# Searching the Nodes of a Graph: Theory and Algorithms 

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#### Abstract

One or more searchers must capture an invisible evader hiding in the nodes of a graph. We study this version of the graph search problem under additional restrictions, such as monotonicity and connectedness. We emphasize that we study node search, i.e., the capture of a node-located evader; this problem has so far received much less attention than edge search, i.e., the capture of an edge-located evader.

We show that in general graphs the problem of node search is easier than that of edge search, Namely, every edge clearing search is also node clearing, but the converse does not hold in general (however node search is NP-complete, just like edge search). Then we concentrate on the internal monotone connected (IMC) node search of trees and show that it is essentially equivalent to IMC edge search; hence Barriere's tree search algorithm [2], originally designed for edge search, can also be used for node search.

We return to IMC node search on general graphs and present (several variants of) a new algorithm: GSST (Guaranteed Search by Spanning Tree). GSST clears a graph G by performing all its clearing moves along a spanning tree $\mathbf{T}$ of $\mathbf{G}$. Because spanning trees can be generated and cleared very quickly, GSST can test a large number of spanning trees and find one which clears $\mathbf{G}$ with a small (though not necessarily minimal) number of searchers. We prove the existence of probabilistically complete variants of GSST (i.e., these variants are guaranteed to find a minimal IMC node clearing schedule if run for sufficiently long time). Our experiments also indicate that GSST can efficiently node-clear large graphs given only a small running time. An implementation of GSST (running on Windows and Linux computers) is also provided and made publicly available.


## 1 Introduction

In this paper we study a variant of graph search. Our motivation comes from applied robotics problems. As an introduction, consider the following two problems, which can both be modeled as graph searches.

1. An evader is hiding in a building and a team of one or more searchers are trying to capture him. The building is represented by a graph $\mathbf{G}$ : nodes are rooms and edges are doors between rooms. Both searchers and evader occupy the nodes of the graph and move from node to node by sliding along the edges. The evader is captured when he is located in the same node as a searcher. It is assumed that the evader has the following characteristics: (a) he wants to avoid capture, (b) he is invisible to the searchers (unless he is located in the same node as one of them), (c) he is always aware of the searchers' locations and (d) he is arbitrarily fast.
2. An evader and a team of searchers are located inside a cave. The cave is again represented by a graph G: edges are tunnels and nodes are the intersections of tunnels. The searchers move from node to node, sliding along the edges; capture takes place if a searcher slides along an edge occupied by the evader, or if the evader moves through a node occupied by a searcher. The previously mentioned properties of the evader are assumed to hold in this case as well.
[^0]The two problems are very similar but differ in one important respect. In the first problem the evader is assumed to reside in the nodes of the graph (the rooms); in the second problem he is assumed to reside in the edges (the tunnels). Accordingly, we will call the first problem node search and the second edge search. Edge- or node-location of the evader is an important aspect of the graph search problem and can be used to categorize the related literature. Historically, research has concentrated on the problem of "edge-located evader". To the best of our knowledge, very little work has been done on the problem of "node-located evader". Interestingly, the term "node search" has been used in the past to denote a version of the graph search for edge-located evader [25]. Hence, to avoid confusion, let us repeat that we use "node search" to denote the search for a node-located evader.

Another important aspect of the graph search problem is the "visibility" of the evader. Namely, we say that the evader is visible if the searchers always know his location; we say that the evader is invisible if the evaders may become aware of his location only when he is in the same node (or edge) as they.

Additional aspects of the graph search include whether it is internal (the searchers can only move along the edges of the graph) or not (a searcher can, in one step, move to any node of the graph - we call this "teleporting"), monotone (the evader can never return to a searched part of the graph) and connected (the searched parts form a connected subgraph). These terms will be defined rigorously in Section 2.

We now present a brief overview of the graph search literature. Early papers include [6] where the problem of cave searching was posed and [31] where the first mathematical study of the problem was presented. The problem was independently rediscovered a little later in [32]. The edge search problem was placed in the context of graph theory in [29]. The already mentioned "node search" variant (actually dealing with edge-located evader) appeared a little earlier in [25]. Another variant, called "mixed search" (and again dealing with edge-located evader) appeared in [5] and was further studied in [35, 38]. The study of connected edge search started relatively recently; see [2, 3, 11, 13, 14, 39 . There is very little published on "true" node search, i.e., the search for a node-located evader. There is a considerable literature on pursuit of visible node-located evaders, which we will not discuss, since visible evader search is outside the scope of the current paper; let us simply mention that an important early paper is [30]. Graph search is also related to several graph parameters, most notably pathwidth and vertex separation. These connections are discussed in many papers, e.g., in [4, 8, (9, 24, 25, 36]. Finally there are several reviews of the graph search literature: an old and deep one is [4], a more recent one is [1] and a very recent and very extensive one is [12].

The above mentioned papers adopt a theoretical point of view. While there is much discussion about graph search algorithms, we have found few actually implemented algorithms which can tackle relatively large graphs (with the exception of tree search algorithms). A much more applied line of research appears in the robotics literature, for example in [15, 16, 17, 27] and the book [28]. These papers present actual implementations of algorithms, as well as numerical and even physical experiments, but they provide little (if any) theoretical justification of their methods. In particular, the distinction between edge and node search often appears to be misunderstood. We attempt to clarify this distinction and also combine the robotics and graph theoretic points in two technical reports we have previously published [20, 23].

In the current paper we study graph search for an invisible, node-located evader. We also examine connections to the problem of invisible, edge-located evader. We do not deal with the case of visible evader. We are mainly interested in internal, monotone, connected (IMC) node search. This version of the problem is the one most relevant to robotics applications which, as already stated, provide our main motivation. In general, we try to provide a balanced combination of theory and implementation.

The main contribution of the paper is the introduction of (several variants of) the node-search algorithm GSST (Guaranteed Search by Spanning Tree). This algorithm is presented in detail, theoret-
ically motivated and analyzed and also practically evaluated by a number of numerical experiments 1 The main idea behind the algorithm is that every IMC node search generates a spanning tree of the graph; conversely, every spanning tree can be used to perform a node search. Trees can be searched much more efficiently than general graphs; GSST exploits this fact to quickly compute a large number of searches and then output the one which requires the least number of searchers ${ }^{2}$. Our experiments also indicate that GSST can efficiently node-clear large graphs in a small running time. From the theoretical point of view, we establish several results about node search and its relation to edge search. These results are used to motivate the GSST algorithm and also to prove that some variants of GSST are probabilistically complete (i.e., they are guaranteed to find a minimal IMC node clearing schedule if given sufficient time).

The paper is organized as follows. Preliminary concepts and notation are introduced in Section 2, edge search and node search are compared in Section 3; IMC node and edge search on trees is studied in Section 4 IMC node search on general graphs is studied in Section 5 where the GSST search algorithm is also introduced; the algorithm is evaluated by numerical experiments in Section 6, conclusions and future research directions appear in Section 7. Some edge search results by Barriere et al. [2, 3] are presented in Appendix A the connection of node search to mixed edge search is discussed in Appendix B) connections to pathwidth are discussed in Appendix C; an implementation of GSST as an actual executable program (which runs on Windows and Linux computers) is discussed in Appendix D.

## 2 Preliminaries

### 2.1 Basic concepts and notation

We denote graphs by boldface letters, e.g. $\mathbf{G}=(V, E)$ where $V$ is the node set and $E$ is the edge set. We will always label the nodes of $\mathbf{G}$ as $V=\{1,2, \ldots, N\}$ (hence the graph contains $N$ nodes, i.e. $|V|=N$ ). Edges are denoted as $\{u, v\}$ where $u, v \in V$; often we will write $u v$ instead, for simplicity, but note that $v u$ and $u v$ are the same edge (i.e. we study undirected graphs). We only consider connected graphs without loops or multiple edges. Also, to avoid some trivial cases, we will always consider graphs with at least one edge (and at least two nodes).

Nodes $u, v$ are said to be neighbors iff $u v \in E$.
Given a node sequence $u_{1} u_{2} \ldots u_{L}$, where $u_{i} u_{i+1} \in E$ for $i=1,2, \ldots, L-1$, we say that $u_{1} u_{2} \ldots u_{L}$ is

1. a path iff $u_{i} \neq u_{j}$ for $i, j \in\{1,2, \ldots, L\}$ and $i \neq j$;
2. a cycle iff $u_{i} \neq u_{j}$ for $i, j \in\{1,2, \ldots, L\}$ (and $i \neq j$ ) except that $u_{1}=u_{L}$.

A tree is a connected graph without any cycles. Equivalent definitions are: a tree is a graph for which there is a unique path between every pair of nodes; a tree is a connected graph with $N$ nodes and $N-1$ edges. The leaves of the tree are the nodes which have exactly one neighbor.

A rooted tree is a tree with a distinguished node $u_{0}$ called the root of the tree. Given a tree $\mathbf{T}=(V, E)$, we will denote the same tree rooted at $u_{0}$ by $\mathbf{T}_{u_{0}}=\left(V, E, u_{0}\right)$. Take any node $u_{L} \neq u_{0}$ and let $u_{0} u_{1} \ldots u_{L-1} u_{L}$ be the unique path from $u_{0}$ to $u_{L}$; then $u_{L-1}$ is the parent of $u_{L} ; u_{L}$ is the child of $u_{L-1}$. Given a node $x$, its children, their children and so on are the descendants of $x$. Given a rooted tree $\mathbf{T}_{x}=(V, E, x)$ and a node $y$, consider the node set

$$
V[y]=\{z: z \text { is } y \text { or a descendant of } y\}
$$

[^1]and the edge set
$$
E[y]=\{z u: z u \in E \text { and } z, u \in V[y]\}
$$
note that both $V[y]$ and $E[y]$ are determined by both $y$ and $x$. The graph $(V[y], E[y])$ is a tree (a sub-tree of $\left.\mathbf{T}_{x}\right)$. The rooted tree $(V[y], E[y], y)$ will be denoted by $\mathbf{T}_{x}[y]$.

A search schedule on the graph $\mathbf{G}=(V, E)$ is a sequence of ordered pairs of nodes:

$$
\mathbf{S}=\left((u(1), v(1)),(u(2), v(2)), \ldots,\left(u\left(t_{f i n}\right), v\left(t_{f i n}\right)\right)\right)
$$

subject to $\{u(t), v(t)\} \in E$ for $t=1,2, \ldots, t_{\text {fin }}$. We call $\mathbf{S}(t)=(u(t), v(t))$ the $t$-th move (or step) of the search schedule. The $t$-th move can be any one of the following

1. placing a searcher at node $v$ (in which case $u(t)=0$ and $v(t) \in V$ ) or
2. sliding a searcher from node $u(t) \in V$ to node $v(t) \in V$ or
3. removing a searcher from node $u$ (in which case $u(t) \in V$ and $v(t)=0$ ).

We will also use the more evocative notation $\mathbf{S}(t)=(u \rightarrow v)$, i.e. a searcher is moved from node $u$ to node $v$. Moves of the form: $0 \rightarrow v$ (a new searcher is introduced in the graph) and $u \rightarrow 0$ (a searcher is removed from the graph) involve the fictitious "source node" 0 (it is not an element of $V$ ), in which searchers are kept whenever they are not actively involved in the graph search; the evader does not have access to the source node.

Given a search $\mathbf{S}$, the number of searchers inside the graph at time $t$ will be denoted by $\operatorname{sn}(\mathbf{S}, t)$. The maximum number of searchers used by $\mathbf{S}$ will be denoted by $\overline{s n}(\mathbf{S})$, i.e.

$$
\overline{s n}(\mathbf{S})=\max _{t} \operatorname{sn}(\mathbf{S}, t)
$$

These numbers must not be confused with the search number of a graph, which will be defined in Section 2.2,

Finally, we say that a node $u$ is guarded at time $t$ iff a searcher is located at $u$ (at time $t$ ); otherwise we say $u$ is unguarded. A path $u_{1} u_{2} \ldots u_{L}$ is called n-unguarded (node unguarded) iff nodes $u_{i}(i=1,2, \ldots, L)$ are unguarded (at time $t$ ); otherwise it is called $n$-guarded. The path is called $e$ unguarded (edge unguarded) iff nodes $u_{i}(i=2,3, \ldots, L-1)$ are unguarded (at time $t$ ) and e-guarded otherwise. It is easy to see that

$$
\begin{aligned}
& u_{1} u_{2} \ldots u_{L} \text { is e-unguarded } \Rightarrow u_{2} u_{3} \ldots u_{L-1} \text { is n-unguarded, } \\
& u_{1} u_{2} \ldots u_{L} \text { is n-unguarded } \Rightarrow u_{1} u_{2} \ldots u_{L} \text { is e-unguarded. }
\end{aligned}
$$

The reason for the two different definitions is that the first pertains to node recontamination and the second to edge recontamination, as will be seen in Section 2.2 .

### 2.2 Node and edge search

We repeat the assumptions introduced in Section 1 regarding the evader. Namely, the evader wants to avoid capture, he is invisible to the searchers (unless located in the same node), he is always aware of the searchers' locations and arbitrarily fast. The net result of all these assumptions is that (in both node and edge search) the evader can (and will) always avoid capture if an escape route is available. Hence, from the searchers' point of view, we can think of graph search as a process of eliminating escape routes. This is expressed as follows:

1. a node / edge is dirty if it can possibly contain the evader and clear otherwise (e.g., a node occupied by a searcher is clear);
2. a previously clear node / edge can become dirty (e.g., when a previously guarded path between a clear and a dirty node becomes unguarded) - this is called recontamination;
3. graph search is the process of gradually decreasing the dirty set (of nodes or edges) until it becomes the empty set (i.e., the evader has no escape route left).

This is a worst case approach which essentially eliminates the evader from the graph search, introducing in his place the dirty set. We can think of node (edge) search as a one-player "node game" ("edge game"). In both these games the player controls all searchers, much like pieces in a game of chess. We now present the rules of the two games. For reasons which will be explained presently, we substitute the terms "clear" and "dirty" with the terms "n-clear" and "n-dirty" (in the node game) and "e-clear" and "e-dirty" (in the edge game).

## Rules of the Node Game

No At time $t=0$ all nodes are $n$-dirty and no searcher is located in the graph.
N1 At every time $t=1,2, \ldots$ the player performs one of the following moves:
N1a place a searcher on a node,
N1b remove a searcher from a node,
N1c slide a searcher along one edge.
N2 An n-dirty node becomes $n$-clear when occupied by a searcher.
N3 An n-clear node $u$ becomes n-dirty when it is connected to to an n-dirty node $v$ by an $n$-unguarded path.

N4 An edge is n-dirty if it is adjacent to an n-dirty node; otherwise it is clear.
N5 The game is concluded when all nodes (and consequently also all edges) are n-clear.

## Rules of the Edge Game

E0 At time $t=0$ all edges are $e$-dirty and no searcher is located in the graph.
E1 At every time $t=1,2, \ldots$ the player performs one of the following moves:
E1a place a searcher on a node,
E1b remove a searcher from a node,
E1c slide a searcher along one edge.
E2 An e-dirty edge becomes e-clear when a searcher slides along the edge.
E3 An e-clear edge $u v$ becomes e-dirty when it is connected to to a e-dirty edge $x y$ by an e-unguarded path.

E4 A node is e-dirty if it is unguarded and adjacent to an e-dirty edge; otherwise it is e-clear.
E5 The game is concluded when all edges (and consequently also all nodes) are e-clear.

Remark 2.1. The reason for using "n-clear" and "e-clear" (instead of simply "clear") is that an edge can be clear in the node game and dirty in the edge game. An example will illustrate this point. Consider the graph of Fig. 1 and the search schedule $0 \rightarrow 1,0 \rightarrow 1,1 \rightarrow 2,2 \rightarrow 4,4 \rightarrow 3$. If this schedule is executed in a node game, after the final move all nodes are n-clear and so all edges are also n -clear. But in an edge game, after the final move, edge $\{1,3\}$ has still not been traversed and hence it is still e-dirty. More generally, while "n-clear" and "e-clear" refer to similar physical situations, there is no a priori reason that their mathematical definitions are equivalent (in Section 4 we will show that they are equivalent provided certain conditions are satisfied).


Figure 1: A graph in which node clearing and edge clearing are not equivalent. For example, placing a searcher into node $0(0 \rightarrow 1)$, then placing a second searcher at 1 and sending him through $2,4,3$ ( $0 \rightarrow 1,1 \rightarrow 2,2 \rightarrow 4,4 \rightarrow 3$ ) is node clearing but not edge clearing.

Remark 2.2. Note that in the edge game clear edges are "recorded" at the completion of each move. This is important to note in the case that an edge is e-cleared and e-dirtied during the same move (i.e. for a single value of $t$ ). To clarify this, suppose that in the graph of Fig. 1 the following moves are executed: $0 \rightarrow 1$ at $t=1 ; 1 \rightarrow 2$ at $t=2$. In this case edge $\{1,2\}$ is first e-cleared (because traversed) and then e-dirtied, both during $t=2$. After completion of the second move, edge $\{1,2\}$ is $e$-dirty.

We now introduce more detailed notation regarding clear/dirty nodes/edges. For the node game we will use the following notation.

$$
\begin{aligned}
V_{N}^{C}(t) & =\{u \in V: \text { at time } t, u \text { is n-clear }\} ; \\
V_{N}^{D}(t) & =\{u \in V: \text { at time } t, u \text { is n-dirty }\} \\
E_{N}^{C}(t) & =\{u v \in E: \text { at time } t, u v \text { is n-clear }\} \\
E_{N}^{D}(t) & =\{u v \in E: \text { at time } t, u v \text { is n-dirty }\} \\
\mathbf{G}_{N}^{C}(t) & =\left(V_{N}^{C}(t), E_{N}^{C}(t)\right) \text { is the } n \text {-clear graph at time } t .
\end{aligned}
$$

We use exactly analogous notation for the edge game, but with an $E$ (rather than $N$ ) subscript: $V_{E}^{C}(t)$, $V_{E}^{D}(t), E_{E}^{C}(t), E_{E}^{D}(t), \mathbf{G}_{E}^{C}(t)=\left(V_{E}^{C}(t), E_{E}^{C}(t)\right)$. Obviously we have

$$
\begin{array}{lll}
V_{N}^{C}(t) \cup V_{N}^{D}(t)=V, & V_{N}^{C}(t) \cap V_{N}^{D}(t)=\emptyset, & E_{N}^{C}(t) \cup E_{N}^{D}(t)=E, \\
V_{E}^{C}(t) \cup V_{E}^{C}(t) \cap E_{N}^{D}(t)=\emptyset, \\
V_{E}^{C}(t) \cap V_{E}^{D}(t)=\emptyset, & E_{E}^{C}(t) \cup E_{E}^{D}(t)=E, & E_{E}^{C}(t) \cap E_{E}^{D}(t)=\emptyset .
\end{array}
$$

Remark 2.3. The n-clear / e-clear graph definitions are consistent, i.e. $\left(V_{N}^{C}(t), E_{N}^{C}(t)\right) /\left(V_{E}^{C}(t), E_{E}^{C}(t)\right)$ are graphs. More specifically:

1. in the node game $\mathbf{G}_{N}^{C}(t)$ consists of the n-clear nodes and all edges between such nodes (these are exactly the set $\left.E_{N}^{C}(t)\right)$;
2. in the edge game $\mathbf{G}_{E}^{C}(t)$ consists of the e-clear edges and none of these can be left "dangling", i.e. if $u v \in E_{E}^{C}(t)$ then $u, v \in V_{E}^{C}(t)$ (we will prove this as Lemma 3.1 in Section 3); in addition $V_{E}^{C}(t)$ may contain guarded nodes adjacent to e-dirty edges; such nodes may appear as disconnected components of $\mathbf{G}_{E}^{C}(t)$.

In the current paper we will concentrate on some restricted versions of the edge and node games. More specifically, the restrictions concern the allowable search schedules and they are the following.

1. We call a search schedule rooted iff searchers can be placed only into a single, prespecified node $u_{0}$, called the root of the search.
2. We call a search schedule internal iff searchers once placed into the graph (a) can only move along the edges and (b) are never removed from the graph. In other words, "teleporting" (the arbitrary movement of a searcher from one node to another, irrespective of the graph connectivity) is not allowed ${ }^{3}$
3. In the node game we call a search schedule $\mathbf{S}$ monotone iff the clear node set is never decreasing: $V_{N}^{C}(t) \subseteq V_{N}^{C}(t+1)$ for all $t$. In other words, once a node is n-cleared it never becomes n-dirty again. The definition is similar for the edge game; i.e., $\mathbf{S}$ is monotone in the edge game iff $E_{E}^{C}(t) \subseteq E_{E}^{C}(t+1)$ for all $t \underbrace{4}_{\square}$
4. In the node game (resp. edge game) we call a search schedule connected iff the clear graph $\mathbf{G}_{N}^{C}(t)$ (resp. $\left.\mathbf{G}_{E}^{C}(t)\right)$ is connected for $t=1,2, \ldots$.

In Section 3 we will study general, unrestricted graph search; in the remaining sections we will concentrate on internal monotone (IM), internal connected (IC) and, especially, internal monotone connected (IMC) node / edge search. Our focus originates in our interest in practical pursuit / evasion problems which arise in robotics. The nature of the problem dictates the restrictions on the search.

1. The internality constraint arises from the fact that a robot cannot "teleport"; it can only move through rooms and doors connecting these rooms; in the graph model of the physical situation this is reflected by the use of edge-sliding moves only.
2. Similarly, rootedness reflects the fact that usualy an environment has a single entrance through which robots must enter.
3. Connectedness and monotonicity are desirable (but not mandatory) characteristics of robotic search. For example, in a hostile environment it is preferrable that the cleared area consists of a single connected region, which is easier to guard and control than multiple regions. Similarly, if a high cost is associated with clearing an area, it is desirable that this area is cleared only once, i.e., that the search is monotone.
[^2]A given graph G with $N$ nodes can always be node-cleared using $N$ searchers: just place one searcher in every node. Usually $\mathbf{G}$ can be cleared with much fewer searchers; the node search number of $\mathbf{G}$, denoted by $s_{N}(\mathbf{G})$, is the minimum number of searchers required to node-clear $\mathbf{G}$. Introducing additional constraints on the search schedule we get additional search numbers:

$$
\begin{array}{ccl}
s_{N}^{i}(\mathbf{G}): & \text { min. nr. of searchers required to clear } \mathbf{G} \text { with } & \text { internal node search schedule; } \\
s_{N}^{i m}(\mathbf{G}): & -/ /- & \text { IM node search schedule; } \\
s_{N}^{i c}(\mathbf{G}): & -/ /- & \text { IC node search schedule; } \\
s_{N}^{i m c}(\mathbf{G}): & -/ /- & \text { IMC node search schedule }
\end{array}
$$

For the edge game we define the corresponding edge search numbers: $s_{E}(\mathbf{G}), s_{E}^{i}(\mathbf{G}), s_{E}^{i m}(\mathbf{G})$, $s_{E}^{i c}(\mathbf{G}), s_{E}^{i m c}(\mathbf{G})$.

In case of a search rooted at node $x$, we will modify the notation as follows: $s_{N}(\mathbf{G} ; x)$ is the minimum number of searchers required to node-clear $\mathbf{G}$ by a search rooted at $x$; similarly for $s_{E}(\mathbf{G} ; x)$, $s_{E}^{i}(\mathbf{G} ; x)$ and so on. If the graph $\mathbf{G}$ is a rooted tree $\mathbf{T}_{x}$, then we will assume it is always searched by rooted searches; hence $s_{E}\left(\mathbf{T}_{x}\right)=s_{E}(\mathbf{T} ; x), s_{E}^{i}\left(\mathbf{T}_{x}\right)=s_{E}^{i}(\mathbf{T} ; x)$ and so on.

A minimal node search schedule is one which clears $\mathbf{G}$ using $s_{N}(\mathbf{G})$ searchers; a minimal IM node search schedule is an IM schedule which clears $\mathbf{G}$ using $s_{N}^{i m}(\mathbf{G})$ searchers and similarly for all other types of either node or edge search.

## 3 Comparison of node and edge search

In this section we compare "unrestricted" edge and node search. In other words, we compare the outcome of a search schedule $\mathbf{S}$ when it is used first in a node game and then in edge game, both played on the same graph $\mathbf{G}$ and without requiring rootedness, internality, monotonicity or connectedness.

Our main result is Theorem 3.3 which, informally, says the following: every search schedule $\mathbf{S}$ gives at least as good results in the node game as in the edge game. More precisely, at every step $t$ of the search, the set of e-clear nodes (and edges) is a subset of the set of n-clear nodes (and edges). An immediate corollary is that an edge clearing search schedule is also node clearing; The converse does not hold, as we have already seen by the example of Remark 2.1.

To prove Theorem 3.3 we will need the following lemmas.
Lemma 3.1. Given a graph $\mathbf{G}$ and a search schedule $\mathbf{S}$, for $t=0,1,2, \ldots$ we have

$$
\begin{align*}
u \in V_{E}^{D}(t) & \Rightarrow\left(\forall u x \in E: u x \in E_{E}^{D}(t)\right)  \tag{1}\\
u v \in E_{E}^{C}(t) & \Rightarrow\left(u, v \in V_{E}^{C}(t)\right) \tag{2}
\end{align*}
$$

In other words: if a node $u$ is e-dirty, then all edges ux, adjacent to $u$, are also e-dirty; and if an edge $u v$ is e-clear, then nodes $u, v$ are e-clear.

Proof. For (1): since $u$ is e-dirty, it must be unguarded. We distinguish two cases.
(1.i) Suppose $u$ is adjacent to a single edge, call it $u v$. If $u v$ is e-clear then $u$ is not adjacent to any e-dirty edges and hence must be e-clear, which is contrary to the original assumption. Hence $u v$ is e-dirty.
(1.ii) On the other hand, if $u$ is adjacent to more than one edges, they cannot all be e-clear (then $u$ would also be e-clear); so at least one edge, call it $u w$, is e-dirty. Suppose there is also an e-clear edge, call it $u v$. But then the e-unguarded path $w u v$ connects e-dirty $w u$ to e-clear $u v$, which leads to contradiction. Hence all edges adjacent to $u$ are e-dirty.

For (2): suppose $u \in V_{E}^{D}(t)$. Then $u$ is unguarded and there exists some $u x \in E_{E}^{D}(t)$ and $x \neq v$. But the path $x u v$ is e-unguarded and so $u v \in E_{E}^{D}(t)$ which is a contradiction.

Lemma 3.2. If a node $u$ is e-cleared exactly at time $t$, then $u$ is entered by a searcher at $t$.
Proof. Suppose $u \in V_{E}^{D}(t-1) \cap V_{E}^{C}(t)$. This can happen in two ways. (i) $u$ was unguarded and e-dirty at $t-1$ and then entered by a searcher at $t$. This is exactly the case described by the theorem. (ii) The other way for $u$ to be cleared exactly at $t$ is if $u$ was unguarded and e-dirty at $t-1$ and in addition a single edge, call it $u x_{0}$, was e-dirty at $t-1$ and e-cleared at $t$ (while all other edges $u x$ were e-clear at $t-1$ and remained e-clear at $t$ ). For this to happen, the move at time $t$ is either $u \rightarrow x_{0}$ or $x_{0} \rightarrow u$. But $u \rightarrow x_{0}$ is not possible, because then $u$ would be guarded and e-clear at $t-1$. Hence the move at $t$ is $x_{0} \rightarrow u$ and $u$ was entered at $t$ (which shows that (ii) is a sub-case of (i)).

Now we are ready to prove the following.
Theorem 3.3. Given a graph $\mathbf{G}$ and a search schedule $\mathbf{S}$, we have

$$
\begin{equation*}
\text { for } t=0,1,2, \ldots: \quad V_{E}^{C}(t) \subseteq V_{N}^{C}(t) \quad \text { and } \quad E_{E}^{C}(t) \subseteq E_{N}^{C}(t) \tag{3}
\end{equation*}
$$

Proof. We will actually show the relationships (equivalent to (3) )

$$
V_{N}^{D}(t) \subseteq V_{E}^{D}(t), E_{N}^{D}(t) \subseteq E_{E}^{D}(t)
$$

The proof is by induction. At $s=0$ all nodes are dirty in both the edge game and the node game, hence $V_{N}^{D}(0)=V_{E}^{D}(0)=V$. Suppose $V_{N}^{D}(s) \subseteq V_{E}^{D}(s)$ for $s=1,2, \ldots, t$. Now we take $s=t+1$ and consider first the cleared and then the recontaminated nodes.
I. Cleared Nodes. By Lemma 3.2, a node becomes e-cleared exactly at $t+1$ only if entered at $t+1$. Hence at most one such node (call it $x_{0}$ ) exists and, since entered at $t+1$, it is also n-clear at $t+1$. In other words: $x_{0} \in V_{E}^{D}(t) \cap V_{E}^{C}(t+1) \Rightarrow x_{0} \in V_{N}^{C}(t+1)$.
II. Recontaminated Nodes. Denote by $A$ the set of nodes recontaminated (in the node game) exactly at $t+1$; i.e., $A=V_{N}^{C}(t) \cap V_{N}^{D}(t+1)$. By the definition of n-dirty node, every $u \in A$ must have a path (n-unguarded at $t+1$ ) to a node $v \in V_{N}^{D}(t+1)$. Furthermore, at least one $u \in A$ must have a path (n-unguarded at $t+1$ ) to a node $v \in V_{N}^{D}(t)$ (if this did not hold for any $u \in A$, then none of the nodes of $A$ would get recontaminated to begin with) and in fact this shows every $u \in A$ must have a path (n-unguarded at $t+1$ ) to a node $v \in V_{N}^{D}(t)$. So take any $u \in A$, then there exists a path $v_{1} v_{2} \ldots v_{K} u$ satisfying:

$$
\begin{align*}
& v_{1} \in V_{N}^{D}(t) \cap V_{N}^{D}(t+1)  \tag{4}\\
& \quad v_{1} v_{2} \ldots v_{K} u \text { is n-unguarded at } t+1 \tag{5}
\end{align*}
$$

Now we will show that $v_{1}$ must belong to $V_{E}^{D}(t) \cap V_{E}^{D}(t+1)$ as well. First, $v_{1} \in V_{N}^{D}(t) \subseteq V_{E}^{D}(t)$. Second, if $v_{1} \notin V_{E}^{D}(t+1)$ then $v_{1}$ was cleared exactly at $t+1$, which (by Lemma 3.2) means a searcher entered $v_{1}$ at $t+1$; but then $v_{1} \notin V_{N}^{D}(t+1)$ which is contrary to (4). Hence $v_{1} \in$ $V_{E}^{D}(t) \cap V_{E}^{D}(t+1)$. Since $v_{1}$ is $e$-dirty at time $t+1, v_{1} v_{2}$ is e-dirty at $t+1$ by Lemma 3.1. Also $v_{1} v_{2} \ldots v_{K} u$ is e-unguarded at $t+1$. Hence $u \in V_{E}^{D}(t+1)$. Since this is true for every $u \in A$, we conclude

$$
\begin{equation*}
A \subseteq V_{E}^{D}(t+1) \tag{6}
\end{equation*}
$$

If the move at $t+1$ was placing or sliding a searcher into node $x_{0}$, let $B=\left\{x_{0}\right\}$; else let $B=\emptyset$. Since $V_{N}^{D}(t) \subseteq V_{E}^{D}(t)$ we have

$$
\begin{equation*}
V_{N}^{D}(t)-B \subseteq V_{E}^{D}(t)-B \tag{7}
\end{equation*}
$$

Since the e-dirty nodes at $t+1$ include all the e-dirty nodes at $t$ except the node $x_{0}$ (which was e-cleared at $t+1$ ), we have

$$
\begin{equation*}
V_{E}^{D}(t)-B \subseteq V_{E}^{D}(t+1) \tag{8}
\end{equation*}
$$

And so we have

$$
\begin{equation*}
V_{N}^{D}(t)-B \subseteq V_{E}^{D}(t+1) . \tag{9}
\end{equation*}
$$

The n-dirty nodes at $t+1$ are exactly the n-dirty nodes at $t$, excluding nodes n-cleared at $t+1$, plus the nodes recontaminated at $t+1$. In other words

$$
\begin{equation*}
V_{N}^{D}(t+1)=\left(V_{N}^{D}(t)-B\right) \cup A \tag{10}
\end{equation*}
$$

Combining eqs. (6), (9) and (10) we get

$$
\begin{equation*}
V_{N}^{D}(t+1)=\left(V_{N}^{D}(t)-B\right) \cup A \subseteq V_{E}^{D}(t+1) . \tag{11}
\end{equation*}
$$

Hence by induction we get $V_{N}^{D}(t) \subseteq V_{E}^{D}(t) \quad$ (and $\left.V_{E}^{C}(t) \subseteq V_{N}^{C}(t)\right)$ for $t=0,1,2, \ldots$.
Let us now consider the edge sets $E_{N}^{D}(t)$ and $E_{E}^{D}(t)$. Take any $t$ and any edge $u v \in E_{N}^{D}(t)$. Without loss of generality we can assume $u \in V_{N}^{D}(t) \subseteq V_{E}^{D}(t)$ and hence, by Lemma 3.1, $u v \in E_{E}^{D}(t)$. Hence, for $t=0,1,2, \ldots$, we have

$$
E_{N}^{D}(t) \subseteq E_{E}^{D}(t) \quad \text { and } \quad E_{E}^{C}(t) \subseteq E_{N}^{C}(t),
$$

which completes the proof.
Corollary 3.4. Given a graph $\mathbf{G}$ and an internal search schedule $\mathbf{S}$, if $\mathbf{S}$ is edge clearing it is also node clearing.

Proof. Suppose the final move of $\mathbf{S}$ took place at time $t_{\text {fin }}$, then $E_{E}^{D}\left(t_{f i n}\right)=\emptyset$. Since $E_{N}^{D}\left(t_{f i n}\right) \subseteq$ $E_{E}^{D}\left(t_{f i n}\right)=\emptyset$ we have $E_{N}^{D}\left(t_{f i n}\right)=\emptyset$ which also means $V_{N}^{D}\left(t_{f i n}\right)=\emptyset$ (if a node $u$ was n-dirty, an adjacent edge $u v$ would also be n -dirty).

Remark 3.5. From Theorem 3.3 we see that edge search is "weaker" than node search, i.e., for every search schedule, and at every time step, the clear set in the edge game is smaller or equal than the one in the node game. There is a variant of edge search, the so-called mixed edge search which, as we will see in Appendix B, is equivalent to node search, i.e. every search schedule produces the same clear and dirty sets at every step of both the edge and node game (this is stated and proved as Theorem B.1). From this fact follow two consequences. First, node search is $N P$-complete (since mixed edge search is NP-complete [5]). Second, Theorem 3.3 can be obtained as a corollary of Theorem B.1. A more detailed discussion of these issues appears in Appendix B.

## 4 Search on Trees

In this section we will focus on rooted IMC (internal, monotone, connected) search on trees and we will compare node and edge search in this context.

Several authors have studied rooted IMC edge search [2, 3, 11, 13, 14, 39]. Our main results in this section are Theorems 4.1 and 4.2 which establish the equivalence of rooted IMC node and edge search on trees. Hence several results already established for edge search also hold for node search (on trees) as will be seen in the sequel.

Theorem 4.1. Given a tree $\mathbf{T}$ and an internal rooted search schedule $\mathbf{S}$, if $\mathbf{S}$ is monotone connected in the node game then

$$
\text { for } t=0,1,2, \ldots: \quad V_{E}^{C}(t)=V_{N}^{C}(t) \quad \text { and } E_{E}^{C}(t)=E_{N}^{C}(t)
$$

and $\mathbf{S}$ is monotone connected in the edge game.

Proof. The proof is inductive. We have $V_{E}^{C}(0)=V_{N}^{C}(0)=\emptyset$ and $E_{E}^{C}(0)=E_{N}^{C}(0)=\emptyset$. Also, since the search is rooted and internal, the move at $t=1$ will be the placement of a searcher into the root $u_{0}$; hence $V_{E}^{C}(1)=V_{N}^{C}(1)=\left\{u_{0}\right\}$ and $E_{E}^{C}(1)=E_{N}^{C}(1)=\emptyset$. Now suppose that at time $t \geq 1$ we have

$$
\begin{equation*}
V_{E}^{C}(t)=V_{N}^{C}(t), \quad E_{E}^{C}(t)=E_{N}^{C}(t) \tag{12}
\end{equation*}
$$

and let us consider the move $u \rightarrow v$, performed at $t+1, t>1$.
We first dispose of the case where the move involves the node 0 . Because of internality, $v \neq 0$ (no searcher is removed from the graph). If the move is $0 \rightarrow v$, by rootedness we must have $v=u_{0}$. Also by rootedness, $u_{0}$ is the first node cleared (at some $t=1$ ). Now, in the node game: (a) by monotonicity, no node is recontaminated at $t+1$ and (b) no node is n-cleared either (since $u_{0}$ remains n-clear, the move $0 \rightarrow u_{0}$ implies $\left.V_{N}^{C}(t+1)-V_{N}^{C}(t)=\emptyset\right)$; hence $V_{N}^{C}(t)=V_{N}^{C}(t+1)$ which also means $E_{N}^{C}(t)=E_{N}^{C}(t+1)$. In the edge game, $0 \rightarrow u_{0}$ (a) does not clear any new edges (so it does not clear any new nodes either) and (b) does not remove any searchers from any node, so neither edge nor node recontamination is possible; hence $V_{E}^{C}(t)=V_{E}^{C}(t+1)$ and $E_{E}^{C}(t)=E_{E}^{C}(t+1)$. In short, if the $(t+1)$-th move involves node 0 , from (12) follows

$$
V_{E}^{C}(t+1)=V_{N}^{C}(t+1), \quad E_{E}^{C}(t+1)=E_{N}^{C}(t+1)
$$

Next we examine the case where the $(t+1)$-th move is $u \rightarrow v$, with $u, v \neq 0$. We will examine the effects of this move separately for the node and edge game.
I. Node Game. $u$ was guarded at time $t$, so $u \in V_{N}^{C}(t)$. Suppose, first, that at $t$ only one searcher was located on $u$, so $u$ is unguarded at $t+1$. However, since $\mathbf{S}$ was supposed monotone in the node game, there is no node $x \in V_{N}^{C}(t) \cap V_{N}^{D}(t+1)$ and, especially, $u$ remains n-clear. Since no node has become n-dirty at $t+1$, no edge has become n-dirty either (an edge becomes n-dirty iff incident on an n -dirty node). As for clearings, $v$ is n-clear at $t+1$, since entered by a searcher; and $u v$ is also n-clear since $\{u, v\} \subseteq V_{N}^{C}(t+1)$. No node other than $v$ was entered at $t+1$, so no node other than $v$ was n-cleared. Is there an edge (other than $u v$ ) which was n-cleared exactly at $t+1$ ? This is only possible if (a) the edge has the form $v x_{1}$ and (b) $v \in V_{N}^{D}(t) \cap V_{N}^{C}(t+1)$ ( $v$ became n-clear exactly at $t+1$ ) and (c) $x_{1} \in V_{N}^{C}(t) \cap V_{N}^{C}(t+1)$. Suppose $x_{1} \neq u$. Since $\mathbf{G}_{N}^{C}(t)$ is connected, there is a path $u x_{L-1} \ldots x_{1}$ which is inside $\mathbf{G}_{N}^{C}(t)$ and hence does not contain $v$. Then the tree $\mathbf{T}$ contains a cycle $u x_{L-1} \ldots x_{1} v u$ which is a contradiction. Hence the only edge possibly n-cleared at $t+1$ (if not already n-clear at $t$ ) is $u v$. In short

$$
\begin{equation*}
V_{N}^{C}(t+1)=V_{N}^{C}(t) \cup\{v\} \text { and } E_{N}^{C}(t+1)=E_{N}^{C}(t) \cup\{u v\} \tag{13}
\end{equation*}
$$

The case that $u$ is still guarded at $t+1$ (i.e. $u$ contained more than one searcher at time $t$ ) is omitted, since it is similar but easier to treat than the case of unguarded $u$.
II. Edge Game. Again, we first examine the case that only one searcher was located in $u$ at time $t$. So $u$ was guarded (and e-clear) at $t$ and is unguarded at $t+1$. We will now show that $u$ remains e-clear at $t+1$; in other words that

$$
\begin{equation*}
u \in V_{E}^{C}(t) \cap V_{E}^{C}(t+1) . \tag{14}
\end{equation*}
$$

Suppose, on the contrary, that we have

$$
\begin{equation*}
u \in V_{E}^{C}(t) \cap V_{E}^{D}(t+1) \tag{15}
\end{equation*}
$$

There are only two ways for (15) to hold, each of which we examine separately.
II. $1 \exists x_{1} \neq v$ such that $u x_{1} \in E_{E}^{D}(t) \cap E_{E}^{D}(t+1)$. Then $u x_{1} \in E_{E}^{D}(t)=E_{N}^{D}(t)$. Since the move at $t+1$ was $u \rightarrow v, u$ was guarded at $t$ and $x_{1} \in V_{N}^{D}(t)$. Now, from $u \rightarrow v$ and $x_{1} \neq v$ we get $x_{1} \in V_{N}^{D}(t+1)$; from this and $u$ unguarded at $t+1$ we conclude that $u \in V_{N}^{D}(t+1)$ which contradicts monotonicity of the node search.
II. 2 Alternatively, there exists a path $u x_{1} \ldots x_{K-1} x_{K}$ (with $K>1$ ) such that $x_{K} x_{K-1} \in E_{E}^{D}(t) \cap$ $E_{E}^{D}(t+1)$ and $u x_{1} \ldots x_{K-1} x_{K}$ is e-unguarded at $t+1$. First note that, since $u x_{1} \ldots x_{K}$ is eunguarded at $t+1$ and the $(t+1)$-th move is $u \rightarrow v$, it follows that $v \notin\left\{x_{1}, \ldots, x_{K-1}\right\}$; and since $\mathbf{T}$ is a tree, $v \neq x_{K}$ too; in short $v \notin\left\{x_{1}, \ldots, x_{K}\right\}$. Also $u \notin\left\{x_{1}, \ldots, x_{K}\right\}$, since $u x_{1} \ldots x_{K}$ is a path. Now

$$
\left.\begin{array}{l}
\text { the }(t+1) \text {-th move is } u \rightarrow v \\
u \notin\left\{x_{1}, \ldots, x_{K}\right\} \\
u x_{1} \ldots x_{K} \text { is e-unguarded at } t+1
\end{array}\right\} \Rightarrow u x_{1} \ldots x_{K} \text { is e-unguarded at } t
$$

and this together with $x_{K} x_{K-1} \in E_{E}^{D}(t)$ implies $x_{K-1} \in V_{E}^{D}(t)=V_{N}^{D}(t)$. From $v \neq x_{K-1}$ follows that $x_{K-1}$ was not entered at $t+1$ and so $x_{K-1} \in V_{N}^{D}(t+1)$. From this and $u x_{1} \ldots x_{K-1}$ being n-unguarded at $t+1$ we conclude that $u \in V_{N}^{D}(t+1)$ which contradicts monotonicity of the node search.

In short, $u$ cannot become e-dirty at $t+1$, i.e., we have proved (14). Using this we now will show that no previously e-clear edge $x y$ can become e-dirty at $t+1$. Because this would require the existence of some edge $p q \in E_{E}^{D}(t) \cap E_{E}^{D}(t+1)$ and a path $x y \ldots p q$ which is e-unguarded at $t+1$ but e-guarded at $t$ (otherwise $x y$ would already be e-dirty at $t$ ). But any such path would have to contain $u$ (no searcher was removed from any other node at $t+1$ ) which would in turn imply $u \in V_{E}^{D}(t+1)$, which contradicts (14). Since no edge is recontaminated no node is recontaminated either (the only node guarded at $t$ and unguarded at $t+1$ is $u$, for which we have (14)).

On the other hand, the only edge possibly e-cleared at time $t+1$ (if not already e-clear at $t$ ) is $u v$, since no other edge was traversed at $t+1$. Hence the only node possibly e-cleared at time $t+1$ (if not already e-clear at $t$ ) is $v$.

In short we have shown

$$
\begin{equation*}
V_{E}^{C}(t+1)=V_{E}^{C}(t) \cup\{v\} \text { and } E_{E}^{C}(t+1)=E_{E}^{C}(t) \cup\{u v\} . \tag{16}
\end{equation*}
$$

The treatment of the case that $u$ is still guarded at $t+1$ (i.e., at $t$ it contained more than one searcher) is omitted, since it is similar but easier to the case of $u$ unguarded. From (12), (13) and (16) we obtain

$$
\begin{equation*}
V_{E}^{C}(t+1)=V_{N}^{C}(t+1) \text { and } E_{E}^{C}(t+1)=E_{N}^{C}(t+1) \tag{17}
\end{equation*}
$$

and, proceeding inductively, we obtain the required result:

$$
\begin{equation*}
\text { for } t=0,1,2, \ldots: \quad V_{E}^{C}(t)=V_{N}^{C}(t) \quad \text { and } \quad E_{E}^{C}(t)=E_{N}^{C}(t) \tag{18}
\end{equation*}
$$

Monotonicity and connectedness in the edge game follow from the fact that these hold in the node game and from eq. (18).

Theorem 4.2. Given a tree $\mathbf{T}$ and an internal rooted search schedule $\mathbf{S}$ with root $u_{0}$, if $\mathbf{S}$ is monotone connected in the edge game and satisfies either of the following constraints

$$
\begin{aligned}
& \text { C1 } E_{E}^{C}(1)=\emptyset, E_{E}^{C}(2)=\left\{u_{0} v\right\}, \\
& \text { C2 } E_{E}^{C}(1)=\emptyset, E_{E}^{C}(2)=\emptyset, E_{E}^{C}(3)=\left\{u_{0} v\right\},
\end{aligned}
$$

then

$$
\begin{equation*}
\text { for } t=0,1,2, \ldots: \quad V_{E}^{C}(t)=V_{N}^{C}(t) \quad \text { and } E_{E}^{C}(t)=E_{N}^{C}(t) \tag{19}
\end{equation*}
$$

and $\mathbf{S}$ is monotone connected in the node game.

Remark 4.3. Before proving the theorem, let us discuss the significance of constraints $\mathbf{C} 1$ and $\mathbf{C} 2$. Three remarks must be made

1. Regarding the kind of searches which will satisfy either of the constraints: $\mathbf{C 1}$ will be satisfied by searches with $\mathbf{S}(1)=0 \rightarrow u_{0}, \mathbf{S}(2)=u_{0} \rightarrow v$ (provided $u_{0}$ has a single neighbor, namely $v$ ); C2 will be satisfied by searches with $\mathbf{S}(1)=0 \rightarrow u_{0}, \mathbf{S}(2)=0 \rightarrow u_{0}, \mathbf{S}(3)=u_{0} \rightarrow v$; in both cases $u_{0}$ is, obviously, the root of the search.
2. The constraints are imposed to exclude situations similar to the one discussed in Remark 2.2, where an edge is e-cleared and e-dirtied during the same time step $t$.
3. Finally, $\mathbf{C} \mathbf{1}$ and $\mathbf{C} \mathbf{2}$ are not exceedingly restrictive; as will be seen later, every "interesting" IMC search on a tree either satisfies $\mathbf{C 1} / \mathbf{C} 2$ or can easily be converted to an equivalent search which does.

Proof of Theorem 4.2. The theorem will be proved by induction. It is easy to check that

$$
\begin{equation*}
\text { for } t \in\left[0, t_{1}\right]: \quad V_{E}^{C}(t)=V_{N}^{C}(t) \text { and } E_{E}^{C}(t)=E_{N}^{C}(t) \tag{20}
\end{equation*}
$$

with $t_{1}=2$ when $\mathbf{C} 1$ holds and $t_{1}=3$ when $\mathbf{C} 2$ holds. Now suppose that at time $t \geq t_{1}$ we have

$$
\begin{equation*}
V_{E}^{C}(t)=V_{N}^{C}(t), \quad E_{E}^{C}(t)=E_{N}^{C}(t) \tag{21}
\end{equation*}
$$

and consider the move performed at $t+1$.
We first dispose of the case where the move involves the node 0 . Because of internality, $v \neq 0$ (no searcher is removed from the graph). If the move is $0 \rightarrow v$, by rootedness we must have $v=u_{0}$. By $\mathbf{C 1}$ and C2, $u_{0}$ is the first node cleared (at $t=1$ ). In the edge game, at $t+1$, no edge is recontaminated (by monotonicity) and no edge is e-cleared (since no edge is traversed); hence $E_{E}^{C}(t)=E_{E}^{C}(t+1)$; since no searcher is removed from a node, $V_{E}^{C}(t)=V_{E}^{C}(t+1)$ as well. In the node game, $u_{0}$ is already n-clear at $t_{1}$ and no other node is entered at $t+1$, so no node is n-cleared; no searcher is removed from the graph, so no node is recontaminated; hence $V_{N}^{C}(t)=V_{N}^{C}(t+1)$ which implies $E_{N}^{C}(t)=E_{N}^{C}(t+1)$ as well. In short, if the $(t+1)$-th move involves node 0 , from (21) and the above arguments follows

$$
V_{E}^{C}(t+1)=V_{N}^{C}(t+1), \quad E_{E}^{C}(t+1)=E_{N}^{C}(t+1)
$$

Next we examine the case where the $(t+1)$-th move is $u \rightarrow v$, with $u, v \neq 0$. We will examine the effects of this move separately for the node and edge game.
I. Edge Game. By monotonicity of the search, no previously e-clear edge becomes e-dirty at $t+1$; and hence the only node that can possibly become e-dirty is $u$. We will show however that $u$ also remains e-clear. Since $E_{E}^{C}\left(t_{1}\right)=\left\{u_{0} v\right\}$ and $t \geq t_{1}$, from edge monotonicity follows that $E_{E}^{C}(t) \neq \emptyset$ for all $t \geq t_{1}$. And, since $u \in V_{N}^{C}(t)=V_{E}^{C}(t)$ and $\mathbf{G}_{E}^{C}(t)$ is connected, there exists $u z \in E_{E}^{C}(t)$; if $u$ becomes e-dirty at $t+1$, then $u z \in E_{E}^{D}(t+1)$, which contradicts monotonicity of the edge search. Hence no previously e-clear node becomes e-dirty at $t+1$.

Regarding new clearings, edge $u v$ and node $v$ may become e-clear at $t+1$ (if they were not already e-clear at $t$ ). No edge other than $u v$ was traversed, hence no edge other than $u v$ (and no node other than $v$ ) can be e-cleared at $t+1$.

In short, in the edge game we have

$$
\begin{equation*}
V_{E}^{C}(t+1)=V_{E}^{C}(t) \cup\{v\} \text { and } E_{E}^{C}(t+1)=E_{E}^{C}(t) \cup\{u v\} . \tag{22}
\end{equation*}
$$

II. Node Game. Again, we first examine the case that only one searcher was located on $u$ at time $t$. So $u$ was guarded (hence n-clear) at $t$ and is unguarded at $t+1$. We will now show that $u$ remains n -clear at $t+1$; in other words that

$$
\begin{equation*}
u \in V_{N}^{C}(t) \cap V_{N}^{C}(t+1) . \tag{23}
\end{equation*}
$$

Suppose on the contrary that we have

$$
\begin{equation*}
u \in V_{N}^{C}(t) \cap V_{N}^{D}(t+1) \tag{24}
\end{equation*}
$$

(24) can happen in only two ways, each of which we examine separately.
II. $1 \exists x_{1} \neq v$ such that $u x_{1} \in E$ and $x_{1} \in V_{N}^{D}(t) \cap V_{N}^{D}(t+1)$. Then $x_{1} \in V_{N}^{D}(t)=V_{E}^{D}(t) \Rightarrow u x_{1} \in$ $E_{E}^{D}(t)$ and hence (since the move at $t+1$ was $u \rightarrow v$ and $\left.x_{1} \neq v\right) u x_{1} \in E_{E}^{D}(t+1)$.
Since $E_{E}^{C}(t) \neq \emptyset$ (for all $t \geq t_{1}$ ) and $u \in V_{N}^{C}(t)=V_{E}^{C}(t)$ and $\mathbf{G}_{E}^{C}(t)$ is connected, there exists $u z \in E_{E}^{C}(t)$. However, since $u x_{1} \in E_{E}^{D}(t+1)$ and $u$ is unguarded at $t+1$, we conclude $u z \in E_{E}^{D}(t+1)$, which contradicts monotonicity of the edge search.
II. 2 Alternatively, there exists a path $u x_{1} \ldots x_{K-1} x_{K}($ with $K>1)$ such that $x_{K} \in V_{N}^{D}(t) \cap V_{N}^{D}(t+1)$ and $u x_{1} \ldots x_{K-1} x_{K}$ is n-unguarded at $t+1$ (so $v \notin\left\{u, x_{1}, \ldots, x_{K}\right\}$ ). First note that, since $u x_{1} \ldots x_{K}$ is a path, $u \notin\left\{x_{1}, \ldots, x_{K}\right\}$. Now

$$
\left.\begin{array}{l}
\text { the }(t+1) \text {-th move is } u \rightarrow v \\
u \notin\left\{x_{1}, \ldots, x_{K}\right\} \\
u x_{1} \ldots x_{K} \text { is n-unguarded at } t+1
\end{array}\right\} \Rightarrow u x_{1} \ldots x_{K} \text { is e-unguarded at } t
$$

Since $x_{K} \in V_{N}^{D}(t)=V_{E}^{D}(t)$ it follows that $x_{K-1} x_{K} \in E_{E}^{D}(t)$; since also $x_{K-1} x_{K}$ is not traversed at $t+1$, it follows that $x_{K-1} x_{K} \in E_{E}^{D}(t+1)$. By the same reasoning as in the previous case, there exists $u z \in E_{E}^{C}(t)$ and, since $u x_{1} \in E_{E}^{D}(t+1)$ and $u$ is unguarded at $t+1$, we conclude $u z \in E_{E}^{D}(t+1)$, which contradicts the edge monotonicity of $\mathbf{S}$.

In short, $u$ cannot become n-dirty at $t+1$, i.e., we have proved 23 . Using this fact and an argument similar to that of Theorem 4.1, we conclude that no previously n-clear node can become n-dirty at $t+1$. Since no node is recontaminated, no edge is recontaminated either.

On the other hand, the only node possibly n-cleared at time $t+1$ is $v$ (no other node was entered) and hence the only edge possibly n-cleared at time $t+1$ is $u v$ (if another edge was cleared, by an analysis similar to that of Theorem 4.1, we conclude that a cycle must exist in $\mathbf{T}$, which is impossible). Hence the only edge possibly n-cleared at $t+1$ is $u v$.

In short, in the node game we have

$$
\begin{equation*}
V_{N}^{C}(t+1)=V_{N}^{C}(t) \cup\{v\} \text { and } E_{N}^{C}(t+1)=E_{N}^{C}(t) \cup\{u v\} \tag{25}
\end{equation*}
$$

The treatment of the case that $u$ is still guarded at $t+1$ (i.e., it contained more than one searcher at $t$ ) is omitted, since it is similar to but easier than the case of $u$ unguarded. From (21), (22) and (25) we obtain

$$
\begin{equation*}
V_{E}^{C}(t+1)=V_{N}^{C}(t+1) \text { and } E_{E}^{C}(t+1)=E_{N}^{C}(t+1) \tag{26}
\end{equation*}
$$

and, proceeding inductively, we obtain the required result

$$
\begin{equation*}
\text { for } t=0,1,2, \ldots: \quad V_{E}^{C}(t)=V_{N}^{C}(t) \quad \text { and } \quad E_{E}^{C}(t)=E_{N}^{C}(t) \tag{27}
\end{equation*}
$$

Monotonicity and connectedness in the node game follow from the fact that these hold in the edge game and eq. (27).

We will now review some results (obtained by Barriere et al. [2, 3]) regarding IMC edge search on trees and will show that similar results hold for IMC node search on trees. In the following presentation the terminology and notation of [2, 3] is somewhat changed, to conform with the one used in the current paper.

The first basic result of [2] is the following.

Theorem 4.4. [2] For every tree $\mathbf{T}$ there is an IMC edge clearing search schedule $\mathbf{S}$ which uses $s_{E}^{i c}(\mathbf{T})$ searchers. Moreover, in $\mathbf{S}$ all searchers are initially placed at the same node $u_{0}$ (i.e., the search schedule is rooted) and the first step (after placing the searchers) consists in clearing an edge incident to $u_{0}$.

Corollary 4.5. For every tree $\mathbf{T}$ we have $s_{E}^{i c}(\mathbf{T})=s_{E}^{i m c}(\mathbf{T})$. There exists a rooted IMC search schedule which achieves this bound.

Theorem 4.4 can be extended to the following sequence of inequalities 3 .
Theorem 4.6. [3] For every tree $\mathbf{T}$ we have.

$$
\begin{equation*}
s_{E}(\mathbf{T})=s_{E}^{i}(\mathbf{T})=s_{E}^{m}(\mathbf{T}) \leq s_{E}^{i m}(\mathbf{T})=s_{E}^{c}(\mathbf{T})=s_{E}^{i c}(\mathbf{T})=s_{E}^{m c}(\mathbf{T})=s_{E}^{i m c}(\mathbf{T}) \leq 2 s_{E}(\mathbf{T})-2 . \tag{28}
\end{equation*}
$$

Furthermore, there are trees $\mathbf{T}$ for which the inequality $s_{E}^{m}(\mathbf{T}) \leq s_{E}^{i m}(\mathbf{T})$ is strict.
Barriere et al. also present the Search algorithm [2], which computes a minimal edge-clearing rooted IMC schedule $\mathbf{S}$ for every tree $\mathbf{T}$. The next Theorem shows that $\mathbf{S}$ is also a minimal nodeclearing rooted IMC schedule.

Theorem 4.7. For every tree $\mathbf{T}$, the search $\mathbf{S}$ produced by the Search algorithm is

1. a minimal edge-clearing IMC search of $\mathbf{T}$, which is also rooted;
2. a minimal node-clearing IMC search of $\mathbf{T}$, which is also rooted.

Proof. Part 1 of the Theorem is proved as Lemma 9 in [2]. Let us now prove part 2. Since $\mathbf{S}$ is an edge-clearing strategy it is also node-clearing. We next show it is minimal.

Let us first consider the case of trees $\mathbf{T}$ such that $s_{E}^{i m c}(\mathbf{T})=1$. It is easy to see that such trees are paths of the form $u_{1} u_{2} \ldots u_{N}$, and Search produces the "obvious" edge clearing schedule: $0 \rightarrow u_{1}$, $u_{1} \rightarrow u_{2}, \ldots$ (or the reverse, starting at $u_{N}$ ) which is also a rooted minimal node clearing IMC schedule.

Suppose now that $s_{E}^{i m c}(\mathbf{T}) \geq 2$. By part 1 , Search will produce a rooted minimal edge clearing IMC schedule $\mathbf{S}$. Also, by relaxing the requirement that all searchers are placed into the graph in the first move (in other words, by interspersing searcher placements with edge slidings) we can obtain from $\mathbf{S}$ a new search schedule $\mathbf{S}^{\prime}$ which

1. uses the same number of searchers as $\mathbf{S}$ (i.e., $\overline{s n}(\mathbf{S})=\overline{s n}\left(\mathbf{S}^{\prime}\right)$ );
2. is rooted (i.e. all searcher placements are into $u_{0}$ );
3. satisfies either $\mathbf{C 1}$ or $\mathbf{C} 2$ of Theorem 4.2;
4. produces the same sequence of clear graphs $\mathbf{G}_{N}^{C}(t)$ as $\mathbf{S}$ (with a slight time adjustment, to account for the changes in searcher placement times).

Hence $\mathbf{S}^{\prime}$ satisfies the conditions of Theorem 4.2 and is IMC in the edge game which means that, using $\mathbf{S}^{\prime}$, we have

$$
\begin{equation*}
\text { for } t=0,1,2, \ldots: \quad V_{E}^{C}(t)=V_{N}^{C}(t) \quad \text { and } E_{E}^{C}(t)=E_{N}^{C}(t) . \tag{29}
\end{equation*}
$$

From this also follows that $\mathbf{S}^{\prime}$ is a node clearing rooted IMC search schedule of $\mathbf{T}$.
Now take another node clearing IMC search of $\mathbf{T}$, call it $\mathbf{S}^{\prime \prime}$, which is minimal for the node game (i.e., $\overline{s n}\left(\mathbf{S}^{\prime \prime}\right)=s_{N}^{i m c}(\mathbf{T})$ ). Then, by Theorem 4.1, $\mathbf{S}^{\prime \prime}$ is also edge clearing, hence

$$
\overline{s n}\left(\mathbf{S}^{\prime \prime}\right) \geq s_{E}^{i m c}(\mathbf{T})=\overline{s n}(\mathbf{S}) .
$$

On the other hand, since $\mathbf{S}^{\prime}$ is node clearing,

$$
\overline{s n}\left(\mathbf{S}^{\prime \prime}\right)=s_{N}^{i m c}(\mathbf{T}) \leq \overline{s n}\left(\mathbf{S}^{\prime}\right)=\overline{s n}(\mathbf{S})
$$

Hence

$$
s_{N}^{i m c}(\mathbf{T})=\overline{s n}\left(\mathbf{S}^{\prime \prime}\right)=\overline{s n}(\mathbf{S})=s_{E}^{i m c}(\mathbf{T})
$$

which completes the proof.
Corollary 4.8. For every tree $\mathbf{T}$ we have $s_{E}^{i m c}(\mathbf{T})=s_{N}^{i m c}(\mathbf{T})$.
Remark 4.9. Theorem4.7shows that the Barriere Search algorithm can be used to compute a rooted IMC node clearing schedule for every tree $\mathbf{T}$. How good is such a schedule? In other words, can we node clear a tree $\mathbf{T}$ with fewer than $s_{N}^{i m c}(\mathbf{T})$ searchers? We can certainly do this with an internal (non-monotone, non-connected) search. For example, let $\mathbf{T}^{(2 M)}$ be the complete binary tree of height $2 M$; then $s_{N}^{i}\left(\mathbf{T}^{(2 M)}\right)=M$, while $s_{N}^{i m c}\left(\mathbf{T}^{(2 M)}\right)=2 M$; but we do not know the values of $s_{N}^{i m}\left(\mathbf{T}^{(2 M)}\right)$ and $s_{N}^{i c}\left(\mathbf{T}^{(2 M)}\right)$. More generally, we do not have the analog for node search of the inequalities (28). It is easy to see that

$$
s_{N}(\mathbf{G})=s_{N}^{i}(\mathbf{G}) \leq s_{N}^{m}(\mathbf{G}) \leq \begin{gather*}
s_{N}^{i m}(\mathbf{G})  \tag{30}\\
s_{N}^{c}(\mathbf{G})=s_{N}^{i c}(\mathbf{G})
\end{gather*} \leq s_{E}^{m c}(\mathbf{G})=s_{E}^{i m c}(\mathbf{G})
$$

for every graph $\mathbf{G}$ and, in particular, for every tree $\mathbf{T}$ (and the inequalities in (30) can be strict). Refining (30) to something like (28) is a subject of our future research. The question is of special interest to us because, as already mentioned, the one absolute requirement for robotic pursuit / evasion is internality; monotonicity and connectedness are desirable but not indispensable.

Remark 4.10. A basic component of Barriere's Search algorithm is the " $\lambda$ labeling" of edges, performed by the auxiliary Label algorithm[2]. As will be seen in the next section, the Barriere $\lambda$ labels are also used by our GSST algorithm to node-clear general graphs.

## 5 Search on Graphs

We now turn to the study of rooted IMC node searches of an arbitrary graph $\mathbf{G}=(V, E)$ (which will always be assumed to have $N$ nodes, i.e., $|V|=N$ ). In every such search, the $t$-th move has the form $\mathbf{S}(t)=u \rightarrow v$, where $u, v \in V$ or, perhaps, $u=0$, the "source" node.

We present several variants of a basic node clearing algorithm. All variants are guaranteed to find a node clearing search schedule; we prove that two of the variants will find a minimal schedule with probability $1-\alpha^{M}$ where $\alpha \in(0,1)$ and $M$ is the number of iterations of the algorithm; even non-minimal search schedules require a reasonably small number of searchers, as will be seen by the experiments of Section 6 .

### 5.1 Motivation

Our basic algorithmic idea is motivated by the following rather simple observation: every rooted IMC node-clearing search of $\mathbf{G}$ generates a spanning tree. This observation can be refined in the form of the following theorem.

Theorem 5.1. Given a graph $\mathbf{G}=(V, E)$ and a rooted IMC node clearing search $\mathbf{S}$ of $\mathbf{G}$. The clearing moves of $\mathbf{S}$ generate a sequence of trees, $\left(\mathbf{T}_{0}, \mathbf{T}_{1}, \mathbf{T}_{2}, \ldots, \mathbf{T}_{N}\right)$, where (for $\left.n=1,2, \ldots, N\right) \mathbf{T}_{n}=\left(V_{n}, E_{n}\right)$ and the following hold:

D1 $\mathbf{T}_{0}$ is the empty graph $\left(V_{0}=\emptyset, E_{0}=\emptyset\right)$;

D2 $\mathbf{T}_{N}$ is a spanning tree of $\mathbf{G}\left(V_{N}=V, E_{N} \subseteq E\right)$;
D3 for $n=1,2, \ldots, N: V_{n-1} \subseteq V_{n}, E_{n-1} \subseteq E_{n}$ (in other words $\mathbf{T}_{n-1}$ is a subtree of $\mathbf{T}_{n}$ );
D4 for $n=1,2, \ldots, N: V_{n}=V_{n-1} \cup\left\{u_{n}\right\}$, and for $n=2,3, \ldots: E_{n}=E_{n-1} \cup\left\{u_{i} u_{n}\right\}$, with $i \in[1, n-1]$.
Proof. Inductively. Since $\mathbf{S}$ is monotone, it involves $N$ clearing moves. $\mathbf{T}_{0}$ is the empty graph. $\mathbf{T}_{1}$ is formed by the first move of $\mathbf{S}$, which consists in placing a searcher at the root node. So $\mathbf{T}_{1}$ is the tree with a single node and trivially has $\mathbf{T}_{0}$ as a subgraph. Suppose D3 and D4 hold up to $m=n$ and consider the $(n+1)$-th clearing move of $\mathbf{S}$. Since $\mathbf{S}$ is connected, we add to $\mathbf{T}_{n}$ one node $u_{n+1}$ and (as explained in the Proof of Theorem 4.1) exactly one edge $u_{i} u_{n+1}$ (with $i \in[1, n]$ ) to obtain a new tree $\mathbf{T}_{n+1}$ (of $n+1$ nodes and $n$ edges) which also satisfies D3 and D4. Hence D3 and D4 hold for $m=1,2, \ldots, N$. At $m=N, V_{N}$ contains $N$ nodes, hence $V_{N}=V ;$ since $\mathbf{T}_{N}$ is a tree, it is a spanning tree of $\mathbf{G}$.

Remark 5.2. An abbreviated statement of Theorem 5.1 could be: "Every rooted IMC node clearing search of $\mathbf{G}$ specifies a spanning tree of $\mathbf{G}$ and an order of clearing the nodes". But the order must be consistent with the edge structure of the graph and the spanning tree (e.g., we cannot use an order of node clearing which requires non-existent edges). This consistence is exactly what conditions D3 and D4 describe.

Theorem 5.1 gives the motivation for our algorithm GSST (Guaranteed $\boldsymbol{S}$ earch with $\boldsymbol{S}$ panning $\boldsymbol{T r e e s}$ ), which is informally described below.

```
Algorithm 1 GSST, Informal Description
    Input: Graph G
    Select a spanning tree \(\mathbf{T}\) of \(\mathbf{G}\) and a root \(u_{0}\) of \(\mathbf{T}\)
    Find a rooted IMC node-clearing search \(\mathbf{S}^{\prime}\) of \(\mathbf{T}\)
    Apply \(\mathbf{S}\) to \(\mathbf{G}\)
    if at some step of \(\mathbf{S}^{\prime}\) a move \(u \rightarrow v\) would result in recontamination in \(\mathbf{G}\) then
        send a "guard" to \(u\)
        execute \(u \rightarrow v\)
    end if
    Output: The search \(\mathbf{S}\) obtained by combining \(\mathbf{S}^{\prime}\) with the guard moves
```

Note that, by construction, all clearing moves take place along edges of the spanning tree $\mathbf{T}$.
In the above description we have used the term "guard". Stated informally, the searchers play two roles: "tree searchers" perform the clearing moves, always along the edges of the spanning tree; "guards" are stationary and block potential recontamination paths. However, note that a particular searcher can change roles during the course of the search.

The main advantage of the GSST algorithm is that it is fast. A random spanning tree $\mathbf{T}$ can be quickly generated and searched. Since $\mathbf{T}$ and $\mathbf{G}$ have the same node set, node-clearing $\mathbf{T}$ in a nodemonotone manner results in node-clearing $\mathbf{G}$ as well. The main issue is: how many guards will be required to block recontamination through non-tree edges of $G$ ? The number usually turns out to be quite reasonable, because (a) guards can be reused and (b) tree searchers can also be used as guards when they do not perform clearing moves.

Both the labeling and the traversal phase of GSST can be executed in either a centralized or distributed manner; the latter is useful for robotic applications, where each robot can share some of the computational load. In the distributed implementation, all searchers share the underlying spanning tree and labeling. When a searcher reaches a node, he checks to see if he can move without recontamination.

If he can, he determines his next move based on the traversal strategy, and he shares this move with the team.

Hence GSST has short execution time and can be run repeatedly (in reasonable time), using many different (randomly selected) spanning trees. Our algorithm depends on the quick discovery of a spanning tree corresponding to a minimal or near-minimal search (Theorem 5.1). Our experiments in Section 6 show that GSST discovers near-minimal searches in only a short time for several families of complex graphs.

An important characteristic of the GSST algorithm is its "anytime-ness". Anytime algorithms return a partial answer before completion and they keep providing improved answers, the improvement increasing with computation time [40. GSST has these characteristics, as will be understood in Section 5.2. Namely, GSST is characterized by monotonicity (the solution only improves over time), recognizable quality (the quality of the solution, i.e. number of searchers, can be determined at run time), consistency (the algorithm will not spend too much time finding a single solution), and interruptibility.

### 5.2 The GSST Algorithms

A detailed description of the GSST algorithm is given by the following pseudocode on p .19 . The notation $\mathbf{S}=\mathbf{S} \mid(u \rightarrow v)$ means that the move $(u \rightarrow v)$ is appended to the previously determined search schedule $\mathbf{S}$ (i.e., becomes the next move of $\mathbf{S}$ ). Several subroutines appearing in the following listing will be discussed presently.

The following remarks explain the operation of the algorithm.

1. Generate a random spanning tree $\mathbf{T}$ by GenerateTree. We have used two different methods of random spanning tree generation.
(a) The uniform method is an implementation of Wilson's 37] loop erased random walk algorithm.
(b) The DFS method selects a root node and randomly moves down the tree in a depth-first manner. At each node, a random incident edge is chosen and set as an edge in the spanning tree. A visited list is maintained, and when a node is visited more than once, the edge used to reach it the second time is set as a non-tree edge. This eliminates cycles in the graph and thus generates a tree. When a leaf is reached, the algorithm recurses to ensure that all nodes are included in the tree (i.e, it is a spanning tree of the original graph). The motivation for this method is to bias towards spanning trees that require fewer guards. The intuition is that a DFS traversal will generate only a few nodes with non-tree edges, thus leading to few required guards.
2. Label the edges of $\mathbf{T}$ by the $\mathbf{R}$-Label algorithm (this, a modification of Barriere's Label algorithm, is listed and discussed in Appendix A).
3. While n-dirty nodes still exist, select an n-dirty edge $u v$ of $\mathbf{T}$ (by SelectEdge, to be discussed presently). Let a searcher traverse $u v$ if this does not cause node recontamination ${ }^{5}$ if no such searcher exists, then use a new searcher (originally placed at the root) to traverse $u v$.
4. Repeat the process until all nodes are n-cleared. (Hence the algorithm does not allow node recontamination and will generate as many searchers as necessary to prevent it).
5. When all nodes have been n-cleared, a rooted IMC node clearing search has been generated, which performs all its clearing moves along the edges of $\mathbf{T}$.
[^3]```
Algorithm 2 GSST
    Input: G: a graph; \(M\) : no. of spanning trees to use.
    \(\mathbf{S}_{\text {min }}=\emptyset\)
    \(s_{\text {min }}=\infty\)
    for \(m=1: M\) do
        \(\mathbf{S}=\emptyset\)
        \(\mathbf{T}=\) GenerateTree \((\mathbf{G})\)
        Randomly choose root \(u_{0}\)
        \(\mathbf{S}=\mathbf{S} \mid\left(0 \rightarrow u_{0}\right)\)
        \(V_{N}^{C}=\left\{u_{0}\right\}, E_{N}^{C}=\emptyset\)
        \(V_{N}^{D}=V-\left\{u_{0}\right\}, E_{N}^{D}=E\)
        \(\mathbf{G}_{N}^{C}=\left(V_{N}^{C}, E_{N}^{C}\right)\)
        \(\lambda=\mathbf{R}-\operatorname{Label}(\mathbf{T})\)
        while \(V_{N}^{D} \neq \emptyset\) do
            \(u v=\operatorname{SelectEdge}\left(\mathbf{G}_{N}^{C}, \mathbf{T}, E_{N}^{D}, \lambda\right)\)
            if a searcher can traverse \(u v\) without recontamination then
                Move that searcher to \(u\) staying inside the clear graph
                \(\mathbf{S}=\mathbf{S} \mid(u \rightarrow v)\)
                \(V_{C}=V_{N}^{C} \cup\{v\}, E_{N}^{C}=E_{C} \cup\{u v\}\)
                \(V_{N}^{D}=V_{N}^{D}-\{v\}, E_{N}^{D}=E_{N}^{D}-\{u v\}\)
                \(\mathbf{G}_{N}^{C}=\left(V_{N}^{C}, E_{N}^{C}\right)\)
            else
                \(\mathbf{S}=\mathbf{S} \mid\left(0 \rightarrow u_{0}\right)\)
            end if
        end while
        if \(\overline{s n}(\mathbf{S})<s_{\text {min }}\) then
            \(\mathbf{S}_{\text {min }}=\mathbf{S}\)
            \(s_{\text {min }}=\overline{s n}(\mathbf{S})\)
        end if
    end for
    Output: Node clearing schedule \(\mathbf{S}_{\text {min }}\).
```

6. Go back to step 1 and repeat the process with a new spanning tree.
7. After the maximum number of trees and (corresponding searches) has been generated, return a search $\mathbf{S}_{\text {min }}$ which attains $s_{\text {min }}$, the minimum value of $\overline{s n}(\mathbf{S})$.

The subroutine SelectEdge chooses an n-dirty edge uv belonging to $\mathbf{T}$ and adjacent to the current clear graph $\mathbf{G}_{\mathbf{N}}^{C}$. There are several ways to perform this selection.

1. Labeled Selection (L). Select the next edge $u v$ of $\mathbf{T}$ to be traversed according to the Barriere $\lambda$ labels (as in algorithm R-Search, presented in Appendix A); however, if traversing $u v$ would cause recontamination then select the next edge in the Barriere sequence; if, at some stage of the search, traversing any edge of $\mathbf{T}$ would cause recontamination (i.e., if all searchers are stuck) then introduce a new searcher at the root.
2. Labeled Selection with Randomized Tie-breaking (LR). Same as the previous except that ties of edge labels are broken randomly.
3. Randomized Selection (R). Choose $u v$ randomly (without making any use of the Barriere labeling) by a uniform probability on all n-dirty edges belonging to $\mathbf{T}$ and adjacent to $\mathbf{G}_{\mathbf{N}}^{C}$ :

$$
\operatorname{Pr}(u v)= \begin{cases}c & \text { if } u \text { is n-clear and } u v \text { is n-dirty and an edge of } \mathbf{T} ; \\ 0 & \text { else. }\end{cases}
$$

4. Labeled Weighted Selection (LW). This rule is intermediate between $R$ and $L$ : edge selection is still random but, instead of a uniform probability distribution, the probability of an edge $u v$ being selected is inversely proportional to its Barriere $\lambda$ label.
5. Label Dominated Selection (LD). This can be done by labeling edges that lead to parts of the graph that are trees (subtrees of the graph). A list of searchers who can move without recontamination can be maintained during search. If an edge adjacent to $V_{N}^{C}(t)$ leads to a subtree of the graph, and enough free searchers are available, clearing this subtree can only improve the search strategies.

By using each of the above rules in the "basic" GSST algorithm, we obtain ten GSST variants: uniform GSST-L, uniform GSST-LR, ... , uniform GSST-LD, DFS GSST-L, ..., DFS GSST-LD. These variants (except for the two GSST-R's) utilize the Barriere labeling, but do not necessarily produce a Barriere traversal of the spanning tre $母^{6}$. In a sense Randomized Selection is the simplest or most naive rule that can be used to select the next move of the search schedule: every n-dirty edge of $\mathbf{T}$ (adjacent to the clear graph) is equally likely to be selected. The remaining three rules can be understood as ways to bias the probability by which edges are selected in some meaningful way. The effectiveness of these rules will be judged by the experiments of Section 6. From the theoretical point of view, we will show in Section 5.3 that the uniform GSST-R and uniform GSST-LD will find a minimal node clearing schedule with probability $1-\alpha^{M}$ where $M$ is the number of iterations and $\alpha \in(0,1)$; we conjecture that this property does not hold for Barriere selection. As a practical matter, the issue is how large $M$ has to be for $1-\alpha^{M}$ to be sufficiently close to 1 . However, the experiments of Section 6 indicate that the above rules find good search schedules in very reasonable time.

Variants of GSST can also be produced by replacing the GenerateTree subroutine with an exhaustive generation of all spanning trees of $\mathbf{G}$ (it can be used in conjunction with any of the above variants of SelectEdge). To do this we have used Char's spanning tree enumeration algorithm [21]. Exhaustive enumeration is feasible only for relatively small graphs.

Finally note that the search schedules produced by (every variant of) GSST are IMC. This holds for the search of both $\mathbf{T}$ and $\mathbf{G}$. Indeed, for the search of $\mathbf{G}$ to work, the search of $\mathbf{T}$ must be IMC. In other words, no obvious modification of GSST will produce, for example, an internal, connected, nonmonotone node-clearing search of $\mathbf{G}$. The question arises: how good is a minimal IMC node clearing of $\mathbf{G}$ (as compared to, for example, an internal but not monotone / connected node clearing)? This is the question already hinted at in Remark 4.9.

Let us close this section by repeating that the basic idea of GSST is to perform all clearing moves along the edges of a spanning tree. This idea exploits the facts that (a) spanning trees can be both generated and searched quickly and (b) blocking recontamination does not require an excessively large number of guards (because a searcher can change roles as a guard and a tree searcher).

### 5.3 Completeness

Definition 5.3. Given a graph $\mathbf{G}$ and a search $\mathbf{S}$ of $\mathbf{G}$, the frontier at $t$ (under $\mathbf{S}$ ) is

$$
V_{N}^{F}(t)=\left\{u: u \in V_{N}^{C}(t) \text { and } \exists v: v \in V_{N}^{D}(t), u v \in E\right\},
$$

[^4]i.e., the $n$-clear nodes which are connected to $n$-dirty nodes.

Lemma 5.4. In a node search, for every $t$, the nodes $u \in V_{N}^{F}(t)$ are guarded.
Proof. A frontier node $u$ is by definition n-clear and adjacent to an n-dirty node $v$. This is only possible if $u$ is guarded.

Definition 5.5. Consider a rooted IMC node clearing search $\mathbf{S}$ of $\mathbf{G}$. Let $t_{0}=0$ and suppose the clearing moves of $\mathbf{S}$ take place at times $t_{1}, \ldots, t_{N}$; let also $t_{0}=0$. The $n$-th phase of $\mathbf{S}($ for $n=1,2, \ldots, N)$ is the time interval $\left[t_{n-1}+1, t_{n}\right]$, i.e. the interval between the $(n-1)$-th and $n$-th clearing move.

The following remarks are rather obvious. For $m=2,3, \ldots$, a target edge $u_{i} u_{m}$ (with $i \in[1, m-1]$ ) corresponds to the $m$-th phase (here we take the root node to be $u_{1}$ ). While $t \in\left[t_{m-1}+1, t_{m}\right]$ the algorithm moves a searcher towards $u_{i} u_{m}$. For $t \in\left[t_{m-1}+1, t_{m}-1\right], u_{i} u_{m}$ is n-dirty, $u_{i}$ is n-clear, $u_{m}$ is n-dirty. At $t=t_{m}$ we have $\mathbf{S}\left(t_{m}\right)=u_{i} \rightarrow u_{m}$ and $u_{i} u_{m}, u_{m}$ are n-cleared.

Lemma 5.6. Given a graph $\mathbf{G}=(V, E)$ and a rooted IMC node clearing search $\mathbf{S}$ of $\mathbf{G}$, produced by either GSST-R or GSST-LW, let $t_{1}, t_{2}, \ldots, t_{N}$ be the times at which clearing moves take place; let also $t_{0}=0$. Then for $n=2, \ldots, N$ :

1. for $t \in\left[t_{n-1}, t_{n}-1\right]: V_{N}^{F}(t)=V_{N}^{F}\left(t_{n-1}\right)$;
2. for $t \in\left[t_{n-1}, t_{n}-1\right]$ : every $u \in V_{N}^{F}(t)$ contains exactly one searcher, except one node $u(t)$ which possibly contains two searchers;
3. for $t=t_{n}$ : every $u \in V_{N}^{F}\left(t_{n}\right)$ contains exactly one searcher.

Proof. The proof is by induction on $n$. Items $1,2,3$ of the theorem hold for $n=1, t \in\left[t_{0}, t_{1}\right]=\{0,1\}$. Suppose they also hold up to the $m$-th phase. In the interval $\left[t_{m}+1, t_{m+1}\right]$ a target edge $u_{i} u_{m+1}$ (with $i \in[1, m])$ is selected and an available searcher is sent towards $u_{i} u_{m+1}$. Because GSST-R / GSST-LW always selects for clearing an edge adjacent to the clear graph, $u_{i} \in V_{N}^{F}\left(t_{m}\right), u_{m+1} \in V_{N}^{D}\left(t_{m}\right)$. There are three cases.
I. $u_{i}$ is neighbor of a single n-dirty node, namely $u_{m+1}$. In this case $t_{m+1}=t_{m}+1$ (i.e., $u_{m+1}$ is n-cleared in one step) and $\left[t_{m}, t_{m+1}\right]=\left\{t_{m}, t_{m}+1\right\}$. Node $u_{i} \notin V_{N}^{F}\left(t_{m+1}\right)$; node $u_{m+1}$ may or may not belong to $V_{N}^{F}\left(t_{m+1}\right)$ but, at any rate, it contains exactly one searcher; no searchers enter or exit any other nodes, hence (by the inductive hypothesis) all $u \in V_{N}^{F}\left(t_{m+1}\right)$ contain exactly one searcher. II. $u_{i}$ is neighbor of more than one n-dirty nodes, one of which is $u_{m+1}$, and there are free searchers. Since $u_{i}$ is a frontier node, by hypothesis it contains a single searcher who, consequently, is stuck. However, if free searchers are available inside n-clear, non-frontier nodes, one of these searchers will be sent to $u_{i} u_{m+1}$ by a sequence of moves. At every $t$ during this sequence, the searcher may enter a frontier node $w$; for that particular $t, w$ will be the only frontier node which contains two searchers. At $t=t_{m+1}-1$ the searcher will be located at $u_{i}$ (which will now contain two searchers) and at $t_{m+1}$ he will enter $u_{m+1}$; this leaves at $t_{m+1}$ every $u \in V_{N}^{F}\left(t_{m}\right)$ with a single searcher and also places a searcher at $u_{m+1}$. The only node which may have been added to the frontier is $u_{m+1}$ which contains a single searcher; every other node $u \in V_{N}^{F}(t+1)$ was previously in the frontier, contained a single searcher and, if a searcher entered $u$ at some $t \in\left[t_{m}+1, t_{m+1}-1\right]$, it exited $u$ at $t+1$; hence every such node at $t_{m+1}$ contains a single searcher.
III. The final case is when all frontier-located searchers are stuck and there are no searchers inside n-clear, non-frontier nodes. In this case a new searcher is placed at the root node and the rest of the analysis is identical to that of Case II. Hence items $1,2,3$ also hold for $\left[t_{m}+1, t_{m+1}\right]$ and we can complete the induction for $n=1,2, \ldots, N$.

Lemma 5.7. Given a graph $\mathbf{G}=(V, E)$ and a tree sequence $\left(\mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{N}\right)$ which satisfies conditions D1-D4 of Theorem 5.1. Then uniform GSST-R /GSST-LW with $M=1$ (i.e., using a single spanning tree) has a nonzero probability of producing a search $\mathbf{S}$ which generates $\left(\mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{N}\right)$.

Proof. The probability that Algorithm 1 generates the tree sequence $\left(\mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{N}\right)$ is

$$
\operatorname{Pr}\left(\mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{N}\right)=\left[\prod_{n=1}^{N} \operatorname{Pr}\left(\mathbf{T}_{n} \mid \mathbf{T}_{N}, \mathbf{T}_{0}, \ldots, \mathbf{T}_{n-1}\right)\right] \operatorname{Pr}\left(\mathbf{T}_{0} \mid \mathbf{T}_{N}\right) \operatorname{Pr}\left(\mathbf{T}_{N}\right)
$$

Note that the conditioning in the above expression always includes $\mathbf{T}_{N}$, since this is the first choice made in running GSST-R / GSST-LW. Now obviously, $\operatorname{Pr}\left(\mathbf{T}_{0} \mid \mathbf{T}_{N}\right)=1$. By Wilson's Theorem 1 [37], $\operatorname{Pr}\left(\mathbf{T}_{N}\right)>0$ for every spanning tree $\mathbf{T}_{N}$. Also, $\operatorname{Pr}\left(\mathbf{T}_{n} \mid \mathbf{T}_{N}, \mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{n-1}\right)$ is the probability of expanding (at the $n$-th step) $\mathbf{T}_{n-1}$ by the edge $u_{i} u_{n} \in \mathbf{E}_{n}-\mathbf{E}_{n-1}$ which, by the construction of both GSST-R and GSST-LW, is always positive. Finally, $\operatorname{Pr}\left(\mathbf{T}_{N} \mid \mathbf{T}_{N}, \mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{N-1}\right)=1$. Hence $\operatorname{Pr}\left(\mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{N}\right)>0$ for every sequence $\mathbf{T}_{1}, \ldots, \mathbf{T}_{N}$.

Lemma 5.8. Given a graph $\mathbf{G}=(V, E)$ and a rooted IMC node clearing search $\mathbf{S}$ of $\mathbf{G}$; let $\left(\mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{N}\right)$ be the tree sequence generated by $\mathbf{S}$. Let $\mathbf{S}^{\prime}$ be a search produced by either GSST-R or GSST-LW and also generating $\left(\mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{N}\right)$. Then $\overline{s n}(\mathbf{S}) \geq \overline{s n}\left(\mathbf{S}^{\prime}\right)$.

Proof. The proof is exactly the same for GSST-R and GSST-LW, so we only prove the first one, by induction. Let $t_{1}, \ldots, t_{N}$ be the clearing times of $\mathbf{S}$ and $t_{1}^{\prime}, \ldots, t_{N}^{\prime}$ be the clearing times of $\mathbf{S}^{\prime}$. Also let $t_{0}=t_{0}^{\prime}=0$.

At $t_{0}=t_{0}^{\prime}=0$ we have $s n\left(\mathbf{S}^{\prime}, 0\right)=s n(\mathbf{S}, 0)=0$.
The only times at which $\operatorname{sn}\left(\mathbf{S}^{\prime}, t\right)$ may change are $1, t_{1}^{\prime}+1, \ldots, t_{N-1}^{\prime}+1$. Suppose that

$$
\operatorname{sn}\left(\mathbf{S}, t_{n}\right) \geq \operatorname{sn}\left(\mathbf{S}^{\prime}, t_{n}^{\prime}\right)
$$

Further, suppose that at $t_{n}^{\prime}+1$ a new searcher is introduced in $\mathbf{S}^{\prime}$. This can only happen (in the $\mathbf{S}^{\prime}$ search) if all of the following hold:

1. at $t_{n}^{\prime}$ exactly $\left|V_{N}^{F}(t)\right|$ searchers exist in $\mathbf{G}$;
2. there are no searchers inside nodes $u \in V_{N}^{C}\left(t_{n}^{\prime}\right)-V_{N}^{F}\left(t_{n}^{\prime}\right)$ (i.e., all searchers are located inside frontier nodes);
3. all searchers are stuck (i.e., moving a searcher out of a frontier node $u$ exposes $u$ to recontamination).

The sequence $\left(\mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{N}\right)$ along with the clearing times determines the frontier $V_{N}^{F}(t)$ for every $t$. Hence $\mathbf{S}^{\prime}$ at $t_{n}^{\prime}$ has the same frontier as $\mathbf{S}$ at $t_{n}$. If conditions 1-3 above hold in $\mathbf{S}^{\prime}$, then every searcher is located in a frontier node and is stuck. It is possible that non-stuck searchers exist in $\mathbf{S}$ (located either in frontier or non-frontier nodes) but this also means that $\operatorname{sn}\left(\mathbf{S}, t_{n}\right) \geq s n\left(\mathbf{S}^{\prime}, t_{n}^{\prime}\right)+1$; hence adding a searcher in $\mathbf{S}^{\prime}$ at $t_{n}^{\prime}+1$ preserves

$$
s n\left(\mathbf{S}, t_{n}\right) \geq s n\left(\mathbf{S}^{\prime}, t_{n}^{\prime}+1\right) .
$$

Since no searchers are added in $\mathbf{S}^{\prime}$ for $t \in\left[t_{n}^{\prime}+2, t_{n+1}^{\prime}\right]$ and no searchers are ever removed in $\mathbf{S}$ (i.e., $\left.s n\left(\mathbf{S}, t_{n+1}\right) \geq \operatorname{sn}\left(\mathbf{S}, t_{n}\right)\right)$ we also get

$$
\operatorname{sn}\left(\mathbf{S}, t_{n+1}\right) \geq \operatorname{sn}\left(\mathbf{S}^{\prime}, t_{n+1}^{\prime}\right) .
$$

From the above inequality inductively we get $s n\left(\mathbf{S}, t_{N}\right) \geq s n\left(\mathbf{S}^{\prime}, t_{N}^{\prime}\right)$ which proves the Lemma.

Theorem 5.9. Given a graph $\mathbf{G}=(V, E)$.

1. Uniform GSST-R will generate a minimal rooted IMC clearing node search of $\mathbf{G}$ with probability greater than or equal to $1-\alpha_{1}^{M}$ where $M$ is the number of iterations and $\alpha_{1} \in(0,1)$.
2. Uniform GSST-LW will generate a minimal rooted IMC clearing node search of $\mathbf{G}$ with probability greater than or equal to $1-\alpha_{2}^{M}$ where $M$ is the number of iterations and $\alpha_{2} \in(0,1)$.

Proof. The proof is exactly the same for GSST-R and GSST-LW, so we only prove the first one. G has at least one minimal rooted IMC node clearing search $\mathbf{S}$ of $\mathbf{G}$. Let $\left(\mathbf{T}_{0}, \mathbf{T}_{1}, \ldots, \mathbf{T}_{N}\right)$ be the tree sequence generated by $\mathbf{S}$. By Lemma 5.7, GSST-R has a nonzero probability, call it $\beta_{1}$, of generating in a single iteration a search $\mathbf{S}^{\prime}$ with the same tree sequence as $\mathbf{S}$. Then, by Lemma 5.8,

$$
\overline{s n}(\mathbf{S}) \geq \overline{s n}\left(\mathbf{S}^{\prime}\right) .
$$

Since $\mathbf{S}$ is minimal, $\overline{s n}(\mathbf{S})=\overline{s n}\left(\mathbf{S}^{\prime}\right)$ and so $\mathbf{S}^{\prime}$ is minimal too. Now, the probability of not generating $\mathbf{S}^{\prime}$ in a single iteration is $\alpha_{1}=1-\beta_{1}$; and the probability of not generating $\mathbf{S}^{\prime}$ in $M$ iterations is $\alpha_{1}^{M}=\left(1-\beta_{1}\right)^{M}$, while the probability of generating $\mathbf{S}^{\prime}$ in $M$ iterations is $1-\alpha_{1}^{M}$.

Finally, let us mention that the GSST algorithm can be modified to produce an edge- rather than node-clearing search using the following theorem.

Theorem 5.10. Given a graph $\mathbf{G}$ and an IMC node clearing search $\mathbf{S}$ using $K$ searchers, there is an edge clearing search $\mathbf{S}^{\prime}$ using either $K$ or $K+1$ searchers.

Proof. Suppose that $\mathbf{G}$ contains $L_{0}$ edges, that the length (i.e., number of moves) of $\mathbf{S}$ is $t_{\text {fin }}$ and that $\overline{s n}(\mathbf{S})=K$. The new search $\mathbf{S}^{\prime}$ will consist of the $\mathbf{S}$ moves (executed at integer times $t=1,2, \ldots, t_{\text {fin }}$ ) combined with the moves of an extra searcher, the "edge cleaner", who will only (if at all) move at fractional time steps of the form $t+\frac{l}{3 L_{0}+1}, l=1,2, \ldots, 3 L_{0}$ (fractional times are introduced to preserve the "alignment" of $\mathbf{S}$ and $\mathbf{S}^{\prime}$, i.e. to ensure $\mathbf{S}(t)=\mathbf{S}^{\prime}(t)$ at integer times; of course, once $\mathbf{S}^{\prime}$ has been obtained, the time scale can be renormalized, so that all moves occur at integer times). We will use the notation $t^{-}=t-\frac{1}{3 L_{0}+1}$.

We will describe the moves of of the edge cleaner on a step-by-step basis, for $t=1,2, \ldots, t_{\text {fin }}$ in such a manner that at the same time we will complete an inductive proof of the fact that
for $t=1,2, \ldots, t_{\text {fin }}+1: \quad V_{N}^{C}(t-1)=V_{E}^{C}(t-1), \quad V_{N}^{F}(t-1)=V_{E}^{F}(t-1), \quad E_{N}^{C}\left(t^{-}\right)=E_{E}^{C}\left(t^{-}\right) ;$ recall that $V_{N}^{F}$ is the frontier, i.e., the set of n-clear nodes connected to n-dirty nodes (and similarly for $V_{E}^{F}$ ) and that the frontier nodes are always guarded in both the node and edge game.

For $t=1$ we have $V_{N}^{C}(0)=V_{E}^{C}(0)=\emptyset, V_{N}^{F}(0)=V_{E}^{F}(0)=\emptyset$, and the edge cleaner is not used, so $E_{N}^{C}\left(1^{-}\right)=E_{N}^{C}(0)=E_{E}^{C}(0)=E_{E}^{C}\left(1^{-}\right)$. Now suppose that

$$
\begin{equation*}
\text { for } s=1,2, \ldots, t \quad V_{N}^{C}(t-1)=V_{E}^{C}(t-1), \quad V_{N}^{F}(t-1)=V_{E}^{F}(t-1), \quad E_{N}^{C}\left(t^{-}\right)=E_{E}^{C}\left(t^{-}\right) \tag{31}
\end{equation*}
$$

and consider $s=t+1$. Let the $(t+1)$-th move of $\mathbf{S}$ be $u \rightarrow v$. We consider three cases.

1. Case I. If $u$ is an interior node (i.e., a non-frontier node: $u \in V_{N}^{C}(t-1)-V_{N}^{F}(t-1)=$ $\left.V_{E}^{C}(t-1)-V_{E}^{F}(t-1)\right)$, then $u v$ is an edge of $\mathbf{G}_{N}^{C}(t)=\mathbf{G}_{E}^{C}(t)$ and no new nodes/edges are
cleared, either in the node or edge game. No path becomes unguarded, and so no recontaminaiton is possible either. Hence

$$
\begin{align*}
& V_{N}^{C}(t)=V_{N}^{C}(t-1)=V_{E}^{C}(t-1)=V_{E}^{C}(t),  \tag{32}\\
& V_{N}^{F}(t)=V_{N}^{F}(t-1)=V_{E}^{F}(t-1)=V_{E}^{F}(t) .
\end{align*}
$$

The edge cleaner is not used, hence also

$$
\begin{equation*}
E_{N}^{E}\left((t+1)^{-}\right)=E_{N}^{E}\left(t^{-}\right)=E_{E}^{E}\left(t^{-}\right)=E_{N}^{E}\left((t+1)^{-}\right) . \tag{33}
\end{equation*}
$$

2. Case II. If $u$ is a frontier node $\left(u \in V_{N}^{F}(t-1)=V_{E}^{F}(t-1)\right)$ which contains a single searcher at $t-1$, then $u$ becomes unguarded at $t$. Clearly

$$
\begin{equation*}
V_{N}^{C}(t)=V_{N}^{C}(t-1) \cup\{v\} . \tag{34}
\end{equation*}
$$

It is also easy to see that there is no edge $u y$ with $y \in V_{N}^{D}(t-1)=V_{E}^{D}(t-1)$ and $y \neq v$ : if such an edge existed, then we would have $y \in V_{N}^{D}(t)(y$ was not entered at $t)$ and so $u \in V_{N}^{D}(t)$; but $u \in V_{N}^{D}(t-1)$ (it was guarded at $\left.t-1\right)$ and so node monotonicity of $\mathbf{S}$ would be violated.
Take any edge $p q \in E_{E}^{C}\left(t^{-}\right)=E_{N}^{C}\left(t^{-}\right)$. Edge $p q$ cannot be recontaminated in the node game (no node was recontaminated). Edge $p q$ cannot be recontaminated in the edge game either; for this to happen there must exist an e-unguarded path from $p q$ to some $x z \in E_{E}^{D}\left(t^{-}\right)$; but all such paths must go through $u$ (no other node became unguarded at $t$ ) and hence must include $u v$ (the only edge e-dirty at $t^{-}$and incident on $u$ ) but $v$ is guarded at $t$. Hence no edge is e-dirtied at $t$ and so no node is e-dirtied either. In other words

$$
\begin{equation*}
V_{E}^{C}(t)=V_{E}^{C}(t-1) \cup\{v\} . \tag{35}
\end{equation*}
$$

which, together with (34) shows that $V_{N}^{C}(t)=V_{E}^{C}(t)$. Also, $V_{N}^{F}(t)=V_{E}^{F}(t)$; in both the node and edge game, $u$ was removed from the frontier and $v$ was possibly added to it. In short, we have established (32) for Case II as well.
From the previous remarks we know that, in the edge game and at time $t$, no edge was recontaminated and edge $u v$ was e-cleared. I.e.,

$$
E_{E}^{C}(t)=E_{E}^{C}\left(t^{-}\right) \cup\{v u\} .
$$

In the node game, on the other hand,

$$
\begin{equation*}
E_{N}^{C}(t)=E_{N}^{C}\left(t^{-}\right) \cup\left\{v x_{0}, v x_{1}, \ldots, v x_{L}\right\}=E_{N}^{C}\left((t+1)^{-}\right) \tag{36}
\end{equation*}
$$

where $L \geq 0, x_{0}=u$ and there may exist other nodes $x_{1}, x_{2}, \ldots, x_{L}$ which must (a) be neighbors of $v$ and (b) belong to $V_{N}^{F}(t-1) \subseteq V_{N}^{C}(t-1)$. Hence $x_{1}, \ldots, x_{L}$ were guarded at $t-1$ and remain so at $t$. In other words, at time $t$ the edges $v x_{1}, \ldots, v x_{L}$ have both endpoints guarded and are n-clear but e-dirty. Now we invoke the edge cleaner who (at times $t+\frac{1}{3 L_{0}+1}, t+\frac{2}{3 L_{0}+1}, \ldots$ ) moves to $v$ and then performs the moves $v \rightarrow x_{1}, x_{1} \rightarrow v, v \rightarrow x_{2}, x_{2} \rightarrow v, \ldots, x_{L} \rightarrow v$. This entire sequence can be performed in no more than $3 L_{0}$ moves, so at $t=(t+1)^{-}$the edges $v x_{0}, v x_{1}, \ldots, v x_{L}$ have been e-cleared and so

$$
\begin{equation*}
E_{E}^{C}\left((t+1)^{-}\right)=E_{N}^{C}\left(t^{-}\right) \cup\left\{v x_{0}, v x_{1}, \ldots, v x_{L}\right\} . \tag{37}
\end{equation*}
$$

Combining (36) and (37) we get $E_{N}^{C}\left((t+1)^{-}\right)=E_{E}^{C}\left((t+1)^{-}\right)$. In short, we have established (33) for Case II as well.
3. Case III. The final case to examine is when $u$ is a frontier node which contains more than one searcher at $t-1$. We omit a detailed treatment because the proof combines elements from the previous two cases; namely, recontamination does not happen (for the same reasons as in Case I) bu the edge cleaner may possibly be required (as in Case II).

Hence, in all three cases considered, starting from (31) we have established (32) and (33). Hence we can complete the induction up to time $t=t_{\text {fin }}+1$ which means

$$
\begin{equation*}
V_{N}^{C}\left(t_{f i n}\right)=V_{E}^{C}\left(t_{f i n}\right), \quad E_{N}^{C}\left(\left(t_{f i n}+1\right)^{-}\right)=E_{E}^{C}\left(\left(t_{f i n}+1\right)^{-}\right) \tag{38}
\end{equation*}
$$

Eqs.(38) imply that all nodes are e-cleared at $t_{\text {fin }}$ but a few extra steps may be required to e-clear all edges (by time $\left(t_{f i n}+1\right)^{-}$at most).

## 6 Graph Search Experiments

In this section we evaluate the performance of the GSST algorithm by numerical experiments. Some of these experiments involve specific graphs (Section 6.1) and others involve families of graphs (Sections 6.2 and 6.3 - in which case we present average results). We use ten variants of GSST, obtained by using two different methods of spanning tree generation (uniform and DFS) and five methods of edge traversal (GSST-L, GSST-R, GSST-LR, GSST-LW, and GSST-LD).

### 6.1 Experiments using Individual Graphs

### 6.1.1 Simple Graph

The first graph we have used appears in Fig 2. This is a relatively simple graph (with IMC node search number $s_{N}^{i m c}=3$ ) which we use to illustrate the basic principles of GSST operation.


Figure 2: A simple graph.
We node-search the graph using the ten GSST variants, each with $M=5000$ generated spanning trees; to each spanning tree corresponds a search $\mathbf{S}_{m}, m=1,2, \ldots, 5000$. In Table 2 we list for each GSST variant: (a) $\min _{m=1,2, \ldots, M} \overline{s n}\left(\mathbf{S}_{m}\right)$ (i.e., the minimum number of searchers achieved by the
specific combination), (b) the proportion of minimal searches out of the the $M$ total searches and (c) the tim\& ${ }^{7}$ (in sec) required to run the $M$ searches.

|  | Uniform ST generation |  |  | DFS ST generation |  |  |
| :--- | ---: | ---: | :--- | ---: | ---: | ---: |
| Edge Traversal | Min | Prop. of min | Time | Min | Prop. of min | Time |
| GSST-L | 3 | 0.1562 | 0.203125 | 3 | 0.1804 | 0.203125 |
| GSST-R | 3 | 0.2031 | 0.171875 | 3 | 0.2848 | 0.203125 |
| GSST-LR | 3 | 0.2656 | 0.265625 | 3 | 0.2096 | 0.265625 |
| GSST-LW | 3 | 0.2292 | 0.218750 | 3 | 0.2854 | 0.250000 |
| GSST-LD | 3 | 0.2031 | 0.218750 | 3 | 0.3020 | 0.234375 |

Table 2. Node-clearing the "simple graph" by the various GSST variants: minimum number of searchers attained and proportion of minimal solutions; number of spanning trees generated is $M=$ $5 \cdot 10^{3}$.

The true node search number (i.e., 3) has been found by every variant of GSST. Generally, the DFS variants perform better than the uniform ones, as can be seen by the higher proportion of minimal solutions achieved. A better understanding of the distribution of the number of searchers required by each search can be obtained by looking at the histogram of the distribution; one such histogram (for the variant with BH edge traversal and uniform spanning tree generation) is plotted in Fig.3. We can see that this simple graph has a high proportion of spanning trees which yield minimal schedules.


Figure 3: The histogram of the distribution of minimum number of searchers required to node clear the "simple graph" by the uniform GSST-L variant.

An additional tool to evaluate the anytime performance of GSST is the plot of $R(m)=\min _{i \leq m} \overline{s n}\left(\mathbf{S}_{i}\right)$ (the minimum node clearing number of searchers achieved by the first $m$ searches). $R(m)$ is decreasing with $m$. The overall minimum achieved by GSST is $R(M)$ (having tried $M$ searches). If this minimum is achieved for a small value of $m$, then the minimal solution has been obtained quickly. A graph of $R(m)$ (for the variant with labeled edge traversal and uniform spanning tree generation) appears in Fig. 4 , as can be seen a minimal solution (clearing the graph with three searchers) is achieved by the seventh computed search, approximately at time $t=0.203125 \cdot 7 / 5000=2.8438 \times 10^{-4} \mathrm{sec}$.

The "simple graph" is simple enough to have a relatively small number of spanning trees (namely 272, as computed by Kirchoff's theorem [18] ) and so we can also run GSST on all spanning trees

[^5]

Figure 4: Plot of $R(m)$ vs. $m$ for the "simple graph" and the uniform GSST-L variant. The $m$ axis is truncated at $m=20$, since the rest of the plot shows no change.
(exhaustive enumeration). The computation takes 0.03125 (for the GSST-L variant) sec and shows that 83 out of the 272 spanning trees (rooted at node 1) yield minimal searches ${ }^{8}$.

### 6.1.2 Tree/Grid Graph

The second graph we have used appears in Fig.5. It consists of a "root" node and two branches under it; the left branch is a tree and the right one a grid; hence the name "tree/grid". The graph has $s_{N}^{i m c}=4$ and we have deliberately designed it to "trick" the GSST algorithm. For the sake of definiteness consider GSST-L. If the root of the search is node 6 , then GSST-L will find a four searcher node clearing IMC schedule. However, if the root is node 1, then GSST-L will only find a five searcher schedule, even after enumerating all spanning trees. A four searcher IMC node clearing schedule is possible from either starting node; but it requires the use of a non-Barriere edge traversal (for example, one produced by the GSST-R variant); but GSST-L will always first send the searchers down the right branch (towards the grid); actually, going first to the left branch, towards the tree, is better (i.e. yields a four searcher schedule).

Table 3 summarizes the results of our experiment, using the ten variants of GSST and $M=5 \cdot 10^{4}$ spanning trees per variant. Note that in this table (as in all others) the starting node is selected randomly. The uniform variants are able to generate minimal search schedules, but not the DFS ones.

|  | Uniform ST generation |  |  | DFS ST generation |  |  |
| :--- | ---: | ---: | :--- | ---: | ---: | ---: |
| Edge Traversal | Min | Prop. of min | Time | Min | Prop. of min | Time |
| GSST-L | 4 | 0.00054 | 5.921875 | 5 | 0.06380 | 6.687500 |
| GSST-R | 4 | 0.00328 | 3.734375 | 5 | 0.06956 | 4.390625 |
| GSST-LR | 4 | 0.00076 | 8.328125 | 5 | 0.06976 | 8.765625 |
| GSST-LW | 4 | 0.00206 | 4.812500 | 5 | 0.07252 | 5.609375 |
| GSST-LD | 4 | 0.00232 | 4.968750 | 5 | 0.09376 | 5.656250 |

Table 3. Node-clearing the "tree/grid" by the various GSST variants: minimum number of searchers attained and proportion of minimal solutions; number of spanning trees generated is $M=5 \cdot 10^{4}$.

[^6]

Figure 5: The "tree/grid" graph.

In this case the uniform variants find four-searcher clearing schedules and hence outperform the DFS generated ones, which can only clear the graph with five or more searchers? Note also the lower proportion of minimal solutions, even for the uniform variants. For example, uniform GSST-L finds $0.00054 \cdot 50000=27$ minimal solutions. This in an indication that this graph is indeed harder than the "simple" one, at least for the GSST algorithm. Also, the tree/grid graph has 31529 spanning trees (and correspondingly many cycles) which is an additional indication that it is (much) harder to search than the simple graph, which has 272 spanning trees. The small number of minimal solutions can also be appreciated by looking at the histogram (for uniform GSST-L variant it is plotted in Fig.6).


Figure 6: The histogram of the distribution of minimum number of searchers required to node clear the "hard graph" by the uniform GSST-L variant. There is a small, barely noticeable, bar at $k=4$, corresponding to the four-searcher clearing schedules.

However, despite the small proportion of minimal solutions, the first one is always found after a relatively small number of iterations. For example, the uniform GSST-L variant finds a four searcher schedule with the 420-th spanning tree generated (out of a total of $5 \cdot 10^{4}$ spanning trees), approximately at time $t=5.921875 \cdot 420 / 50000=4.9744 \times 10^{-2} \mathrm{sec}$.

[^7]
### 6.1.3 NSH Graph

The next graph we use has been obtained by discretization of an actual floorplan, namely the first floor of the Newel-Simon building in the Carnegie Mellon University campus. In Fig 7 we present the actual floorplan and its discretization; in Fig. 8 we present the resulting graph (the node numbers in Fig 8 correspond to the cell numbers in Fig 7 ).


Figure 7: The NSH floorplan.


Figure 8: The NSH graph.

This graph, with 60 nodes and 64 edges, appears to be more complex than the previous ones; however it has a relatively small number of spanning trees (namely 3604, as computed by Kirchoff's theorem). By visual inspection it appears very unlikely that the graph can be node-cleared with two searchers; on the other hand, as will be seen presently, GSST can compute three-searcher clearing schedules. Hence we conjecture that the search number $s_{N}^{i m c}$ is three.

Applying the "standard" variants of GSST and using $M=5 \cdot 10^{4}$ spanning trees per variant we obtain the results of Table 4.

|  | Uniform ST generation |  |  | DFS ST generation |  |  |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: |
| Edge Traversal | Min | Prop. of min | Time | Min | Prop. of min | Time |
| GSST-L | 3 | 0.00224 | 14.265625 | 3 | 0.01038 | 14.578125 |
| GSST-R | 3 | 0.00002 | 9.406250 | 3 | 0.00006 | 10.140625 |
| GSST-LR | 3 | 0.00478 | 21.937500 | 3 | 0.01026 | 21.390625 |
| GSST-LW | 3 | 0.00328 | 11.546875 | 3 | 0.00608 | 12.593750 |
| GSST-LD | 3 | 0.00544 | 12.625000 | 3 | 0.00880 | 13.578125 |

Table 4. Node-clearing the NSH graph by the various GSST variants: minimum number of searchers attained and proportion of minimal solutions; number of spanning trees generated is $M=5 \cdot 10^{4}$.

We see that all the GSST variants using uniform spanning tree generation achieve the true search number, namely three. The random edge traversal variants perform poorest of all: the uniform GSST-R finds only one minimal search and the DFS GSST-R variant three. Generally, while the time required to complete $M$ searches is higher for the NSH graph than for the "tree/grid" graph, the proportion of correct solutions is in some cases higher (at least for the uniform variants). A histogram of the searchers required appears in Fig 9 , for the uniform GSST-L variant; for this variant the first minimal search schedule is computed at step 845 (out of 50000), approximately at time $14.265625 \cdot 845 / 50000=$ 0.24109 sec .


Figure 9: The histogram of the distribution of minimum number of searchers required to node clear the NSH graph by the uniform GSST-L variant.

### 6.1.4 National Art Gallery Graph

The next graph we use has also been obtained by discretization of an actual floorplan, namely the first floor of the National Gallery of Art, in Washington, DC. In Fig 10 we present the actual floorplan and the discretization we have used; in Fig. 11 we plot the corresponding graph.


Figure 10: The National Gallery of Art floorplan.


Figure 11: The National Gallery of Art graph.

This graph, has 70 nodes and 93 edges and is more complex than any of the previously used graphs. In particular, it is more complex than the NSH graph. While the National Art Gallery graph has about $50 \%$ more edges than the NSH graph, it has a much larger number of spanning, approximately $5.3 \cdot 10^{14}$, as compared to 3604 ; this precludes use of the GSST with exhaustive enumeration of the spanning trees. Also, we have no obvious way to compute the true node search number. Nevertheless, we proceed to apply the ten GSST variants using $M=1.5 \cdot 10^{5}$ spanning trees per combination. The results obtained appear in Table 5.

|  | Uniform ST generation |  |  | DFS ST generation |  |  |
| :--- | ---: | ---: | :--- | ---: | ---: | ---: |
| Edge Traversal | Min | Prop. of min | Time | Min | Prop. of min | Time |
| GSST-L | 5 | 0.0000066 | 74.378237 | 5 | 0.0004466 | 77.214947 |
| GSST-R | 6 | 0.0013133 | 45.500000 | 5 | 0.0004800 | 50.984375 |
| GSST-LR | 5 | 0.0000066 | 78.172631 | 5 | 0.0002000 | 87.250000 |
| GSST-LW | 5 | 0.0000200 | 62.859375 | 5 | 0.0007600 | 77.500000 |
| GSST-LD | 5 | 0.0000066 | 64.656250 | 5 | 0.0009800 | 77.546875 |

Table 5. Node-clearing the National Art Gallery graph by the various GSST variants: minimum number of searchers attained and proportion of minimal solutions; number of spanning trees generated is $M=1.5 \cdot 10^{5}$.

We see that all the GSST variants ${ }^{10}$ find node clearing searches with five searchers. While we cannot be sure that the true node search number of the National Art Gallery is five, we have been unable to find a lower search number using any method (including extensive inspection by the authors). In addition, the middle part of the graph resembles a five-by-six grid, which is known to have a node search number of five (see Section 6.3). A histogram of the searchers required appears in 12, for the uniform GSST-L variant; for this variant the first minimal search schedule is found at step 101823 (out of a total $1.5 \cdot 10^{5}$ ). Similar results hold for the other variants.


Figure 12: The histogram of the distribution of minimum number of searchers required to node clear the National Gallery of Art graph by the uniform GSST-L variant.

The National Art Gallery graph results (as well as the NSH ones) show the applicability of GSST on graphs that are derived from representations of real indoor environments (which has been the main motivation for our research). The methods that incorporate Barriere labeling improve performance on the NSH map because it is similar to a tree (it becomes a tree if we remove only a few edges). In this case, Barriere labeling, which is based on trees, helps to improve the schedules generated by GSST.

[^8]The National Art Gallery, on the other hand, is more similar to a grid, which lessens the advantage of using Barriere labeling to guide traversal (though a significant improvement is still obtained).

### 6.2 Interval Graphs

The next experiment presented involves, unlike the ones of Section 6.1, a large number of graphs belonging to the same family. Our goal is to evaluate the average performance of the GSST algorithm. While, strictly speaking, the results are specific to the family of interval graphs, they also suggest general properties of the GSST algorithm.

To evaluate the performance of GSST we must have some estimate of the actual search number of each graph we use; then we can compare this search number with the minimum $\overline{s n}(\mathbf{S})$ achieved by GSST. To satisfy this condition, we will work with interval graphs. Briefly, an interval graph is obtained from a system of intervals, i.e. a collection of intervals of real numbers; each interval corresponds to a graph node and two nodes are connected by an edge iff the corresponding intervals intersect. It is known [10, 22] that the (non-monotone, non-connected) edge search number of an interval graph $\mathbf{G}$ (i.e., $s_{E}(\mathbf{G})$ ) is equal to its interval width, which is defined to be the size of the largest clique of the graph. It is also known that the interval width of an interval graph $\mathbf{G}$ of $N$ nodes can be computed in time $\mathrm{O}(N)$ [10, 22]. Of course, while we know the edge search number $s_{E}(\mathbf{G})$ exactly, this only provides an estimate of the IMC node-search number $s_{N}^{i m c}(\mathbf{G})$, in which we are really interested. There is no strict inequality connecting $s_{E}(\mathbf{G})$ and $s_{N}^{i m c}(\mathbf{G})$; we just know that $s_{N}(\mathbf{G}) \leq s_{E}(\mathbf{G})$ and $s_{N}(\mathbf{G}) \leq s_{N}^{i m c}(\mathbf{G})$. In general, we expect that the discrepancy between $s_{E}(\mathbf{G})$ and $s_{N}^{i m c}(\mathbf{G})$ is not too large and hence $s_{E}(\mathbf{G})$ can be used to evaluate GSST performance.

We randomly generate interval systems (and the corresponding interval graphs) by the following mechanism. First we select two parameters of the family: $N$, the number of intervals, and $\Delta$, the average interval length. Then we generate $N$ intervals, the $n$-th interval having its left endpoint at $n$ and the right endpoint at $n+\delta$, where $\delta$ follows an exponential probability law $f(\delta) \sim e^{-\delta / \Delta}$. We form the interval graph $\mathbf{G}$ corresponding to this interval system and compute its interval width. We repeat the process 100 times to obtain 100 interval graphs; these form a family characterized by the parameters $N, \Delta$ (specified by us) and also by the average number of edges and the average interval width. We repeat the process for five different choices of $(N, \Delta)$; this information is summarized in Table 6.

| $N$ | $\Delta$ | Average num. of Edges | Average Interval Width |
| ---: | ---: | ---: | ---: |
| 30 | 3 | 74.22 | 5.49 |
| 30 | 5 | 113.06 | 7.66 |
| 25 | 7 | 182.01 | 9.51 |
| 35 | 10 | 231.84 | 11.68 |
| 40 | 15 | 366.79 | 15.53 |

Table 6. Characteristics of the various families of interval graphs used for the experiments.

Finally, we apply the ten GSST variants to each of the five familes (using $M=2 \cdot 10^{4}$ spanning trees per variant). The results are summarized in Tables 7.a (for the uniform variants) and 7.b (for the DFS variants). In these tables every row corresponds to a family, the families being indexed by their average interval widths (appearing in the first column). Each of the remaining columns corresponds to one GSST variant and lists the average (over the 100 graphs) minimum number of searchers achieved by the respective variant. Hence, an estimate of the efficiency of the GSST variants can be obtained by comparing the first column of each table to the remaining ones. For example, in the first row of Table 7.a we see that all uniform GSST variants node clear the graphs of the first family with fewer searchers (on the average) than those "predicted" by the interval width of the graphs (e.g., $5.15<5.49$ ). As
we proceed down the rows of Table 7.a to graphs of higher interval width (and, presumably, of greater complexity) the average number of searchers required to clear a graph increases above the average interval width. For example, in the last row the average interval width is 15.53 and the average minimum number of searchers required by the uniform GSST-L variant is 19.05; in other words the uniform GSST-L requires $\frac{19.05-15.53}{15.53}=22.66 \%$ more searchers than expected by the interval width estimate. Things get better with the DFS variants; for example, for the ( $N=40, \Delta=15$ ) family, the searcher overhead incurred by the DFS GSST-L variant is $\frac{17.68-15.53}{15.53}=13.84 \%$, which is actually quite good for graphs of such high complexity.

| Av. Int. Width | GSST-L | GSST-R | GSST-LR | GSST-LW | GSST-LD |
| ---: | ---: | ---: | ---: | ---: | ---: |
| 5.49 | 5.15 | 5.32 | 5.36 | 5.14 | 5.16 |
| 7.66 | 7.82 | 8.08 | 8.03 | 7.80 | 7.87 |
| 9.51 | 11.01 | 11.42 | 11.45 | 11.02 | 11.30 |
| 11.68 | 13.61 | 13.91 | 13.91 | 13.60 | 13.63 |
| 15.53 | 19.05 | 19.37 | 19.50 | 19.06 | 19.25 |

Table 7.a: Average minimum number of searchers required to node-clear each family of interval graphs by the uniform GSST variants; number of spanning trees generated is $M=2 \cdot 10^{4}$.

| Av. Int. Width | GSST-L | GSST-R | GSST-LR | GSST-LW | GSST-LD |
| ---: | ---: | ---: | ---: | ---: | ---: |
| 5.49 | 5.04 | 5.13 | 5.04 | 5.08 | 5.06 |
| 7.66 | 7.34 | 7.47 | 7.39 | 7.44 | 7.36 |
| 9.51 | 10.23 | 10.34 | 10.25 | 10.25 | 10.24 |
| 11.68 | 12.70 | 12.80 | 12.64 | 12.73 | 12.62 |
| 15.53 | 17.68 | 17.87 | 17.75 | 17.72 | 17.66 |

Table 7.b: Average minimum number of searchers required to node-clear each family of interval graphs by the DFS GSST variants; number of spanning trees generated is $M=2 \cdot 10^{4}$.

The results on interval graphs show that GSST can yield near-minimal schedules on a large class of graphs. They also demonstrate that GSST scales well with increasing complexity both in performance and computation time. This is a direct result of the linear scalability of the algorithm in the number of nodes in the environment.

### 6.3 Grid Graphs

### 6.3.1 Full Grids

We now present some experiments involving grid graphs, i.e. graphs with nodes located at points with integer coordinates (an example appears in Fig. 13). It is easily seen that a grid graph of dimension $J_{1} \times J_{2}$ (i.e. containing nodes with coordinates $\left.\left(j_{1}, j_{2}\right) \in\left\{1,2, \ldots, J_{1}\right\} \times\left\{1,2, \ldots, J_{2}\right\}\right)$ can be node cleared using $J_{0}=\min \left(J_{1}, J_{2}\right)$ searchers; the corresponding search schedule is obvious ${ }^{111}$. However, for a general graph search algorithm (such as GSST) which must work without assuming any special structure of the graph, grid graphs are potentially extremely hard, because they have many cycles and many spanning trees, only a few of which correspond to minimal search schedules.

In this experiment we use 6 grid graphs, with dimensions starting at $5 \times 5$ and going up to $10 \times 10$. To each such graph we apply the ten GSST variants; in Tables 8.a (uniform ST generation variants) and 8.b (DFS ST generation variants) we list the minimum search number attained by each variant on

[^9]each graph ( $M$, the number of spanning trees generated per graph is also listed, in the last column). Let us also note that the total execution time for this experiment is approximately 25 hours, reflecting (a) the large number of graphs used, (b) the high complexity of many graphs, (c) the large number of spanning trees used (the much smaller execution time for each individual graph and GSST variant is not listed, for economy of space).

| Uniform random spanning tree generation |  |  |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | :---: |
| Graph Dim. | GSST-L | GSST-LR | GSST-R | GSST-LW | GSST-LD | No.Trees |  |
| $5 \times 5$ | 5 | 5 | 5 | 5 | 5 | $10^{5}$ |  |
| $6 \times 6$ | 7 | 7 | 7 | 7 | 7 | $2 \cdot 10^{5}$ |  |
| $7 \times 7$ | 8 | 8 | 8 | 8 | 8 | $3 \cdot 10^{5}$ |  |
| $8 \times 8$ | 10 | 9 | 9 | 9 | 9 | $3 \cdot 10^{5}$ |  |
| $9 \times 9$ | 11 | 11 | 11 | 10 | 11 | $4 \cdot 10^{5}$ |  |
| $10 \times 10$ | 13 | 13 | 13 | 13 | 13 | $5 \cdot 10^{5}$ |  |

Table 8.a. Minimum search number attained by the uniform GSST variants on the full-grid graphs.

| DFS random spanning tree generation |  |  |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | :---: |
| Graph Dim. | GSST-L | GSST-LR | GSST-R | GSST-LW | GSST-LD | No.Trees |  |
| $5 \times 5$ | 6 | 6 | 6 | 6 | 6 | $10^{5}$ |  |
| $6 \times 6$ | 7 | 7 | 8 | 7 | 7 | $2 \cdot 10^{5}$ |  |
| $7 \times 7$ | 9 | 9 | 9 | 9 | 9 | $3 \cdot 10^{5}$ |  |
| $8 \times 8$ | 11 | 11 | 11 | 11 | 11 | $3 \cdot 10^{5}$ |  |
| $9 \times 9$ | 12 | 13 | 12 | 13 | 12 | $4 \cdot 10^{5}$ |  |
| $10 \times 10$ | 14 | 14 | 14 | 14 | 15 | $5 \cdot 10^{5}$ |  |

Table 8.b. Minimum search number attained by the DFS GSST variants on the full-grid graphs.

We see that, on the grids, the uniform variants perform better than the DFS ones; this is exactly the opposite situation from what happens in interval graphs. With few exceptions, the uniform variants (Table 8.a) either attain the actual minimum search number (for the $5 \times 5$ graph) or incur a small overhead of 1 or 2 extra searchers. The situation changes with the $10 \times 10$ graph, where the minimum attained search number is 13 , with a $30 \%$ overhead over the true $s_{N}^{i m c}=10$. However, the $10 \times 10$ grid graph is very complex, with 100 nodes, 180 edges and $5.6943 \cdot 10^{42}$ spanning trees. Judging from the distribution of the attained search numbers (which, for economy of space, is not displayed here) lower search numbers can be attained but they require a much larger number of spanning trees. To get an idea of the complexity of the full grid graphs, we list in Table 8.c. the number of spanning trees for each of the six graphs used. This provides a measure of the hardness of node-clearing the graph. This actually quite reasonable: a graph with many cycles (and many spanning trees) is harder to clear because it contains many escape routes for the evader.

| Graph Dim. | No. of spaning trees |
| ---: | ---: |
| $5 \times 5$ | $5.57560 \cdot 10^{8}$ |
| $6 \times 6$ | $3.2566 \cdot 10^{13}$ |
| $7 \times 7$ | $1.9872 \cdot 10^{19}$ |
| $8 \times 8$ | $1.2623 \cdot 10^{26}$ |
| $9 \times 9$ | $8.3266 \cdot 10^{33}$ |
| $10 \times 10$ | $5.6943 \cdot 10^{42}$ |

Table 8.c. Number of spanning trees for each of the full-grid graphs

On the positive side, we see that node-clearing schedules can be computed for the full grid graphs using slightly more than the minimum number of searchers and in reasonable computation time. For example, the uniform GSST-L variant finds the first 8 -searcher clearing schedule for the $7 \times 7$ graph in under 2 mins and an 11-searcher clearing schedule for the $9 \times 9$ graph in under 10 mins. Considering that (due to the large number of cycles) grid graphs are potentially some of the hardest graphs for GSST ${ }^{12}$, we find these results to be quite satisfactory.

### 6.3.2 Depleted Grids

Our final experiment involves what we call "depleted grid graphs". These are obtained as follows. First we choose two parameters, $J_{1}$ and $J_{2}$, the length and width of the grid. Just like with the full grid graphs, we place nodes at the positions $\left(j_{1}, j_{2}\right) \in\left\{1,2, \ldots, J_{1}\right\} \times\left\{1,2, \ldots, J_{2}\right\}$. We also connect all nearest neighbor nodes along the following lines

1. one horizontal line: $\left\{\left(j_{1}, 1\right)\right\}_{j_{1} \in\left\{1,2, \ldots, J_{2}\right\}}$,
2. $J_{1}$ vertical lines: $\left\{\left(j_{1}, j_{2}\right)\right\}_{j_{2} \in\left\{1,2, \ldots, J_{2}\right\}}$, for $j_{1}=1,2, \ldots, J_{1}$,
obtaining a tree of the form indicated by the solid edges in Fig 13 . Finally, we consider all pairs of nearest neighbors $\left(i_{1}, i_{2}\right),\left(j_{1}, j_{2}\right)$ which are not already connected and add to the graph an edge connecting each such pair with probability $p$.


Figure 13: Generating a depleted grid graph. The solid lines correspond to edges which are always present in the graph. Each dotted line becomes and edge with probability $p$.

The resulting graph $\mathbf{G}$ is a subgraph of the full $J_{1} \times J_{2}$ grid and has $s_{N}^{i m c}(\mathbf{G}) \leq \min \left(J_{1}, J_{2}\right)$; in other words, $\min \left(J_{1}, J_{2}\right)$ yields an upper bound (useful for estimating the performance of GSST) of $s_{N}^{i m c}(\mathbf{G})$. It seems reasonable that, on the average, $s_{N}^{i m c}(\mathbf{G})$ is an increasing function of $p$ and for $p=1$ the bound is tight, i.e. $s_{N}^{i m c}(\mathbf{G})=\min \left(J_{1}, J_{2}\right)$; it will be useful to keep this in mind when evaluating the results of the experiment. Note also that, as $p$ approaches 0 , the graph becomes more similar to a tree (has fewer cycles).

We generate twelve families of depleted grids; namely we use dimensions $5 \times 5,7 \times 7,8 \times 8,10 \times 10$ and $p$ values $0.4375,0.7500$ and 0.9375 , yielding twelve combinations. We generate fifty graphs from each family and apply to each of these the ten GSST variants, using $2 \cdot 10^{5}$ spanning trees for each graph. The results appear in Tables 9.a (uniform variants) and 9.b (DFS variants). Namely, for each

[^10]combination we present the minimum search number attained, averaged over the fifty graphs of the corresponding family. It can be seen that the resulting numbers are quite low, especially for the uniform variants, often going under the $\min \left(J_{1}, J_{2}\right)$ bound.

| Graph Dim. | $p$ | GSST-L | GSST-R | GSST-LR | GSST-LW | GSST-LD |
| ---: | :---: | ---: | ---: | ---: | ---: | ---: |
| $5 \times 5$ | 0.4375 | 2.86 | 2.86 | 2.86 | 2.86 | 2.86 |
| $5 \times 5$ | 0.7500 | 3.68 | 3.68 | 3.50 | 3.48 | 3.48 |
| $5 \times 5$ | 0.9375 | 4.53 | 4.53 | 4.12 | 4.18 | 4.20 |
| $7 \times 7$ | 0.4375 | 3.63 | 3.56 | 3.64 | 3.57 | 3.62 |
| $7 \times 7$ | 0.7500 | 5.24 | 5.14 | 5.32 | 5.08 | 5.06 |
| $7 \times 7$ | 0.9375 | 6.62 | 6.40 | 6.56 | 6.30 | 6.28 |
| $8 \times 8$ | 0.4375 | 4.22 | 4.14 | 4.08 | 4.08 | 4.00 |
| $8 \times 8$ | 0.7500 | 6.10 | 6.14 | 5.90 | 5.86 | 5.84 |
| $8 \times 8$ | 0.9375 | 7.70 | 7.76 | 7.52 | 7.40 | 7.46 |
| $10 \times 10$ | 0.4375 | 5.44 | 5.50 | 5.32 | 5.30 | 5.28 |
| $10 \times 10$ | 0.7500 | 7.94 | 8.02 | 7.84 | 7.68 | 7.80 |
| $10 \times 10$ | 0.9375 | 10.20 | 10.40 | 10.24 | 10.06 | 10.94 |

Table 9.a. Average minimum number of searchers required to node-clear each family of depleted-grid graphs by the uniform GSST variants; number of spanning trees generated is $M=2 \cdot 10^{5}$.

| Graph Dim. | $p$ | GSST-L | GSST-R | GSST-LR | GSST-LW | GSST-LD |
| ---: | :---: | ---: | ---: | ---: | ---: | ---: |
| $5 \times 5$ | 0.4375 | 3.36 | 3.36 | 3.36 | 3.36 | 3.36 |
| $5 \times 5$ | 0.7500 | 4.38 | 4.36 | 4.36 | 4.36 | 4.36 |
| $5 \times 5$ | 0.9375 | 5.04 | 5.04 | 5.02 | 5.04 | 4.98 |
| $7 \times 7$ | 0.4375 | 4.24 | 4.24 | 4.24 | 4.24 | 4.24 |
| $7 \times 7$ | 0.7500 | 5.82 | 5.78 | 5.84 | 5.80 | 5.77 |
| $7 \times 7$ | 0.9375 | 7.24 | 7.30 | 7.28 | 7.28 | 7.32 |
| $8 \times 8$ | 0.4375 | 4.68 | 4.66 | 4.66 | 4.63 | 4.64 |
| $8 \times 8$ | 0.7500 | 6.56 | 6.62 | 6.62 | 6.64 | 6.62 |
| $8 \times 8$ | 0.9375 | 8.58 | 8.52 | 8.62 | 8.54 | 8.56 |
| $10 \times 10$ | 0.4375 | 5.80 | 5.80 | 5.78 | 5.78 | 5.80 |
| $10 \times 10$ | 0.7500 | 8.62 | 8.58 | 8.68 | 8.72 | 8.62 |
| $10 \times 10$ | 0.9375 | 11.34 | 11.38 | 11.54 | 11.38 | 11.26 |

Table 9.b.Average minimum number of searchers required to node-clear each family of depleted-grid graphs by the DFS GSST variants; number of spanning trees generated is $M=2 \cdot 10^{5}$.

The depleted grid results show the high performance of GSST on graphs with large numbers of spanning trees and many cycles. Better results are obtained for smaller values of $p$, i.e. when the graph becomes more similar to a tree; this is to be expected, since a basic component of the GSST algorithm is search along a spanning tree. It is interesting to note that even for a quite high value of $p$, namely 0.9375 , results (at least for uniform GSST variants) are quite good; taking into account also the results of the full grids, it appears that the increase of attained search number over $\min \left(J_{1}, J_{2}\right)$ happens only when $p$ gets very close to 1 . Once again we observe that the uniform variants perform better than the DFS ones.

### 6.4 Discussion

Our experiments demonstrate the performance of GSST on several different classes of graphs. On many complex graphs, GSST was able to find a minimal search schedule in reasonable time. On all graphs, at least one near-minimal schedule is computed early in the execution of the algorithm. In fact, the "anytime" operation of GSST (see also the discussion in Section 5.1) yields reasonably good solutions in very short time and then keeps improving on these solutions as long as additional computation time is available. While the minimal schedules form a small percentage of the computed solutions, many more near-minimal solutions are available. These properties of GSST are highly desirable, especially given the fact that (as far as we are aware) no other algorithm has been implemented which can tackle graphs of the size and complexity presented here.

We can highlight some specific conclusions supported by the experiments on the various graphs considered in Section 6.1-6.3.

1. The NSH and National Art Gallery graphs show the application of GSST on graphs that are derived from representations of indoor environments. The methods that incorporate Barriere labeling improve performance on the NSH map because it is similar to a tree. The National Art Gallery, on the other hand, is more similar to a grid, which lessens the advantage of using Barriere labeling to guide traversal.
2. The results on interval graphs show that GSST can yield near-minimal schedules on a large class of graphs. They also demonstrate that, with increasing complexity, GSST scales well both in performance and computation time.
3. Finally, the grid results show the high performance of GSST on graphs with large numbers of spanning trees and many cycles. These are potentially some of the hardest graphs for GSST. Also, it appears that the the increase of attained search number over $\min \left(J_{1}, J_{2}\right)$ happens only when $p$ gets very close to 1 .

Comparing spanning tree generation methods, we see that DFS generation performs better on all graphs considered except for grid graphs (and the tree/grid graph) where the uniform ST generation is better. At this point we do not have a good explanation but we intend to further research this fact because we believe it will give us a better understanding of which are the "good" spanning trees, i.e. the ones associated with minimal searches. See Section 7 for some additional remarks on this issue.

Regarding edge traversal methods, there is no single GSST variant which consistently outperforms all others. However, uniform GSST-L never does too poorly and is fast; GSST-LW has the same advantages and, in addition, is provably complete. Hence, in case node-clearing schedules must be quickly produced (i.e. computation time is at premium), one can use a "reduced suite" of GSST variants, consisting of GSST-L and GSST-LW (with both uniform and DFS spanning tree generation) and expect to obtain nearly as good results (at a fraction of the computing time) as when using the "full" suite of the ten GSST variants.

Let us also stress that all variants of GSST can attain lower search numbers than the ones presented here if sufficient running time is available.

## 7 Conclusion

Motivated by the problem of robotic pursuit / evasion, in the current paper we have investigated node search, i.e., the capture by a team of searchers of an invisible evader located in the nodes of a graph. This problem has so far received little attention in the graph theoretic literature, with most related publications concentrating on the problem of edge-located evader. The basic contributions of the current paper are of two kinds.

1. From the theoretical point of view, we have shown that, in general graphs, the problem of node search is easier than that of edge search, in the sense that every edge clearing search is also a node clearing one; but the converse does not hold in general. We have then concentrated on the internal monotone connected (IMC) node search of trees and shown that it is essentially equivalent to edge search under the same restrictions; hence Barriere's tree search algorithm, originally designed for edge search, can also be used for node search.
2. From the algorithmic point of view, we have presented GSST, a new algorithm which performs IMC search on general graphs. This algorithm is based on the fact that every node clearing search generates a spanning tree; hence node-clearing a preselected spanning tree (by the use of tree searchers) and simultaneously blocking recontamination (by the use of guards) monotonically node-clears the graph. Because spanning tree generation and search can both be performed very efficiently, a large number of spanning trees can be tried by GSST until the one yielding the minimum number of searchers is discovered. Experiments have shown that GSST can quickly node-clear large and complex graphs using a small number of searchers.

Many issues touched upon in the current paper require additional research. We conclude by listing some future research directions, categorized as follows.

## 1. Theory.

(a) Optimality. Given a graph $\mathbf{G}$ with IMC node search number $s_{N}^{i m c}(\mathbf{G})$. Suppose we have $K$ searchers with $K \geq s_{N}^{i m c}(\mathbf{G})$. What is the best we can do? Of course "best" depends on some optimality criterion. For example we may want to clear the graph in the minimum number of steps. Or, we may want a search strategy which minimizes the cumulative "area" of the dirty set (i.e., $\sum_{t=1}^{T} V_{N}^{D}(t)$ ). We would like to obtain algorithms which solve such optimization problems.
(b) Our experiments show that the uniform GSST variants perform better than DFS on grids and grid-like graphs. We want to discover a mathematical explanation of this fact, e.g., to find necessary and / or sufficient conditions under which uniform GSST outperforms DFS GSST. A result of this type will probably have useful computational implications - see also item 2.(a) below.
(c) We want to establish a sequence of inequalities between node search numbers. In other words, to refine (30), either for a general graph $\mathbf{G}$, or for the special case when $\mathbf{G}$ is a tree. For the latter case, we conjecture that we can re-establish the Barriere et al. inequalities (28) so as to hold for node- rather than edge-search numbers.
(d) Finally, we are interested in questions of consistency. Suppose that we are given a graph along with a configuration of each nodes (i.e., a specification of the clear and dirty nodes). Can this configuration be obtained by a sequence of moves in the node game? In the edge game? Also, given a configuration of clear and dirty nodes and another one of clear and dirty edges, are these configurations compatible in the node game? In the edge game?
2. Computation. The GSST algorithm searches the space of spanning trees in a random manner. Is there a way to bias the search in some useful way? For this we need a characterization of "good" spanning trees, i.e., trees which yield a low number of searchers (this includes tree searchers and guards). We have no such characterization at present. Perhaps we can obtain it by understanding why uniform variants perform better than DFS on grid graphs (and conversely on interval graphs).
3. Extensions. We want to study theoretically and develop algorithms for the following variants of the search problem.
(a) For less restricted types of node search, e.g., internal monotone (but not connected), internal connected (but not monotone); or for completely unrestricted node search.
(b) For different types of pursuer / evader behavior; e.g., for pursuers with extended (non-local) visibility, evaders with finite speed etc.
(c) How does the graph search problem look from the evader's point of view? Given a graph $\mathbf{G}$ and $K$ searchers, what is the best the evader can do if $K<s(\mathbf{G})$ ? What if $K \geq s(\mathbf{G})$ ? In the latter case capture is guaranteed, but can the evader, for example, maximize the number of steps until capture? Many other interesting problems can be posed from the evader's point of view.

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## A Appendix: Edge Search on Trees

In this Appendix we present the R-Label and R-Search algorithms, which perform rooted IMC search of trees and are also used by the GSST algorithm. R-Label and R-Search are simplified versions of corresponding algorithms presented by Barriere et al. [2]; some theoretical results from [2] are also reviewed. In our presentation the terminology and notation of [2] are somewhat changed, to conform with the current paper.

The next Lemma and Definition play a crucial role in computing the IMC search number of a tree.
Lemma A.1. [2] Given a rooted tree $\mathbf{T}_{x}$ and a node $y$ of $\mathbf{T}_{x}$, if $y$ has more than one children, enumerate them as $y_{1}, y_{2}, \ldots, y_{K}$, so that they satisfy $s_{E}^{i m c}\left(\mathbf{T}_{x}\left[y_{k}\right]\right) \geq s_{E}^{i m c}\left(\mathbf{T}_{x}\left[y_{k+1}\right]\right)$ for $k=1,2, \ldots, K-1$. Then

$$
\begin{equation*}
s_{E}^{i m c}\left(\mathbf{T}_{x}[y]\right)=\max \left(s_{E}^{i m c}\left(\mathbf{T}_{x}\left[y_{1}\right]\right), s_{E}^{i m c}\left(\mathbf{T}_{x}\left[y_{2}\right]\right)+1\right) . \tag{39}
\end{equation*}
$$

Definition A.2. [2] Given a tree $\mathbf{T}=(V, E)$ and a node $x \in V$, consider all the edges $x y$ incident on $x$ and label them as follows.

1. If $y$ is a leaf, then $\lambda_{x}(x y)=1$.
2. If $y$ is not a leaf, let $z_{1}, z_{2}, \ldots, z_{K}$ be the neighbors of $y$ other than $x$, enumerated so that $\lambda_{y}\left(y z_{k}\right) \geq$ $\lambda_{y}\left(y z_{k+1}\right)($ for $k=1,2, \ldots, K-1)$ and define

$$
\begin{equation*}
\lambda_{x}(x y)=\max \left(\lambda_{y}\left(y z_{1}\right), \lambda_{y}\left(y z_{2}\right)+1\right) \tag{40}
\end{equation*}
$$

Note that the above definition assigns two labels to each edge $x y$, namely $\lambda_{x}(x y)$ and $\lambda_{y}(x y)$. Intuitively, $\lambda_{x}(x y)$ is the number of searchers that must cross the directed $x y$ edge in an IMC edgeclearing of the tree. In [2] Barriere et al. present the Label algorithm which computes the labels $\left\{\lambda_{x}(x y)\right\}_{x y \in E}$. Furthermore they prove the following.

Lemma A.3. [2] For every tree $\mathbf{T}$ and every edge $x y$ of $\mathbf{T}$, we have $s_{E}^{i m c}\left(\mathbf{T}_{x}[y]\right)=\lambda_{x}(x y)$.
Lemma A.4. [2] Given a tree $\mathbf{T}=(V, E)$, for every $x \in V$ define $\mu(x)$ as follows

$$
\mu(x)= \begin{cases}\lambda_{x}\left(x y_{1}\right) & \text { if } x \text { has a single neighbor } y_{1} \\ \max \left(\lambda_{x}\left(x y_{1}\right), \lambda_{x}\left(x y_{2}\right)+1\right) & \text { otherwise }\end{cases}
$$

in the second line of the definition, $y_{1}, y_{2}, \ldots$ are the neighbors of $x$, enumerated so that they satisfy $s_{E}^{i m c}\left(\mathbf{T}_{x}\left[y_{k}\right]\right) \geq s_{E}^{i m c}\left(\mathbf{T}_{x}\left[y_{k+1}\right]\right)$ for $k=1,2, \ldots$. Then

$$
s_{E}^{i m c}\left(\mathbf{T}_{x}\right)=\mu(x) \text { and } s_{E}^{i m c}(\mathbf{T})=\min _{x \in V} \mu(x)
$$

The algorithms and results of [2] are geared towards computing $s_{E}^{i m c}(\mathbf{T})$. The GSST algorithm, on the other hand, requires the computation of $s_{E}^{i m c}\left(\mathbf{T}_{x}\right)$ and rooted searches. Hence we introduce the R-Label and R-Search algorithms, which are simplified, "rooted" versions of the corresponding Barriere algorithms. The pseudocode of these algorithms is listed in the next page.

The R-Label algorithm takes as input a tree $\mathbf{T}=(V, E)$ and a root node $u_{0} \in V$. The algorithm is a straightforward implementation of Definition A.2. R-Label does not compute all the labels $\left\{\lambda_{x}(x y)\right\}_{x y \in E}$ but only the ones for which $y$ is a child of $x$ in the rooted tree $\mathbf{T}_{u_{0}}$; for these edges, $\mathbf{R}$-Label produces the same $\lambda$ labels as Label. R-Label uses two subroutines: given that $u_{0}$ is the root, the subroutine $\operatorname{Depth}\left(\mathbf{T}, u_{0}\right)$ returns a partition $\left\{V^{(l)}\right\}_{l=1}^{L}$ of the node set $V$, where $V^{(l)}$ contains all nodes of depth $l$; the subroutine $\operatorname{SortChildren}\left(y, \mathbf{T}, u_{0}, \lambda\right)$ returns the children of $y$ sorted in decreasing order of their $\lambda$ labels.

The R-Search algorithm uses the $\lambda$ labels to produce a rooted IMC (edge and node) clearing search of T. R-Search is almost identical to Barriere's Search, the only difference being that the starting node $u_{0}$ is given, rather than chosen by the algorithm (this implies that $\mathbf{R}$-Search produces a minimal rooted IMC search of $\mathbf{T}_{u_{0}}$ ). R-Search uses Barriere's subroutine Move. The notation $\mathbf{S}=\mathbf{S} \mid(u, v, J)$ means: append $J$ moves of the form $u \rightarrow v$ to the search schedule $\mathbf{S}$.

```
Algorithm 3 R-Label(T, \(u_{0}\) )
    Input: Tree \(\mathbf{T}=(V, E)\), start node \(u_{0}\)
    \(\left\{V^{(l)}\right\}_{l=1}^{L}=\operatorname{Depth}\left(\mathbf{T}, u_{0}\right)\)
    for \(l=L-1\) with step -1 until 1 do
        for \(x \in V^{(l)}\) do
            for \(y \in \operatorname{SortChildren}\left(x, \mathbf{T}, u_{0}, \lambda\right)\) do
                \(\left[z_{1}, z_{2}, \ldots, z_{K}\right]=\operatorname{SortChildren}\left(y, \mathbf{T}, u_{0}, \lambda\right)\)
                if Length \(\left(\left[z_{1}, z_{2}, \ldots, z_{K}\right]\right)=0\) then
                    \(\lambda_{x}(x y)=1\)
                else if Length \(\left(\left[z_{1}, z_{2}, \ldots, z_{K}\right]\right)=1\) then
                    \(\lambda_{x}(x y)=\lambda_{y}\left(y z_{1}\right)\)
                else
                    \(\lambda_{x}(x y)=\max \left(\lambda_{y}\left(y z_{1}\right), \lambda_{y}\left(y z_{2}\right)+1\right)\)
                end if
            end for
        end for
    end for
    Output: Edge labeling \(\left\{\lambda_{x}(x y)\right\}_{x y \in E}\)
```

```
Algorithm 4 R-Search \(\left(\mathbf{T}, u_{0}, \lambda\right)\)
    Input: Tree \(\mathbf{T}=(V, E)\), start node \(u_{0}\), edge labeling \(\lambda\)
    \(\mathbf{S}=\emptyset\)
    \(\left[y_{1}, y_{2}, \ldots, y_{K}\right]=\operatorname{SortChildren}\left(u_{0}, \mathbf{T}, u_{0}, \lambda\right)\)
    for \(k=K\) with step -1 until 1 do
        \(\mathbf{S}=\operatorname{Move}\left(x, y_{k}, \lambda_{x}\left(x y_{k}\right), \mathbf{S}\right)\)
    end for
    Output: Search \(\mathbf{S}\)
```

```
Algorithm 5 Move \((u, v, J, \mathbf{S})\)
    Input: Tree \(\mathbf{T}=(V, E)\), start node \(u_{0}\), edge labeling \(\lambda\)
    \(\mathbf{S}=\mathbf{S} \mid(u, v, J)\)
    \(\left[w_{1}, w_{2}, \ldots, w_{K}\right]=\operatorname{SortChildren}\left(v, \mathbf{T}, u_{0}, \lambda\right)\)
    for \(k=K\) with step -1 until 1 do
        \(\mathbf{S}=\operatorname{Move}\left(v, w_{k}, \lambda_{y}\left(y w_{k}\right), \mathbf{S}\right)\)
    end for
    \(\mathbf{S}=\mathbf{S} \mid(v, u, J)\)
    Output: Search \(\mathbf{S}\)
```


## B Appendix: The relationship between node search and mixed edge search

Theorem 3.3 tells us that edge search is "weaker" than node search in the sense that, for every graph and every search schedule, we have $V_{E}^{C}(t) \subseteq V_{N}^{C}(t)$ and $E_{E}^{C}(t) \subseteq E_{N}^{C}(t)$. We will now consider a variant of edge search, the so-called mixed edge search which, as we will see, is equivalent to node search.

Mixed edge search is obtained by augmenting the clearing rules. In the "Mixed Search Edge Game" an edge can be cleared not only by sliding, but also by guarding both its endpoints. More precisely, to obtain the rules of the mixed edge search game, take the rules of the edge game (presented in Section 22, change the terms "e-clear" and "e-dirty" to "m-clear" and "m-dirty" and add the following rule:

E2 ${ }^{\prime}$ An m-dirty edge $u v$ becomes m-clear when both $u$ and $v$ are occupied by searchers.
In other words, we can clear an edge either by traversing it or by guarding both its endpoints.
We will use $V_{M}^{C}(t)$ to denote the set of m-clear nodes at time $t$, and will analogously use the notations $E_{M}^{C}(t), V_{M}^{D}(t), E_{M}^{D}(t), \mathbf{G}_{M}^{C}(t)$ etc.

We emphasize that, in the mixed search edge game, the evader resides in the edges of the graph and recontamination occurs by the same rules as in the "plain" edge game.

Theorem B.1. Given a graph $\mathbf{G}$ and an internal search schedule $\mathbf{S}$. Then, we have

$$
\begin{equation*}
\text { for } t=0,1,2, \ldots: \quad V_{N}^{C}(t)=V_{M}^{C}(t) \quad \text { and } \quad E_{N}^{C}(t)=E_{M}^{C}(t) . \tag{41}
\end{equation*}
$$

Proof. The proof is inductive and consists of several steps.
I. At $t=0$ we have

$$
\begin{equation*}
V_{N}^{C}(0)=V_{M}^{C}(0)=\emptyset \quad \text { and } \quad E_{N}^{C}(0)=E_{M}^{C}(0)=\emptyset \tag{42}
\end{equation*}
$$

Suppose that we have

$$
\begin{equation*}
\text { for } s=0,1, \ldots, t: \quad V_{N}^{C}(s)=V_{M}^{C}(s) \quad \text { and } \quad E_{N}^{C}(s)=E_{M}^{C}(s) \tag{43}
\end{equation*}
$$

and let us next consider time $t+1$.
II. We will now prove that

$$
\begin{equation*}
E_{N}^{C}(t+1) \subseteq E_{M}^{C}(t+1) \tag{44}
\end{equation*}
$$

To this end we will show two things.
II. a If a previously $n$-dirty edge becomes $n$-clear at $t+1$, then it also becomes $m$-clear. Suppose that $u v \in E_{N}^{D}(t) \cap E_{N}^{C}(t+1)$. In other words, $u v$ is n-cleared exactly at $t+1$; this means that one of $u, v$ was already n-clear by $t$ and the other was n-cleared exactly at $t+1$. Without loss of generality, assume

$$
\begin{equation*}
u \in V_{N}^{C}(t) \cap V_{N}^{C}(t+1), \quad v \in V_{N}^{D}(t) \cap V_{N}^{C}(t+1) \tag{45}
\end{equation*}
$$

Since a node can be n-cleared only by moving into it, the $(t+1)$-th move must be $w \rightarrow v$. We distinguish the following cases.
(i.1) Suppose $w \neq u$. Since $v \in V_{N}^{D}(t)$ and $u \in V_{N}^{C}(t), u$ was guarded at $t$; since $w \neq u, u$ remains guarded at $t+1$. Hence, at $t+1, u v$ has both ends guarded and so $u v \in E_{M}^{C}(t+1)$. (Note that this analysis holds even when $w=0$, i.e., when the move was to place a new searcher).
(i.2) Suppose $w=u$ and $u$ is still guarded at $t+1$ (i.e., at $t$ there was more than one searcher in $u$ ). Then at $t+1$ both $u$ and $v$ are guarded and $u v \in E_{M}^{C}(t+1)$.
(i.3) The remaining possibility is that $w=u$ and $u$ is unguarded at $t+1$ (i.e., at $t$ there was exactly one searcher in $u$ ). Note that $u \rightarrow v$ at $t+1$ means that $u$ is guarded at $t$ and hence $u \in V_{M}^{C}(t)$. If $u \in V_{M}^{C}(t+1)$ as well, then $u v \in V_{M}^{C}(t+1)$. So we now will show that $u \in V_{M}^{D}(t+1)$ is not possible. Indeed $u \in V_{M}^{D}(t+1)$ can only happen in one of the following two ways.
(i.3.1) There exists a node $x_{1} \neq v$ such that $u x_{1} \in E_{M}^{D}(t) \cap E_{M}^{D}(t+1)$. But $u x_{1} \in E_{M}^{D}(t)=$ $E_{N}^{D}(t)$ implies $x_{1} \in V_{N}^{D}(t)$ (because $u$ is guarded at $t$ ) and $x_{1} \in V_{N}^{D}(t)$ implies $x_{1} \in V_{N}^{D}(t+1)$ (because $x_{1}$ was not entered at $t+1$ ). Finally, from $x_{1} \in V_{N}^{D}(t+1)$ and $u$ unguarded at $t+1$, we conclude $u \in V_{N}^{D}(t+1)$ which contradicts assumption (45).
(i.3.2) Alternatively, there exists a path $v u x_{1} \ldots x_{K}$ (with $K \geq 2$ ), e-unguarded at $t+1$ and with $x_{K-1} x_{K} \in E_{M}^{D}(t) \cap E_{M}^{D}(t+1)$ (note also that, since $v u x_{1} \ldots x_{K}$ is a path, $\{u, v\} \cap\left\{x_{1}, x_{2}, \ldots, x_{K}\right\}=$ $\emptyset$ ). In this case, we have

$$
\left.\begin{array}{l}
v u x_{1} \ldots x_{K} \text { e-unguarded at } t+1 \\
u \rightarrow v \text { at } t+1 \\
u \notin\left\{x_{1}, x_{2}, \ldots, x_{K}\right\}
\end{array}\right\} \Rightarrow u x_{1} \ldots x_{K} \text { is e-unguarded at } t ;
$$

and

$$
\left.\begin{array}{l}
u x_{1} \ldots x_{K} \text { e-unguarded at } t \\
x_{K-1} x_{K} \in E_{M}^{D}(t)
\end{array}\right\} \Rightarrow x_{K-1} \in V_{M}^{D}(t)=V_{N}^{D}(t)
$$

and finally

$$
\left.\left.\begin{array}{l}
x_{K-1} \in V_{N}^{D}(t) \\
u \rightarrow v \text { at } t+1 \\
v \notin\left\{x_{1}, x_{2}, \ldots, x_{K}\right\} \\
\quad u x_{1} \ldots x_{K-1} \text { is n-unguarded at } t+1
\end{array}\right\} \Rightarrow x_{K-1} \in V_{N}^{D}(t+1)\right\} \Rightarrow u \in V_{N}^{D}(t+1)
$$

But $u \in V_{N}^{D}(t+1)$ contradicts assumption (45). Hence $u \notin V_{M}^{D}(t+1)$.
In short, by examining cases (i.1)-(i.3) we have shown that

$$
\begin{equation*}
u v \in E_{N}^{D}(t) \cap E_{N}^{C}(t+1) \Rightarrow u v \in E_{M}^{C}(t+1) . \tag{46}
\end{equation*}
$$

i.e., if at $t+1$ some $u v$ is n -cleared it is also m-cleared.
II.b If a previously m-clear edge becomes m-dirty at $t+1$, then it also becomes $n$-dirty. Suppose that $u v \in E_{M}^{C}(t) \cap E_{M}^{D}(t+1)$. Without loss of generality, we can assume that there exists a $x_{K-1} x_{K} \in E_{M}^{D}(t) \cap E_{M}^{D}(t+1)$ and a path $u v x_{1} \ldots x_{K}$ which was e-guarded at $t$ but became eunguarded at $t+1$. Since the path became e-unguarded at $t+1$, exactly one node in it was guarded at $t$ (and became unguarded at $t+1$ ). Call this node $x_{\bar{k}}$, with $\bar{k}=0$ if the node in question is $v$ (i.e., $x_{0}=v$ ) and $\bar{k} \in\{1,2, \ldots, K-1\}$ otherwise. The move at $t+1$ is $x_{\bar{k}} \rightarrow z$, where $z \notin\left\{v, x_{1}, \ldots, x_{K-1}\right\}$ (since $u v x_{1} \ldots x_{K}$ is e-unguarded at $t+1$ ). Since $x_{K-1} x_{K} \in E_{M}^{D}(t)=E_{N}^{D}(t)$, either $x_{K-1} \in V_{N}^{D}(t)$ or $x_{K} \in V_{N}^{D}(t)$. We consider the following cases.
(ii.1) If $x_{K-1} \in V_{N}^{D}(t)$ then $x_{K-1} \in V_{N}^{D}(t+1)$ as well (it was not entered) and the path $v x_{1} \ldots x_{K-1}$ is n-unguarded at $t+1$. So $v \in V_{N}^{D}(t+1)$ and $u v \in E_{N}^{D}(t+1)$.
(ii.2) If $x_{K-1} \notin V_{N}^{D}(t)$, then $x_{K} \in V_{N}^{D}(t)$ and so $x_{K-1}$ is the guarded node in the path $u v x_{1} \ldots x_{K}$ (i.e., $\bar{k}=K-1$ ). We distinguish two subcases.
(ii.2.1) The move at $t+1$ is $x_{K-1} \rightarrow z$ and $z \neq x_{K}$. Then $x_{K} \in V_{N}^{D}(t+1)$ and so $x_{K-1} \in$ $V_{N}^{D}(t+1)$ and, by the same argument as in (ii.1), we get $u v \in E_{N}^{D}(t+1)$.
(ii.2.2) The move at $t+1$ is $x_{K-1} \rightarrow x_{K}$. Since we have assumed $x_{K-1} x_{K} \in E_{M}^{D}(t) \cap E_{M}^{D}(t+1)$, $x_{K-1} x_{K}$ must be adjacent to some other edge $x_{K-1} y \in E_{M}^{D}(t) \cap E_{M}^{D}(t+1)$; so we can use $y$ in place of $x_{K}, u v x_{1} \ldots x_{K-1} y$ in place of $u v x_{1} \ldots x_{K}$ and conclude by the reasoning of case (ii.2.1) that $u v \in E_{N}^{D}(t+1)$.

In short, we have shown that

$$
\begin{equation*}
u v \in E_{M}^{C}(t) \cap E_{M}^{D}(t+1) \Rightarrow u v \in E_{N}^{D}(t+1) \tag{47}
\end{equation*}
$$

i.e., if at $t+1$ some $u v$ is m-dirtied then it is also n-dirtied.

Hence every edge added at $t+1$ to $E_{N}^{C}(t)$ is also added to $E_{M}^{C}(t)$ and every edge removed at $t+1$ from $E_{M}^{C}(t)$ is also removed from $E_{N}^{C}(t)$. These facts, combined with $E_{N}^{C}(t)=E_{M}^{C}(t)$, yield

$$
\begin{equation*}
E_{N}^{C}(t+1) \subseteq E_{M}^{C}(t+1) \tag{48}
\end{equation*}
$$

III. To strengthen 48 to set equality, we need the reverse set inclusion. In fact we can prove

$$
\begin{equation*}
\text { for } s=0,1, \ldots, t, t+1, \ldots: \quad E_{M}^{C}(s) \subseteq E_{N}^{C}(s) \tag{49}
\end{equation*}
$$

by following the proof of Theorem 3.3 and replacing "e-clear", "e-dirty", ... with "m-clear", "m-dirty", $\ldots$ (analogs of Lemmas 3.1, 3.2 can also be proved easily) From (48) and 49) we obtain

$$
\begin{equation*}
E_{M}^{D}(t+1)=E_{N}^{D}(t+1) \tag{50}
\end{equation*}
$$

IV. Now suppose $u \in V_{M}^{D}(t+1)$. Then $u$ is unguarded and adjacent to some $u v \in E_{M}^{D}(t+1)=$ $E_{N}^{D}(t+1)$. Hence either $u$ or $v$ is n-dirty and so (since $u$ is unguarded) $u \in V_{N}^{D}(t+1)$. On the other hand, if $u \in V_{N}^{D}(t+1)$, then $u$ is unguarded and there exists a $u v \in E_{N}^{D}(t+1)=E_{M}^{D}(t+1)$ and (since $u$ is unguarded) $u \in V_{M}^{D}(t+1)$. It follows that

$$
V_{M}^{D}(t+1)=V_{N}^{D}(t+1)
$$

V. In short, we have established that

$$
\begin{equation*}
\binom{E_{M}^{C}(t)=E_{N}^{C}(t)}{V_{M}^{C}(t)=V_{N}^{C}(t)} \Rightarrow\binom{E_{M}^{C}(t+1)=E_{N}^{C}(t+1)}{V_{M}^{C}(t+1)=V_{N}^{C}(t+1)} . \tag{51}
\end{equation*}
$$

Eq. 51] and the fact $V_{M}^{C}(0)=V_{N}^{C}(0)$ and $E_{M}^{C}(0)=E_{N}^{C}(0)$ complete the proof of the theorem.
Several facts follow from TheoremB.1. First, it is well known that mixed edge search is NP-complete [9] hence node search is also NP-complete.

Second, Theorem B.1 can be used to obtain a Theorem 3.3 as a corollary with a short proof (which, of course, presupposes the lengthy proof of Theorem B.1. We give a sketch of such a proof. Suppose a graph $\mathbf{G}$ and a search schedule $\mathbf{S}$ are given. Then, by Theorem B.1 we have (for every $t$ ) $V_{N}^{C}(t)=V_{M}^{C}(t)$ and $E_{N}^{C}(t)=E_{M}^{C}(t)$. Now suppose $\mathbf{S}$ is applied to $\mathbf{G}$ under the rules of the edge game and, up to time $t$ we have $V_{E}^{C}(t) \subseteq V_{M}^{C}(t)$ and $E_{E}^{C}(t) \subseteq E_{M}^{C}(t)$. Then, at time $t+1$, every e-cleared edge is also m -cleared (since the mixed game has all the clearing rules of the edge game and an additional one) and every m-dirtied edge is also e-dirtied (since the edge and mixed games have the same recontamination rules) except if an edge was already e-dirty. Hence $E_{E}^{C}(t+1) \subseteq E_{M}^{C}(t+1)$, which also shows that $V_{E}^{C}(t+1) \subseteq V_{M}^{C}(t+1)$.

Finally, the above Theorem B.1] and the monotonicity result proved in [5] ("if there is a mixed edge clearing search of $\mathbf{G}$ using $\leq K$ guards, there is a monotone mixed edge clearing search of $\mathbf{G}$ using $\leq K$ guards") makes the following conjecture seem almost trivially true.

[^11]Conjecture B.2. If there is a node clearing search $\mathbf{S}$ of $\mathbf{G}$ using at most $K$ guards, there is a monotone node clearing search $\mathbf{S}^{\prime}$ of $\mathbf{G}$ using at most $K$ guards.

However some additional work is required to prove the conjecture, because of the following detail: edge monotonicity does not necessarily imply node monotonicity. In other words, the implication

$$
\begin{equation*}
E_{N}^{C}(t) \subseteq E_{N}^{C}(t+1) \Rightarrow V_{N}^{C}(t) \subseteq V_{N}^{C}(t+1) \tag{52}
\end{equation*}
$$

is not necessarily true! Consider again the search schedule of Remark 2.2. In this case $E_{N}^{C}(1)=$ $E_{N}^{C}(2)=\emptyset$, but $V_{N}^{C}(1) \subseteq V_{N}^{C}(2)$ does not hold. It is true however (and easy to prove) that

$$
\begin{equation*}
\text { (strong edge monotonicity) } \Rightarrow \text { (node monotonicity). } \tag{53}
\end{equation*}
$$

Unfortunately, the result proved by Bienstock in 5] concerns "simple", not strong monotonicity. Hence one of our future research goals is to extend Bienstock's result to node search and thus prove that $s_{N}(\mathbf{G})=s_{N}^{m}(\mathbf{G})$.

## C Appendix: Vertex separation and pathwidth

Vertex separation and pathwidth are strongly (but not obviously) related to graph search and search number, as discussed in (among other papers) [29, 12], where the author can find a more detailed discussion; here we just give the basic definitions and the main theorem.

Definition C.1. Given a graph $\mathbf{G}=(V, E)$, a path decomposition of $\mathbf{G}$ is a pair $(X, \mathbf{P})$, where $X=\left\{X_{1}, \ldots, X_{M}\right\}$ is a family of subsets of $V$ and $\mathbf{P}$ is a path whose nodes are the subsets $X_{i}$ and they satisfy the following properties

1. $\cup_{m=1}^{M} X_{i}=V$.
2. For every edge $v w$ in $E$, there is a subset $X_{m}$ that contains both $v$ and $w$.
3. If $1 \leq i \leq j \leq k \leq M$ then $X_{i} \cap X_{k} \subseteq X_{j}$.

Definition C.2. The width of a path decomposition ( $X, \mathbf{P}$ ) (with $X=\left\{X_{1}, \ldots, X_{n}\right\}$ ) is denoted by $p d w(X, \mathbf{P})$ and is defined by

$$
p d w(X, \mathbf{P})=\max _{1 \leq i \leq n}\left|X_{i}\right| .
$$

Definition C.3. The pathwidth of a graph $\mathbf{G}=(V, E)$ is denoted by $p w(\mathbf{G})$ and defined by

$$
p w(\mathbf{G})=\min _{\text {all path decompositions }(X, \mathbf{P}) \text { of } \mathbf{G}} p d w(X, \mathbf{P}) .
$$

Definition C.4. Given a graph $\mathbf{G}=(V, E)$ (with $|V|=N)$ and a permutation $\mathcal{P}$ of $\{1,2, \ldots, N\}$. The vertex separation of $\mathbf{G}$ with respect to $\mathcal{P}$ is denoted by vs $(\mathbf{G}, \mathcal{P})$ and defined by

$$
\text { vs }(\mathbf{G}, \mathcal{P})=\max _{1 \leq n \leq N} \mid\{u \in V: \mathcal{P}(u) \leq n \text { and } \exists v \text { such that: } \mathcal{P}(v)>n \text { and } u v \in E\} \mid .
$$

The vertex separation of $G$ is denoted by $v s(\mathbf{G})$ and defined by

$$
\text { vs }(\mathbf{G})=\min _{\text {all permutations } \mathcal{P}} v s(\mathbf{G}, \mathcal{P}) .
$$

The following theorem shows the connection between $v s(\mathbf{G})$ and $p w(\mathbf{G})$ and search number.
Theorem C.5. For every graph $\mathbf{G}$, vs $(\mathbf{G})=p w(\mathbf{G})$ and $\left|v s(\mathbf{G})-s_{E}(\mathbf{G})\right| \leq 1$.

## D Appendix: Implementations of the GSearch Algorithm

We have implemented the GSST algorithm in two forms which are publicly available.

1. As a command line executable, which runs on Windows and Linux computers. The program and supporting material are available at the URL
```
http://www.frc.ri.cmu.edu/~gholling/home/software.html.
```

2. As a graphical user interface (GUI), available at
http://users.auth.gr/~kehagiat/KehagiasSoftware.htm.

We discuss each of these implementations separately.

## D. 1 The Command Line program

This is an executable, gsearch.exe, which has the following usage

```
USAGE: gsearch -m [graph] -n [no.trees] -s [startnode] -g [gen.tree]
    -y [wr-search] -w [wr-tree] -v [visualize] -t [edge traversal]
    -i [improve tree] -l [low number] -r [redundancy check]
EXAMPLE: gsearch -m graphs/Edge01.txt -g exhaustive -s 1 -n 500000 -t bh
```

The various options available are as follows.

```
-m [graph]: string, name of file with edge list of the graph.
-n [no.trees]: int, how many sp.trees to generate (DEFAULT is 1).
-s [start node]: int, which node to start
    (DEFAULT is 1, random choice is 0)
-g [gen.tree]: string, method of generating spanning trees
    (acceptable values: readtree, exhaustive, uniform, dfsrand;
        DEFAULT is uniform)
-y [wr-search]: string, how many best searches to write
    (acceptable values: one, all, none; DEFAULT is one)
-w [wr-tree]: string, how many best sp.trees to write
    (acceptable values: one, all, none; DEFAULT is one)
-t [traversal]: string, how to break edge label ties
    (acceptable values: bh, random, bhrand, bhweight, bhdom;
        DEFAULT is bh).
-i [improve tree]: boolean, use tree improve technique or not.
    (acceptable values are 0 / 1; DEFAULT is 0: do not use it)
-r [redundancy check]: boolean, check for redundant trees
    (acceptable values are 0 / 1; DEFAULT 1:
        check for redundancy)
-l [low number]: break if a tree is found with this number of
        searchers (DEFAULT is 0: do not break)
-v [visualize]: boolean, use visualizer (only supported
        on linux) (DEFAULT is 0: do not use)
-h [help]
```

The -m option indicates the file which contains the graph description. This must be an ascii (plain text) file containing a list of the edges of the graph, one edge per line, indicated as a pair of nodes. The nodes must be continuously numbered from 1 to $N$ and these numbers are used as labels. $N$ is assumed to be the largest number appearing in the edge list (the graph is assumed to be undirected and connected). Examples of edge lists can be found in the graphs directory.

The remaining options of gsearch correspond to the description of the algorithm in Section 5. The -n option corresponds to the $M$ parameter of GSST (number of spanning trees), -s is the root of the search, -g describes the uniform and DFS methods for generating spanning trees (there are also options for exhaustive enumeration of all spanning trees and for reading a specific spanning tree from file) and the -t options (bh, random, bhrand, bhweight, bhdom) corresponds to the traversal methods (L, R, LR, LW and LD, respectively). The $-\mathrm{i},-\mathrm{r}$ and -1 options are self explanatory. The -y option writes one or more minimal strategies in the file(s) output/strat*.txt; each row of the file corresponds to one step of the strategy and shows the nodes in which the searchers are currently located. The -w option writes the rooted tree(s) corresponding to optimal strategies, in the file(s) output/tree*.txt; each row of the file shows one parent and her child. The -v visualization option only works on Linux computers (for a visualization of the search on Windows computers use the GUI).

The command line gsearch was implemented by G. Hollinger in ANSI C using the gcc 4.1 compiler.

## D. 2 The Graphical Interface

This is an executable, gsearchGUI.exe, which corresponds closely to the command line program. (In fact the GUI is a front end for the command line program.) Launching the program brings up the window of Figure 14 . On the left side we see several input boxes. All of these correspond to abovementioned options of the command line program with one exception: in addition to the edge list file (graphs/Edge11.txt in Figure 14) there is also a file containing the $x$ - and $y$-coordinates of the nodes (graphs/Node11.txt in Figure 14 - this is required for visualization). The GUI starts with the default selection of Edge01.txt and Node01.txt ; if you want a different graph, type the corresponding file names in the input boxes.


Figure 14: A screenshot of the gsearch GUI.

Having completed the input boxes on the left side of the window, you can now use the buttons on the right side.

1. The Search button performs the graph search (it actually invokes the gsearch executable and stores results in the output directory). Time to complete the search depends on the size of the graph and the number of spanning trees used. Do not use the exhaustive spanning tree enumeration option on large graphs because the program may take too long to terminate.
2. The Plot Graph button plots the graph. An image of the graph is generated and an image viewer is invoked which can be used to view the image file (this is always named graph.jpg). A screenshot of the image viewer appears in Figure 15.
3. The Plot Search button plots the graph search. Actually, first the search strategy files are used to generate a sequence of image files, one image corresponding to each step of the search (these files are stored in the directory pix) and then the image viewer is invoked to view the files. The user can step through the images using the arrow keys, or use the start / stop buttons to run a slide show of the search. A screenshot of the image viewer appears in Figure 15.


Figure 15: A screenshot of the image viewer.
Nota bene: do not run Plot Search before running Search! The Plot Search function uses the latest results of Search; if these correspond to an earlier graph (from a previous run of the GUI) the plotted search will produce nonsense reults.

The GUI was implemented by Ath. Kehagias in MS Visual Basic 5.0 (and a little bit of C, using the dev-cpp 4.9.9.2 environment). The image viewer is a freeware program called AutoPics and made available by the company Mydesktophelp (at http://www.mydesktophelp.com). The GD graphics library (version gd-2.0.34-win32) has also been used; this library is available at http://www.libgd.org/.

Finally it must be stressed that the current version, gsearchGUI v.0.9, is still beta and requires further development. However we make it publicly available in the hope that it will prove useful to the graph search community.

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[^1]:    ${ }^{1}$ An implementation of the algorithm (for Windows and Linux computers) is available in the public domain; for details see Appendix D
    ${ }^{2}$ In fact GSST is an "anytime" algorithm [40]. Namely, the longer it runs the better solutions are provided, but a reasonable solution becomes available even in the initial stages of the algorithm.

[^2]:    ${ }^{3}$ Since no searchers are ever removed during an internal search $\mathbf{S}$, we have $\overline{s n}(\mathbf{S})=s n\left(\mathbf{S}, t_{f i n}\right)$ where $t_{f i n}$ is the length of the search.
    ${ }^{4}$ An important detail is worth mentioning at this point. There is a property, call it strong monotonicity, which characterizes schedules in which a traversed edge never becomes e-dirty again. The schedule of Remark 2.2 is monotone (since $\left.E_{E}^{C}(1)=E_{E}^{C}(2)=\emptyset\right)$ but not strongly monotone. We stress this detail mostly for the sake of completness; in what follows we will not use strong monotonicity.

[^3]:    ${ }^{5}$ The line "Move that searcher to $u$ staying inside the clear graph" is actually a simplification, i.e., in the interest of brevity, we do not indicate how such a path is obtained; however this is always possible, usually in more than one ways.

[^4]:    ${ }^{6}$ This is true even of GSST-L, because an edge which would be next in Barriere's traversal order may be temporarily skipped if its traversal (at the current stage of the search) would cause recontamination.

[^5]:    ${ }^{7}$ All computations were performed by the gsearch.exe program (see Appendix D), on a PC with Intel Dual Core E7500 CPU , running at 2.93 GHz with 3 GB RAM; computation time is reported in seconds, unless otherwise indicated.

[^6]:    ${ }^{8}$ Note that all GSST variants, except GSST-L, use a randomized element in the order of edge traversal - hence multiple runs with the same spanning tree can yield different searches.

[^7]:    ${ }^{9}$ We have experimented with larger values of $M$ (e.g., $M=10^{6}$ ) but the DFS variants still cannot achieve a four searcher clearing schedule.

[^8]:    ${ }^{10}$ Except for uniform GSST-R; however 5-searcher clearing schedules can also be found by this variant if it is run for a sufficiently large $M$.

[^9]:    ${ }^{11}$ Namely (assuming that the height $J_{1}$ of the grid is less than or equal to the width $J_{2}$ ) place $J_{1}$ searchers on one of the vertical sides of the grid and slide them horizontally to the other vertical side.

[^10]:    ${ }^{12}$ It is interesting to note that a somewhat related fact is well known in the probabilistic inference literature: inference is easier on graphs with low tree-width (7].

[^11]:    ${ }^{13}$ Basically the proof remains valid because the recontamination rules remain the same.

