Analysis, Synchronization and Microcontroller Implementation of a Generalized Hyperjerk System, with Application to Secure Communications Using a Descriptor Observer

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***Abstract*—This work proposes a novel hyperjerk system, as a generalization of a hyperjerk system proposed by Dalkiran and Sprott. Initially an analysis of the dynamical behavior of the system is performed. Then, the system is applied to the problem of secure communications. This is done by considering the transmitting signal as an additional state, leading to a rectangular descriptor system, for which an observer is designed, thus achieving synchronization and safe input reconstruction. The design is illustrated through numerical simulations and a microcontroller implementation.**

***Index Terms*—Chaos synchronization, hyperjerk system, observer design, descriptor systems, microcontroller implementation**

I. Introduction

Chaos is a ubiquitous phenomenon that governs many physical or mechanical systems. Chaotic systems are systems with high sensitivity to initial conditions, so small perturbations in the initial conditions will lead to completely different trajectories in the solution space. Thus, due to their high complexity, chaotic systems are extremely difficult to control and predict, making them suitable in applications of security, encryption, secure communications, robotics and more.

In 1996, Gottlieb showed that it is possible to express some 3D chaotic systems in the form of a single third-order ordinary differential equation, termed as a jerk differential equation [1]. Jerk differential equations arise in many areas of physics and engineering, such as circuits [2], oscillators [3], and more. In classical mechanics, a (hyper)jerk system is a system described by

$$\frac{d^{(n)}x}{dt^{(n)}} = f\left(x, \frac{dx}{dt}, \frac{d^{(n-1)}x}{dt^{(n-1)}}\right), \;\; n \geq 3 \tag{1}$$

or as a system of n differential equations of first order

$$\dot{x}_1 = x_2, \; ..., \; \dot{x}_{n-1} = x_n, \;\; \dot{x}_n = f(x_1, ..., x_n) \tag{2}$$

where for $n = 3$, (1) is called a jerk system and for $n \geq 4$ a hyperjerk. Hyperjerk systems have gained considerable attention, due to their simple structure and complex dynamical properties [4]–[6].

In an approach to create novel chaotic systems, many researchers consider enriching the complexity of existing systems by modifying their nonlinear terms, aiming at systems with more complex dynamics, that are more suitable for applications. The optimization of chaotic systems is usually performed by the following two modifications. In the first approach the nonlinear term is modified to a higher order nonlinear term, such as changing the product term to an exponential or logarithmic function [7], while in the second approach simple adjustments to the nonlinear terms without affecting its order [8], [9] are made.

In 2016, Dalkiran and Sprott [10] proposed a simple chaotic hyperjerk system having only one nonlinear term, which is an exponential term. In this paper, the aforementioned system is improved in order to obtain a more complex chaotic system by adding a constant term in the exponent of the nonlinear term. The novel proposed system has a higher complexity and is thus more suitable for consideration in applications.

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A trending application of chaos theory is synchronization. This refers to the problem of designing a suitable feedback control law, so that the trajectories of two (or more) coupled chaotic systems converge. The problem of synchronization has been extensively studied, since it finds applications in numerous fields, including physics, engineering, encryption, secure communications, biology and many more, see for example [11] (and the references therein). In this work, we consider the application of the proposed chaotic system to the problem of secure comminications. The problem posed here is to mask an information signal by transmitting it through a chaotic system, having the role of the master system, and then retrieving it back after appropriate signal processing at the receiver end, which plays the role of the slave system. To address this problem, a reformulation to descriptor form is considered, as in [11]–[16]. To achieve synchronization and input reconstruction, the input here is considered as an additional state for the system. This leads to a reformulation of the system in rectangular descriptor form. Now, by designing an observer for the augmented system, it is possible to achieve synchronization and information signal reconstruction simultaneously.

In addition, to further enhance the potential implementability of the proposed design, apart from numerical simulations in Matlab, a microcontroller implementation of the system is considered, using Arduino boards. The simulation results showcase the effectiveness of the design.

The rest of the paper is structured as follows. In Section II, the proposed hyperjerk chaotic system is presented, along with its simulation and analysis. In Section III, the problem of observer design and secure communications is considered. Section V concludes the paper.

II. The Proposed Dynamical System

In this work, the following 4D autonomous nonlinear dynamical system (3) is considered,

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -cx_1 - bx_2 - ax_4 - e^{x_3+d} \end{cases} \tag{3}$$

which has been produced by adding the term d in the aforementioned system proposed by Dalkiran and Sprott [10]. Also, in system (3) the parameters are kept as $a = 1$, $b = 3$ and $c = 1$, as in [10], while parameter d plays the bifurcation parameter of the system. So, for the chosen set of parameters the systems equilibrium point is $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)^T = (-1, 0, 0, 0)^T$.

Thus the Lyapunov exponents of the hyperjerk system proposed by Dalkiran and Sprott ($d = 0$) are calculated by using Wolf's algorithm [17] as: $LE_1 = 0.21268$, $LE_2 = 0$, $LE_3 = -0.34648$ and $LE_4 = -1.30880$. Also, the Kaplan-Yorke dimension [18],

$$D_{KY} = j + \frac{\sum_{i=1}^{j} L_i}{|L_{j+1}|} \tag{4}$$

where j is the index for which $\sum_{i=1}^{j} L_i > 0$ and $\sum_{i=1}^{j+1} L_i < 0$, is a useful tool in order to determine the fractal dimension of the corresponding attractor. In the case of the hyperjerk chaotic system (3) of order $n = 4$, with $d = 0$, has been calculated as: $D_{KY} = 2.61235$.

For revealing system's (3) dynamics, regarding the value of parameter d, the system's bifurcation diagram of x_1 versus d, which is produced when the trajectory crosses the section plane $x_2 = 0$ with $dx_2/dt > 0$, has been plotted in Fig. 1(a). As illustrated, system (3) inserts to a wide chaotic region, which interrupted by small periodic windows, through a period-doubling route, as the parameter d decreased.

By plotting the diagram of Kaplan-Yorke dimension of system (3) versus the parameter d (Fig. 1(b)), the effect of adding the constant term d in the exponent of the nonlinear term is revealed. For a wide range of parameter's d values system (3) has greater value of Kaplan-Yorke dimension in regards to the system of Dalkiran and Sprott. Especially, for $d = -0.154$, system (3) has the greatest value of D_{KY}, which is equal to 2.7866 and the chaotic attractors in various phase planes for this value of parameter d are depicted in Fig. 2. So, the complexity of the system, proposed by Dalkiran and Sprott, has been increased by adding the constant term d in the exponent of the system's nonlinear term.

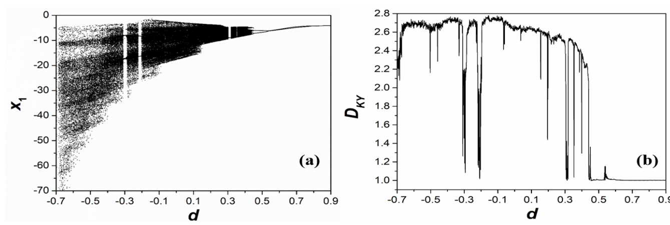

Fig. 1. (a) Bifurcation diagram of x_1 versus parameter d. (b) Diagram of system's (3) Kaplan-Yorke dimension versus parameter d.

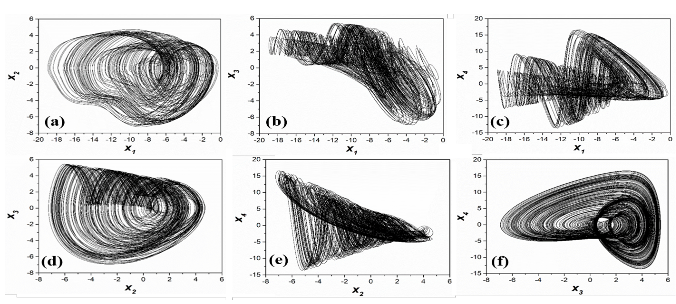

Fig. 2. Phase portraits, for $a = 1$, $b = 3$, $c = 1$ and $d = -0.154$, in (a) $x_1 - x_2$ plane, (b) $x_1 - x_3$ plane, (c) $x_1 - x_4$ plane, (d) $x_2 - x_3$ plane, (e) $x_2 - x_4$ plane and (f) $x_3 - x_4$ plane.

III. Synchronization and Secure Communication

A. Problem Formulation

Now, consider the non-autonomous case of (3)

$$\begin{cases} \dot{x}_1(t) = x_2(t) + a_1 s(t) \\ \dot{x}_2(t) = x_3(t) + a_2 s(t) \\ \dot{x}_3(t) = x_4(t) + a_3 s(t) \\ \dot{x}_4(t) = -cx_1(t) - bx_2(t) - ax_4(t) - e^{x_3(t)+d} + a_4 s(t) \\ y(t) = C_1 x(t) + D s(t) \end{cases} \tag{5}$$

where $s(t)$ is the transmitted signal that is injected linearly in the system, $y(t)$ is the measurable output, a_1, a_2, a_3 are the coefficients the transmitted signal and C_1, D are matrices of appropriate dimensions. To design an observer that can simultaneously achieve synchronization and accurate reconstruction of the transmitted signal, we first consider the transmitted signal $s(t)$ as an additional state of the system. Thus, the system (5) is rewritten as a rectangular descriptor system of the form

$$E\dot{x} = Ax + f(x) \tag{6a}$$

$$y(t) = Cx(t) \tag{6b}$$

where

$$E = (I_4, 0_{4\times 1}), A = \begin{pmatrix} 0 & 1 & 0 & 0 & a_1 \\ 0 & 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & 1 & a_3 \\ -c & -b & 0 & -a & a_4 \end{pmatrix}, C = (C_1, D)$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ s(t) \end{pmatrix}, f(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -e^{x_3(t)+d} \end{pmatrix}$$

So now, by designing an observer for the system (6), we can estimate the augmented state $x(t)$, which is equivalent to estimating the original system states and the output as well, achieving synchronization and secure input reconstruction simultaneously. For this, the following assumptions are required on the system operators and nonlinear parts:

- $rank(E^T, C^T)^T = n$, where n is equal to the dimension of the state space, given by the number of columns of E.
- The nonlinear part satisfies the Lipschitz property, that is, there exists a positive scalar $\gamma > 0$ such that

$$||f(x) - f(y)|| \leq \gamma ||x - y||, \forall x, y \in \mathbb{R}^n \tag{7}$$

The first assumption is common for observer design. The second assumption on the type of nonlinearity f covers a wide range of systems. Even for systems that do not satisfy this property globally, a constant γ can be found so that the system satisfies the Lipschitz property locally, which is useful for applications.

The desired observer for system (6) is the following

$$\dot{z}(t) = Nz(t) + Ly(t) + Rf(\hat{x}) \tag{8a}$$

$$\hat{x}(t) = z(t) + My(t) \tag{8b}$$

where N, L, R, M are to be computed, so that the state $\hat{x}$ approximates x, that is $||\hat{x} - x|| \to 0$ as $t \to \infty$, $\forall x(0), z(0)$. Following the work of [15], [19], the following steps are applied for computing the observer matrices.

1) Compute the matrix R, following the procedure given in [19, Appendix A.].
2) Compute the matrices $\tilde{E} = RE$, $\tilde{A} = RA$ that correspond to the system

$$\tilde{E}\dot{x} = \tilde{A} + Rf(x), \; y = Cx$$

and check the solvability of the LMI, for $P > 0$ and $\tilde{K}$,

$$\begin{pmatrix} \tilde{A}^T P + P\tilde{A} - C^T \tilde{K}^T - \tilde{K}C + \gamma^2 I & PR \\ R^T P & -I \end{pmatrix} < 0 \tag{9}$$

3) If (9) is solvable, the observer matrices are computed from the following equations

$$K = P^{-1}\tilde{K}, \quad N = \tilde{A} - KC \tag{10}$$

$$\tilde{E} = I - MC, \quad L = K + NM \tag{11}$$

and the observer (8) can synchronise with system (6) and also recover the transmitted signal $s(t)$.

Remark 1: [20] The detectability condition

$$rank \begin{pmatrix} sE - A \\ C \end{pmatrix} = n, \; \forall s \in \bar{\mathbb{C}}^+ \tag{12}$$

is necessary for the solvability of the LMI (9), where $\bar{\mathbb{C}}^+ = \{s | s \in \mathbb{C}, Re(s) \geq 0\}$. This is important, since it directly affects the choice of the output matrix C and the parameters a_1, a_2, a_3.

B. Simulation Results

For simulation of the above system, we consider parameters $a = 1$, $b = 3$, $c = 1$, $d = -0.154$ and $a_1 = a_2 = 0$, $a_4 = 1$, $a_3 = 1$. The output matrix is taken as $C = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \end{pmatrix}$. The information signal is an arbitrary binary signal with period 0.2s. The initial conditions are chosen as $x_1(0) = 0.2$, $x_2(0) = 0.2$, $x_3(0) = 0.2$, $x_4(0) = 0.2$, $z_1(0) = 1$, $z_2(0) = 1$, $z_3(0) = 1$, $z_4(0) = 1$, $z_5(0) = 1$. The simulation is performed in Matlab 2015a, using the LMI toolbox and ode45, with a step size of 10^{-5}. The result is shown in Fig. 3.

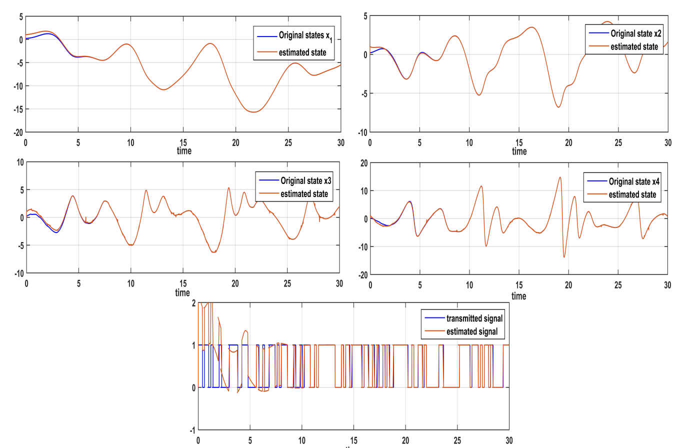

Fig. 3. Time response of original and estimated states and transmitted signal.

C. Microcontroller Implementation

To further showcase the applicability of the proposed design, a microcontroller implementation is considered. For this purpose, we used two Arduino boards (Uno REV3), so that the first one is programmed to generate the original states and the transmitted signal, while the second is programmed to simulate the observer. Also, in order to exploit signals generated by the Arduino boards, a low pass filter is used to convert the signals from PWM to analog (see Fig. 4). The system structure is shown in Fig. 5. After experimentation, the Euler method is chosen for integration, with step size equal to 0.001. The results are shown in Fig. 6, with initial conditions chosen as $x_1(0) = 0.2$, $x_2(0) = 0.2$, $x_3(0) = 0.2$, $x_4(0) = 0.2$, $z_1(0) = 5$, $z_2(0) = 5$, $z_3(0) = 5$, $z_4(0) = 5$, $z_5(0) = 5$ and the same information signal. The simulation time is 40secs.

IV. Conclusions

In this work, a modification of a hyperjerk system is proposed, having higher complexity than the original. This system was then applied to the problem of secure communication through observer design and the system was simulated using microcontrollers. Future aspects will consider a wireless implementation of the design, as well as the application of the hyperjerk system to other encryption problems.

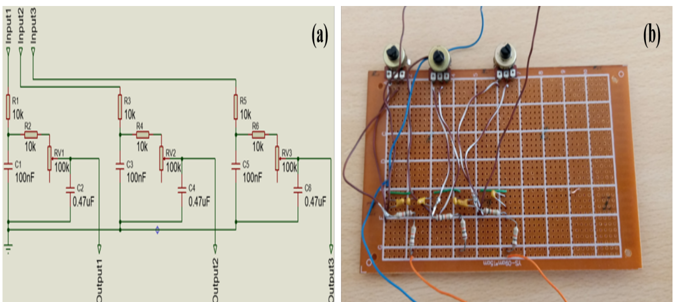

Fig. 4. (a) Low pass filter schematic. (b) Low pass filter circuitry realization.

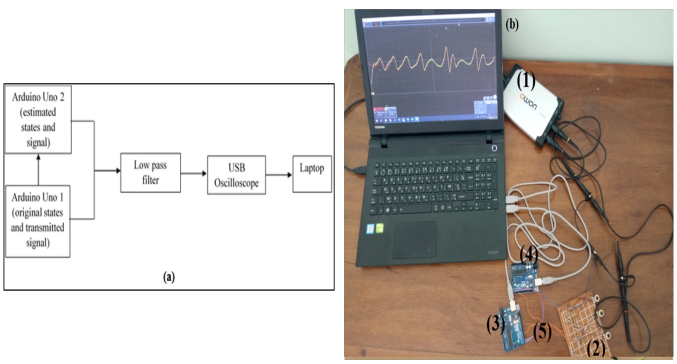

Fig. 5. (a) The diagram of the system. (b) The physical realization: (1) USB oscilloscope, (2) low pass filter circuit, (3) Arduino Uno to generate original states and transmitted signal, (4) Arduino Uno to generate the observer, (5) Wires to provide the electric power and transfer information between the two boards.

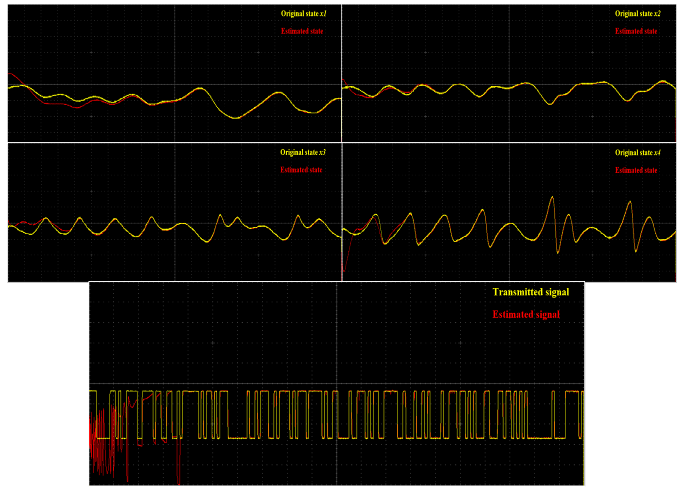

Fig. 6. Microcontroller simulation results.

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