# Demand Functions and a Characterization of Reflexivity

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**Abstract**

We show that in reflexive Banach spaces there are two classes of closed cones (consumption sets) i.e. cones whose any budget set is bounded and cones whose any budget set is unbounded. We prove that in the first category of these cones the demand correspondence of any upper semicontinuous preference relation exists and we show also that this property characterizes the reflexive spaces. Moreover we prove that in this class of cones, the demand correspondence of any convex, continuous preference relation is upper hemicontinuous. To prove these results we use some tools from the theory of Banach spaces.

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## 1 Introduction

In the theory of finite dimensional competitive economies, budget sets corresponding to strictly positive price vectors are always bounded and therefore compact, hence the demand correspondence (single valued or multivalued) of any continuous preference always exists and this is one of the fundamental properties of this theory.

On the contrary, in the theory of infinite dimensional competitive economies the existence of the demand correspondence is not ensured. There are known many examples of preference relations $\succeq$ defined on a closed cone $P$ of a normed space $X$ which does not attain maximum in some budget set of $P$. But it is not known (at least to us) an example of an infinite dimensional closed cone of a normed space $X$ so that any continuous preference relation $\succeq$ defined on $P$, attains its maximum in any budget set of $P$. To study the existence of the demand correspondences we suppose that $P$ is a closed cone of a normed space $X$ and we study necessary conditions for the existence of the demand correspondence of any preference relation of $P$. So in Theorem 9, we show that if the demand correspondence of any preference relation of $P$ which is defined by a continuous, linear functional (utility function) of $X$ exists, then the cone $P$ has a bounded budget set and so we give an important necessary condition of these cones. Of course, if the demand correspondence of any continuous preference relation of $P$ exists, then $P$ has again a bounded budget set.

In the sequel we show that in reflexive Banach spaces does not exist a closed cone with a bounded base and an other unbounded base. In the language of Mathematical Economics this means that if $P$ is a closed cone (consumption set) of a reflexive Banach space, the budget sets of $P$ are of the same type with respect to the boundedness, i.e. any budget set of $P$ is bounded or any budget set of $P$ is unbounded. We show that in the first category of these cones the demand correspondence of any upper semicontinuous preference relation exists. This is an important property because if $P$ is a cone of a reflexive space with a bounded budget set then the demand correspondence of any upper semicontinuous preference relation of $P$ exists. This property is useful for applications because it is easy to check if $P$ has a bounded budget set or not. Indeed if we choose a price vector $p$, it is easy to check if the corresponding budget set is bounded or not. If the budget set is bounded then any budget set is also bounded and the demand correspondence of any upper semicontinuous preference relation of $P$ exists.

We show also that the above property of cones characterizes the reflexive spaces. Especially we prove, Theorem 11, that a Banach space $X$ is reflexive if and only if $X$ has the property: if $P$ is a closed cone of $X$ with a bounded budget set, then for any continuous preference relation $\succeq$ of $P$ which is defined by a positive linear functional, the demand correspondence of $\succeq$ exists.

In the sequel we show, Theorem 13, that if a closed cone of a reflexive Banach space has a bounded budget set, then the demand correspondence of any continuous, convex, preference relation which satisfies the Walras' Law is upper hemicontinuous. Recall that if $\succeq$ is locally non-satiated[1] then the demand correspondence of $\succeq$, whenever exists, satisfies the Walras' Law.

We prove also that in dual spaces we have a similar classification of closed cones. Especially we show, Theorem 5, that if $P$ is a weak-star closed cone of the dual $X^*$ of a normed space $X$, then all the bases for $P$ which are defined by elements of $X$ are bounded or all these bases for $P$ are unbounded.

At this point we mention the work of A. Araujo, [**?**], where it is supposed that the commodity set is a closed, convex subset of a Banach space $X$, $u : A \longrightarrow \mathbb{R}$ is a continuous strictly quasi-concave utility function and for any $a \in \mathbb{R}$, any strictly positive linear functional $p \in X^*$ and any real number $w > 0$ it is denoted by $C_a$ the set $C_a = \{x \in A | u(x) \geq a\}$, by $w_p$ the real number $w_p = \inf\{p(y)|y \in C_a\}$ and by $\phi(p, w)$ the unique solution of the maximization problem $\max\{u(x)|x \in A$ with $p(x) = w\}$ whenever such a solution exists. In Theorem 2 of [**?**] it is proved: If $A$ is bounded, $u$ is uniformly continuous so that for some $a \in \mathbb{R}$ the set $C_a$ has non empty interior and for any price vector $p \in X^*$ with $\|p\| = 1$, either there exists $x \in A$ with $p(x) < w_p = \inf\{p(y)|y \in C_a\}$ and there exists $\phi(p, w_p)$ or $\inf\{p(y)|y \in C_a\}$ is attained, then $X$ is reflexive. In the same theorem it is also remarked that if $X$ is reflexive, $A$ is bounded and weakly closed and $u$ is weakly continuous then $\phi(p, w_p)$ exists for any $(p, w_p)$. So a characterization of reflexive spaces is proved. But the results of

[1] If $\succeq$ is monotone or if $\succeq$ has an extremely desirable bundle then $\succeq$ is locally non-satiated.

important article of Araujo and the results of our paper are independent. In the present article the consumption sets are unbounded (cones) and also we do not suppose that the interior of $C_a$ is non empty.

Finally note that in the theory of Banach spaces, the characterizations of the differed kind of spaces is one of the most important problems of this subject. Our characterization of reflexivity connects the demand theory with the geometry of Banach spaces.

## 2 Bases for cones - Budget sets

Let $X$ be a normed space. Denote by $X^*$ the norm dual of $X$ and by $\mathbb{R}_+$ the set of positive real numbers $\lambda \geq 0$. A nonempty, convex subset $P$ of $X$ is a **cone** (or a wedge of $X$) if $\lambda P \subseteq P$ for each $\lambda \in \mathbb{R}_+$. If moreover $P \cap (-P) = \{0\}$ the cone $P$ is **pointed**. The cone $P$ is **generating** if $X = P - P$ and the cone $P$ is **normal** if a real number $a > 0$ exists so that for any $x, y \in X$, $0 \leq x \leq y$ implies $\|x\| \leq a\|y\|$. The set $P^0 = \{f \in X^* \mid f(x) \geq 0 \text{ for each } x \in P\}$, is the **dual cone** of $P$ in $X^*$. Suppose that $X$ is ordered by the cone $P$, i.e for any $x, y \in X$ we have $x \geq y$ if and only if $x - y \in P$. Then for any $x, y \in X$ with $x \leq y$ the set $[x, y] = \{z \in X \big| x \leq z \leq y\}$ is the order interval $x, y$. If $x \in P$ so that $X = \cup_{n=1}^{\infty} n[-x, x]$, $x$ is an **order unit** of $X$. Suppose also that $X^*$ is ordered by $P^0$. So a linear functional $f$ of $X$ is **positive** (on $P$) if $f(x) \geq 0$ for each $x \in P$, **strictly positive** (on $P$) if $f(x) > 0$ for each $x \in P, x \neq 0$ and **uniformly monotonic** (on $P$) if $f(x) > a\|x\|$, for each $x \in P$, where $a$ is a constant real number $a > 0$. If a strictly positive linear functional exists, the cone $P$ is pointed. A convex subset $B$ of $P$ is a **base for the cone** $P$ if for each $x \in P, x \neq 0$ a unique real number $f(x) > 0$ exists such that $\frac{x}{f(x)} \in B$. Then the function $f$ is additive and positively homogeneous on $P$ and $f$ can be extended to a linear functional on $P - P$ by the formula $f(x_1 - x_2) = f(x_1) - f(x_2), x_1, x_2 \in P$, and in the sequel this linear functional can be extended to a linear functional on $X$. So we have that $B$ is a base for the cone $P$ if and only if a strictly positive (not necessarily continuous) linear functional $f$ of $X$ exists so that,

$$B = \{x \in P \mid f(x) = 1\}.$$

Then we say that the base $B$ is defined by the functional $f$. If $B$ is a base for the cone $P$ with $0 \notin \overline{B}$, where $\overline{B}$ is the closed hull of $B$, then a continuous linear functional $f$ of $X$ separating $\overline{B}$ and $0$ exists. Then the functional $f$ is strictly positive and the cone $P$ has a base defined by $f$. If $B$ is bounded the base defined by $f$ is also bounded. So we have:

*The cone $P$ has a base defined by a continuous linear functional $f$ of $X$ if and only if $P$ has a base $B$ with $0 \notin \overline{B}$. If moreover the base $B$ is bounded the base for $P$ defined by $f$ is also bounded.*

The above notions and properties are connected with the demand theory as follows: Suppose that in a competitive economy the commodity-price duality is the ordered dual system $\langle X, X^* \rangle$, i.e. the commodity space is a normed space $X$ and the topological dual $X^*$ of $X$ is the price space. Suppose that the consumption set is a cone $P$ of $X$ and that $X$ is ordered by the cone $P$. Then any strictly positive (on $P$) and continuous linear functional of $X$ is a **price vector**. For any $f \in X^*$, strictly positive on $P$ and for any real number (wealth level) $w > 0$, the set $B_w(f) = \{x \in P \big| f(x) \leq w\}$ is the **budget set** corresponding to $f$ and $w$ and the set $L = \{x \in P \big| f(x) = w\}$ is **the budget line** of $B_w(f)$. Of course $L$ is the base for the cone $P$ defined by the continuous, linear functional $g = \frac{f}{w}$. Therefore any budget set defines a base for the cone $P$ (the budget line) which is defined by a continuous linear functional. Conversely any base $K$ for $P$ which is defined by a continuous linear functional $f$ of $X$ defines the budget set $B_1(f) = \{x \in P \big| f(x) \leq 1\}$ whose budget line is the base $K$. So there exists an one-to-one correspondence between budget sets of $P$ and bases for $P$ which are defined by continuous linear functionals therefore, in the sense of this correspondence, we may identify budget sets of $P$ with bases for $P$ defined by continuous linear functionals.

Finally if we suppose that $\omega \in P$ is the initial endowment (of a consumer), the budget set corresponding to the price vector $f$ is also denoted by $B_\omega(f)$, i.e. $B_\omega(f) = \{x \in P \big| f(x) \leq f(\omega)\}$.

For ordered spaces and bases for cones we refer to the book, [**?**], for Banach spaces to [**?**] and for notions on economics we refer to the books [**?**] and [**?**].

The following is a useful result for our study. For its proof see in [**?**], Theorem 3.8.4.

**Theorem 1.** *A cone $P$ of a normed space $X$ has a bounded base $B$ with $0 \notin \overline{B}$ if and only if the dual cone $P^0$ of $P$ in $X^*$ has interior points.*

We give below some examples of bases for cones (budget sets). Recall that for any real number $1 \leq p < +\infty$, $\ell_p$ is the space of real sequences $x = (x_i)$ so that $\sum_{i=1}^{\infty} |x_i|^p < +\infty$ with norm $\|x\| = \big(\sum_{i=1}^{\infty} |x_i|^p\big)^{\frac{1}{p}}$, and $c_0$, respectively $\ell_\infty$, is the space of convergent to zero, respectively of bounded, real sequences $x = (x_i)$ with norm $\|x\| = sup_{i\in\mathbb{N}}|x_i|$. Recall also the following property of ordered spaces which is used in the examples below: any positive linear functional of a Banach space $X$ ordered by a closed, generating cone $P$ is continuous, [**?**], Corollary 3.5.6.

**Example 2.** (i) *The positive cone $\ell_1^+ = \{x = (x_i) \in \ell_1 \big| \ x_i \geq 0$ for any $i\}$ of $\ell_1$, has a bounded and an unbounded base (a bounded and an unbounded budget set). Any element of $c_0$ defines an unbounded base for $\ell_1^+$, hence any price vector $f \in c_0$ defines an unbounded budget set in $\ell_1^+$.*

Indeed, the set
$$B = \{x \in \ell_1^+ \mid u(x) = 1\},$$
where $u = (u_i) \in \ell_\infty$ with $u_i = 1$ for each $i$, is a bounded base for the cone $\ell_1^+$ with $\|x\| = 1$ for each $x \in B$. Any base $K$ for $\ell_1^+$ defined by an element $f = (f_1, f_2, ...)$ of $c_0$ is unbounded because $\frac{e_n}{f_n} \in K$ for each $n$, where $\{e_n\}$ is the usual Schauder basis of $\ell_1$ and $\|\frac{e_n}{f_n}\| = \frac{1}{f_n} \longrightarrow +\infty$.

(ii) *Any base for the positive cone $\ell_p^+ = \{x = (x_i) \in \ell_p \big| \ x_i \geq 0$ for any $i\}$ of $\ell_p$ with $1 < p < +\infty$, is unbounded, hence any price vector $f \in \ell_q^+$ defines an unbounded budget set in $\ell_p^+$.* Indeed if we suppose that
$$B = \{x \in \ell_p^+ \mid f(x) = 1\},$$

is a base for $\ell_p^+$ defined by the linear functional $f$ of $\ell_p$, then as we have noted above, $f$ is continuous, therefore $f = (f_1, f_2, ...) \in \ell_q$. Also $\frac{e_n}{f_n} \in B$ for each $n$, where $\{e_n\}$ is the usual Schauder basis of $\ell_p^+$, therefore any base for $\ell_p^+$ is unbounded.

(iii) We give an example of a closed cone without a base. Suppose that $\ell_2(\Gamma)$, where $\Gamma$ is an uncountable set, is the space of real valued functions $x : \Gamma \longrightarrow \mathbb{R}$ so that $\sum_{i\in\Gamma} x^2(i) < +\infty$ with norm $\|x\| = (\sum_{i\in\Gamma} x^2(i))^{\frac{1}{2}}$, ordered by the cone $\ell_2^+(\Gamma) = \{x \in \ell_2(\Gamma) \big| \; x(i) \geq 0$ for each $i\}$. For any element $x$ of this space we have that $x(i) \neq 0$ for at most a countably many $i$. $\ell_2(\Gamma)$ does not have strictly positive linear functionals because if we suppose that $f$ is such a linear functional, we have that $f \in \ell_2(\Gamma)$ with $f(i) = f(e_i) > 0$ for each $i \in \Gamma$, a contradiction. Hence does not exist a base for the cone $\ell_2^+(\Gamma)$.

(iv) *Any normed space $X$ has an infinite dimensional closed cone $P$ with a bounded base defined by a continuous linear functional of $X$ and the dual $X^*$ of $X$ has an infinite dimensional, weak star closed cone with a bounded base defined by an element of $X$.* This holds because if we suppose that $x_0 \in X$ with $x_0 \neq 0$, $U$ is a closed ball of $X$ with center $x_0$ not containing $0$ and $P = \{\lambda x \mid \lambda \in \mathbb{R}_+, x \in U\}$, is the cone of $X$ generated by $U$, then $P$ is closed and it is easy to show that any continuous linear functional of $X$ separating $0$ and $U$ defines a bounded base for the cone $P$. Suppose now that $x_0^* \in X^*$ with $x_0^* \neq 0$ and suppose also that $V$ is a closed ball of $X^*$ with center $x_0^*$, not containing $0$. Then the cone $C = \{\lambda x^* \mid \lambda \in \mathbb{R}_+, x^* \in V\}$ is weak star closed and any element $y$ of $X$ separating $0$ and $V$ (such an element exists because $V$ is weak-star compact) defines a bounded, weak star closed base for the cone $C$. As we will show in Theorem 5 of this article, the cone $C$ cannot have an unbounded base defined by an element of $X$.

(v) *Any Banach space $X$ has a closed cone with an unbounded base defined by a continuous linear functional.*

Since any Banach space has a basic sequence, [**?**], Theorem 4.1.30, we may suppose that $E$ is a closed subspace of $X$ with a Schauder basis $\{b_n\}$ with $\|b_n\| = 1$ for each $n$ and suppose that $P = \{x = \sum_{i=1}^{\infty} \lambda_i b_i \mid \lambda_i \in \mathbb{R}_+ \;$ for each $\; i\}$ is the positive cone of the basis $\{b_n\}$. The cone $P$ is closed and suppose that $\{f_n\}$ is the sequence of the coefficient functionals of the basis. The functional $f = \sum_{i=1}^{\infty} \frac{f_i}{2^i \|f_i\|}$ is strictly positive on $P$ and it defines an unbounded base $B$ for the cone $P$ because $2^i \|f_i\| b_i \in B$ for each $i$.

We close this introductional section by the next two results which are useful for our study. We give their proof but we do not present these results as new.

**Proposition 3.** *Suppose that $B$ is a base for $P$ defined by the linear functional $f$. Then the base $B$ is bounded if and only if the functional $f$ is uniformly monotonic.*

*Proof.* If we suppose that $\|x\| \leq M$ for each $x \in B$, then for each $x \in P, x \neq 0$ we have $\|\frac{x}{f(x)}\| \leq M$, therefore $\|x\| \leq M f(x)$, for each $x \in P$, hence $f$ is uniformly monotonic. For the converse suppose that $f(x) \geq a\|x\|$ for each $x \in P$, where $a$ is a real number $a > 0$. Then for each $x \in B$ we have $1 = f(x) \geq a\|x\|$, therefore the base $B$ is bounded. ∎

**Proposition 4.** *Any base for a finite dimensional closed cone $P$ of a normed space $X$ is bounded.*

*Proof.* Suppose that $B$ is a base for $P$ defined by the linear functional $f$ and $x_n \in B$ with $\|x_n\| \longrightarrow \infty$. Then $f\big(\frac{x_n}{\|x_n\|}\big) \longrightarrow 0$. Since the set $P \cap U$, where $U$ is the closed unit ball of $X$ is compact, a subsequence of $\{\frac{x_n}{\|x_n\|}\}$ exists which converges to an element $x_0$ of $P$. Then we have that $\|x_0\| = 1$ and $f(x_0) = 0$, contradiction because $f$ is strictly positive on $P$. ■

A natural question arising by this property of finite dimensional cones is if the converse is also true, i.e. *if any base for a closed cone $P$ of a normed space $X$ is bounded, is the cone $P$ finite dimensional?* In Example 6 below, we show that the answer to this problem is negative.

# 3 Demand functions and reflexivity

Suppose that $X$ is a normed space. The map $x \longrightarrow \hat{x}$, for any $x \in X$ where $\hat{x} \in X^{**}$ so that $\hat{x}(x^*) = x^*(x)$ for any $x^* \in X^*$ is the **natural map** of $X$ into $X^{**}$. If the natural map is onto $X^{**}$ then the space $X$ is **reflexive**. Of course any reflexive space is a Banach space.

In this section whenever we say that a base for a cone of a normed space is bounded, we will mean of course that it is bounded in the norm topology of the space.

If $P$ is a cone of $X^*$, we say that an element $x$ of $X$ defines a base for the cone $P$ if $x$, as a linear functional on $X^*$, is strictly positive on $P$, i.e. $x(x^*) > 0$, for each $x^* \in P, x^* \neq 0$. Then $B = \{x^* \in P \mid x(x^*) = 1\}$, is the base for $P$ defined by $x$.

**Theorem 5.** *The following statements are true:*

*(i) The dual $X^*$ of a normed space $X$ does not have a weak star closed cone $P$ with a bounded base and an other unbounded base defined (the bases for $P$) by elements of $X$,*

*(ii) any reflexive Banach space $X$ does not have a closed cone $P$ with a bounded base and an other unbounded base defined (the bases for $P$) by continuous linear functionals.*

In the next example we show that in any reflexive space the class of infinite dimensional closed cones whose any base defined by a continuous linear functional is bounded, is nonempty.

**Example 6.** Suppose that $X$ is a reflexive space, and suppose that $P$ is an infinite dimensional, closed cone of $X$ with a closed, bounded base. Such a cone exists by Example 3, (iv). By the previous Theorem, any base for $P$ which is defined by a continuous linear functional is unbounded.

Suppose that $P$ is a closed cone of a normed space $X$ and that $\succeq$ is a **preference relation** on $P$, i.e. $\succeq$ is reflexive, complete and transitive. $\succeq$ is **upper semicontinuous** if for each $x \in P$ the set $\{y \in P | y \succeq x\}$ is closed, $\succeq$ is **lower semicontinuous** if for each $x \in P$ the set $\{y \in P | y \preceq x\}$ is closed and $\succeq$ is **continuous** if it is lower and upper semicontinuous. Also $\succeq$ is **locally non-satiated** if for any $x \in P$ and any real number $\varepsilon > 0$ there exists $y \in P$ so that $y \succ x$ and $\|y - x\| < \varepsilon$. Recall that if $\succeq$ is strictly monotone (i.e. for any $x, y \in P$, $x > y$ implies $x \succ y$) or if $\succeq$ has an extremely desirable bundle (i.e. there exists $u \in P$ so that $x + \lambda u \succ x$ for any $x \in P$ and any real number $\lambda > 0$) then $\succeq$ is locally non-satiated.

For any $p \in X^*$, strictly positive on $P$ and any real number $w > 0$ denote by $\mathbf{x}(p, w)$ the set of elements of the budget set $B_w(p)$ at which $\succeq$ takes maximum on $B_w(p)$, i.e.

$$\mathbf{x}(p, w) = \{x \in B_w(p) \big| x \succeq y \text{ for any } y \in B_w(p)\}.$$

If for any strictly positive $p \in X^*$ and any $w > 0$ the set $\mathbf{x}(p, w)$ is nonempty, then $(p, w) \longrightarrow \mathbf{x}(p, w)$ is the **demand correspondence** of $\succeq$ and we say that the demand correspondence of $\succeq$ exists. If $p(x) = w$ for any $x \in \mathbf{x}(p, w)$ we say that the demand correspondence (whenever exists) satisfies the **Walras' Law**. If for any $p$ and $w$ the set $\mathbf{x}(p, w)$ is a singleton, the demand correspondence defines **the demand function** of $\succeq$.

If a linear functional $q$ of $X$ exists so that for any $x, y \in P$ we have $x \succeq y$ if and only if $q(x) \geq q(y)$, we will say that $\succeq$ **is defined by the linear functional** $q$ of $X$. If $q$ is continuous (on $X$) we say that $\succeq$ is defined by a continuous, linear functional. If $\succeq$ is defined by the linear functional $q$ we have: $\succeq$ is continuous if and only if $q$ is continuous on $P$. Therefore for the continuity of $\succeq$ only the continuity of $q$ on $P$ and not on the whole space $X$ is needed. The next is an example of a linear functional $f$ which is continuous on $P$ but it isn't not continuous in the whole space $X$.

For the sake of completeness we give below some known properties of the demand correspondences. The proof of $(i)$ and $(ii)$ is quite analogous with the proof of these properties in the finite dimensional case. For the proof of (iii) see in [**?**], Theorem 1.2.2.

**Theorem 7.** *Suppose that $P$ is a cone of a normed space $X$ and that $\succeq$ is a preference relation defined on $P$. Then for any $p \in X^*$, strictly positive on $P$ and any real number $w > 0$ we have:*

*(i) If $\succeq$ is locally non-satiated then $p(x) = w$ for any $x \in \mathbf{x}(p, w)$,*

*(ii) if $\succeq$ is upper semicontinuous, then $\mathbf{x}(p, w)$ is closed and*

*(iii) if for some topology $\tau$ of $P$, $\succeq$ is $\tau$-upper semicontinuous and the budget set $B_w(p)$ is $\tau$-compact, then $\mathbf{x}(p, w) \neq \emptyset$.*

**Theorem 8.** *Suppose that $P$ is a closed cone of a reflexive Banach space $X$.*

*If the cone $P$ has a closed and bounded base, or equivalently, if $P$ has a bounded budget set, then for any upper semicontinuous preference relation $\succeq$ defined on $P$, the demand correspondence of $\succeq$ exists.*

**Theorem 9.** *Suppose that $P$ is a closed cone of a normed space $X$ and suppose that at least one strictly positive (on $P$) and continuous linear functional of $X$ exists.*

*(i) If for any preference relation $\succeq$ defined on $P$ by a continuous linear functional of $X$ the demand correspondence of $\succeq$ exists, then the cone $P$ has a bounded budget set.*

(ii) *If $P^0 - P^0 = X^*$ (especially if $P$ is normal or if $P^0$ has interior points, then $P^0 - P^0 = X^*$) and for any preference relation $\succeq$ defined on $P$ by a continuous and positive (on $P$) linear functional of $X$ the demand correspondence of $\succeq$ exists, then the cone $P$ has a bounded budget set.*

In 1964, D. Milman and V. Milman, stated in [**?**] (see also in [**?**], Theorem 2.9) the following characterization of non reflexive spaces:

**Theorem 10.** *A Banach space $X$ is non reflexive if and only if the positive cone of $\ell_1$, is embeddable in $X$.*

This was one of the first results connecting cones and geometry of Banach spaces. Recall that a cone $P$ of a normed space $X$ is **isomorphic (or locally-isomorphic )** to a cone $Q$ of a normed space $E$ if an additive, positively homogeneous, [2] one-to-one, map $T$ of $P$ onto $Q$ exists such that $T$ and $T^{-1}$ are continuous in the induced topologies. Then we say that the cone $P$ is **embeddable** in $E$ or that $T$ is an **isomorphism** of $P$ onto $Q$. Suppose that $T$ is an isomorphism of $P$ onto $Q$. By using the continuity of $T$ and $T^{-1}$ at zero, we can easily show that

$$\frac{1}{B}\|x\| \leq \|Tx\| \leq A\,\|x\| \text{ for each } x \in P,$$

where $A = sup\{\|Tx\| \mid x \in P, \|x\| \leq 1\}$ and $B = sup\{\|T^{-1}y\| \mid y \in Q, \|y\| \leq 1\}$. Then $T$ can be extended to a linear, one-to-one operator of $P-P$ onto $Q-Q$ by taking $T(x_1 - x_2) = T(x_1) - T(x_2)$, for any $x_1, x_2 \in P$ but the continuity of $T$ and $T^{-1}$ it is not ensured.

For the history of the cone characterization of reflexivity we refer the result of [**?**] that *a Banach space $X$ is non reflexive if and only if $X$ has a closed cone with an unbounded, closed, dentable*[3] *base* and the result of [**?**], that
*a Banach space $X$ is reflexive if and only if for each cone $Q$ of the norm dual $X^*$ of $X$ with a closed, bounded base, the dual cone $Q_0 = \{x \in X \mid x^*(x) \geq 0$ for each $x^* \in Q\}$ of $Q$ in $X$ has interior points.*

**Theorem 11.** *A Banach space $X$ is reflexive if and only if $X$ has the property: for any closed cone $P$ of $X$ with a bounded budget set and for any continuous preference relation $\succeq$ defined on $P$ by a positive linear functional of $X$ the demand correspondence of $\succeq$ exists.*

The next is an example of a closed cone $P$ of a Banach space $X$ with the property: There exists a linear functional $p$ of $X$ so that the restriction of $p$ on $P$ is continuous but $p$ fails to be continuous on $X$. So the preference relation $\succeq$ defined on $X$ by $p$ is continuous on $P$ but not continuous on the whole space $X$.

**Example 12.** Suppose that $X = c_0$. The sequence $\{b_n\}$ with $b_n = \sum_{i=1}^{n} e_i$ is the summing basis of $c_0$ and the positive cone $P$ of this basis is the set of decreasing

[2]A map $T : P \longrightarrow Q$ is additive and positively homogeneous if $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ for each $\lambda, \mu \in \mathbb{R}_+$ and $x, y \in P$.

[3]A subset $K$ of a Banach space $X$ is dentable if for each real number $\varepsilon > 0$ there exists a point $x_\varepsilon$ of $K$ which does not belong to the closure of the convex hull of the set $\{x \in K | \; \|x - x_\varepsilon\| \geq \varepsilon\}$.

convergent to zero real sequences. Then $P$ is a closed subcone of $c_0^+$ and $P - P$ is dense in $c_0$ because for any $x = (x_i) \in c_0$ we have that For any $x = (x_i) \in P$ we put $\lambda_1 = x_1$ and $\lambda_n = x_n - x_{n+1}$ for each $n > 1$. Then $x_n = \sum_{i=n}^{\infty} \lambda_i$, for each $n$. $T : \ell_1^+ \longrightarrow P$, with $T(\lambda) = (\sum_{i=n}^{\infty} \lambda_i)$ is an isomorphism of $\ell_1^+$ onto $P$. If we suppose that any positive linear functional of $c_0$, restricted on $P$ is continuous on $P$, with respect to the induced topology of $P$ we have a contradiction as follows: Suppose that $Tx = Tx^+ - Tx^-$ for any $x \in \ell_1$ is an extension of $T$ on $\ell_1$ Then $T$ is continuous because for any $x \in \ell_1$ we have

$$\|T(x)\| = \|T(x^+) - T(x^-)\| \leq \|T(x^+)\| + \|T(x^-)\| \leq A(\|x^+\| + \|x^-\|) = A\|x\|.$$

Then the adjoint $T^*$ of $T$ is a linear operator from $\ell_1$ into $\ell_\infty$. This operator is onto because for each $\xi \in \ell_\infty^+$ we have that the linear functional $f$ of $c_0$ with $f(T(h)) = \xi(h)$, for each $h \in \ell_1^+$ is continuous on $P$ because $T$ is an isomorphism therefore it is continuous on $c_0$ and $T^*(f) = \xi$. Since $\ell_\infty^+$ is generating $T^*$ is onto. Hence $T^*$ is an isomorphism of $\ell_1$ onto $\ell_\infty$.

A correspondence (multivalued function) $\varphi$ from a topological space $F$ into the subsets of a topological space $G$ is **upper hemicontinuous** at a point $x \in F$ if for any open neighborhood $V$ of $\varphi(x)$ the upper inverse[4] $\varphi^u(V)$ of $V$ is a neighborhood of $x$. $\varphi$ is upper hemicontinuous if it is upper hemicontinuous at any point of $F$. Recall also that by the Closed Graph Theorem, [**?**] Theorem 16.12, we have: if the range space $G$ of $\varphi$ is compact and Hausdorff, $\varphi(x)$ is closed for any $x \in F$ and if $\varphi$ has closed graph, then the correspondence $\varphi$ is upper hemicontinuous. For notions and terminology on multivalued functions we refer to [**?**].

In the next result we prove that in closed cones of reflexive Banach spaces with bounded budget sets, the demand correspondence is upper hemicontinuous. To show this result we use again Theorem 5 in order to prove that any strictly positive and continuous linear functional $p$ of $X$ is an interior point of the order interval $[\frac{p}{2}, \frac{3p}{2}]$ of $X^*$. This enable us to show that the restriction of the demand correspondence on this interval takes values in a compact, Hausdorff, topological space and so we can apply the closed graph theorem for multivalued functions. This is the crucial point of this proof.

Suppose that $\succeq$ is a preference relation defined on a cone $P$ of a normed space $X$. If for each $x, y \in P$ and for any sequence $\{\lambda_n\}$ of positive real numbers which converges to $\lambda$ we have: $x \succeq \lambda_n y$ for each $n$ implies $x \succeq \lambda y$, then we will say that $\succeq$ is **radially lower semicontinuous**. Of course any lower semicontinuous preference relation is radially lower semicontinuous. Similarly if $\lambda_n y \succeq x$ for each $n$ implies $\lambda y \succeq x$, we will say that $\succeq$ is radially upper semicontinuous and if $\succeq$ is radially lower and radially upper semicontinuous, $\succeq$ is radially continuous.

In the next theorem we suppose that $\succeq$ is an upper semicontinuous and radially lower semicontinuous preference relation of $P$. Of course any continuous preference relation of $P$ satisfies these properties.

Finally note that if $\succeq$ is locally non-satiated then $\mathbf{x}(p, w)$ is a subset of the budget line of $B_w(p)$, therefore the demand correspondence of $\succeq$, whenever exists, satisfies

[4]The upper inverse $\varphi^u(V)$ of $V$ is the set of vectors $x$ of $F$ so that $\varphi(x) \subseteq V$.

the Walras' Law. As we have noted before, if $\succeq$ is monotone or if $\succeq$ has an extremely desirable bundle then $\succeq$ is locally non-satiated.

Recall also the following result which we will use below: in normed spaces convex sets have the same closure in the weak and in the norm topology, [**?**], Theorem 2.5.16.

**Theorem 13.** *Suppose that $P$ is a closed cone of a reflexive Banach space $X$, $P$ has a closed bounded base and that $\succeq$ is a convex, upper semicontinuous and radially lower semicontinuous preference relation of $P$. If the demand correspondence satisfies the Walras' Law, then it is (the demand correspondence) upper hemicontinuous. If $\mathbf{x}(p,w)$ is a singleton for any $p, w$, the demand function of $\succeq$ is continuous.*

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