The Lattice of Fuzzy Intervals and Sufficient Conditions for Its Distributivity

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September 15, 2002

Abstract

Given a *reference* lattice $(X, \sqsubseteq)$, we define *fuzzy intervals* to be the fuzzy sets such that their p-cuts are crisp closed intervals of $(X, \sqsubseteq)$. We show that: given a complete lattice $(X, \sqsubseteq)$ the collection of its fuzzy intervals is a complete lattice. Furthermore we show that: if $(X, \sqsubseteq)$ is completely distributive then the lattice of its fuzzy intervals is distributive.

Keywords: Algebra, Fuzzy Algebras, Fuzzy Lattices.

1 Introduction

The following is a small sample of the large literature on fuzzy algebras. Rosenfeld wrote the first paper on *fuzzy groups* [10]; a recent review is [3]. *Fuzzy rings* and *fuzzy ideals of rings* are studied in [19, 5, 2, 20]. Seselja, Tepavcevska and others have presented a far reaching famework of L-fuzzy and P-fuzzy algebras [11, 12, 13].

Fuzzy lattices are a particular type of fuzzy algebras. A fuzzy lattice is a fuzzy set such that its cuts are sublattices of a "reference lattice" $(X, \sqsubseteq)$. Relatively little has been published on fuzzy lattices. Yuan and Wu introduced the concept [17] and Ajmal studied it in greater detail [1]. Swamy and Raju [14] and, more recently, Tepavcevska and Trajkovski [15] studied *L-fuzzy lattices.*[1].

In this note we introduce *fuzzy intervals* within the context of fuzzy lattices. I.e. a fuzzy interval is defined to be a fuzzy set such that its cuts are closed intervals of a reference lattice $(X, \sqsubseteq)$. It appears that fuzzy intervals (in this lattice theoretic sense) have not been studied previously. A special case which has been extensively studied is that of fuzzy intervals with the reference lattice $(X, \sqsubseteq)$ being a set of *real numbers* [7]. Some connections between this special case and the more general case studied here will be discussed briefly in Section 5..

As mentioned, our study of fuzzy intervals is lattice theoretic. We establish some basic properties of fuzzy intervals and we show the following: given a complete lattice $(X, \sqsubseteq)$, the collection of its fuzzy intervals is a complete lattice; if $(X, \sqsubseteq)$ is *completely distributive* then the lattice of its fuzzy intervals is distributive.

[1]Two additional senses of the term "fuzzy lattice" should also be mentioned. Kaburlasos and Petridis use *fuzzy inclusion measures* [6, 8, 9] to introduce a concept of "fuzzy lattice" which is different from the one used in the previously mentioned works; however there is an interesting connection between the two approaches through the concept of *fuzzy orders.* In addition, [16, 18, 21] and many others use the term "fuzzy lattice" to denote a quite different mathematical concept, namely a completely distributive lattice with an order reversing involution.

2 Preliminaries

In what follows, the closed unit interval is denoted by $L \doteq [0,1] \subseteq R$. The usual order of real numbers is denoted by $\leq$; the maximum (resp. minimum) of x, y is denoted by $x \vee y$ (resp. $x \wedge y$). Given a set $P \subseteq L$, $\vee P$ (resp. $\wedge P$) denotes the supremum (resp. the infimum) of P. $(L, \leq, \vee, \wedge)$ is a totally ordered set.

The *reference lattice* is denoted by $(X, \sqsubseteq, \sqcup, \sqcap)$ and it is assumed to be complete. Hence, for every $Y \subseteq X$ the elements $\sqcap Y$, $\sqcup Y$ exist; in particular, there exist $\sqcap X$ (the minimum element of X) and $\sqcup X$ (the maximum element of X), hence we can write $X = [\sqcap X, \sqcup X]$.

Definition 2.1 *A* fuzzy set *is a function $M : X \to L$. The collection of all fuzzy sets (from X to L) will be denoted by $\mathbf{F}(X, L)$ or simply by $\mathbf{F}$.*

In a standard manner, we introduce an order on $\mathbf{F}$ using the "pointwise" order of $(L, \leq, \vee, \wedge)$. The symbols $\leq, \vee, \wedge$ will be used without danger of confusion.

Definition 2.2 *For $M, N \in \mathbf{F}$ we write $M \leq N$ iff for all $x \in X$ we have: $M(x) \leq N(x)$.*

Definition 2.3 *For $M, N \in \mathbf{F}$: we define the fuzzy set $M \vee N$ by: $(M \vee N)(x) \doteq M(x) \vee N(x)$; we define the fuzzy set $M \wedge N$ by: $(M \wedge N)(x) \doteq M(x) \wedge N(x)$.*

It is well known [7] that $\leq$ is an order on $\mathbf{F}$ and that $(\mathbf{F}, \leq, \vee, \wedge)$ is a complete and distributive lattice with $\sup(M, N) = M \vee N$, $\inf(M, N) = M \wedge N$.

Definition 2.4 *Given a fuzzy set $M : X \to L$, the p-*cut *of M is denoted by M_p and defined by $M_p \doteq \{x : M(x) \geq p\}$.*

We will need some properties of p-cuts, summarized in the following propositions. Their proofs can be found in [7].

Proposition 2.5 *Take any $M \in \mathbf{F}$ with p-cuts $\{M_p\}_{p \in L}$ and $N \in \mathbf{F}$ with p-cuts $\{N_p\}_{p \in L}$. Then $M = N$ iff for all $p \in L$ we have $M_p = N_p$.*

Proposition 2.6 *Take any $M \in \mathbf{F}$ with p-cuts $\{M_p\}_{p \in L}$. Then we have the following.*
(i) For all $p, q \in L$ we have: $p \leq q \Rightarrow M_q \subseteq M_p$.
(ii) For all $P \subseteq L$ we have: $\cap_{p \in P} M_p = M_{\vee P}$.
(iii) $M_0 = X$.

Proposition 2.7 *Consider a family of sets $\{\widetilde{M}_p\}_{p \in L}$ which satisfy the following.*
(i) For all $p, q \in L$ we have: $p \leq q \Rightarrow \widetilde{M}_q \subseteq \widetilde{M}_p$.
(ii) For all $P \subseteq L$ we have: $\cap_{p \in P} \widetilde{M}_p = \widetilde{M}_{\vee P}$.
(iii) $\widetilde{M}_0 = X$.
Define the fuzzy set $M(x) = \vee\{p : x \in \widetilde{M}_p\}$. Then for all $p \in L$ we have $M_p = \widetilde{M}_p$.

We will also need some well-known properties of (crisp) closed intervals in a lattice.

Definition 2.8 *Given $x_1, x_2 \in X$, with $x_1 \sqsubseteq x_2$, the* closed interval *$[x_1, x_2]$ is defined by $[x_1, x_2] \doteq \{z : x_1 \sqsubseteq z \sqsubseteq x_2\}$.*

We consider the empty set $\emptyset$ to be a closed interval, the so called *empty interval.* This can also be denoted as $[x_1, x_2]$ with any x_1, x_2 such that $x_1 \not\sqsubseteq x_2$. Denote by $\mathbf{I}$ the collection of (crisp) closed intervals of X (including the empty interval). The structure $(\mathbf{I}, \subseteq)$ is an ordered set. In fact it is a lattice, as the following propositions show (proofs are omitted for brevity; they follow from the fact that being a closed interval is a *closure property* on $(\mathbf{I}, \subseteq)$ [4]).

Proposition 2.9 *Given any nonempty interval $A = [a_1, a_2] \subseteq X$, we have $a_1 = \sqcap A$, $a_2 = \sqcup A$.*

Proposition 2.10 *Given any family of closed intervals $\mathbf{J} \subseteq \mathbf{I}$ the set $\cap_{[a_1,a_2]\in\mathbf{J}}[a_1, a_2]$ is a closed interval; more specifically, we have*

$$\cap_{[a_1,a_2]\in\mathbf{J}}[a_1, a_2] = [\sqcup_{[a_1,a_2]\in\mathbf{J}}a_1, \sqcap_{[a_1,a_2]\in\mathbf{J}}a_2]$$

and this is the largest closed interval contained in every member of $\mathbf{J}$.

Definition 2.11 *Given $A, B \in \mathbf{I}$, define $\mathbf{S}(A, B) \doteq \{C\colon C \in \mathbf{I},\ A \subseteq C, B \subseteq C\}$. Then we define*

$$A \dot{\cup} B \doteq \cap_{C\in\mathbf{S}(A,B)}C.$$

Proposition 2.12 *The structure $(\mathbf{I}, \subseteq, \dot{\cup}, \cap)$ is a lattice with respect to the $\subseteq$ order (i.e. set theoretic inclusion). Given any intervals $A = [a_1, a_2] \in \mathbf{I}$, $B = [b_1, b_2] \in \mathbf{I}$, $\sup(A, B) = A\dot{\cup}B = [a_1 \sqcap b_1, a_2 \sqcup b_2]$, $\inf(A, B) = A\cap B = [a_1 \sqcup b_1 , a_2 \sqcap b_2]$.*

Remark. In other words, given any intervals $A = [a_1, a_2]$, $B = [b_1, b_2]$, $[a_1 \sqcap b_1, a_2 \sqcup b_2]$ is the smallest closed interval which contains both A and B and $[a_1 \sqcup b_1 , a_2 \sqcap b_2]$ is the largest closed interval contained in both A and B.

We define *fuzzy sublattices* and *fuzzy convex sublattices* in terms of their p-cuts; this is different from, but equivalent to Ajmal's approach [1].

Definition 2.13 *We say $M : X \to L$ is a* fuzzy sublattice *of $(X, \sqsubseteq)$ iff $\forall p \in L$ the set M_p is a sublattice of $(X, \sqsubseteq)$.*

Definition 2.14 *We say $M : X \to L$ is a* fuzzy convex sublattice *of $(X, \sqsubseteq)$ iff $\forall p \in L$ the set M_p is a convex sublattice of $(X, \sqsubseteq)$; (i.e. $\forall p \in L, \forall x, y \in M_p$ we have $[x \sqcap y, x \sqcup y] \subseteq M_p$).*

Proposition 2.15 *$M : X \to L$ is a fuzzy sublattice of $(X, \sqsubseteq)$ iff*

$$\forall x, y \in X : \quad M(x \sqcap y) \wedge M(x \sqcup y) \geq M(x) \wedge M(y).$$

Proof. See [15]. ■

Proposition 2.16 *Let $M : X \to L$ be a fuzzy sublattice of $(X, \sqsubseteq)$. It is a fuzzy convex sublattice of $(X, \sqsubseteq)$ iff*

$$\forall x, y \in X, \forall z \in [x \sqcap y, x \sqcup y] : \quad M(z) \geq M(x \sqcap y) \wedge M(x \sqcup y) = M(x) \wedge M(y). \tag{1}$$

Proof. (i) Assume M is a fuzzy convex sublattice. Choose any $x, y \in X$. Set $p_1 = M(x \sqcap y)$, $p_2 = M(x \sqcup y)$; then $x \sqcap y$, $x \sqcup y \in M_{p_1 \wedge p_2}$. Take any $z \in [x \sqcap y, x \sqcup y]$. Since M is a fuzzy convex sublattice: $z \in M_{p_1 \wedge p_2} \Rightarrow M(z) \geq p_1 \wedge p_2 = M(x \sqcap y) \wedge M(x \sqcup y)$. Since $x, y \in [x \sqcap y, x \sqcup y]$ we have $M(x) \geq M(x \sqcap y) \wedge M(x \sqcup y)$, $M(y) \geq M(x \sqcap y) \wedge M(x \sqcup y)$; and so $M(x) \wedge M(y) \geq M(x \sqcap y) \wedge M(x \sqcup y)$. On the other hand, since M is a fuzzy sublattice, from Proposition 2.15 we have $M(x \sqcap y) \wedge M(x \sqcup y) \geq M(x) \wedge M(y)$. Hence $M(x \sqcap y) \wedge M(x \sqcup y) = M(x) \wedge M(y)$.

(ii) Conversely, assume (1) holds. Take any $p \in L$. If M_p is empty, then it is a convex sublattice. If M_p is not empty, take any $x, y \in M_p$. Set $p_1 = M(x)$, $p_2 = M(y)$. We have $x \in M_p \Rightarrow p_1 = M(x) \geq p$, $y \in M_p \Rightarrow p_2 = M(y) \geq p$. From (1) we have $M(x \sqcap y) \geq M(x) \wedge M(y) = p_1 \wedge p_2 \geq p \Rightarrow x \sqcap y \in M_p$. Similarly $x \sqcup y \in M_p$ and so M_p is a sublattice. Set $q_1 = M(x \sqcap y)$, $q_2 = M(x \sqcup y)$. Now take any $z \in [x \sqcap y, x \sqcup y]$. From (1) we have $M(z) \geq q_1 \wedge q_2 = p_1 \wedge p_2 \geq p \Rightarrow z \in M_p$. Hence M_p is a convex sublattice for all $p \in L$, i.e. M is a fuzzy convex sublattice. ∎

3 The Lattice of Fuzzy Intervals

We now introduce *fuzzy intervals*.

Definition 3.1 *We say $M : X \to L$ is a* fuzzy interval *of $(X, \sqsubseteq)$ iff*

$$\forall p \in L : \quad M_p \text{ is a closed interval of } (X, \leq).$$

The collection all fuzzy intervals will be denoted by $\widetilde{\mathbf{I}}(X, L)$ or simply by $\widetilde{\mathbf{I}}$.

The following proposition will be often used in the sequel. It states that an arbitrary intersection of fuzzy intervals yields a fuzzy interval.

Proposition 3.2 *For all $\widetilde{\mathbf{J}} \subseteq \widetilde{\mathbf{I}}$ we have: $\wedge_{M \in \widetilde{\mathbf{J}}} M \in \widetilde{\mathbf{I}}$*

Proof. Choose any $\widetilde{\mathbf{J}} \subseteq \widetilde{\mathbf{I}} \subseteq \mathbf{F}$. The fuzzy set $\wedge_{M \in \widetilde{\mathbf{J}}} M$ is well defined, in view of the fact that $(\mathbf{F}, \leq, \vee, \wedge)$ is a complete lattice. Choose any $p \in L$. It is easy to show that $(\wedge_{M \in \widetilde{\mathbf{J}}} M)_p = \cap_{M \in \widetilde{\mathbf{J}}} M_p$. Then for every $M \in \widetilde{\mathbf{J}}$, the cut M_p will be a closed interval (perhaps the empty interval). From Proposition 2.10, an arbitrary intersection of closed intervals yields a closed interval. Hence, for every $p \in L$ the set $(\wedge_{M \in \widetilde{\mathbf{J}}} M)_p$ is a closed interval, i.e. $\wedge_{M \in \widetilde{\mathbf{J}}} M$ is a fuzzy interval. ∎

Since $\widetilde{\mathbf{I}} \subseteq \mathbf{F}$, it follows that $(\widetilde{\mathbf{I}}, \leq)$ is an ordered set. We now establish (using Proposition 3.2) that $(\widetilde{\mathbf{I}}, \leq)$ is a lattice.

Definition 3.3 *For all $M, N \in \widetilde{\mathbf{I}}$ we define $M \dot{\vee} N$ as follows. We define $\widetilde{\mathbf{S}}(M, N) \doteq \{A : A \in \widetilde{\mathbf{I}}$, $M \leq A, N \leq A\}$ and then define*

$$M \dot{\vee} N \doteq \wedge_{A \in \widetilde{\mathbf{S}}(M,N)} A.$$

Proposition 3.4 $(\widetilde{\mathbf{I}}, \leq, \dot{\vee}, \wedge)$ *is a complete lattice.*

Proof. (i) $M \wedge N$ is the infimum in $\mathbf{F}$ of M and N. From Proposition 3.2 we have $M \wedge N \in \widetilde{\mathbf{I}}$, hence $M \wedge N$ is also the infimum of M and N in $\widetilde{\mathbf{I}}$.

(ii) For all $A \in \widetilde{\mathbf{S}}(M, N)$ we have $M \leq A$ and so $M \leq \wedge_{A \in \widetilde{\mathbf{S}}(M,N)} A = M \dot{\vee} N$; similarly $N \leq M \dot{\vee} N$. Furthermore, if there is some $B \in \widetilde{\mathbf{I}}$ such that $M \leq B$, $N \leq B$, then $B \in \widetilde{\mathbf{S}}(M, N)$. Hence $M \dot{\vee} N =$

$\wedge_{A\in\widetilde{\mathbf{S}}(M,N)}A \leq B$. Finally, since $\widetilde{\mathbf{S}}(M,N) \subseteq \widetilde{\mathbf{I}}$, we have $M \dot{\vee} N = \wedge_{A\in\widetilde{\mathbf{S}}(M,N)}A \in \widetilde{\mathbf{I}}$. Hence $M \dot{\vee} N$ is the supremum in $\widetilde{\mathbf{I}}$ of M and N.

(iii) To establish completeness of $(\widetilde{\mathbf{I}}, \leq, \dot{\vee}, \wedge)$ we must show that any $\widetilde{\mathbf{J}} \subseteq \widetilde{\mathbf{I}}$ has an infimum and a supremum in $\widetilde{\mathbf{I}}$. We have already remarked (Proposition 3.2) that, for any $\widetilde{\mathbf{J}} \subseteq \widetilde{\mathbf{I}}$, the set $\wedge_{M\in\widetilde{\mathbf{J}}}M$ is a well defined fuzzy interval. Since $\wedge\widetilde{\mathbf{J}} = \wedge_{M\in\widetilde{\mathbf{J}}}M$ is the infimum of $\widetilde{\mathbf{J}}$ in $\mathbf{F}$, it will also be the infimum of $\widetilde{\mathbf{J}}$ in $\widetilde{\mathbf{I}} \subseteq \mathbf{F}$. Regarding the supremum, we must define appropriately $\dot{\vee}\widetilde{\mathbf{J}}$. Define a set $\widetilde{\mathbf{S}}(\widetilde{\mathbf{J}}) = \{A \in \widetilde{\mathbf{I}} : \forall M \in \widetilde{\mathbf{J}}$ we have $M \leq A\}$. Define $\dot{\vee}\widetilde{\mathbf{J}} \doteq \wedge_{A\in\widetilde{\mathbf{S}}(\widetilde{\mathbf{J}})}A$. Then $\dot{\vee}\widetilde{\mathbf{J}} \in \widetilde{\mathbf{I}}$ (as an intersection of fuzzy intervals), and it is easy to show that: $\forall M \in \widetilde{\mathbf{J}}$ we have $M \leq \dot{\vee}\widetilde{\mathbf{J}}$, $\forall A \in \widetilde{\mathbf{S}}(\widetilde{\mathbf{J}})$ we have $\dot{\vee}\widetilde{\mathbf{J}} \leq A$. Hence $\dot{\vee}\widetilde{\mathbf{J}}$ is the supremum of $\widetilde{\mathbf{J}}$ and completeness has been established. ■

The following propositions establish some properties of fuzzy intervals.

Definition 3.5 *For every fuzzy set M we define $L_M \doteq \{p : M_p \neq \emptyset\}$.*

Proposition 3.6 *(i) Let M be a fuzzy convex sublattice. If we have*

$$\forall p \in L_M : M(\sqcap M_p) \geq \wedge_{x\in M_p}M(x), \quad M(\sqcup M_p) \geq \wedge_{x\in M_p}M(x), \tag{2}$$

then M is a fuzzy interval.

(ii) If M is a fuzzy interval, then it is a fuzzy convex sublattice and we have

$$\forall p \in L_M : M(\sqcap M_p) \geq \wedge_{x\in M_p}M(x), \quad M(\sqcup M_p) \geq \wedge_{x\in M_p}M(x).$$

Proof. (i) Assume (2) holds. Choose any $p \in L_M$. Now, by completenes of $(X, \sqsubseteq)$, $\sqcap M_p$ and $\sqcup M_p$ exist. Clearly $M_p \subseteq [\sqcap M_p, \sqcup M_p]$. On the other hand, from (2), $M(\sqcap M_p) \geq \wedge_{x\in M_p}M(x) \geq p \Rightarrow \sqcap M_p \in M_p$, i.e. M_p contains its infimum. Similarly $M(\sqcup M_p) \geq \wedge_{x\in M_p}M(x) \geq p \Rightarrow \sqcup M_p \in M_p$. Since M_p is a convex sublattice and $\sqcap M_p, \sqcup M_p \in M_p$, it follows that $[\sqcap M_p, \sqcup M_p] \subseteq M_p$. Hence for all $p \in L_M$ we have that $M_p = [\sqcap M_p, \sqcup M_p]$. Further, for all $p \in L - L_M$, M_p is the empty set, which is considered a closed interval. Hence for all $p \in L$ the set M_p is a closed interval, i.e. M is a fuzzy interval.

(ii) If M is a fuzzy interval then for all $p \in L_M$ we have $M_p = [\sqcap M_p, \sqcup M_p]$, which is a closed interval and *a fortiori* a convex sublattice. Hence M is a fuzzy convex sublattice. Furthermore, $M_p = [\sqcap M_p, \sqcup M_p] \Rightarrow \sqcap M_p \in M_p \Rightarrow M(\sqcap M_p) \geq \wedge_{x\in M_p}M(x)$. Similarly, $\sqcup M_p \in M_p \Rightarrow M(\sqcup M_p) \geq \wedge_{x\in M_p}M(x)$. ■

Corollary 3.7 *If M is a fuzzy interval, then $\forall p \in L_M$ we have $M(\sqcap M_p) \wedge M(\sqcup M_p) = \wedge_{x\in M_p}M(x)$.*

Corollary 3.8 *Let X be finite. Then every fuzzy convex sublattice is a fuzzy interval and conversely.*

Proposition 3.9 *If M is a fuzzy interval, then $\forall p \in L_M$ we have $M_p = M_{p_1\wedge p_2}$, where $p_1 = M(\sqcap M_p)$, $p_2 = M(\sqcup M_p)$.*

Proof. Choose any $p \in L_M$. Since M is a fuzzy interval, we have $M_p = [\sqcap M_p, \sqcup M_p]$. Set $p_1 = M(\sqcap M_p) \geq p$, $p_2 = M(\sqcup M_p) \geq p$. Then $M(\sqcap M_p) = p_1 \geq p_1 \wedge p_2$ and so $\sqcap M_p \in M_{p_1\wedge p_2}$. Similarly $\sqcup M_p \in M_{p_1\wedge p_2}$. Since M is a fuzzy interval (and so a fuzzy convex sublattice) it follows that $[\sqcap M_p, \sqcup M_p] \subseteq M_{p_1\wedge p_2}$. On the other hand $p_1 \wedge p_2 \geq p \Rightarrow M_{p_1\wedge p_2} \subseteq M_p = [\sqcap M_p, \sqcup M_p]$. Hence $M_{p_1\wedge p_2} = M_p$. ■

4 Distributivity

In all of this section we assume $(X, \sqsubseteq, \sqcup, \sqcap)$ to be *completely distributive* according to the following definition.

Definition 4.1 *The lattice* $(X, \sqsubseteq, \sqcup, \sqcap)$ *is said to be* completely distributive*, iff for every set* $Y \subseteq X$ *we have* $x \sqcup (\sqcap_{y \in Y} y) = \sqcap_{y \in Y}(x \sqcup y)$, $\qquad x \sqcap (\sqcup_{y \in Y} y) = \sqcup_{y \in Y}(x \sqcap y)$.

Let M, N be fuzzy intervals. Our first task is to establish some properties of the cuts $(M \wedge N)_p$ and $(M \dot{\vee} N)_p$. From Proposition 3.4 we see that $M \wedge N$ and $M \dot{\vee} N$ are fuzzy intervals; hence $\forall p \in L$ the cuts $(M \wedge N)_p$ and $(M \dot{\vee} N)_p$ are (crisp) closed intervals.

Definition 4.2 *For all* $M, N \in \widetilde{\mathbf{I}}$ *and for all* $p \in L$ *we define* $C_p(M, N) = M_p \cap N_p$.

Proposition 4.3 *For all* $M, N \in \widetilde{\mathbf{I}}$ *and for all* $p \in L$ *we have:* $(M \wedge N)_p = C_p(M, N)$.

Proof. Take any $M, N \in \widetilde{\mathbf{I}}$, any $p \in L$. We have $x \in (M \wedge N)_p \Leftrightarrow (M \wedge N)(x) \geq p \Leftrightarrow M(x) \wedge N(x) \geq p \Leftrightarrow (M(x) \geq p$ and $N(x) \geq p) \Leftrightarrow (x \in M_p$ and $x \in N_p) \Leftrightarrow x \in M_p \cap N_p = C_p(M, N)$. ■

Proposition 4.4 *Take any* $M, N \in \widetilde{\mathbf{I}}$. *We have:*
(i) $\forall p, q \in L : p \leq q \Rightarrow C_q(M, N) \subseteq C_p(M, N)$,
(ii) $\forall P \subseteq L$: $\cap_{p \in P} C_p(M, N) = C_{\vee P}(M, N)$.
(iii) $C_0(M, N) = X$.

Proof. These properties follow from the fact that for all $p \in L$ we have $C_p(M, N) = (M \wedge N)_p$, i.e. the family $\{C_p(M, N)\}_{p \in L}$ is a family of cuts. ■

Hence we have characterized the cuts of $M \wedge N$ in terms of the cuts of M and N. We will now do the same for the cuts of $M \dot{\vee} N$. However, before proceeding we need some auxiliary definitions and propositions.

Definition 4.5 *For every* $M \in \widetilde{\mathbf{I}}$, *we define the functions* $\underline{M} : L \to X$, $\overline{M} : L \to X$ *as follows. For* $p \in L_M$, $\underline{M}(p) \doteq \sqcap M_p$, $\overline{M}(p) \doteq \sqcup M_p$; *for* $p \in L - L_M$, $\underline{M}(p) \doteq \sqcup X$, $\overline{M}(p) \doteq \sqcap X$.

Remark. Hence we can write $M_p = [\underline{M}(p), \overline{M}(p)]$ for *every* $p \in L$. Because: if $p \in L_M$, then $M_p = [\sqcap M_p, \sqcup M_p] = [\underline{M}(p), \overline{M}(p)]$; if $p \in L - L_M$, then $M_p = \emptyset = [\sqcup X, \sqcap X] = [\underline{M}(p), \overline{M}(p)]$.

Proposition 4.6 *Take any* $M \in \widetilde{\mathbf{I}}$ *and for all* $p \in L$ *set* $M_p = [\underline{M}(p), \overline{M}(p)]$. *Then*
(i) $\forall p, q \in L : p \leq q \Rightarrow \left(\underline{M}(p) \sqsubseteq \underline{M}(q), \overline{M}(p) \sqsupseteq \overline{M}(q)\right)$.
(ii) $\forall P \subseteq L : \sqcup_{p \in P} \underline{M}(p) = \underline{M}(\vee P)$, $\sqcap_{p \in P} \overline{M}(p) = \overline{M}(\vee P)$.

Proof. (i) Since $\{M_p\}_{p \in P}$ are cuts, from Prop.2.6.(i) we have: $p \leq q \Rightarrow M_q \subseteq M_p \Rightarrow [\underline{M}(q), \overline{M}(q)] \subseteq [\underline{M}(p), \overline{M}(p)] \Rightarrow (\underline{M}(p) \leq \underline{M}(q), \overline{M}(p) \geq \overline{M}(q))$. Note in particular that: if $q \notin L_M$, then $\underline{M}(p) \sqsubseteq \underline{M}(q) = \sqcup X$ and $\overline{M}(p) \sqsupseteq \overline{M}(q) = \sqcap X$.

(ii) Since $\{M_p\}_{p \in P}$ are cuts, from Prop.2.6.(ii) we have: $\cap_{p \in P} M_p = M_{\vee P}$. But $M_{\vee P} = [\underline{M}(\vee P), \overline{M}(\vee P)]$ and (Proposition 2.10) $\cap_{p \in P} M_p = [\sqcup_{p \in P} \underline{M}(p), \sqcap_{p \in P} \overline{M}(p)]$ which yields the required result. Note in particular that: if there exists some $q \in P$ such that $q \in L - L_M$, then $M_q = \emptyset$, $\cap_{p \in P} M_p = \emptyset$, and $M_{\vee P} = \emptyset = [\underline{M}(\vee P), \overline{M}(\vee P)]$ with $\underline{M}(\vee P) = \sqcup X$, $\overline{M}(\vee P) = \sqcap X$. Also, in this case $\underline{M}(q) = \sqcup X$, $\sqcup_{p \in P} \underline{M}(p) = \sqcup X$, $\overline{M}(q) = \sqcap X$, $\sqcap_{p \in P} \underline{M}(p) = \sqcap X$. ■

Proposition 4.7 *(i) Take any* $P \subseteq L$ *and any functions* $F : L \to X$, $G : L \to X$ *which satisfy*

$$p \leq q \Rightarrow F(p) \sqsubseteq F(q), \qquad \sqcup_{p\in P} F(p) = F(\vee P),$$
$$p \leq q \Rightarrow G(p) \sqsubseteq G(q), \qquad \sqcup_{p\in P} G(p) = G(\vee P).$$

Then $\sqcup_{p\in P} (F(p) \sqcap G(p)) = F(\vee P) \sqcap G(\vee P)$.

(ii) Take any $P \subseteq L$ *and any functions* $F : L \to X$, $G : L \to X$ *which satisfy*

$$p \leq q \Rightarrow F(p) \sqsupseteq F(q), \qquad \sqcap_{p\in P} F(p) = F(\vee P),$$
$$p \leq q \Rightarrow G(p) \sqsupseteq G(q), \qquad \sqcap_{p\in P} G(p) = G(\vee P).$$

Then $\sqcap_{p\in P} (F(p) \sqcup G(p)) = F(\vee P) \sqcup G(\vee P)$.

Proof. For (i), take any $p \in P$. Then $F(p) \sqcap G(p) \sqsubseteq F(p)$. Hence $\sqcup_{p\in P} (F(p) \sqcap G(p)) \sqsubseteq \sqcup_{p\in P} F(p)$ $= F(\vee P)$. Similarly $\sqcup_{p\in P} (F(p) \sqcap G(p)) \sqsubseteq \sqcup_{p\in P} G(p) = G(\vee P)$. It follows that

$$\sqcup_{p\in P} (F(p) \sqcap G(p)) \sqsubseteq F(\vee P) \sqcap G(\vee P). \tag{3}$$

On the other hand, using complete distributivity, we have $\sqcup_{p\in P, q\in P} (F(p) \sqcap G(q)) = \sqcup_{p\in P} (F(p) \sqcap (\sqcup_{q\in P} G(q)))$ $= \sqcup_{p\in P} (F(p) \sqcap G(\vee P)) = (\sqcup_{p\in P} F(p)) \sqcap G(\vee P) = F(\vee P) \sqcap G(\vee P)$. In short

$$F(\vee P) \sqcap G(\vee P) = \sqcup_{p\in P, q\in P} (F(p) \sqcap G(q)) \tag{4}$$

Finally, since $(L, \leq)$ is totally ordered, P is a sublattice of $(L, \leq)$; so for any $p, q \in P$ we have $p \vee q \in P$. Then $(p \leq p \vee q,\ q \leq p \vee q) \Rightarrow F(p) \sqcap G(q) \sqsubseteq F(p \vee q) \sqcap G(p \vee q)$. So $\sqcup_{p\in P, q\in P} (F(p) \sqcap G(q)) \sqsubseteq$ $\sqcup_{p\in P, q\in P} (F(p \vee q) \sqcap G(p \vee q)) \sqsubseteq \sqcup_{r\in P} (F(r) \sqcap G(r))$. Hence

$$\sqcup_{p\in P, q\in P} (F(p) \sqcap G(q)) \sqsubseteq \sqcup_{p\in P} (F(p) \sqcap G(p)) \tag{5}$$

From (3), (4), (5) follows that $\sqcup_{p\in P} (F(p) \sqcap G(p)) = F(\vee P) \sqcap G(\vee P)$ and (i) has been proved; (ii) is proved dually. ∎

Now we return to the cuts of $M \dot{\vee} N$.

Definition 4.8 *For all* $M, N \in \widetilde{\mathbf{I}}$ *and for all* $p \in L$ *we define* $D_p(M, N) = M_p \dot{\cup} N_p$.

Proposition 4.9 *Take any* $M, N \in \widetilde{\mathbf{I}}$ *. We have*

(i) $\forall p, q \in L$*:* $p \leq q \Rightarrow D_q(M, N) \subseteq D_p(M, N)$,

(ii) $\forall P \subseteq L : \cap_{p\in P} D_p(M, N) = D_{\vee P}(M, N)$.

(iii) $D_0(M, N) = X$.

Proof. (i) Assume $p \leq q$. Then $(M_q \subseteq M_p, N_q \subseteq N_p) \Rightarrow M_q \dot{\cup} N_q \subseteq M_p \dot{\cup} N_p \Rightarrow D_q(M, N)$ $\subseteq D_p(M, N)$.

(ii) Take any $P \subseteq L$ and any $p \in P$. We have $D_p(M, N) = [\underline{M}(p) \sqcap \underline{N}(p), \overline{M}(p) \sqcup \overline{N}(p)]$, hence

$$\cap_{p\in P} D_p(M, N) = [\sqcup_{p\in P} (\underline{M}(p) \sqcap \underline{N}(p)), \sqcap_{p\in P} (\overline{M}(p) \sqcup \overline{N}(p))]. \tag{6}$$

Also

$$D_{\vee P}(M, N) = [\underline{M}(\vee P) \sqcap \underline{N}(\vee P), \overline{M}(\vee P) \sqcup \overline{N}(\vee P)]. \tag{7}$$

Use Proposition 4.7.(i) with $F(p) = \underline{M}(p)$ and $G(p) = \underline{N}(p)$. Then

$$\sqcup_{p\in P}(\underline{M}(p) \sqcap \underline{N}(p)) = \underline{M}(\vee P) \sqcap \underline{N}(\vee P). \tag{8}$$

Use Proposition 4.7.(ii) with $F(p) = \overline{M}(p)$ and $G(p) = \overline{N}(p)$. Then

$$\sqcap_{p\in P}(\overline{M}(p) \sqcup \overline{N}(p)) = \overline{M}(\vee P) \sqcup \overline{N}(\vee P). \tag{9}$$

Eqs.(6–9) yield the required result.

(iii) $D_0(M,N) = M_0 \dot\cup N_0 = X \dot\cup X = X$. ■

Proposition 4.10 *For all $M, N \in \widetilde{\mathbf{I}}$ and for all $p \in L$ we have: $(M \dot\vee N)_p = D_p(M,N)$*

Proof. From Proposition 4.9 follows that $\{D_p(M,N)\}_{p\in L}$ is a family of cuts. Hence, if we define a fuzzy set $(M \veebar N)$ by setting

$$\forall x \in X : (M \veebar N)(x) \doteq \vee \{p : x \in D_p(M,N)\}$$

then $\forall p \in L$ we will have $(M \veebar N)_p = D_p(M,N)$ (Proposition 2.7). From this also follows that $(M \veebar N)$ is a fuzzy interval (since $\forall p \in L$ we have $(M \veebar N)_p = D_p(M,N) = M_p \dot\cup N_p$). Now choose any $p \in L$; we will show that $(M \dot\vee N)_p = (M \veebar N)_p$.

First, $(M \dot\vee N)_p$ is a (crisp) closed interval. Also, $x \in M_p \Rightarrow (M \dot\vee N)(x) \geq M(x) \geq p \Rightarrow x \in (M \dot\vee N)_p$. So $M_p \subseteq (M \dot\vee N)_p$. Similarly $N_p \subseteq (M \dot\vee N)_p$. Hence $(M \dot\vee N)_p \in \mathbf{S}(M_p, N_p)$ which implies that $(M \veebar N)_p = D_p(M,N) = M_p \dot\cup N_p = \cap_{A\in\mathbf{S}(M_p,N_p)} A \subseteq (M \dot\vee N)_p$.

Second, choose any $x \in X$ and set $p = M(x)$. Then $x \in M_p \subseteq D_p(M,N) = (M \veebar N)_p$. Hence $(M \veebar N)(x) \geq p = M(x)$; similarly $(M \veebar N)(x) \geq N(x)$. Since $M \dot\vee N = \sup(M,N)$, it follows that $(M \veebar N)(x) \geq (M \dot\vee N)(x)$ and so $(M \veebar N)_p \supseteq (M \dot\vee N)_p$.

So we have $(M \veebar N)_p = (M \dot\vee N)_p$ which (Proposition 2.5) implies $M \veebar N = M \dot\vee N$. ■

Proposition 4.11 *$(\widetilde{\mathbf{I}}, \leq, \dot\vee, \wedge)$ is a distributive lattice.*

Proof. We must show that for any $A, B, C \in \widetilde{\mathbf{I}}$ we have $(A \dot\vee B) \wedge C = (A \wedge C) \dot\vee (B \wedge C)$ and $(A \wedge B) \dot\vee C = (A \dot\vee C) \wedge (B \dot\vee C)$. We will show this by showing equality of the p-cuts.

Indeed, choose any $p \in L$ and set $A_p = [a_1, a_2]$, $B_p = [b_1, b_2]$, $C_p = [c_1, c_2]$ (in case any of these intervals is empty, denote it by $[\sqcap X, \sqcup X]$). Now

$$\left((A \dot\vee B) \wedge C\right)_p = (A \dot\vee B)_p \cap C_p = (A_p \dot\cup B_p) \cap C_p =$$

$$([a_1, a_2] \dot\cup [b_1, b_2]) \cap [c_1, c_2] = [a_1 \sqcap b_1, a_2 \sqcup b_2] \cap [c_1, c_2] =$$

$$[(a_1 \sqcap b_1) \sqcup c_1, (a_2 \sqcup b_2) \sqcap c_2] = [(a_1 \sqcup c_1) \sqcap (b_1 \sqcup c_1), (a_2 \sqcap c_2) \sqcup (b_2 \sqcap c_2))] =$$

$$[a_1 \sqcup c_1, a_2 \sqcap c_2] \dot\cup [b_1 \sqcup c_1, b_2 \sqcap c_2] = ([a_1, a_2] \cap [c_1, c_2]) \dot\cup ([b_1, b_2] \cap [c_1, c_2]) =$$

$$(A_p \cap C_p) \dot\cup (B_p \cap C_p) = (A \wedge C)_p \dot\cup (B \wedge C)_p = \left((A \wedge C) \dot\vee (B \wedge C)\right)_p.$$

Since for all $p \in L$ we have $\left((A \dot\vee B) \wedge C\right)_p = \left((A \wedge C) \dot\vee (B \wedge C)\right)_p$, it follows that $(A \dot\vee B) \wedge C = (A \wedge C) \dot\vee (B \wedge C)$. Dually we show that $(A \wedge B) \dot\vee C = (A \dot\vee C) \wedge (B \dot\vee C)$. ■

5 Discussion

In this paper we have introduced fuzzy intervals and obtained some of their basic properties. The method we have used is rather standard in the study of fuzzy algebras – in particular we have obtained several properties of fuzzy intervals by studying their p-cuts. This method can be used to obtain further properties of fuzzy intervals.

In our analysis we have made several assumptions, the most prominent ones being that: (a) L is $[0,1]$ and (b) X is complete and completely distributive. To what extent can these assumptions be relaxed?

Regarding L, the analysis remains unchanged if $(L,\leq,\vee,\wedge)$ is simply a chain. But it does not seem obvious how to generalize our results to L-fuzzy lattices, because Proposition 4.7 requires that for every $P\subseteq L$, and for all $p,q\in P$, we have $p\vee q\in P$; for this to be true for arbitrary $P\subseteq L$, $(L,\leq)$ must be a chain.

The completeness of $(X,\sqsubseteq,\sqcup,\sqcap)$ is also essential. Obviously, if $(X,\sqsubseteq,\sqcup,\sqcap)$ is not complete, there is no guarantee that an infinite union of fuzzy intervals will be a fuzzy interval. Regarding complete distributivity, it has only been used in Section 4, but there it plays an essential role in the proof of Proposition 4.7. Let us note that in the important special case where X has finite cardinality, completeness is automatically satisfied and complete distributivity is equivalent to distributivity (which clearly is a minimum requirement for the lattice of fuzzy intervals to be distributive).

Finally, let us discuss briefly the important special case when $(X,\sqsubseteq,\sqcup,\sqcap)=(R,\leq,\vee,\wedge)$. In this case we obtain the "classical" notion of a fuzzy interval, i.e. a fuzzy set such that its p-cuts are closed intervals on the real line (compare [7, p.37, p.48]. It is worth noting that, taking $X=R^n$, the notion of a fuzzy convex sublattice also specializes to that of a "classical" convex fuzzy set [7, p.41]. Fuzzy intervals and convex fuzzy sets in this "classical" sense have been studied extensively. It appears worthwhile to study "classical" fuzzy intervals from the lattice theoretic point of view. Conversely, they can serve as a source of inspiration for generalizations (especially of convexity results) in the context of a general lattice $(X,\sqsubseteq,\sqcup,\sqcap)$.

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