Problems of Adaptive Optimization in Multiclass M/GI/1 Queues with Bernoulli Feedback[1]

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Abstract

Adaptive algorithms are obtained for the solution of separable optimization problems in multiclass $M/GI/1$ queues with Bernoulli feedback. Optimality of the algorithms is established by modifying and extending methods of stochastic approximation. These algorithms, can be used as a basis for designing policies for semi-separable and approximate lexicographic optimization problems and in the case of $M/GI/1$ queues without feedback, they also provide a simple policy for lexicographic optimization. The results obtained on stochastic approximation imply convergence of classical recursions such as Robbins-Monroe in cases where the conditional second moment of their increments is not finite.

Keywords: Stochastic Scheduling, Queueing Systems, Adaptive Control, Stochastic Approximation.

1 Introduction

An $M/GI/1$ queue with Bernoulli feedback consists of a single server and a set of N queueing nodes denoted by $\mathcal{N} := \{1, \ldots, N\}$. Exogenous arrivals to the queues are Poisson with rate λ_i for node $i \in \mathcal{N}$. The service requirement of each job in node $i \in \mathcal{N}$ is independent of all else and has distribution $B_i(\cdot)$. The service in each queue is first-come-first-served and upon completion at node $i \in \mathcal{N}$ a job is routed to node $j \in \mathcal{N}$ with probability p_{ij} and leaves the system with probability $p_{i0} := 1 - \sum_{j \in \mathcal{N}} p_{ij}$.

For this model, we address the problem of allocating the server to the nodes in order to satisfy certain design criteria. The service allocation policies we consider are nonidling, nonpreemptive for each job in each node, and nonanticipative. The last term means that scheduling decisions do not depend on the arrival times of jobs that will arrive in the future and on the service times of jobs that will complete service in the future. Call such policies admissible and denote their set by Π. To date, only policies that minimize the weighted sum of average delays have been identified, see Klimov [17],[18] and Tcha and Pliska [26]. While this may be appropriate for some situations, there are applications (see [2]) for which it is natural to consider more general functions of average delays. In this paper, we address a class of such optimization problems and provide optimal scheduling policies.

For $u \in \Pi$, $i \in \mathcal{N}$ and $n \geq 1$, denote by $R_i^u(n)$ the delay of the nth job in node i under policy u, to arrive either externally or from a node in $\mathcal{N} \setminus \{i\}$. Also, set $R^u(n) := (R_1^u(n), \ldots, R_N^u(n))^T$. We first consider the following problem.

Separable minimization Consider real valued, convex and continuously differentiable functions $\{\phi_i(\cdot)\}_{i \in \mathcal{N}}$ on $\mathbf{R}$ with the property that for $M > 0$ and $\theta, h \in \mathbf{R}^N$ such that $|\theta| \leq M$, $|h| \leq 1$, there exists a constant $C \geq 0$, depending on M, such that

$$|\phi_i(\theta_i + h_i) - \phi_i(\theta_i) - \phi_i'(\theta_i)h_i| \leq C h_i^2, \quad i \in \mathcal{N}. \tag{1}$$

For such functions consider the following problem.

Problem (S) $\inf \{\limsup_{n \to \infty} \sum_{i \in \mathcal{N}} \phi_i \left((1/n) \sum_{k=1}^n R_i^u(k)\right) \ : \ u \in \Pi\}$.

The condition in (1) is satisfied if the ϕ_i's are twice continuously differentiable. This somewhat weaker form is imposed in order to accommodate integrals of piece-wise linear functions arising in applications (see [2]).

We provide simple adaptive policies that solve this problem. The policies are of the following type. At each decision instant, the delays at the various nodes are estimated and then used to determine the priorities of the nodes that remain fixed until the next decision instant. The choice of the decision instants is fairly arbitrary and the only statistical parameters that are used are the first moments of the service times and the routing probabilities.

We develop techniques of Stochastic Approximation to prove the optimality of the policies under minimal statistical assumptions, namely finiteness of the second moment of service times.

Problem (S) can be used as a basis for the solution of semi-separable and approximate lexicographic and min-max optimization problems that arise naturally in applications. In the special case of models without feedback, a simple policy for lexicographic optimization can easily be derived. While we have strong reasons to believe that the same policy is optimal for the general problem of queues with feedback, we lack a proof at this time. For details on these variants of problem (S), the reader is referred to [2].

The optimal policies for the problem of minimizing linear costs of average delays in queues with feedback are strict priority rules. When there is no feedback, the policy becomes particularly simple. See [28] and references therein for related work, which relies mainly on methods of dynamic programming and moment generating functions. The linearity of the cost is heavily used in these approaches and it is difficult to see how to use these techniques for the more general problems considered in this paper. Our approach here is rather different; it relies on the structure of the space of achievable mean delays and uses techniques of Stochastic Approximation.

For multi-class M/GI/1 queues without feedback, the set of achievable vectors of mean delays was characterized by Coffman and Mitrani [6], Gelenbe and Mitrani [11] and extended by Georgiadis and Viniotis [12]. The result was based on the conservation law of Kleinrock [16]. The set is a polymatroid whose vertices can be achieved by strict priority policies. A simple derivation of this fact was obtained in Shanthikumar and Yao [25]. They also show that this property holds for several other models which, however, do not include models with feedback considered in this paper.

Several problems of deterministic optimization can be solved efficiently by special methods when the constraint set is a polymatroid; see [10] and [15]. In particular, it is well known that linear programs can be solved by a "greedy" procedure. In [8], [9], [22] and [23] it is proposed to utilize the solution of the deterministic optimization problem in order to obtain scheduling policies for M/GI/1 queues without feedback. The computation of a dynamic priority policy of the type of Section 3.7 in [16] is presented in [8]. To implement this approach for non-linear costs, one would have to estimate the coefficients of the achievable polymatroid. For some of the models in [25] however, some of the coefficients have not been computed to date. In [9] an approximation is used to circumvent this problem. In addition, the approach requires the estimation of the arrival rates and of the first and second moments of the service times. Two computations must then be performed on these estimates. The first one obtains the optimal vector of delays while the second one determines the dynamic priority that achieves it. These computations must be repeated as the model parameters fluctuate and the estimates are updated. Randomized policies of the type suggested on p.205 of [11] can be used instead of dynamic priority policies but they are usually considered impractical because the variance of the delays may be unacceptably high.

The introduction of feedback in the model, entails considerable complications. The achievable region of mean delays for this model was determined in Tsoucas [27]. The set is again a polytope whose vertices can be achieved by strict priority policies. It is no longer a polymatroid, however. Linear programs on this polytope can be solved efficiently by means of Klimov's algorithm [17]. More general deterministic optimization problems on this polytope are considered in Bhattacharya et al [1]. The proposed algorithm for computing node priorities at the decision instants is based on Klimov's algorithm [17]. An alternative algorithm which has appeared in the literature as a solution to linear optimization problems for queues with feedback, [20], and has certain computational advantages in some situations can also be used (see [2]). In contrast to the previous approaches, our policies do not require the computation of the optimal point or the knowledge of the boundaries of the polytope; the only statistical parameters used are the first moment of the service times and the routing probabilities.

In our earlier work [3], we analyzed adaptive policies that solve lexicographic optimization problems with linear cost functions for queues without feedback. Besides addressing a considerably more general problem, our technique here departs from the one in [3] in two significant respects. First, our policies in [3] were analyzed in the case where updates were performed only at the beginning of busy periods. As a consequence, the updates become infrequent for large utilizations, which leads to large variability of delays. For the policy presented in the paper, the only restriction on the time between updates is that this time is no more than 2L busy periods, where L is fixed but arbitrary. Hence, even if a busy period is long, updates can be done at short intervals, for example at the completion of every service or at short time intervals, and the policy would still be optimal. The variance of our optimal policy can be controlled in this way. Second, in [3] we required that the fourth moment of the service time distribution be finite. Here, finiteness of only the second moment is required. This condition cannot be weakened further. Measurements in existing systems suggest that service requirements can be quite volatile. It is therefore desirable to establish the applicability of our policies under the weakest possible assumptions. The generality of model and of the update instants, together with the weakening of the moment assumptions lead to considerable complications and require new techniques.

The difficulty of analyzing queueing systems under policies with general update rules seems to be an inherent one. It also appears in a class of stochastic gradient algorithms which give rise to one-dimensional recursions. Chong and Ramadge [4] were able to deal with this difficulty. Our approach is similar in spirit but the models and the problem formulations are quite different and require substantially different techniques. Our results were derived independently. Furthermore, one should be able to use our method to reduce the requirement in [4] that no moment of the service time distribution lower than the sixth be infinite. This is straightforward for [5].

It turns out that our method for reducing moment requirements has broader applicability. Methods of analyzing stochastic recursions typically require finiteness of the conditional second moment of their increments. We replace this requirement by a stochastic dominance condition and finiteness of the conditional first moment. The relevant result is Theorem 3 in §4.2.

The remaining of this paper is organized as follows. The optimal policy for problem (S) is described in §2. Optimality is established in §3. Results on stochastic approximation that are used in §3 are proved in §4.

A few words on notation: It is assumed that all processes considered in this paper are constructed on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Almost sure (a.s.) convergence is with respect to $\mathcal{P}$. We do not distinguish between random variables that are equal almost surely and drop the indication "a.s." The symbol C is used in inequalities to denote a sufficiently large positive constant. Its value is immaterial and may be different in different inequalities. The derivative of a function $f(x)$ is denoted as $f'(x)$. Finally, for $x \in \mathbf{R}$, define $x^+ := \max\{x, 0\}$.

2 A policy for separable minimization

2.1 Preliminaries

Let $\lambda := (\lambda_1, \ldots, \lambda_N)^T$ and denote by P the matrix with elements $\{p_{ij}\}_{i,j \in \mathcal{N}}$. We make the following assumptions:

(C.1) For each node $i \in \mathcal{N}$ there exists a node $j \in \mathcal{N}$ such that $\lambda_j > 0$ and with positive probability a job in j will visit i before leaving the system.

(C.2) The matrix $I - P$ is invertible.

(C.3) The service time distributions, $B_i(\cdot)$, $i \in \mathcal{N}$, have finite second moments, i.e. $\int_0^\infty t^2 dB_i(t) < \infty$.

(C.4) With $\beta_i := \int_0^\infty t \, dB_i(t)$, $i \in \mathcal{N}$, $\beta := (\beta_1, \cdots, \beta_N)^T \in \mathbf{R}^N$ and $\lambda := (\lambda_1, \cdots, \lambda_N)^T \in \mathbf{R}^N$, the stability condition $\lambda^T (I - P)^{-1} \beta < 1$ holds.

Conditions (C.1) and (C.2) imply that there is a unique and positive $\alpha \in \mathbf{R}_+^N$ that solves the flow equations

$$\alpha = \lambda + \alpha P. \tag{2}$$

As in [27] it can be shown that for $u \in \Pi$ the sequence of delay vectors $\{R^u(n)\}_{n \geq 1}$ satisfies certain linear constraints that are imposed by the work conserving property of u. To describe them two sets of constants are needed. First, for $S \subseteq \mathcal{N}$ and $i \in S$ let a_i^S denote the expected total amount of service that a job starting in node i receives before exiting S for the first time. In matrix notation first step equations give

$$a^S = (I - P_{SS})^{-1} \beta_S. \tag{3}$$

Second, non-negative constants $\{F(S) : S \subset \mathcal{N}\}$ exist, that are independent of $u \in \Pi$; by convention set $F(\emptyset) = 0$. With $\{\alpha_i\}_{i \in \mathcal{N}}$ as defined in (2), the constraints are written as a polytope

$$\mathcal{A} := \left\{ x \in \mathbf{R}^N \ : \ \sum_{i \in \mathcal{N}} a_i^{\mathcal{N}} \alpha_i x_i = F(\mathcal{N}); \quad \sum_{i \in S} a_i^S \alpha_i x_i \geq F(S), \quad S \subset \mathcal{N} \right\}.$$

Lemma 1 *(a) For $u \in \Pi$ and $S \subset \mathcal{N}$,*

$$\lim_{n \to \infty} \sum_{i \in \mathcal{N}} a_i^{\mathcal{N}} \alpha_i \frac{1}{n} \sum_{k=1}^{n} R_i^u(k) = F(\mathcal{N}), \tag{4}$$

$$\liminf_{n \to \infty} \sum_{i \in S} a_i^S \alpha_i \frac{1}{n} \sum_{k=1}^{n} R_i^u(k) \geq F(S). \tag{5}$$

Equality obtains and the limit exists in (5) if u gives priority to nodes in S over nodes in $\mathcal{N} \backslash S$.

(b) For x in $\mathcal{A}$ there exists a policy $u \in \Pi$ such that $\lim_{n \to \infty} (1/n) \sum_{k=1}^n R^u(k) = x$.

As the above result shows, the constants $\{F(S) : S \in \mathcal{N}\}$ are fundamental in characterizing the system. While they are hard to compute in general (and not known todate for some models), it is worth noting that the policy proposed here does not use the knowledge of these constants.

The next lemma shows that the infimum in Problem (S) can be achieved and establishes a connection with an associated deterministic problem. For $x \in \mathbf{R}^N$ set $\phi(x) := \sum_{i \in \mathcal{N}} \phi_i(x_i)$, for $n \geq 1$ set $\bar{R}^u(n) := (1/n) \sum_{k=1}^n R^u(k)$, and denote $y^* := \min\{\phi(x) : x \in \mathcal{A}\}$.

Lemma 2 *There exists a policy v in Π that solves Problem (S) and furthermore,*

$$\lim_{n \to \infty} \phi\left(\bar{R}^v(n)\right) = y^*. \tag{6}$$

Proof Since $\mathcal{A}$ is compact and $\phi(\cdot)$ is continuous, there exists $x^* \in \mathcal{A}$ such that $\phi(x^*) = y^*$. This and Lemma 1(b) imply that

$$\inf\left\{\limsup_{n\to\infty} \phi\left(\bar{R}^u\left(n\right)\right) : u \in \Pi\right\} \leq y^*.$$

The result will follow if we show that for $u \in \Pi$,

$$\liminf_{n\to\infty} \phi\left(\bar{R}^u(n)\right) \geq y^*. \tag{7}$$

For $\delta > 0$ consider the set

$$\mathcal{A}^\delta := \left\{x \in \mathbf{R}^N : \left|\sum_{i\in\mathcal{N}} a_i^{\mathcal{N}} \alpha_i x_i - F\left(\mathcal{N}\right)\right| \leq \delta, \quad \sum_{i\in S} a_i^S \alpha_i x_i \geq F(S) - \delta, \quad S \subset \mathcal{N}\right\}.$$

The continuity of $\phi(\cdot)$ and the compactness of $\mathcal{A}$ imply that for any $\epsilon > 0$, $\delta > 0$ can be chosen so that $\min\left\{\phi(x) \ : \ x \in \mathcal{A}^\delta\right\} \geq y^* - \epsilon$. Lemma 1 and the continuity of $\phi(\cdot)$ imply that $\liminf_{n\to\infty} \phi\left(\bar{R}^u(n)\right) \geq y^* - \epsilon$, and this establishes (7). □

We next present a policy that solves Problem (S).

2.2 Optimal policy

The scheduling policies used in computer operating systems and communication networks are usually of the following kind: A fixed scheduling rule is employed for a certain time interval at the end of which the rule is updated in order to incorporate new information such as system performance, traffic fluctuations, etc. The frequency of updates is restricted by the overhead of monitoring and limited computational resources. The policy we consider is similar. It employs a fixed priority rule that is updated at instants that can also be chosen quite arbitrarily.

Specifically, for $u \in \Pi$ let $\{\mathcal{F}_t^u\}_{t\geq 0}$ be the σ-field generated by the vector of queue lengths $\{\eta^u(s)\}_{0\leq s\leq t}$ which we take to be right continuous. The a policy $\pi \in \Pi$ that we propose here specified by

- a sequence of update instances $\{\sigma_n^\pi\}_{n=1}^\infty$: these are $\mathcal{F}_t^\pi$-stopping times (set $\sigma_0^\pi = 0$);
- a permutation $\pi\left(\sigma_n^\pi\right)$ of $\mathcal{N}$ that determines a fixed priority rule with which jobs are served in the interval $[\sigma_n^\pi, \sigma_{n+1}^\pi)$, $n = 0, 1, \ldots$. A permutation π on $\mathcal{N}$ is identified with a fixed priority rule where node π_i has priority over node π_j for $i < j$.

The sequence of permutations $\{\pi\left(\sigma_n^\pi\right)\}_{n=0}^\infty$ is determined from Klimov's [17] algorithm which is given next.

Algorithm (A.1) Its input is a vector $c \in \mathbf{R}^N$ and its output consists of all permutations π on $\mathcal{N}$ that can be obtained via the following two steps.

Step 1. Set $S_N := \mathcal{N}$ and

$$\nu_N := \min\left\{ \frac{c_i}{a_i^{\mathcal{N}}} \ : \ i \in \mathcal{N} \right\}.$$

Pick

$$\pi_N \in \operatorname{argmin}\left\{ \frac{c_i}{a_i^{\mathcal{N}}} \ : \ i \in \mathcal{N} \right\}. \tag{8}$$

Step 2. For $k = 0, \ldots, N-2$, set $S_{N-(k+1)} := S_{N-k} \setminus \{\pi_{N-k}\}$, and

$$\nu_{N-(k+1)} := \min\left\{ \frac{c_i - \sum_{l=0}^{k} a_i^{S_{N-l}} \nu_{N-l}}{a_i^{S_{N-(k+1)}}} \ : \ i \in S_{N-(k+1)} \right\}.$$

Pick

$$\pi_{N-(k+1)} \in \operatorname{argmin}\left\{ \frac{c_i - \sum_{l=0}^{k} a_i^{S_{N-l}} \nu_{N-l}}{a_i^{S_{N-(k+1)}}} \ : \ i \in S_{N-(k+1)} \right\}. \tag{9}$$

To indicate dependence on c we will write $\{\nu_i(c), S_i(c), \pi_i(c)\}_{i \in \mathcal{N}}$.

The input to algorithm (A.1) at the decision instant σ_n^π is determined as follows. Let $A_i^\pi(t)$ be the number of jobs served at node i by time t under policy π, incremented by one. The additional job is included in order to have a positive quantity for all $t \geq 0$. Also, set

$$\theta_i(t) := \frac{1}{A_i^\pi(t)} \int_0^t \eta_i^\pi(s)\, ds, \qquad \xi_i(t) := \frac{t}{A_i^\pi(t)} \phi_i'(\theta_i(t)). \tag{10}$$

Policy π uses permutation $\pi(\sigma_n^\pi) := \pi(\xi(\sigma_n^\pi))$, $n = 0, 1, \ldots$, i.e., an output of Algorithm (A.1) with input $\{\xi_i(\sigma_n^\pi)\}_{i \in \mathcal{N}}$.

It is clear that no optimality property can hold unless an upper bound is imposed on the lengths of intervals between updates $\{\sigma_{n+1}^\pi - \sigma_n^\pi\}_{n=0}^\infty$. Assume that the system is empty at $t = 0$, set $T_0 = 0$ and by T_n, $n \geq 1$, denote the end of the nth busy period. We impose on π

Condition (U) For some $L \geq 1$ for each $n \geq 0$ there is $m \geq 1$ such that

$$T_{nL} \leq \sigma_m^\pi < T_{(n+1)L}.$$

Consequently, successive updates are no more than $2L$ busy periods apart.

3 Optimality of policy π

3.1 Preliminaries

In this section and the next we embark on a somewhat lengthy proof of the fact that policy π solves Problem (S).

Recall that for $n \geq 1$, T_n denotes the end of the nth busy period. Since the server does not idle whenever there are jobs in the system, it is clear that the length of a busy period and hence T_n, is invariant over policies $u \in \Pi$. The same is true for $A_i^u(T_n)$ and we can therefore set $A_{ni} := A_i^u(T_n)$. For $n \geq 1$ and $i \in \mathcal{N}$ define the indices

$$\theta_{ni} := \theta_i(T_{nL}) = \frac{1}{A_{nL,i}} \int_0^{T_{nL}} \eta_i^\pi(s)\, ds \qquad \xi_{ni} := \xi_i(T_{nL}) = \frac{T_{nL}}{A_{nL,i}} \phi_i'(\theta_{ni}). \tag{11}$$

If it can be shown that $\lim_{n\to\infty} \phi(\theta_n) = y^*$ then it will follow that $\lim_{n\to\infty} \phi\left(\bar{R}^\pi(n)\right) = y^*$. This implication can be established easily by using the continuity of ϕ, the compactness of $\mathcal{A}$ and the identity

$$\sum_{k=1}^{A_i^u(T_n)-1} R_i^u(k) = \int_0^{T_n} \eta_i(s)\, ds, \quad i \in \mathcal{N},\ n \geq 0,\ u \in \Pi. \tag{12}$$

So we only need to prove

Theorem 1 $\lim_{n\to\infty} \phi(\theta_n) = y^*$.

The proof begins with the derivation of a recursion for $\{\theta_n\}_{n=1}^\infty$. Set $\tau := E[T_1]$ and recall that $\{\alpha_i\}_{i\in\mathcal{N}}$ is the solution of the throughput equation (2). Note that, by (C.1), $\alpha_i > 0$, $i \in \mathcal{N}$. For $n = 0, 1, \ldots$ and $i \in \mathcal{N}$ set

$$K_{n+1,i} := \frac{A_{(n+1)L,i} - A_{nL,i}}{L\tau\alpha_i},\ a_{n+1,i} := \frac{A_{(n+1)L,i}}{(n+1)L\tau\alpha_i},\ J_{n+1,i}^\pi := \frac{1}{L\tau\alpha_i}\int_{T_{nL}}^{T_{(n+1)L}} \eta_i^\pi(s)ds. \tag{13}$$

It can be easily verified that for $i \in \mathcal{N}$ and $n = 1, 2, \ldots$,

$$\theta_{n+1,i} = \theta_{ni} + \frac{1}{n+1}\left[J_{n+1,i}^\pi - \theta_{ni}K_{n+1,i}\right] + \frac{1}{n+1}\left(\frac{1}{a_{n+1,i}} - 1\right)\left[J_{n+1,i}^\pi - \theta_{ni}K_{n+1,i}\right]. \tag{14}$$

Observe that by (11), $\theta_0 = 0$. More generally we will consider sequences $\{\theta_n\}_{n=0}^\infty$ generated by (13) with arbitrary initial condition $\theta_0 \in \mathbf{R}^N$.

Almost sure convergence properties of this recursion can be studied in the framework of stochastic approximation. To this end, for $\rho > 0$ write in the obvious correspondence with (14)

$$\theta_{n+1} := \theta_n + \frac{1}{n+1} f^{(1)}_{n+1} + \frac{1}{(n+1)^{1+\rho}} f^{(2)}_{n+1}, \tag{15}$$

and set

$$f_{n+1} := f^{(1)}_{n+1} + \frac{f^{(2)}_{n+1}}{(n+1)^{\rho}}, \quad n \geq 0. \tag{16}$$

We will see later that $f^{(2)}_{n+1} = O(1)$ in a sense that will be made precise, and that the convergence properties of (14) are determined primarily by $f^{(1)}_{n+1}$.

Consider the following Liapunov-type function.

$$\theta^* \in \operatorname{argmin} \{\phi(\theta) : \theta \in \mathcal{A}\}, \tag{17}$$

$$V(\theta) := \sum_{i \in \mathcal{N}} [\phi_i(\theta_i) - \phi_i(\theta_i^*)]. \tag{18}$$

We use the term "Liapunov-type", since $V(\theta) \geq 0$ only when $\theta \in \mathcal{A}$. As shown in the proof of Lemma 2,

$$\liminf_{n \to \infty} V(\theta_n) \geq 0. \tag{19}$$

Theorem 1 will then follow if it is shown that

$$\lim_{n \to \infty} (V(\theta_n))^+ = 0. \tag{20}$$

This will be our objective for the remainder of this section.

Before proceeding with a rigorous description of our technique it seems worthwhile to devote a few paragraphs to the heuristic considerations that have motivated it. To obtain a recursion for $\{V(\theta_n)\}_{n=0}^{\infty}$, we expand $V(\theta_{n+1})$ about $V(\theta_n)$ with remainder r_{n+1}.

$$V(\theta_{n+1}) = V(\theta_n) + \frac{1}{n+1} \left\langle \nabla V(\theta_n), f^{(1)}_{n+1} \right\rangle + \frac{1}{(n+1)^{1+\rho}} \left\langle \nabla V(\theta_n), f^{(2)}_{n+1} \right\rangle + r_{n+1}, \quad n \geq 0, \tag{21}$$

where by (1), (15) and (16),

$$|r_{n+1}| \leq \frac{C}{(n+1)^2} |f_{n+1}|^2, \tag{22}$$

on $\{|\theta_n| \leq M\} \cap \{|f_{n+1}| \leq n+1\}$, $n \geq 0$.

Central in most a.s. convergence results of stochastic approximation is Theorem 1 of Robbins and Siegmund [21]. The next lemma is a slight extension of that result. Although it is not used until §4.1 we state it here because it provides a useful guide to the developments that follow.

Lemma 3 *On $(\Omega, \mathcal{F}, \mathcal{P})$ consider a nondecreasing family of σ-fields $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{F}$ and $\mathcal{G}_n$-measurable random variables $\{z_n, \beta_n, \xi_n, \zeta_n\}_{n \geq 0}$ such that β_n, ξ_n and ζ_n are non-negative and*

$$E\left[z_{n+1} \middle| \mathcal{G}_n\right] \leq z_n \left(1 + \beta_n\right) + \xi_n - \zeta_n. \tag{23}$$

Then, $\lim_{n\to\infty} z_n$ *exists and is finite, and* $\sum_{n=0}^{\infty} \zeta_n < \infty$ *on the set*

$$\left\{ \sum_{n=0}^{\infty} \beta_n < \infty, \ \sum_{n=0}^{\infty} \xi_n < \infty \right\} \cap \left\{ \liminf_{n\to\infty} z_n > -\infty \right\}.$$

Recall the definition of the σ-fields $\{\mathcal{F}_t^\pi\}_{t \geq 0}$ and set

$$\mathcal{F}_n := \mathcal{F}_{T_{nL}}^\pi. \tag{24}$$

Since $\{T_n\}_{n=0}^\infty$ are invariant with respect to policies $u \in \Pi$, π has been dropped from the notation.

The deterministic counterpart in [1] and an analogy with [3] suggest an application of this lemma with $z_n = V(\theta_n)$, $\beta_n = 0$, $\mathcal{G}_n = \mathcal{F}_n$ and

$$\begin{aligned} \zeta_n &= -\frac{1}{n+1} E\left[\left\langle \nabla V(\theta_n), f_{n+1}^{(1)} \right\rangle \middle| \mathcal{F}_n\right] \\ &= -\frac{1}{n+1} \left\langle \nabla V(\theta_n), E\left[f_{n+1}^{(1)} \middle| \mathcal{F}_n\right] \right\rangle \end{aligned}$$

The last equality follows since θ_n is $\mathcal{F}_n$-measurable. Two difficulties arise which are absent in [3].

(a) It must be shown that $\left\langle \nabla V(\theta_n), E\left[f_{n+1}^{(1)} \middle| \mathcal{F}_n\right] \right\rangle \leq 0$, $n \geq 0$. We will see that this is true only on certain events and whenever $V(\theta_n) > 0$.

(b) To apply Lemma 3 one must take the conditional expectation of $\left\langle \nabla V(\theta_n), f_{n+1}^{(2)} \right\rangle$ with respect to $\mathcal{F}_n$. However, this may be infinite according to the results in Section 9.10 of Wolff [29], since assumption (C.3) of §2.1 only requires finiteness of second moments of the service time distributions. The same problem arises in taking the conditional expectation of $|r_{n+1}|$.

Existing results in stochastic approximation seem inadequate for handling difficulties (a) and (b) which turn out to be substantial. For this reason we have formulated and proved a convergence result which applies to this situation. We state it next and show that it implies (20) and hence Theorem 1. Its proof, which is based on Lemma 3, is deferred until §4.

3.2 A result on stochastic approximation

In $\mathbf{R}^N$ consider a sequence $\{\theta_n\}_{n=0}^\infty$ which for $\rho > 0$ satisfies the recursion

$$\theta_{n+1} = \theta_n + \frac{1}{n+1} f_{n+1}^{(1)} + \frac{1}{(n+1)^{1+\rho}} f_{n+1}^{(2)}, \quad n \geq 0. \tag{25}$$

Also consider a non-decreasing family of σ-fields $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}$ such that for $n \geq 1$, $f_n^{(1)}$ and $f_n^{(2)}$ are measurable with respect to $\mathcal{F}_n$. Assume that the following hold.

(SA.1) There exists a compact set $\mathcal{A} \subset \mathbf{R}^N$ such that $\{\theta_n\}_{n=0}^\infty$ converges to $\mathcal{A}$ a.s., i.e.,

$$\lim_{n\to\infty} \inf \{|\theta_n - y| : y \in \mathcal{A}\} = 0.$$

In what follows fix $M > 0$ such that $\mathcal{A} \subseteq \left\{\theta \in \mathbf{R}^N : |\theta| \leq M\right\}$.

(SA.2) There exist

- a sequence of events $\{E_{1n}\}_{n=1}^\infty$ such that $E_{1n} \in \mathcal{F}_n$ for $n \geq 1$ and $\mathcal{P}\left(\liminf_{n\to\infty} E_{1,n}\right) = 1$ and
- a random variable $X \geq 0$ with $E[X] < \infty$

such that for $i = 1, 2$ and on $\{|\theta_n| \leq M\}$,

$$\mathcal{P}\left\{1_{E_{1,n+1}} \left|f_{n+1}^{(i)}\right| > x \middle| \mathcal{F}_n\right\} \leq C\mathcal{P}\{X > x\}, \quad x \geq 0. \tag{26}$$

(SA.3) There exists a continuously differentiable Liapunov-type function $V : \mathbf{R}^N \to \mathbf{R}$ such that for $\theta \in \mathbf{R}^N$ with $|\theta| \leq M$ and $h \in \mathbf{R}^N$ with $|h| \leq 1$,

$$|V(\theta + h) - V(\theta) - \langle \nabla V(\theta), h \rangle| \leq C |h|^2. \tag{27}$$

Also, there exist events $\{E_{2,n}\}_{n=1}^\infty$ with $E_{2,n} \in \mathcal{F}_n$ for $n \geq 1$, such that

$$\mathcal{P}\left(\liminf_{n\to\infty} E_{2,n}\right) = 1, \tag{28}$$

and for $\epsilon > 0$ there exists $\bar{\epsilon} > 0$ and $m \geq 0$ such that for $n \geq m$ and on $E_{2,n} \cap \{|\theta_n| \leq M\} \cap \{V(\theta_n) > \epsilon\}$

$$\left\langle \nabla V(\theta_n), E\left[f_{n+1}^{(1)} \middle| \mathcal{F}_n\right] \right\rangle < -\bar{\epsilon}. \tag{29}$$

Theorem 2 *Under conditions (SA.1)-(SA.3),*

$$\lim_{n\to\infty} (V(\theta_n))^+ = 0.$$

3.3 Verification of conditions (SA.1)-(SA.3)

Consider recursions (14) and (25) which as the notation suggests, are identified term-wise. The associated σ-field is $\{\mathcal{F}_n\}_{n=0}^{\infty}$ as defined in (24).

Conditions (SA.1) and (SA.2)

Condition (SA.1) is a simple consequence of Lemma 1. For (SA.2) recall the definition of K_{ni} from (13) and for $n \geq 0$ set

$$N_{n+1} := \frac{T_{(n+1)L} - T_{nL}}{L\tau}, \qquad K_{n+1} := \sum_{i \in \mathcal{N}} K_{n+1,i}. \tag{30}$$

Note that $J_{ni}^{\pi} \leq L\tau K_{ni} N_n$ for $i \in \mathcal{N}$. This implies that on $\{|\theta_n| \leq M\}$

$$\left| f_{n+1}^{(1)} \right| \leq N \max\{L\tau, M\} \left(K_{n+1} N_{n+1} + K_{n+1} \right).$$

To bound $\left| f_{n+1}^{(2)} \right|$ recall the definition of a_{ni} from (13) and for $n \geq 1$ set

$$b_n := \frac{T_{nL}}{nL\tau}.$$

For $0 < \rho < 1/2$ define the event

$$E_{1,n} := \left\{ \max\left\{ a_{ni}, \frac{1}{a_{ni}} \right\} \leq 1 + \frac{1}{n^{\rho}},\ i \in \mathcal{N} \right\} \cap \left\{ \max\left\{ b_n, \frac{1}{b_n} \right\} \leq 1 + \frac{1}{n^{\rho}} \right\}. \tag{31}$$

On $\{|\theta_n| \leq M\}$ we get

$$1_{E_{1,n+1}} \left| f_{n+1}^{(2)} \right| \leq N \max\{L\tau, M\} \left(K_{n+1} N_{n+1} + K_{n+1} \right).$$

That $\mathcal{P}\left(\liminf_{n\to\infty} E_{1n}\right) = 1$ follows easily from the law of the iterated logarithm. See e.g. p.374 of Dudley [7].

Note that the sequences $\{K_n\}_{n=1}^{\infty}$, $\{N_n\}_{n=1}^{\infty}$ are i.i.d. random variables and that $K_{n+1},\ N_{n+1}$ are independent of $\mathcal{F}_n$. Furthermore, recall condition (C.3) of §2.1 which requires that the distributions of service times have finite second moments. By virtue of Section 9.10 in [29] this implies that $E\left[K_1^2\right]$ and $E\left[N_1^2\right]$ are finite and by the Cauchy-Schwartz inequality,

$$E\left[K_1 N_1\right] \leq E\left[K_1^2\right]^{1/2} E\left[N_1^2\right]^{1/2} < \infty.$$

We can therefore pick

$$X := N \max\{L\tau, M\} \left(K_1 N_1 + K_1 \right) \tag{32}$$

and the verification of (SA.2) is complete.

Condition (SA.3)

We verify now (SA.3) for the Liapunov-type function V given by (18). Condition (27) immediately follows from (1) of §1. It remains to show that (28) and (29) are satisfied. This is the lengthiest part of the proof and occupies the rest of this section.

We begin by upper-bounding the term $\left\langle \nabla V(\theta_n), E\left[f_{n+1}^{(1)} \middle| \mathcal{F}_n\right]\right\rangle$. In addition to the properties of $\{K_n\}_{n=1}^{\infty}$ mentioned in the paragraph containing (32), note that $E[K_{1i}] = 1$, $i \in \mathcal{N}$. For $u \in \Pi$, $i \in \mathcal{N}$ and $n \geq 0$ set

$$J_{n+1,i}^u := \frac{1}{L\tau\alpha_i}\int_{T_{nL}}^{T_{(n+1)L}} \eta_i^u(s)\,ds, \qquad \bar{J}_{n+1}^u := E\left[J_{n+1}^u \middle| \mathcal{F}_n\right]. \tag{33}$$

From (14) and (15) we obtain

$$\left\langle \nabla V(\theta_n), E\left[f_{n+1}^{(1)} \middle| \mathcal{F}_n\right]\right\rangle = \sum_{i\in\mathcal{N}} \phi_i'(\theta_{ni})\left[\bar{J}_{n+1,i}^{\pi} - \theta_{ni}\right]. \tag{34}$$

From the convexity of $\{\phi_i\}_{i\in\mathcal{N}}$ one gets for $i \in \mathcal{N}$, $\theta \in \mathbf{R}^N$,

$$\phi_i'(\theta_i)(\theta_i^* - \theta_i) \leq \phi_i(\theta_i^*) - \phi_i(\theta_i).$$

This and (18) imply that

$$\begin{aligned}
\sum_{i\in\mathcal{N}} \phi_i'(\theta_{ni})\left[\bar{J}_{n+1,i}^{\pi} - \theta_{ni}\right] &= \sum_{i\in\mathcal{N}} \phi_i'(\theta_{ni})\left[\bar{J}_{n+1,i}^{\pi} - \theta_i^*\right] + \sum_{i\in\mathcal{N}} \phi_i'(\theta_{ni})(\theta_i^* - \theta_{ni}) \\
&\leq \sum_{i\in\mathcal{N}} \phi_i'(\theta_{ni})\left[\bar{J}_{n+1,i}^{\pi} - \theta_i^*\right] - V(\theta_n), \quad n \geq 0. \qquad (35)
\end{aligned}$$

Next, recall the definition of ξ_n from (11) and consider the variables $\{\nu_i(\xi_n), S_i(\xi_n), \pi_i(\xi_n)\}_{n\geq 0}$ in Algorithm (A.1). Whenever there is no possibility of confusion we will drop their dependence on ξ_n altogether. From (A.1) and the definition of policy π observe that

$$\frac{T_{nL}}{A_{nL,\pi_{N-k}}}\phi_{\pi_{N-k}}'(\theta_{n,\pi_{N-k}}) = \sum_{l=0}^{k} a_{\pi_{N-k}}^{S_{N-l}}\nu_{N-l}, \quad k = 0,\ldots,N-1.$$

This implies

$$\begin{aligned}
\sum_{i\in\mathcal{N}} \phi_i'(\theta_{ni})\left[\bar{J}_{n+1,i}^{\pi} - \theta_i^*\right] &= \sum_{l=0}^{N-1}\nu_{N-l}\sum_{i\in S_{N-l}} a_i^{S_{N-l}}\alpha_i\left[\bar{J}_{n+1,i}^{\pi} - \theta_i^*\right] \\
&\quad + \sum_{l=0}^{N-1}\nu_{N-l}\sum_{i\in S_{N-l}} a_i^{S_{N-l}}\alpha_i\left(\frac{a_{ni}}{b_n} - 1\right)\left[\bar{J}_{n+1,i}^{\pi} - \theta_i^*\right], \qquad (36)
\end{aligned}$$

which together with (34) and (35) implies

$$
\begin{aligned}
\left\langle \nabla V\left(\theta_n\right), E\left[f_{n+1}^{(1)} \middle| \mathcal{F}_n\right] \right\rangle \quad \leq \quad & \sum_{l=0}^{N-1} \nu_{N-l} \sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \left[\bar{J}_{n+1,i}^{\pi} - \theta_i^*\right] \\
& + \sum_{l=0}^{N-1} \nu_{N-l} \sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \left(\frac{a_{ni}}{b_n} - 1\right) \left[\bar{J}_{n+1,i}^{\pi} - \theta_i^*\right] \\
& - V(\theta_n). \qquad (37)
\end{aligned}
$$

We will show that for appropriate choice of events $E_{2,n}$, the first two terms in (37) can be upper bounded by an arbitrarily small positive quantity as $n \to \infty$ on the set $E_{2,n} \cap \{|\theta_n| \leq M\} \cap \{V(\theta_n) > \epsilon\}$, which implies (29). We will need the following fact.

Lemma 4 $\bar{J}_{n+1}^{\pi} \in \mathcal{A}$, $n \geq 0$.

Proof The truth of this statement will be intuitively clear from the argument given below. A proof in exacting detail using the measure-theoretic definition of conditional probability, will be tedious and lengthy given the complexity of the model and not pertinent to our objective here.

Consider policy u which agrees with π in $[0, T_{(n+1)L})$ and in the kth cycle, $[T_{kL}, T_{(k+1)L})$, $k \geq n+1$, acts as policy π would, if this (kth cycle) was the nth cycle. Since arrivals are Poisson and service times are mutually independent and independent of the arrivals it follows that given $\mathcal{F}_n$, i.e., the history up to time T_{nL}, the cycles $\left\{[T_{kL}, T_{(k+1)L})\right\}_{k \geq n}$ are i.i.d. and the process $\{\eta^u\left(t + T_{nL}\right) : t \geq 0\}$ is regenerative with respect to the cycle process. Therefore for $i \in \mathcal{N}$,

$$
\lim_{k \to \infty} \frac{1}{A_{kL,i}} \int_0^{T_{kL}} \eta_i^u\left(t\right) dt = \frac{1}{L\tau\alpha_i} E\left[\int_{T_{nL}}^{T_{(n+1)L}} \eta_i^u\left(t\right) dt \middle| \mathcal{F}_n\right]. \qquad (38)
$$

Note that $E\left[\int_{T_{nL}}^{T_{(n+1)L}} \eta^u\left(t\right) dt \middle| \mathcal{F}_n\right] = E\left[\int_{T_{nL}}^{T_{(n+1)L}} \eta^{\pi}\left(t\right) dt \middle| \mathcal{F}_n\right]$ since u and π agree on $[0, T_{(n+1)L})$. To conclude note that u belongs in the set of admissible policies Π and by Lemma 1(a), the lhs of (38) belongs in $\mathcal{A}$. □

Consider first the second term on the rhs of (37). Recall from (11) that $\xi_{ni} = \left(b_n/a_{ni}\right) \phi_i'\left(\theta_{ni}\right)/\alpha_i$. Also recall the definition of $E_{1,n}$ from (31) and note that $b_n/a_{ni} \leq 4$ on E_{1n}. Finally note that from the assumptions leading to (1), $\{\phi_i'\left(\cdot\right)\}_{i \in \mathcal{N}}$ are non-negative and continuous. These observations imply that $|\xi_n| \leq \sqrt{N} C_M$ on $\{|\theta_n| \leq M\} \cap E_{1,n}$, where

$$
C_M := 4 \max_{i \in \mathcal{N}} \left\{ \frac{1}{\alpha_i} \max\left\{\phi_i'\left(\theta_i\right) : |\theta| \leq M\right\} \right\}. \qquad (39)
$$

The following result is immediate.

Lemma 5 *For $\epsilon > 0$ there exists m such that for all $n \geq m$ one has on $E_{1,n} \cap \{|\theta_n| \leq M\}$,*

$$\sum_{l=0}^{N-1} \nu_{N-l} \sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \left(\frac{a_{ni}}{b_n} - 1 \right) \left[\bar{J}^{\pi}_{n+1,i} - \theta_i^* \right] < \epsilon.$$

Proof From (31) and Lemma 4 it can be seen that the lhs is bounded above by

$$\frac{6}{n^\rho} N \max_{i \in \mathcal{N}} \left\{ \sup \left\{ \nu_i(\xi) : |\xi| \leq \sqrt{N} C_M \right\} \right\} \max_{S \subseteq \mathcal{N}} \left\{ \max \left\{ \sum_{i \in S} a_i^S \alpha_i x_i : x \in \mathcal{A} \right\} \right\},$$

which tends to zero as $n \to \infty$, since $\nu_i(\cdot)$ is bounded on bounded sets as can be easily seen from its definition in Algorithm (A.1). □

Let us turn our attention now to the first sum on the rhs of (37). Observe that $\sum_{i \in \mathcal{N}} a_i^{S_N} \alpha_i \theta_i^* = F(\mathcal{N})$ since $\theta^* \in \mathcal{A}$. As lemma 4 shows, the same is true for $\bar{J}^{\pi}_{n+1}$. Thus, the first term on the rhs of (37) becomes

$$\sum_{l=1}^{N-1} \nu_{N-l} \sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \left[\bar{J}^{\pi}_{n+1,i} - \theta_i^* \right], \quad n \geq 0. \tag{40}$$

In the special case where updates are performed only at instants $\{T_{kL}\}_{k \geq 0}$, policy π serves nodes in the order $\{\pi_1(\xi_k), \ldots, \pi_N(\xi_k)\}$ during the interval $[T_{nL}, T_{(n+1)L})$, $k \geq 0$, and the situation is the same as in [3]. From Lemma 1(a) and because $\theta^* \in \mathcal{A}$ one has

$$\sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \bar{J}^{\pi}_{n+1,i} = F(S_{N-l}) \leq \sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \theta_i^* \quad l = 1, \ldots, N-1. \tag{41}$$

Since it can be easily verified from (A.1) that $\nu_{N-l} \geq 0$ for $l = 1, \ldots, N-1$, (41) implies that (40) is non-positive, as desired. However, when updates are performed during busy cycles the situation is considerably more delicate. Two observations are crucial. The first one is that the contribution of indices l with small values of ν_{N-l} is correspondingly small.

Lemma 6 *For $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{\{l \geq 1 : \nu_{N-l} \leq \delta\}} \nu_{N-l} \sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \left[\bar{J}^{\pi}_{n+1,i} - \theta_i^* \right] < \epsilon.$$

Proof Indeed, by Lemma 4 and the fact that $\mathcal{A}$ is compact the lhs is bounded from above by

$$2N\delta \max_{S \subseteq \mathcal{N}} \max_{x \in \mathcal{A}} \sum_{i \in S} a_i^S \alpha_i x_i =: C\delta,$$

and we can pick $\delta < \epsilon / C$. $\square$

The second observation is that for $\delta > 0$ and for indices l such that $\nu_{N-l} > \delta$, with high probability as $n \to \infty$, policy π gives priority to nodes $\{\pi_1, \ldots, \pi_{N-l}\}$ over nodes $\{\pi_{N-l+1}, \ldots, \pi_N\}$ in the interval $[T_{nL}, T_{(n+1)L})$. By Lemma 1(a) the terms

$$\sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \left[\bar{J}_{n+1,i}^{\pi} - \theta_i^* \right]$$

should be bounded from above by some small positive quantity. This is the content of Lemma 10 below. A series of definitions and auxiliary results leads to it. We begin with two lemmas which establish the continuity properties of $\nu(\cdot)$ and $\pi(\cdot)$. Their proofs are deferred to the Appendix, as it would be distractive to give them here.

Lemma 7 *For $\{c_i\}_{i \in \mathcal{N}}$ and $\delta > 0$ the exists $\epsilon > 0$ such that*

$$\left| \nu_l(c) - \nu_l(c') \right| < \delta, \; l = 1, \ldots, N, \text{ whenever } \max_{i \in \mathcal{N}} \left| c_i - c_i' \right| < \epsilon.$$

Some more notation is needed in order to formulate the continuity property of $\pi(\cdot)$. For scalars $\{c_i\}_{i \in \mathcal{N}}$ and $\delta \geq 0$ define the ordered partition of $\mathcal{N}$ into δ-clusters $\mathcal{U}^{\delta}(c) := \left\{ \mathcal{U}_i^{\delta}(c) \right\}_{i=1}^{M^{\delta}(c)}$ by requiring that

- for $i_1, i_2 \in \mathcal{N}$ with $i_1 < i_2$, nodes π_{i_1}, π_{i_2} belong to the same δ-cluster, say $\mathcal{U}_k^{\delta}(c)$ iff $\nu_l(c) \leq \delta$ for $l = i_1, \ldots, i_2 - 1$,
- the clusters are numbered so that $\pi_N(c) \in \mathcal{U}_1^{\delta}(c)$ and, for $i_1, i_2 \in \mathcal{N}$ with $i_1 < i_2$, $\pi_{i_1} \in \mathcal{U}_{k_1}^{\delta}(c)$ and $\pi_{i_2} \in \mathcal{U}_{k_2}^{\delta}(c)$ with $k_1 \neq k_2$ implies $k_1 > k_2$.

For $\delta = 0$ denote this partition by $\mathcal{U}(c) := \{\mathcal{U}_i(c)\}_{i=1}^{M(c)}$.

Next, consider an ordered partition $\mathcal{U} = \{\mathcal{U}_i\}_{i=1}^{M}$ of $\mathcal{N}$. Say that a permutation u on $\mathcal{N}$ is of type $\mathcal{U}$ if for $1 \leq k < m \leq M$ and $i, j \in \mathcal{N}$, $u_i \in \mathcal{U}_k$ and $u_j \in \mathcal{U}_m$ implies that $i > j$. Similarly say that a policy $u \in \Pi$ is of type $\mathcal{U}$ if for $1 \leq k < m \leq M$, it always gives priority to nodes in $\mathcal{U}_m$ over nodes in $\mathcal{U}_k$.

Lemma 8 *For a compact set $K \subset \mathbf{R}^N$ and $\delta > 0$ there exists $\epsilon > 0$ such that, for all $c \in K$,*

$$\pi(c') \text{ is of type } \mathcal{U}^{\delta}(c) \text{ whenever } \max_{i \in R} \left| c_i - c_i' \right| < \epsilon.$$

For $n \geq 1$ we now seek to find an event on which policy π is of type $\mathcal{U}^{\delta}(\xi_n)$ in the interval $[T_{(n-1)L}, T_{(n+1)L})$. For the rest of the paper, whenever there is no possibility of confusion, the intersection of sets, $A \cap B$, will be written as AB. Recalling the definition of C_M from (39), we have the following result, whose proof is provided in the Appendix.

Lemma 9 *Take $\tilde{\epsilon} > 0$ satisfying Lemma 8 for the compact set $\left\{|\xi| \leq \sqrt{N} C_M\right\}$ and $\delta > 0$. Then there exists $\epsilon' > 0$ such that with the definition*

$$D_{n+1} := \{K_{n+1} + N_{n+1} + K_{n+1}N_{n+1} < \epsilon' n\}, \tag{42}$$

one has that

$$|\xi_i(t) - \xi_{ni}| < \tilde{\epsilon}, \qquad t \in [T_{(n-1)L}, T_{(n+1)L}),\ i \in \mathcal{N}, \tag{43}$$

on $D_n D_{n+1} E_{1,n-1} E_{1,n} \cap \{|\theta_n| \leq M\}$. In particular, on this event and for $t \in [T_{(n-1)L}, T_{(n+1)L})$, policy π is of type $\mathcal{U}^{\delta}(\xi_n)$.

Consider now any policy $v \in \Pi$ of type $\mathcal{U}^{\delta}(\xi_n)$ with the following property: In the interval $[T_{nL}, T_{(n+1)L})$, $n \geq 0$, v follows policy π until the first time in this interval (if it exists) that π is not of type $\mathcal{U}^{\delta}(\xi_n)$. As a consequence of Lemmas 8 and 9, v agrees with π on $E_{1,n-1}E_{1,n}D_nD_{n+1} \cap \{|\theta_n| \leq M\}$. In addition, if $\nu_{N-l}(\xi_n) > \delta$ for some $l \geq 1$ then v gives priority to nodes in $S_{N-l}(\xi_n)$ over nodes in $\mathcal{N} \backslash S_{N-l}(\xi_n)$. Then, as in the proof of Lemma 4 and by Lemma 1(a)

$$\sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \left[\bar{J}^{v}_{n+1,i} - \theta_i^*\right] \leq 0. \tag{44}$$

Recalling that our goal is to bound from above the quantity

$$\sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \left[\bar{J}^{\pi}_{n+1,i} - \theta_i^*\right],$$

we turn our attention to the quantity $\left\{\bar{J}^{\pi}_{n+1,i} - \bar{J}^{v}_{n+1,i}\right\}_{i \in \mathcal{N}}$. On $E_{1,n-1}E_{1,n} \cap \{|\theta_n| \leq M\}$,

$$\begin{aligned} \left|\bar{J}^{\pi}_{n+1,i} - \bar{J}^{v}_{n+1,i}\right| &\leq 2L\tau E\left[1_{(D_nD_{n+1})^c} K_{n+1}N_{n+1} \middle| \mathcal{F}_n\right] \\ &\leq 2L\tau \left(1_{D_n^c} E\left[K_1N_1\right] + E\left[1_{\{K_1N_1 + K_1 + N_1 \geq \epsilon' n\}} K_1N_1\right]\right). \end{aligned} \tag{45}$$

We can now obtain the desired bound.

Lemma 10 *For $\delta > 0$ let $\tilde{\epsilon}, \epsilon'$ and $\{D_n\}_{n=1}^{\infty}$ be as in Lemma 9. Then for $\epsilon'' > 0$ there exists $m_0 \geq 0$ such that for $n \geq m_0$ and on $D_nE_{1,n-1}E_{1,n} \cap \{|\theta_n| \leq M\}$,*

$$\sum_{\{l \geq 1: \nu_{N-l} > \delta\}} \nu_{N-l} \sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \left[\bar{J}^{\pi}_{n+1,i} - \theta_i^*\right] < \epsilon''. \tag{46}$$

Proof Recall from the discussion preceding (39) that $|\xi_n| \le \sqrt{N} C_M$ on $\{|\theta_n| \le M\} \cap E_{1,n}$. From (44), (45), it follows that on $D_n E_{1,n-1} E_{1,n} \cap \{|\theta_n| \le M\}$,

$$\sum_{\{l \ge 1 : \nu_{N-l} > \delta\}} \nu_{N-l} \sum_{i \in S_{N-l}} a_i^{S_{N-l}} \alpha_i \left[\bar{J}^{\pi}_{n+1,i} - \theta^*_i \right] \le CE \left[1_{\{K_1 N_1 + K_1 + N_1 \ge \epsilon' n\}} K_1 N_1 \right],$$

where

$$C := 2L\tau N^2 \max_{l=1,\dots,N-1} \left\{ \sup \left\{ \nu_{N-l}(\xi) : |\xi| \le \sqrt{N} C_M \right\} \right\} \max_{S \subseteq \mathcal{N}} \left\{ \max_{i \in \mathcal{N}} a_i^S \alpha_i \right\};$$

note that $C < \infty$ because of the boundedness of $\nu_i(\cdot)$ on bounded sets. The result follows since $E[K_1 N_1] < \infty$ and by Lebesgue's dominated convergence theorem

$$\lim_{n \to \infty} E \left[1_{\{K_1 N_1 + K_1 + N_1 \ge \epsilon' n\}} K_1 N_1 \right] = 0.$$

□

There is only a short step remaining to complete the verification of Assumption (SA.3). Fix $\epsilon > 0$ and pick $\delta > 0$ such that the statement of Lemma 6 holds for $\epsilon/4$. With this choice of $\delta > 0$ and as prescribed by Lemmas 9, 10 take $m_0 \ge 0$ such that (46) holds for $\epsilon'' = \epsilon/4$. Then take $m_1 \ge m_0$ such that Lemma 5 holds for $\epsilon/4$. Because of (37), the above choices imply that (29) holds for $\bar{\epsilon} := \epsilon/4$, $E_{2,n} := E_{1,n-1} E_{1,n} D_n$ and $m := m_1$. Finally, observe that by the Borel-Cantelli lemma $\mathcal{P}(\liminf_{n\to\infty} D_n) = 1$. It was remarked earlier that $\mathcal{P}(\liminf_{n\to\infty} E_{1,n}) = 1$ from the law of iterated logarithm. Thus (28) holds and this completes our proof of Theorem 1 provided that Lemmas 7, 8 and Theorem 2 can be established.

4 Two results on stochastic approximation

Our main objective here is to prove Theorem 2 of §3.2. Three features of the result are noteworthy from the point of view of stochastic approximation. First, assumption (SA.3) requires uniform negativity of the inner product in (29) only outside the level sets of V. This is a weakening of the usual assumption, see e.g. Gladyshev [13] and was necessary for our problem. The weakened assumption may also be useful in applications where the drift depends on time. Theorem 12.2, p.69 of Métivier [19] is similar to ours but as it is stated, is not correct. The attempted proof illustrates the difficulties that are circumvented by our technique. Second, assumption (SA.2) replaces the requirement for finiteness of the second moment of f_{n+1}, defined in (16), by the rather weak stochastic bound of (26). Third, by exploiting assumption (SA.1) which requires that $\{\theta_n\}_{n=1}^{\infty}$ converges to a compact set, no growth conditions are required on $E[|f_{n+1}| \,|\mathcal{F}_n]$ in terms of $|\theta_n|$. See p.243 of [21] for a discussion of this requirement. Our observation may be useful more generally since only stability considerations are required to establish Assumption (SA.1).

It is natural to investigate if, in the usual results of stochastic approximation, the requirements of finiteness of the second moment of f_{n+1} can be replaced by our (SA.2). In §4.2 we present such a result. It includes variants of the classical theorems of Robbins-Monroe and Kiefer-Wolfowitz and should also be useful in queueing applications when (SA.1) does not hold.

4.1 Proof of Theorem 2

We start by restating and proving Lemma 3. The difference between this lemma and theorem 1 of [21] is that non-negativity of z_n is not assumed and convergence is proved restricted to the set $\{\liminf_{n\to\infty} z_n > -\infty\}$.

Lemma 3 *On $(\Omega, \mathcal{F}, \mathcal{P})$ consider a nondecreasing family of σ-fields $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{F}$ and $\mathcal{G}_n$-measurable random variables $\{z_n, \beta_n, \xi_n, \zeta_n\}_{n\geq 0}$ such that β_n, ξ_n and ζ_n are non-negative and*

$$E\left[z_{n+1}\middle|\mathcal{G}_n\right] \leq z_n(1+\beta_n) + \xi_n - \zeta_n. \tag{47}$$

Then, $\lim_{n\to\infty} z_n$ exists and is finite, and $\sum_{n=0}^{\infty} \zeta_n < \infty$ on the set

$$\left\{\sum_{n=0}^{\infty} \beta_n < \infty,\ \sum_{n=0}^{\infty} \xi_n < \infty\right\} \cap \left\{\liminf_{n\to\infty} z_n > -\infty\right\}.$$

Proof Because of the similarity with [21] only a brief sketch will be given. Following their reduction steps we can take $\beta_n = 0$, $n \geq 0$ with no loss of generality. For a, b non-negative rationals define the $\mathcal{F}_n$-stopping times

$$s_b := \min\{n \geq 0 : z_n < -b\},$$

$$t_a := \min\left\{n \geq 0 : \sum_{k=0}^{n} \xi_k > a\right\}$$

and set $T_{ab} := t_a \wedge s_b$. Then set $u_n := z_n - \sum_{k=0}^{n-1}(\xi_k - \zeta_k)$, $n = 1, 2, \ldots$, and show that $\{u_{n\wedge T_{ab}}\}_{n=1}^{\infty}$ is a supermartingale bounded from below. From the supermartingale convergence theorem and by taking unions over $a \in \mathbf{Q}_+$ we obtain that $\lim_{n\to\infty} u_{n\wedge s_b}$ exists and is finite on $\{\sum_{k=0}^{\infty} \xi_k < \infty\}$. From this and by unraveling definitions we get that $\lim_{n\to\infty} z_n$ exists and is finite and $\sum_{k=0}^{\infty} \zeta_k < \infty$ a.s. on $\{s_b = \infty\}$. Taking unions over $b \in \mathbf{Q}_+$ completes the proof. □

We will need a consequence of this result.

Lemma 11 *On $(\Omega, \mathcal{F}, \mathcal{P})$ consider a nondecreasing family of σ-fields $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{F}$ and consider $\mathcal{G}_n$-measurable random variables $\{z_n, \beta_n, \xi_n, \zeta_n\}_{n\geq 0}$ such that z_n and β_n are nonnegative and*

$$z_{n+1} = z_n(1+\beta_n) + \xi_{n+1} - \zeta_{n+1}, \quad n \geq 0.$$

In addition assume that there exist sequences of events $\{A_n, B_n\}_{n\geq 0}$ with A_n and B_n in $\mathcal{G}_n$ and nonnegative $\mathcal{G}_n$-measurable random variables $\left\{\tilde{\xi}_n, \tilde{\zeta}_n\right\}_{n\geq 0}$ such that

$$E\left[1_{A_{n+1}}\xi_{n+1}\middle|\mathcal{G}_n\right] \leq \tilde{\xi}_n,$$

$$E\left[1_{B_{n+1}}\zeta_{n+1}\middle|\mathcal{G}_n\right] = \tilde{\zeta}_n.$$

Then, $\lim_{n\to\infty} z_n$ exists and is finite, and $\sum_{n=0}^{\infty} \tilde{\zeta}_n < \infty$ a.s. on

$$\left\{\sum_{n=0}^{\infty} \tilde{\xi}_n < \infty\right\} \cap \liminf_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n.$$

Proof We can again take $\beta_n = 0$, $n \geq 0$. Next, for $n_0 \geq 0$ define $\tilde{z}_{n_0} := z_{n_0}$ and for $n \geq n_0$

$$\tilde{z}_{n+1} := \tilde{z}_n + 1_{A_{n+1}}\xi_{n+1} - 1_{B_{n+1}}\zeta_{n+1}.$$

From the assumptions of the theorem, taking conditional expectations yields

$$E\left[\tilde{z}_{n+1}\middle|\mathcal{G}_n\right] \leq \tilde{z}_n + \tilde{\xi}_n - \tilde{\zeta}_n, \quad n \geq n_0.$$

Lemma 3 is now applicable and implies that $\lim_{n\to\infty} \tilde{z}_n$ exists and is finite, and $\sum_{n=0}^{\infty} \tilde{\zeta}_n < \infty$ a.s. on $\left\{\sum_{n=0}^{\infty} \tilde{\xi}_n < \infty\right\} \cap \{\liminf_{n\to\infty} \tilde{z}_n > -\infty\}$. Observe that for $n \geq n_0$ $\tilde{z}_n = z_n$ on $\cap_{n\geq n_0} A_n \cap_{n\geq n_0} B_n$. It follows that $\lim_{n\to\infty} z_n$ exists and is finite, and $\sum_{n=0}^{\infty} \tilde{\zeta}_n < \infty$ a.s. on

$$\left\{\sum_{n=0}^{\infty} \tilde{\xi}_n < \infty\right\} \cap \liminf_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n.$$

□

Recall that our goal is to show $\lim_{n\to\infty} (V(\theta_n))^+ = 0$ a.s. Not being able to prove this directly we will show that for all $\epsilon > 0$, $\lim_{n\to\infty} \left((V(\theta_n) - \epsilon)^+\right)^2 = 0$ a.s. We first need the following lemma. Recall also that $M > 0$ is the constant fixed in assumption (SA.1) of §3.2.

Lemma 12 *For $\epsilon > 0$ there exists $C \geq 0$ such that for $\theta \in \mathbf{R}^N$ with $|\theta| \leq M$ and $h \in \mathbf{R}^N$ with $|h| \leq 1$*

$$\left|(V(\theta+h) - \epsilon)^2 - (V(\theta) - \epsilon)^2 - 2(V(\theta) - \epsilon)\langle\nabla V(\theta), h\rangle\right| \leq C|h|^2.$$

Proof From (27) it is easy to see that proving the case $\epsilon = 0$ is sufficient. Write

$$\begin{aligned}\left|V^2(\theta+h) - V^2(\theta) - 2V(\theta)\langle\nabla V(\theta), h\rangle\right| &\leq 2|V(\theta)||V(\theta+h) - V(\theta) - \langle\nabla V(\theta), h\rangle| \\ &\quad + |V(\theta+h) - V(\theta)|^2. \end{aligned} \tag{48}$$

For the first term on the right hand side note that V is continuous and hence bounded for $|\theta| \leq M$. The desired bound is then implied by (27). For the second term on the right hand side we make use of the facts that V is continuously differentiable and that the set $\left\{\theta \in \mathbf{R}^N : |\theta| \leq M+1\right\}$ is convex. The intermediate value theorem in Rudin [24] p.218 is therefore applicable and gives

$$|V(\theta+h) - V(\theta)| \leq \sup\{|\nabla V(\theta)| : |\theta| \leq M+1\}|h|.$$

The desired bound follows. □

We derive next a recursion for $\left((V(\theta_n) - \epsilon)^+\right)^2$. As a consequence of previous lemma we can write the expansion

$$\begin{aligned}(V(\theta_{n+1}) - \epsilon)^2 &= (V(\theta_n) - \epsilon)^2 + \frac{2}{n+1}(V(\theta_n) - \epsilon)\left\langle\nabla V(\theta_n), f_{n+1}^{(1)}\right\rangle \\ &\quad + \frac{2}{(n+1)^{1+\delta}}(V(\theta_n) - \epsilon)\left\langle\nabla V(\theta_n), f_{n+1}^{(2)}\right\rangle + r_{n+1}, \end{aligned} \tag{49}$$

where, recalling (16), one has on $\{|\theta_n| \leq M\} \cap \{|f_{n+1}| \leq n+1\}$

$$|r_{n+1}| \leq \frac{C}{(n+1)^2}|f_{n+1}|^2. \tag{50}$$

For $n \geq 0$ define the event $G_n := \{V(\theta_n) > \epsilon\}$ and note that

$$\left((V(\theta_n) - \epsilon)^+\right)^2 = (V(\theta_n) - \epsilon)^2 1_{G_n}, \tag{51}$$

$$1_{G_{n+1}} = 1_{G_n} + \left(1_{G_n^c G_{n+1}} - 1_{G_n G_{n+1}^c}\right). \tag{52}$$

Multiplication of (49) by $1_{G_{n+1}}$ and the definitions

$$\begin{aligned} z_n &:= \left((V(\theta_n) - \epsilon)^+\right)^2 \\ \zeta_{n+1} &:= -\frac{2}{n+1}(V(\theta_n) - \epsilon)\left\langle\nabla V(\theta_n), f_{n+1}^{(1)}\right\rangle 1_{G_n} \\ \xi_{n+1} &:= (V(\theta_n) - \epsilon)^2\left(1_{G_n^c G_{n+1}} - 1_{G_n G_{n+1}^c}\right) \\ &\quad + \frac{2}{n+1}(V(\theta_n) - \epsilon)\left\langle\nabla V(\theta_n), f_{n+1}^{(1)}\right\rangle\left(1_{G_n^c G_{n+1}} - 1_{G_n G_{n+1}^c}\right) \\ &\quad + \frac{2}{(n+1)^{1+\delta}}(V(\theta_n) - \epsilon)\left\langle\nabla V(\theta_n), f_{n+1}^{(2)}\right\rangle 1_{G_{n+1}} \\ &\quad + r_{n+1}1_{G_{n+1}}, \end{aligned} \tag{53}$$

yield the recursion

$$z_{n+1} = z_n + \xi_{n+1} - \zeta_{n+1}.$$

With a view toward applying Lemma 11 set $B_{n+1} := E_{2,n} \cap \{|\theta_n| \leq M\}$. Then, assumption (29) implies that there exist $\bar{\epsilon} > 0$ and $m \geq 0$ such that for $n \geq m$

$$\tilde{\zeta}_n := E\left[1_{B_{n+1}}\zeta_{n+1}\middle|\mathcal{F}_n\right] \geq 0. \tag{54}$$

We now proceed to bound ξ_{n+1} in (53). Note that on $G_n G_{n+1}^c$, $0 < V(\theta_n) - \epsilon \leq V(\theta_n) - V(\theta_{n+1})$ and hence

$$|V(\theta_n) - \epsilon| \, 1_{G_n G_{n+1}^c} \leq |V(\theta_{n+1}) - V(\theta_n)| \, .$$

Similarly one obtains

$$|V(\theta_n) - \epsilon| \, 1_{G_n^c G_{n+1}} \leq |V(\theta_{n+1}) - V(\theta_n)| \, .$$

As was observed in Lemma 12, continuous differentiability of $V(\cdot)$ implies that $V(\theta_n)$ and $\nabla V(\theta_n)$ are bounded on $\{|\theta_n| \leq M\} \cap \{|f_{n+1}| \leq n+1\}$ and

$$|V(\theta_{n+1}) - V(\theta_n)| \leq \frac{C}{n+1} |f_{n+1}|$$

Taking also (50) into account in (53) one obtains on $\{|\theta_n| \leq M\} \cap \{|f_{n+1}| \leq n+1\}$

$$\xi_{n+1} \leq \frac{C}{(n+1)^2} |f_{n+1}|^2 + \frac{C}{(n+1)^{1+\delta}} \left|f_{n+1}^{(2)}\right|.$$

The following result is crucial in applying Lemma 11 without requiring the finiteness of second moments of ξ_{n+1}.

Lemma 13 *Consider a nondecreasing family of σ-fields $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{F}$, and a sequence of random variables $\{Y_n\}_{n=0}^{\infty}$ such that for $n \geq 0$ Y_n is $\mathcal{G}_n$-measurable. Assume that there exists a random variable $Y \geq 0$ with $E[Y] < \infty$ such that for all $x \geq 0$,*

$$\mathcal{P}\left\{|Y_{n+1}| > x \middle| \mathcal{G}_n\right\} \leq C\mathcal{P}\{Y > x\}.$$

Then

(a) $\sup_n E\left[|Y_{n+1}| \middle| \mathcal{G}_n\right] < \infty$,

(b) $\mathcal{P}\{|Y_n| > n \text{ } infinitely \text{ } often\} = 0,$

(c) *for $k > 1$*

$$\sum_{n=1}^{\infty} \frac{1}{n^k} E\left[|Y_n|^k 1\{|Y_n| \leq n\} \middle| \mathcal{G}_{n-1}\right] < \infty.$$

Proof Part (a) is immediate and part (b) follows from the Borel-Cantelli lemma. Part (c) can be established with minor modifications of the proof of Theorem 2.19, p.36 in Hall and Heyde [14]. □

Part (a) of Lemma 13 and assumption (SA.2) imply that on $\{|\theta_n| \le M\}$,

$$\sup_n E\left[1_{E_{1,n+1}}\left|f_{n+1}^{(2)}\right|\Big|\mathcal{F}_n\right] < \infty.$$

We can thus set

$$\begin{aligned} A_{n+1} &:= E_{1,n+1} \cap \{|f_{n+1}| \le n+1\} \cap \{|\theta_n| \le M\} \\ \tilde{\xi}_n &:= C\left(\frac{1}{(n+1)^{1+\delta}} + \frac{1}{(n+1)^2} E\left[1_{A_{n+1}} |f_{n+1}|^2 \Big| \mathcal{F}_n\right]\right) \\ &\ge E\left[\xi_{n+1} 1_{A_{n+1}} \Big| \mathcal{F}_n\right]. \end{aligned}$$

With this choice and (54) Lemma 11 implies that $\lim_{n\to\infty}\left((V(\theta_n)-\epsilon)^+\right)^2$ exists and is finite, and $\sum_{n=0}^{\infty} \tilde{\zeta}_n < \infty$ a.s. on

$$\left\{\sum_{n=0}^{\infty} \tilde{\xi}_n < \infty\right\} \cap \liminf_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n.$$

But (SA.2) implies that for all $x \ge 0$,

$$\mathcal{P}\left\{1_{E_{1,n+1}} |f_{n+1}| > x \Big| \mathcal{F}_n\right\} \le C\mathcal{P}\{2X > x\}.$$

Part (c) of Lemma 13 implies that $\sum_{n=0}^{\infty} \tilde{\xi}_n < \infty$ a.s. on $\liminf_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n$. Thus, it will follow that $\lim_{n\to\infty}(V(\theta_n)-\epsilon)^+$ exists and is finite if it is shown that

$$\mathcal{P}\left(\liminf_{n\to\infty} A_n\right) = \mathcal{P}\left(\liminf_{n\to\infty} B_n\right) = 1.$$

But this follows easily from (SA.1)-(SA.3) and part (b) of Lemma 13. To conclude our proof of Theorem 2 note that if $\mathcal{P}\{\lim_{n\to\infty} V(\theta_n) > \epsilon\} > 0$ then (29) implies that $\mathcal{P}\left\{\sum_{n=0}^{\infty} \tilde{\zeta}_n = \infty\right\} > 0$, a contradiction. We have thus shown that $\lim_{n\to\infty}(V(\theta_n)-\epsilon)^+ = 0$, a.s. □

4.2 A variant of Theorem 2

For $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}$ a non-decreasing family of σ-fields, $\{Y_n\}_{n=1}^{\infty}$ a sequence of $\mathbf{R}^K$ valued random variables such that Y_n is $\mathcal{F}_n$-measurable for $n \ge 1$, and Borel-measurable functions $f_n : \mathbf{R}^N \times \mathbf{R}^K \longrightarrow \mathbf{R}^N$, $n \geq 1$, consider the sequence $\{\theta_n\}_{n=1}^{\infty}$ in $\mathbf{R}^N$ that satisfies the recursion

$$\theta_{n+1} = \theta_n + \frac{1}{n+1} f_{n+1}(\theta_n, Y_{n+1}), \quad n \geq 0, \tag{55}$$

and assume that the following hold.

- **(H.1)** For $\theta \in \mathbf{R}^N$ and $y \in \mathbf{R}^K$,

$$\left| f_{n+1}(\theta, y)^2 \right| \leq C \left(1 + |\theta|^2 + \left| y^2 \right| \right), \quad n \geq 0. \tag{56}$$

- **(H.2)** There exists a random variable $X \geq 0$ with $E[X] < \infty$ such that for $n \geq 0$,

$$\mathcal{P}\{|Y_{n+1}| > x | \mathcal{F}_n\} \leq C\mathcal{P}\{X > x\}, \quad x \geq 0. \tag{57}$$

- **(H.3)** There exists a non-negative, differentiable Liapunov-type function $V : \mathbf{R}^N \longrightarrow \mathbf{R}$ such that for $\theta, h \in \mathbf{R}^N$,

$$|V(\theta + h) - V(\theta) - \langle \nabla V(\theta), h \rangle| \leq C|h|^2 \tag{58}$$

$$\left| \theta^2 \right| \leq CV(\theta). \tag{59}$$

  Also, for $n = 0, 1, \ldots$,

$$\langle \nabla V(\theta_n), E[f_{n+1} | \mathcal{F}_n] \rangle \leq 0,$$

  and for $\epsilon > 0$, there exists $\bar{\epsilon} > 0$ and $m \geq 0$ such that for $n \geq m$ and on $\{V(\theta_n) > \epsilon\}$,

$$\langle \nabla V(\theta_n), E[f_{n+1} | \mathcal{F}_n] \rangle < -\bar{\epsilon}.$$

Theorem 3 *Under these assumptions*

$$\lim_{n \to \infty} V(\theta_n) = 0.$$

A proof of this can be easily obtained from Lemmas 11 and 13. Conditions (58) and (59) required on $V(\cdot)$ are somewhat stringent but cannot be weakened. They are satisfied by quadratic functions.

APPENDIX

A Proof of Lemmas 7 and 8

In this appendix, we provide a proof of Lemmas 7 and 8, which are restated for convenience. We begin with some notation. The R-restriction of the network on a set of nodes $R \subseteq \mathcal{N}$ is a network obtained from the original one by removing all nodes in $\mathcal{N} \backslash R$ and routing to the outside jobs that would be routed to nodes in set $\mathcal{N} \backslash R$ from nodes in the set R. For scalars $\{c_i\}_{i \in R}$ denote the variables of Algorithm (A.1) by $\left\{\nu_i^R(c), S_i^R(c), \pi_i^R(c)\right\}_{i \in R}$ and the δ-cluster partition of R by $\mathcal{U}^{R,\delta}(c) := \left\{\mathcal{U}_i^{R,\delta}(c)\right\}_{i=1}^{M^{R,\delta}(c)}$ and set $\mathcal{U}^R(c) := \mathcal{U}^{R,0}(c)$.

Lemma 7 *For all* $\{c_i\}_{i \in \mathcal{N}}$ *and* $\delta > 0$ *the exists* $\epsilon > 0$ *such that*

$$\left|\nu_l(c) - \nu_l(c')\right| < \delta, \ l = 1, \ldots, N, \text{ whenever } \max_{i \in \mathcal{N}} \left|c_i - c_i'\right| < \epsilon.$$

Proof We prove by induction on k

(I1) For $k = 1, \ldots, N$, for $R \subseteq \mathcal{N}$ such that $|R| = k$, for $\{c_i\}_{i \in R}$, and for $\delta > 0$ there exists $\epsilon > 0$, depending on δ and $\{c_i\}_{i \in R}$, such that

$$\left|\nu_l^R(c) - \nu_l^R(c')\right| < \delta, \ l = 1, \ldots, k, \text{ whenever } \max_{i \in R} \left|c_i - c_i'\right| < \epsilon.$$

The statement is trivially true for $k = 1$. We assume its truth for $k = n - 1 < N$ and prove it for $k = n$. We first show that for $\delta > 0$ there exists $\epsilon_1 > 0$ such that,

$$\pi_n^R(c') \in \mathcal{U}_1^R(c), \text{ whenever } \max_{i \in R} \left|c_i - c_i'\right| < \epsilon_1. \tag{60}$$

This is trivially true if $\mathcal{U}_1^R(c) = R$ so assume the opposite and take $i \in R \backslash \mathcal{U}_1^R(c)$. The definition of $\mathcal{U}_1^R(c)$ implies that there exists $\delta' > 0$ such that $c_i - c_m a_i^R / a_m^R \geq \delta'$ for all $m \in \mathcal{U}_1^R(c)$. Note that the choice of δ' depends on $\{c_i\}_{i \in R}$. Therefore,

$$c_i' - c_m' \frac{a_i^R}{a_m^R} = c_i' - c_i + \left(c_i - c_m \frac{a_i^R}{a_m^R}\right) + \left(c_m - c_m'\right) \frac{a_i^R}{a_m^R} \geq \delta' - \frac{2B^R}{b^R} \max_{i \in R} \left|c_i - c_i'\right|, \tag{61}$$

where

$$b^R := \min_{i \in R} a_i^R, \ B^R := \max_{i \in R} a_i^R. \tag{62}$$

It suffices to take $\epsilon_1 < b^R \delta' / 2B^R$ for (60) to hold.

Take $\{c'\}_{i\in R}$ such that $\max_{i\in R}|c_i - c'_i| < \epsilon_1$, set $j := \pi_n^R(c')$ and for $i \in \tilde{R} := R\backslash\{j\}$ define

$$\tilde{c}_i := c_i - c_j \frac{a_i^R}{a_j^R}, \quad \tilde{c}'_i := c'_i - c'_j \frac{a_i^R}{a_j^R}. \tag{63}$$

Note that Algorithm (A.1) on $\tilde{R}$ with inputs $\{\tilde{c}_i\}_{i\in\tilde{R}}$ and $\{\tilde{c}'_i\}_{i\in\tilde{R}}$ gives outputs $\nu_l^{\tilde{R}}(\tilde{c}) = \nu_l^R(c)$ and $\nu_l^{\tilde{R}}(\tilde{c}'_i\} = \nu_l^R(c')$ respectively for $l = 1,\ldots,n$. Statement (I1) for $\tilde{R}$ and $\{\tilde{c}_i\}_{i\in\tilde{R}}$ and $\delta > 0$ implies that there exists $\tilde{\epsilon} > 0$ such that

$$\left|\nu_l^{\tilde{R}}(\tilde{c}) - \nu_l^{\tilde{R}}(\tilde{c}')\right| < \delta, \; l = 1,\ldots,n, \text{ whenever } \max_{i\in\tilde{R}}|\tilde{c}_i - \tilde{c}'_i| < \tilde{\epsilon}.$$

From the easily verified inequality

$$|\tilde{c}_i - \tilde{c}'_i| \leq \frac{2B^R}{b^R}\max_{i\in R}|c_i - c'_i|, \tag{64}$$

taking $\epsilon_2 := b^R\tilde{\epsilon}/2B^R$ and $\max_{i\in R}|c_i - c'_i| < \min\{\epsilon_1,\epsilon_2\}$ gives $\left|\nu_l^R(c) - \nu_l^R(c')\right| < \delta$, $l = 1,\ldots,n-1$. Finally, it is easily seen that

$$\left|\nu_n^R(c) - \nu_n^R(c')\right| \leq \frac{1}{b^R}\max_{i\in R}|c_i - c'_i| \tag{65}$$

and thus for $\epsilon_3 := b^R\delta$ it suffices to take $\epsilon := \min\{\epsilon_1,\epsilon_2,\epsilon_3\}$ in order to satisfy (I1) for $k = n$. □

Lemma 8 *For a compact set $K \subset \mathbf{R}^N$ and $\delta > 0$ there exists $\epsilon > 0$ such that, for $c \in K$,*

$$\pi(c') \text{ is of type } \mathcal{U}^\delta(c) \text{ whenever } \max_{i\in R}|c_i - c'_i| < \epsilon.$$

Proof It will be sufficient to prove

(I2) For $k = 1,\ldots,N$, for $R \subseteq \mathcal{N}$ such that $|R| = k$, for a compact set $K \subset \mathbf{R}^k$ and for $\delta > 0$, there exists $\epsilon > 0$, depending only on K and δ, such that for $c \in K$ with $\left|\mathcal{U}_1^{R,\delta}(c)\right| = m$, $m = 1,\ldots,k$, one has that

$$\pi^R(c') \text{ is of type } \mathcal{U}^{R,\delta}(c) \text{ whenever } \max_{i\in R}|c_i - c'_i| < \epsilon.$$

We employ induction on k and m and observe that (I2) is trivially true for $m = k = 1$. Assuming its truth for $m = 1$, $k = 1,\ldots,n-1 < N$ we proceed to prove it for $m = 1$, $k = n$.

As in the proof of (60) we first show that there exists $\epsilon_1 > 0$ such that

$$\pi_n^R(c') = \pi_n^R(c) \text{ whenever } \max_{i\in R}|c_i - c_i'| < \epsilon_1. \tag{66}$$

Take $\{c_i'\}_{i\in R}$ such that $\max_{i\in R}|c_i - c_i'| < \epsilon_1$ and define $\{\tilde{c}_i\}_{i\in\tilde{R}}$ as in (63). Observe that since $c \in K$, $\tilde{c}$ will belong in a compact set $\tilde{K} \subset \mathbf{R}^{k-1}$. On $\tilde{R}$, $\{\tilde{c}_i\}_{i\in\tilde{R}}$, $\tilde{K}$ and $\delta > 0$, (I2) for $m = 1$, $k = n-1$ implies that there exists $\tilde{\epsilon} > 0$ such that $\pi^{\tilde{R}}(\tilde{c}')$ is of type $\mathcal{U}^{\tilde{R},\delta}(\tilde{c})$ whenever $\max_{i\in\tilde{R}}|\tilde{c}_i - \tilde{c}_i'| < \tilde{\epsilon}$. Note that $\mathcal{U}^{\tilde{R},\delta}(\tilde{c})$ and $\pi^{\tilde{R}}(\tilde{c}')$ are the restrictions of $\mathcal{U}^{R,\delta}(c)$ and $\pi^R(c')$, respectively, on $\tilde{R}$, i.e.,

$$\mathcal{U}^{R,\delta}(c) = \left\{\mathcal{U}^{\tilde{R},\delta}(\tilde{c}), \mathcal{U}_1^{R,\delta}(c)\right\},$$

$$\pi^R(c') = \left\{\pi^{\tilde{R}}(\tilde{c}'), \pi_N^R(c')\right\}.$$

From (64) and (66), taking $\epsilon_2 < b^R\tilde{\epsilon}/2B^R$ and $\epsilon := \min\{\epsilon_1, \epsilon_2\}$ establishes (I2) for $m = 1$, $k = n$.

To complete the proof assume that (I2) holds for $k = n$, $m = 1, \ldots, p-1 < k$ and prove it for $k = n$, $m = p$. There is nothing to prove if $p = n$ so take $p < n$. Fix a number $0 < \delta_1 < \delta$ whose value will be determined later and set $p_1 := \left|\mathcal{U}_1^{R,\delta_1}(c)\right|$. Two cases ensue.

Case 1: $p_1 < p$. Here the result follows immediately since $\delta_1 < \delta$ and (I2) applied for $k = n$, R, K, δ_1 and $m = p_1$ implies the existence of $\epsilon_1 > 0$ such that $\pi^R(c')$ is of type $\mathcal{U}^{R,\delta_1}(c)$ (and hence of type $\mathcal{U}^{R,\delta}(c)$) whenever $\max_{i\in R}|c_i - c_i'| < \epsilon_1$.

Case 2: $p_1 = p$. Then, by definition,

$$c_i - \sum_{l=0}^{p-1} a_i^{S_{n-l}^R(c)} \nu_{n-l}^R(c) > \delta a_i^{\tilde{R}}, \quad i \in \tilde{R} := R \backslash \mathcal{U}_1^{R,\delta}(c), \tag{67}$$

$$\nu_{n-l}^R(c) \le \delta_1, \quad l = 1, \ldots, p-1. \tag{68}$$

We first show that there exists $\epsilon_1 > 0$ such that

$$\pi^R(c') \text{ is of type } \left\{\tilde{R}, \mathcal{U}_1^{R,\delta}(c)\right\} \text{ whenever } \max_{i\in R}|c_i - c_i'| < \epsilon_1. \tag{69}$$

For $i \in \tilde{R}$ write

$$c_i' - \sum_{l=0}^{p-1} a_i^{S_{n-l}^R(c')} \nu_{n-l}^R(c') \quad = \quad c_i - \sum_{l=0}^{p-1} a_i^{S_{n-l}^R(c)} \nu_{n-l}^R(c)$$

$$+c_i' - c_i + a_i^R \left(\nu_n^R(c) - \nu_n^R(c')\right)$$
$$+\sum_{l=1}^{p-1} \left(a_i^{S_{n-l}^R(c)} \nu_{n-l}^R(c) - a_i^{S_{n-l}^R(c')} \nu_{n-l}^R(c')\right), \tag{70}$$

where we have used the fact that $S_n^R(c) = S_n^R(c') = R$. From (68) we have the inequality

$$\begin{aligned}\left|\sum_{l=1}^{p-1} \left(a_i^{S_{n-l}^R(c)} \nu_{n-l}^R(c) - a_i^{S_{n-l}^R(c')} \nu_{n-l}^R(c')\right)\right| &\leq \sum_{l=1}^{p-1} \left|a_i^{S_{n-l}^R(c)} - a_i^{S_{n-l}^R(c')}\right| \nu_{n-l}^R(c) \\ &\quad + \sum_{l=1}^{p-1} a_i^{S_{n-l}^R(c')} \left|\nu_{n-l}^R(c) - \nu_{n-l}^R(c')\right| \\ &\leq n\left(B^R - b^R\right)\delta_1 \\ &\quad + nB^R \max_{l=1,\ldots,p-1} \left|\nu_{n-l}^R(c) - \nu_{n-l}^R(c')\right| \end{aligned} \tag{71}$$

Due to Lemma 6 there exists $\epsilon_2 > 0$ such that, for all $c \in K$, $\max_{l=1,\ldots,p-1} \left|\nu_{n-l}^R(c) - \nu_{n-l}^R(c')\right| < \delta_1$, whenever $\max_{i\in R} |c_i - c_i'| < \epsilon_2$. Consider such $\{c_i'\}_{i\in R}$. In (70), the bounds from (67), (68), (65), and (71) imply that

$$c_i' - \sum_{l=1}^{p-1} a_i^{S_{n-l}^R(c')} \nu_{n-l}^R(c') > \delta b^R - \frac{2B^R}{b^R} \max_{i\in R} |c_i - c_i'| - \left(2B^R - b^R\right) n\delta_1 > 0,$$

if, recalling (62), we take

$$\delta_1 < \delta \min\left\{1, \frac{b^{\tilde{R}}}{2\left(2B^R - b^R\right)n}\right\}, \tag{72}$$

and whenever $\max_{i\in R} |c_i - c_i'| < \min\{\epsilon_2, \epsilon_3\}$, with $\epsilon_3 := b^{\tilde{R}} b^R \delta / 4B^R$. We can therefore take $\epsilon_1 := \min\{\epsilon_2, \epsilon_3\}$ for (69) to hold and henceforth consider $\{c_i'\}_{i\in R}$ such that $\max_{i\in R} |c_i - c_i'| < \epsilon_1$.

For $i \in \tilde{R}$ set

$$\begin{aligned}\tilde{c}_i &:= c_i - \sum_{l=0}^{p-1} a_i^{S_{n-l}^R(c)} \nu_{n-l}^R(c) \\ \tilde{c}_i' &:= c_i' - \sum_{l=0}^{p-1} a_i^{S_{n-l}^R(c')} \nu_{n-l}^R(c').\end{aligned}$$

Observe that since $c \in K$ Lemma 6 implies that $\tilde{c}$ will belong in a compact set $\tilde{K} \subset \mathbf{R}^{n-p}$. Now (I2) for $k = n - p$, $\tilde{R}, \tilde{K}$, δ and $m \leq n - p$ implies the existence of $\tilde{\epsilon} > 0$ such that

$\pi^{\tilde{R}}(\tilde{c}')$ is of type $\mathcal{U}^{\tilde{R},\delta}(\tilde{c})$ whenever $\max_{i\in\tilde{R}}|\tilde{c}_i-\tilde{c}'_i|<\tilde{\epsilon}$. From (65) and (71) we obtain

$$\begin{aligned} |\tilde{c}_i-\tilde{c}'_i| &\leq |c_i-c'_i|+a_i^R\left|\nu_n^R(c)-\nu_n^R(c')\right|+\sum_{l=1}^{p-1}\left|a_i^{S_{n-l}^R(c)}\nu_{n-l}^R(c)-a_i^{S_{n-l}^R(c')}\nu_{n-l}^R(c')\right| \\ &\leq \frac{2B^R}{b^R}\max_{i\in R}|c_i-c'_i|+\left(2B^R-b^R\right)n\delta_1. \end{aligned} \tag{73}$$

Strengthening (72) require that $\delta_1<\min\{\delta,\tilde{\epsilon}\}\min\left\{1,\min\left\{1,b^{\tilde{R}}\right\}/\left(2\left(2B^R-b^R\right)n\right)\right\}$ and set $\epsilon_4:=b^R\tilde{\epsilon}/4B^R$. Then, (73) implies that $\max_{i\in\tilde{R}}|\tilde{c}_i-\tilde{c}'_i|<\tilde{\epsilon}$ and therefore that $\pi^{\tilde{R}}(\tilde{c}')$ is of type $\mathcal{U}^{\tilde{R},\delta}(\tilde{c})$ whenever $\max_{i\in R}|c_i-c'_i|<\min\{\epsilon_1,\epsilon_4\}$. Also note that $\mathcal{U}^{\tilde{R},\delta}(\tilde{c})$ and $\pi^{\tilde{R}}(\tilde{c}')$ are the restrictions on $\tilde{R}$ of $\mathcal{U}^{R,\delta}(c)$ and $\pi^R(c')$, respectively. Thus, the choice $\epsilon:=\min\{\epsilon_1,\epsilon_4\}$ satisfies (I2) for $m=p$, $k=n$ and the proof is complete. □

B Proof of Lemma 9

We restate the lemma here for convenience.

Lemma 9 *Take $\tilde{\epsilon}>0$ satisfying Lemma 8 for the compact set $\left\{|\xi|\leq\sqrt{N}C_M\right\}$ and $\delta>0$. Then there exists $\epsilon'>0$ such that with the definition*

$$D_{n+1}:=\left\{K_{n+1}+N_{n+1}+K_{n+1}N_{n+1}<\epsilon' n\right\}, \tag{74}$$

one has that

$$|\xi_i(t)-\xi_{ni}|<\tilde{\epsilon}, \qquad t\in[T_{(n-1)L},T_{(n+1)L}),\ i\in\mathcal{N}, \tag{75}$$

on $D_nD_{n+1}E_{1,n-1}E_{1,n}\cap\{|\theta_n|\leq M\}$. In particular, on this event and for $t\in[T_{(n-1)L},T_{(n+1)L})$, policy π is of type $\mathcal{U}^\delta(\xi_n)$.

Proof We will only consider the case where $t\in[T_{nL},T_{(n+1)L})$; the case $t\in[T_{(n-1)L},T_{nL})$ is similar.

For $t_1\geq 0$, $t_2\geq 0$, $i\in\mathcal{N}$ consider the equality

$$\theta_i(t_2)-\theta_i(t_1)=\frac{L\tau\alpha_i}{A_i^\pi(t_2)}\left[\frac{1}{L\tau\alpha_i}\int_{t_1}^{t_2}\eta_i^\pi(s)\,ds-\frac{(A_i^\pi(t_2)-A_i^\pi(t_1))}{L\tau\alpha_i}\theta_i(t_1)\right].$$

It implies that for $t\in[T_{nL},T_{(n+1)L})$ and on $E_{1,n}\cap\{|\theta_n|\leq M\}$

$$\begin{aligned} |\theta_i(t)-\theta_{ni}| &\leq \frac{2}{n}\left[K_{n+1,i}N_{n+1}L\tau+MK_{n+1,i}\right] \\ &\leq \frac{2\max\{L\tau,M\}}{n}\left[K_{n+1}N_{n+1}+K_{n+1}\right]. \end{aligned} \tag{76}$$

From the bound

$$\frac{T_{nL}}{A_{(n+1)L,i}} \le \frac{t}{A_i^{\pi}(t)} \le \frac{T_{(n+1)L}}{A_{nL,i}}, \quad i \in \mathcal{N},$$

and the definition of $\xi_i(\cdot)$ in (10) one gets for $t \in [T_{nL}, T_{(n+1)L})$

$$\begin{aligned} |\xi_i(t) - \xi_{ni}| &\le \frac{T_{(n+1)L}}{A_{nL,i}} |\phi_i'(\theta_{ni}) - \phi_i'(\theta_i(t))| \\ &\quad + \left(\frac{T_{(n+1)L}}{A_{nL,i}} - \frac{T_{nL}}{A_{(n+1)L,i}} \right) \phi_i'(\theta_{ni}). \end{aligned} \tag{77}$$

We will bound each term on the rhs by $\tilde{\epsilon}/2$.

For the first term note that

$$\frac{T_{(n+1)L}}{A_{nL,i}} = \frac{1}{\alpha_i a_{ni}} \left[b_n + \frac{N_{n+1}}{n} \right] \le 6 \max_{i \in \mathcal{N}} \left\{ \frac{1}{\alpha_i} \right\},$$

the inequality holding on $E_{1,n} \cap \{N_{n+1} < n\}$. From the continuity of $\{\phi_i'(\cdot)\}_{i \in \mathcal{N}}$, there exists $\epsilon_1 > 0$ such that on $\{|\theta_n| \le M\}$,

$$|\theta_i(t) - \theta_{ni}| < \epsilon_1 \text{ implies } |\phi_i'(\theta_i(t)) - \phi_i'(\theta_{ni})| < \frac{\tilde{\epsilon}}{12} \min_{i \in \mathcal{N}} \{\alpha_i\}, \quad i \in \mathcal{N}.$$

Then, by (76) the first term on the rhs of (77) is bounded from above by $\tilde{\epsilon}/2$ on $E_{1,n} \cap \{|\theta_n| \le M\} \cap \{K_{n+1}N_{n+1} + K_{n+1} + N_{n+1} < n\epsilon_2\}$, where $\epsilon_2 := \min\{1, \epsilon_1/(2\max\{L\tau, M\})\}$.

To get an upper bound of $\tilde{\epsilon}/2$ on the second term of the rhs of (77), verify that

$$\frac{T_{(n+1)L}}{A_{nL,i}} - \frac{T_{nL}}{A_{(n+1)L,i}} \le \frac{1}{\alpha_i a_{ni}} \left[\frac{b_n}{a_{ni}} \frac{K_{n+1}}{n} + \frac{N_{n+1}}{n} \right] \le 8 \max_{i \in \mathcal{N}} \left\{ \frac{1}{\alpha_i} \right\} \frac{K_{n+1} + N_{n+1}}{n},$$

the second inequality holding on $E_{1,n}$. The desired bound holds on $E_{1,n} \cap \{|\theta_n| \le M\} \cap \{K_{n+1} + N_{n+1} < n\epsilon_3\}$ where

$$\epsilon_3 := \frac{\tilde{\epsilon}}{16} \frac{\min_{i \in \mathcal{N}} \{\alpha_i\}}{\max_{i \in \mathcal{N}} \max\{\phi_i'(\theta) : |\theta| \le M\}}.$$

Repeating the above for the case $t \in [T_{(n-1)L}, T_{nL})$, we have thus shown that the choice $\epsilon' := \min\{\epsilon_2, \epsilon_3\}$ satisfies the statement of the lemma. □

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