector $\mathbf{R} \in \mathcal{R}^F(\mathbf{g})$, when the channel is in state $\mathbf{g}$. Also let $t(\mathbf{g})$ denote the amount of time up to time t that the system is in state $\mathbf{g}$. Following the steps of Section 3.1.2, it is easy to derive the achievable region and to state an optimal policy (i.e., a policy that solves (P3)) for the system under consideration. In particular, the achievable region $\mathcal{R}$, is the set

$$\mathcal{R} = \left\{ \mathbf{r} : \mathbf{r} = \sum_{\mathbf{g}} \mathbf{r}(\mathbf{g}) \overline{\Pr}(\mathbf{g}), \ \mathbf{r}(\mathbf{g}) \in \mathcal{R}(\mathbf{g}) \right\},$$

where $\mathcal{R}(\mathbf{g})$ is the convex hull of all transmission vectors $\mathbf{R} \in \mathcal{R}^F(\mathbf{g})$.

The optimal policy u^*, is

Policy u^*. At time slot t, select the transmission rate vector $\mathbf{R}^*$ that is the solution to the optimization problem

$$\text{maximize } \sum_{i=1}^{N} c_i R_i,$$
$$\text{subject to} : \mathbf{R} \in \mathcal{R}\left(\mathbf{g}\left(t\right)\right),$$

where $c_i = f_i'(r_i^{u^*}(t))$, $i = 1, ..., N$.

Since $\mathcal{R}\left(\mathbf{g}\left(t\right)\right)$ is the convex hull of all vectors in $\mathcal{R}^F\left(\mathbf{g}(t)\right)$ it is sufficient to maximize $\sum_{i=1}^{N} c_i R_i$ over all vectors in $\mathcal{R}^F\left(\mathbf{g}(t)\right)$. This latter problem is a form of the Knapsack problem and hence NP-complete. In spite of this, there are pseudopolynomial time algorithms for finding an optimal solution, which work well in practice [14]. There are also fully polynomial time approximations to this problem. If in order to ensure small running time such a polynomial time approximation algorithm is chosen (rather than an exact pseudopolynomial algorithm), then the problem arises whether this approximate solution to the linear problem guarantees an approximate solution to the original problem and in what sense. This issue requires extensions to the framework presented in this paper.

5 Relaxing the Assumptions

The basic policy design steps developed in Section 2 rely on certain simplifying assumptions which permitted us to reveal the essential features of policies involved. However, in certain systems some of these assumptions may be restrictive. Dealing with these cases requires a more general version of the stochastic approximation Theorem 1 and several technical details that depend on the system under consideration. There are various versions of the stochastic approximation theorem with varying degrees of generality. A fairly general one can be found in [4, Section 3.2] which we restate in Appendix A.3 so that we can refer to it in the discussion below. In this section we examine some of the assumptions made in the previous sections and discuss the manner that may be weakened.

1. *Assumptions on* $\{f_i\left(\cdot\right)\}_{i=1}^{N}$: These can be weakened somewhat. Mainly, these assumptions are imposed so that the Lyapunov function condition (38), as well as (40) is satisfied. Specifically, for the maximization problem, it can be assumed that $f_i(x)$, $i = 1, .., N$, are concave, continuously differentiable and satisfy condition (38) (with the obvious correspondence). Hence it is not required that the second derivative of $f_i(x)$ to be continuous. For example, it suffices that $f_i'(x)$ is a piecewise linear function.

2. *Boundedness of the drift* $X\left(t+1\right) - x\left(t\right)$: This assumption may be removed based on condition (SA.2). For example, in [4, Section 3.2] it was sufficient to make certain

assumptions on the moments of the input parameters for the system under consideration.

3. *Condition (C):* This condition may not be true when one looks at the "one step" conditional expectation. For the systems considered in Sections 3.1, 3.2 and 4, if we remove the assumption that the system state process consists of i.i.d. variables Condition (C) will not hold. However, it may still hold when one looks at longer time intervals. The more general version of the theorem in Appendix A.3 allows us to deal with such cases. We outline below the approach to be followed for the system in Section 3.1 if the fading process is regenerative.

   Let us make the (unrealistic, see next item) assumption that the updates of average user powers take place at the regeneration instants T_n of the fading process, and that $E[T_1] < \infty$. These updates determine the coefficients $f'(p_i(T_n))$ of the linear optimization during the whole interval $[T_n, T_{n+1})$. Until the next regeneration instant T_{n+1}, the permutation employed in every slot is chosen so that (22) is satisfied with numerators fixed to $f_i'(p_i(T_n))$. That is, at time t, $t = T_n, ...T_{n+1} - 1$, we select the permutation π^* so that

$$\frac{f'_{\pi^*(1)}\left(p_{\pi^*(1)}\left(T_n\right)\right)}{h_{\pi^*(1)}\left(t+1\right)} \geq .. \geq \frac{f'_{\pi^*(N)}\left(p_{\pi^*(1)}\left(T_n\right)\right)}{h_{\pi^*(N)}\left(t+1\right)}.$$

   Setting $p_{i,n} = p_i(T_n)$, $D_{n+1} = T_{n+1} - T_n$ and $\widehat{P}_i(T_{n+1}) = \sum_{\tau=T_n+1}^{T_{n+1}} P_i(\tau)$ we can write

$$\begin{aligned} p_{i,n+1} \quad &= \quad p_{i,n} + \frac{1}{n+1}\left(\frac{\widehat{P}_i(T_{n+1})}{E\left[T_1\right]} - \frac{D_n P_{in}}{E\left[T_1\right]}\right) + \\ &\frac{1}{(n+1)^{1+\rho}}\left(n+1\right)^{\rho}\left(\frac{E\left[T_1\right](n+1)}{T_{n+1}} - 1\right)\left(\frac{\widehat{P}_i(T_{n+1})}{E\left[T_1\right]} - \frac{D_n P_{in}}{E\left[T_1\right]}\right). \end{aligned} \quad (31)$$

   In this form, there is an obvious correspondence between (31) and (36). Since by the regenerative assumption T_n is the sum of i.i.d. variables, i.e., $T_n = \sum_{m=1}^{n} D_n$, it holds,

$$\lim_{n\to\infty} \frac{E\left[T_1\right](n+1)}{T_{n+1}} = 1.$$

   It turns out the the term in (31) corresponding to $\mathbf{G}_{n+1}^{(2)}$ in (36) is small and satisfies the requirements of the stochastic approximation theorem. The main inequality to obtain is (40). For the latter inequality, following the general approach outlined in

Section 2 it can be seen that it is sufficient to show that the vector

$$\left(E\left[\frac{\widehat{P}_i(T_{n+1})}{E\left[T_1\right]} \middle| \mathbf{p}_n \right] \right)_{i=1}^{N},$$

satisfies Condition (C). This can be done in a similar manner as in Section 3.1 after observing that since the fading process is regenerative it holds

$$\overline{\Pr}(\mathbf{h}) = \frac{E\left[T_1(\mathbf{h})\right]}{E\left[T_1\right]},$$

where $T_1(\mathbf{h})$ is the number of slots in $[0, T_1)$ that the fading process is in state $\mathbf{h}$.

4. To deal with regenerative fading process above, we made the unrealistic assumption that updates of average user powers take place at regeneration instants. This assumption may also be removed. In [4, Section 3.2] it was shown for the corresponding system that is it sufficient to make updates so that no more that $2L$ regeneration points occur between two updates, where L is any fixed number. The proof of this requires the full power of the theorem in Appendix A.3, and is using the fact that the policy is of index type. Since the policy determined by (22) is also of index type, the proof should be adaptable to the system of Section 3.1. The main idea is that if enough history is used for the updates (i.e., time is long enough), then updates within a regeneration period will not change the numerators in inequalities (22) by much and the resulting policy will be close to the one where updates take place only at regeneration instants.

5. The derivation of the achievable region usually requires minimal assumptions. For example, observe that for the arguments leading to the achievable region in Section 3.1, only require that the ergodicity limits (19) hold.

Finally we discuss the practical issue of ability of the system to adapt to changes. Measurements of the type (1) become insensitive to changes as time grows. Hence in practice, either the measurements are confined in a window $(t - T, t)$, or old measurements are weighed out, i.e., (18) is replaced with,

$$\widehat{x}_i^u\left(t+1\right) = \left(1-\beta\right)\sum_{\tau=1}^{t+1} \beta^{t-\tau} X_i^u\left(\tau\right) = \beta \widehat{x}_i^u\left(t\right) + \left(1-\beta\right) X_i^u\left(t+1\right).$$

For these latter updates it was shown in [3] that for β close to 1, the system performance remains close to the optimal one. Simulation experiments in [19] showed that for the wireless system under consideration such a choice provides a good compromise between convergence and adaptivity.

6 Conclusions

We presented a framework for designing optimal adaptive policies, which are related to fairness, in wireless networks. The proposed framework is especially useful when the policies for the solution of related linear optimization problems are relatively simple. This situation arises in several interesting applications. Moreover, it turns out that in certain systems the linear optimization policies depend minimally on system parameters. In this case, the policies designed within the proposed framework inherit the same property.

The essential steps involved in the policy design are simple. Taking into account general assumptions about system parameters and statistics may require considerable technical details, which obscure the essential simplicity of the method. However, there are several general theorems based on which the problems arising from the generality of the assumptions can be addressed.

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A Appendix

A.1 Proof of Inequality (5)

In this appendix we provide a proof of inequality (5) in a slightly more general setting.

Lemma 2 *Let $\mathcal{A}$ be a compact set and $F(\mathbf{x})$ a continuous function in $\Re^N$. Assume that for a sequence $\mathbf{x}(t) \in \Re^N$ it holds*

$$\lim_{t\to\infty} \left(\inf\left\{|\mathbf{x}(t) - \mathbf{y}| : \mathbf{y} \in \mathcal{A}\right\}\right) = 0. \tag{32}$$

Then

$$\limsup_{t\to\infty} F(\mathbf{x}(t)) \leq F(\mathbf{x}^*).$$

where $\mathbf{x}^*$ is an optimal solution to the problem

$$\begin{gathered} \text{maximize } F(\mathbf{x}) \\ \text{subject to } \mathbf{x} \in \mathcal{A}. \end{gathered}$$

Proof. The compactness of $\mathcal{A}$ ensures that $\mathbf{x}^*$ exists. Therefore, it suffices to show that for any $\delta > 0$ there is a t_δ such that

$$F(\mathbf{x}(t)) \leq F(\mathbf{x}^*) + \delta, \text{ for all } t \geq t_\delta.$$

To show this, note first that the continuity of $F(\mathbf{x})$ and the compactness of $\mathcal{A}$ imply the uniform continuity of $F(\mathbf{x})$ in a compact neighborhood [17],

$$\mathcal{A}_a = \left\{\mathbf{z} : |\mathbf{z} - \mathbf{y}| \leq a \text{ for some } \mathbf{y} \in \mathcal{A}\right\}, \ a > 0.$$

Hence, for the given δ, there is and $\varepsilon_\delta > 0$ such that

$$|F(\mathbf{x}) - F(\mathbf{y})| < \delta, \text{ whenever } |\mathbf{x} - \mathbf{y}| \leq \varepsilon_\delta \text{ and } \mathbf{x}, \mathbf{y} \in \mathcal{A}_a. \tag{33}$$

Let now $\varepsilon = \min\left\{a, \varepsilon_\delta\right\}$. Assumption (32) implies that for the given ε there is a t_0 such that

$$\inf\left\{|\mathbf{x}(t) - \mathbf{y}| : \mathbf{y} \in \mathcal{A}\right\} < \varepsilon, \ t \geq t_0.$$

Since the function $e(\mathbf{y}) = |\mathbf{x}(t) - \mathbf{y}|$ is continuous in $\mathbf{y}$ and $\mathcal{A}$ is compact, the infimum is achieved for a point $\mathbf{y}(t) \in \mathcal{A}$. That is, we have

$$|\mathbf{x}(t) - \mathbf{y}(t)| < \varepsilon, \ t \geq t_0, \ \mathbf{y}(t) \in \mathcal{A}. \tag{34}$$

Inequality (34) implies that $\mathbf{x}(t) \in \mathcal{A}_a$ and since obviously $\mathbf{y}(t) \in \mathcal{A}_a$, we have from (33) that

$$F(\mathbf{x}(t)) \leq F(\mathbf{y}(\mathbf{t})) + \delta \leq F(\mathbf{x}^*) + \delta, \ t \geq t_0,$$

as desired. ■

Notes:

1. If $\mathbf{x}^*$ minimizes $F(\mathbf{x})$, then in an analogous fashion it can be shown that
$$\liminf_{t\to\infty} F(\mathbf{x}(t)) \geq F(\mathbf{x}^*).$$

2. It is easily seen by the proof that it suffices to assume that $F(\mathbf{x})$ is continuous in a neighborhood $\mathcal{A}_a$ of the set $\mathcal{A}$.

A.2 Stability Region of the Problem in Section 3.1.2

We have to prove that
$$\lim_{t\to\infty} \left(\inf \left\{ \left| \mathbf{p}^u(t) - \mathbf{p} \right| : \mathbf{p} \in \mathcal{P} \right\}\right) = 0, \tag{35}$$
where $\mathcal{P} = \left\{ \mathbf{p} : \mathbf{p} = \sum_{\mathbf{h}} \mathbf{p}(\mathbf{h}) \overline{\Pr}(\mathbf{h}),\ \mathbf{p}(\mathbf{h}) \in \mathcal{C}(\mathbf{h}) \right\}$. From (20) we have that
$$\mathbf{p}(t) = \sum_{\mathbf{h}} \sum_{\pi} \mathbf{P}(\pi, \mathbf{h}) \frac{t(\pi, \mathbf{h})}{t(\mathbf{h})} \overline{\Pr}(\mathbf{h}),$$
belongs to $\mathcal{P}$. For this particular $\mathbf{p}(t)$, consider the difference
$$\begin{aligned}
& \left| \mathbf{p}^u(t) - \mathbf{p}(t) \right| \\
= & \left| \sum_{\mathbf{h}} \frac{t(\mathbf{h})}{t} \sum_{\pi} \mathbf{P}(\pi, \mathbf{h}) \frac{t(\pi, \mathbf{h})}{t(\mathbf{h})} - \sum_{\mathbf{h}} \sum_{\pi} \mathbf{P}(\pi, \mathbf{h}) \frac{t(\pi, \mathbf{h})}{t(\mathbf{h})} \overline{\Pr}(\mathbf{h}) \right| \\
= & \left| \sum_{\mathbf{h}} \left(\frac{t(\mathbf{h})}{t} - \overline{\Pr}(\mathbf{h}) \right) \sum_{\pi} \mathbf{P}(\pi, \mathbf{h}) \frac{t(\pi, \mathbf{h})}{t(\mathbf{h})} \right| \\
\leq & \max \left\{ |\mathbf{p}| : \mathbf{p} \in \cup_{\mathbf{h}} \mathcal{C}(\mathbf{h}) \right\} \sum_{\mathbf{h}} \left| \left(\frac{t(\mathbf{h})}{t} - \overline{\Pr}(\mathbf{h}) \right) \right|.
\end{aligned}$$

Using (19) and the last inequality we conclude that $\lim_{t\to\infty} \left| \mathbf{p}^u(t) - \mathbf{p}(t) \right| = 0$. Since $\mathbf{p}(t) \in \mathcal{P}$, (35) follows.

A.3 A Stochastic Approximation Theorem

In $\Re^N$ consider a sequence $\mathbf{y}_n,\ n = 1, 2, \ldots$ which for $\rho > 0$ satisfies the recursion
$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{n+1} \mathbf{G}_{n+1}^{(1)} + \frac{1}{(n+1)^{1+\rho}} \mathbf{G}_{n+1}^{(2)}, \quad n \geq 1. \tag{36}$$

Also consider a non-decreasing family of σ-fields $\mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{G}$ such that for $n \geq 2$, $\mathbf{G}_n^{(1)}$ and $\mathbf{G}_n^{(2)}$ are measurable with respect to $\mathcal{G}_n$. Assume that the following hold.

(SA.1) There exists a compact set $\mathcal{A} \subset \Re^N$ such that $\mathbf{y}_n,\ n = 1, 2, ...$ converges to $\mathcal{A}$ a.s., i.e.,

$$\lim_{n\to\infty} \left(\inf\left\{|\mathbf{y}_n - \mathbf{y}| : \mathbf{y} \in \mathcal{A}\right\}\right) = 0.$$

In what follows fix $M > 0$ such that $\mathcal{A} \subseteq \left\{\mathbf{y} \in \Re^N : |\mathbf{y}| \le M\right\}$.

(SA.2) There exist

- a sequence of events $\{E_{1,n}\}_{n=2}^{\infty}$ such that $E_{1,n} \in \mathcal{G}_n$ for $n \ge 2$ and

$$\Pr\left(\liminf_{n\to\infty} E_{1,n}\right) = 1$$

  and

- a random variable $X \ge 0$ with $E[X] < \infty$

such that for $i = 1, 2$ and on $\{|\mathbf{y}_n| \le M\}$,

$$\Pr\left\{1_{E_{1,n+1}} \left|\mathbf{G}_{n+1}^{(i)}\right| > x \middle| \mathcal{G}_n\right\} \le C \Pr\{X > x\}, \quad x \ge 0. \tag{37}$$

(SA.3) There exists a continuously differentiable Lyapunov-type function $V : \Re^N \to \Re$ such that for $\mathbf{y} \in \Re^N$ with $|\mathbf{y}| \le M$, and $\mathbf{h} \in \Re^N$ with $|\mathbf{h}| \le 1$,

$$|V(\mathbf{y} + \mathbf{h}) - V(\mathbf{y}) - \langle \nabla V(\mathbf{y}), \mathbf{h}\rangle| \le C |\mathbf{h}|^2. \tag{38}$$

Also, there exist events $\{E_{2,n}\}_{n=2}^{\infty}$ with $E_{2,n} \in \mathcal{G}_n$ for $n \ge 2$, such that

$$\Pr\left(\liminf_{n\to\infty} E_{2,n}\right) = 1, \tag{39}$$

and for $\varepsilon > 0$ there exists $\overline{\varepsilon} > 0$ and $m \ge 1$ such that for $n \ge m$ and on $E_{2,n} \cap \{|\mathbf{y}_n| \le M\} \cap \{V(\mathbf{y}_n) > \varepsilon\}$ it holds

$$\left\langle \nabla V(\mathbf{y}_n), E\left[\mathbf{G}_{n+1}^{(1)} \middle| \mathcal{G}_n\right]\right\rangle < -\overline{\varepsilon}. \tag{40}$$

Theorem 3 *Under conditions (SA.1)-(SA.3),*

$$\lim_{n\to\infty} \left(V(\mathbf{y}_n)\right)^+ = 0.$$