Lattice-ordered join space: an Applications-Oriented Example

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15 June 2000

Abstract

In this paper we present an example of a lattice-ordered join space, i.e. a structure $(\mathbf{L}, \leq, \cdot)$ where $(\mathbf{L}, \leq)$ is a lattice, $(\mathbf{L}, \cdot)$ is a join space and the $\cdot$ hyperoperation is compatible with the $\leq$ order. Our example is obtained from a join hyperoperation which is frequently used in machine learning applications. We study the basic properties of the join hyperoperation and the associated extension hyperoperation.

1 Introduction

The study of lattice ordered *groups* is a well established branch of classical algebraic theory (for instance, see [1]). The study of ordered *hypergroupoids* and *hypergroups* appears in the work of Konstantinidou and Serafimidis [16, 17, 24, 25, 27]. Join spaces were introduced by Prenowitz and their properties were explored by Prenowitz and Jantosciak (see particularly [21, 22]); join spaces are a special, geometrically motivated class of hypergroups; this has been pointed out by Jantosciak [13] and by Corsini [9]; in addition Corsini and Leoreanu [10, 11] and Zahedi and Ameri [36] have explored the connection of join spaces to fuzzy sets.

By *lattice-ordered join space* we mean a structure $(\mathbf{L}, \leq, \cdot)$ where $(\mathbf{L}, \leq)$ is a lattice, $(\mathbf{L}, \cdot)$ is a join space and the $\cdot$ hyperoperation is compatible with the $\leq$ order. The exact nature of the compatibility will be explained in the sequel.

To the best of our knowledge, a study of *lattice-ordered join spaces* has not been undertaken previously and appears particularly interesting. However, in this paper we set a more modest goal: we study an *example* of lattice-ordered join space. The particular example which is the object of our study is frequently used in *machine learning applications.* In particular, the *join* hyperoperation which forms the basis of our investigation is defined on pairs (a, b) (where $a, b \in \mathbf{L}$, $(\mathbf{L}, \leq)$ is a lattice) as follows

$$a \cdot b \doteq \{x : a \wedge b \leq x \leq a \vee b\}$$

or, which amounts to the same thing, as $a \cdot b = [a \wedge b, a \vee b]$. We also introduce the associated *extension* hyperoperation, and study the properties of the resulting structure $(\mathbf{L}, \leq, \cdot)$. This is a useful prelude to the study of general lattice-ordered join spaces, which we plan to undertake in a future publication. In addition, as already mentioned, the hyperoperations considered in this paper are related to several *machine learning applications*; a study of their properties may have practical implications.

Let us explain the connection of the $a \cdot b$ hyperoperation to machine learning. First note that, when $\mathbf{L}$ is a Euclidean space $\mathbf{R}^N$, $a \cdot b$ is the N-dimensional hyperbox with lowest vertex at the point $a \wedge b$ and highest vertex at the point $a \vee b$. Such hyperboxes are used extensively for *classification* and *clustering*

[0]Ath. Kehagias thanks V.G. Kaburlasos for introducing him to lattice theory in general, and to lattice hyperbox operations in particular. He thanks Karen VanDyck, V.G. Kaburlasos and V. Petridis for many stimulating conversations.

by structures such as *classification, regression* and *decision trees* [3, 23] as well as in *neural* [4, 18] and *fuzzy* [12] extensions of such structures. The use of hyperboxes is implicit in the above cited examples; there are also cases [5, 6, 7, 8, 28, 29] where the hyperbox terminology is used explicitly. Similarly, hyperbox terminology appears in the theory of *computational learning theory* [2, 33].

A hyperbox in a Euclidean space is a special case of an *interval* in a lattice. This is explicitly stated in the literature: in a recent series of papers [14, 15, 19, 20] Kaburlasos and Petridis generalize the classification and clustering theory from Euclidean spaces to complete lattices and present a collection of related algorithms.

Given the ubiquity of the "hyperbox join", it is desirable to formulate a general theory of its properties. Such a theory can be used for the design of efficient machine learning algorithms, as well as for establishing their theoretical properties (convergence, optimality); in addition, as already mentioned such a theory can be seen as a preliminary exploration of the properties of general lattice-ordered join spaces.

Remark. *In this paper we are predominantly concerned with* **distributive** *lattices, in accordance with the assumptions of the motivating papers [14, 15, 19, 20]. Additional properties (for instance completeness) will occasionally be assumed. In the sequel* $(\mathbf{L}, \leq)$ *will denote a* **distributive** *lattice; if this assumption is strengthened or weakened, this will be explicitly stated.*

Remark. To the best of our knowledge, the structure $(\mathbf{L}, \leq, \cdot)$ (where $(\mathbf{L}, \leq)$ is a distributive lattice and $\cdot$ is the hyperbox join, as defined above) first appears in the hyperstructures literature in Corsini's paper [10]. Corsini briefly remarks that $(\mathbf{L}, \leq, \cdot)$ is a join space, anticipating some of the results of the present paper.

2 The Hyperbox Join Hyperoperation

2.1 Definitions and Basic Properties

Let $(\mathbf{L}, \leq)$ be a **distributive** lattice with the *sup* and *inf* binary operations denoted by $\wedge$ and $\vee$, respectively. For every set $A \subseteq \mathbf{L}$ we will write $\vee A$ to denote the supremum and $\wedge A$ to denote the infimum of A (when each of these exists). The notation $a \geq b$ is equivalent to $b \leq a$. We denote the *power set* of $\mathbf{L}$ by $\wp(\mathbf{L})$. We use the standard notation of algebraic hyperstructures, whereby for any operation or hyperoperation $*$:$\mathbf{L} \times \mathbf{L} \rightarrow \wp(\mathbf{L})$, and for all $A, B \subset \mathbf{L}$, we define $A * B \doteq \cup_{a \in A, b \in B} a * b$.We will write $A \sim B$ to denote that $A \cap B \neq \emptyset$.

The concept of *lattice interval* will be repeatedly used in this paper.

Definition 1 *The class of intervals of elements of* $\mathbf{L}$ *is denoted by* $I(\mathbf{L})$*, i.e.* $I(\mathbf{L}) \doteq \{[a, b] : a, b \in \mathbf{L}, a \leq b\}$.

The following properties of *inf* and *sup* acting on intervals will be repeatedly used in the sequel.

Proposition 2 *For all* $a, b, x, y \in \mathbf{L}$ *such that* $x \leq y$, $a \leq b$ *we have:*

(i) $a \vee [x, y] = [a \vee x, a \vee y]$;

(ii) $a \wedge [x, y] = [a \wedge x, a \wedge y]$;

(iii) $[a, b] \vee [x, y] = [a \vee x, b \vee y]$;

(iv) $[a, b] \wedge [x, y] = [a \wedge x, b \wedge y]$.

Proof. (i.1) Take any $c \in a \vee [x,y]$. There is a $z \in [x,y]$ such that $c = a \vee z$. We have

$$x \leq z \leq y \Rightarrow a \vee x \leq a \vee z = c \leq a \vee y \Rightarrow c \in [a \vee x, a \vee y] \Rightarrow a \vee [x,y] \subseteq [a \vee x, a \vee y]. \tag{1}$$

(i.2). Take any $c \in [a \vee x, a \vee y]$. Define $z = (c \wedge y) \vee x = (c \vee x) \wedge y$ (by distributivity). We have

$$x \leq (c \wedge y) \vee x = z = (c \vee x) \wedge y \leq y \Rightarrow z \in [x,y].$$

and

$$z \vee a = (c \wedge y) \vee x \vee a = (c \vee x \vee a) \wedge (y \vee x \vee a) = c \wedge (y \vee x \vee a) = c$$

(where we used $a \vee x \leq c$ and $c \leq y \vee a = x \vee y \vee a$). In short, we have proved that $z \in [x,y]$ and $z \vee a = c$; hence $c \in a \vee [x,y]$ and so

$$[a \vee x, a \vee y] \subseteq a \vee [x,y]. \tag{2}$$

From (1) and (2) follows that $a \vee [x,y] \ = [a \vee x, a \vee y]$. This completes the proof of (i); (ii) is proved dually; (iii) and (iv) are proved similarly. These proofs are omitted for the sake of brevity.■

We now introduce the join hyperoperation.

Definition 3 *The* join *hyperoperation is denoted by $a \cdot b$ ($a, b \in \mathbf{L}$) and is defined by:*

$$a \cdot b \doteq \{x : a \wedge b \leq x \leq a \vee b\} = [a \wedge b, a \vee b].$$

Remark. As already remarked, if $\mathbf{L}$ is a Euclidean space $\mathbf{R}^n$, then $a \cdot b$ is the hyperbox with lowest vertex $a \wedge b$ and highest vertex $a \vee b$. We retain this interpretation for a general $\mathbf{L}$ and will sometimes refer to $a \cdot b$ as the *hyperbox join*.

It is seen immediately that for any $a, b \in \mathbf{L}$ we have $a, b \in a \cdot b$, hence the hyperbox join is well defined (never results in the empty set). We will usually write ab instead of $a \cdot b$. The next proposition states that the class of intervals of $\mathbf{L}$ is exactly the same as the set of all joins of elements of $\mathbf{L}$.

Proposition 4 $I(\mathbf{L}) = \{a \cdot b : a, b \in \mathbf{L}\}$.

Proof. Take any interval $[a,b] \in I(\mathbf{L})$. By definition, $a \leq b$, so $ab = [a \wedge b, a \vee b] = [a,b]$. On the other hand, any ab is an interval by definition.■

The following property of the join will be repeatedly used in the sequel.

Proposition 5 *For all $a, b, x, y \in \mathbf{L}$ such that $x \leq y$, $a \leq b$ we have*

(i) $a \cdot [x,y] = [a \wedge x, a \vee y]$.

(ii) $[a,b] \cdot [x,y] = [a \wedge x, b \vee y]$.

Proof. (i) We have $a[x,y] = \cup_{x \leq z \leq y} az$. Pick any $u \in a[x,y]$, then there is a $z_u \in [x,y]$ such that $a \wedge z_u \leq u \leq a \vee z_u$. Also, $x \leq z_u \Rightarrow a \wedge x \leq a \wedge z_u$ and $y \geq z_u \Rightarrow a \vee y \geq a \vee z_u$. Hence we have $a \wedge x \leq u \leq a \vee y$ which implies that $u \in [a \wedge x, a \vee y]$; hence $a[x,y] \subseteq [a \wedge x, a \vee y]$.

On the other hand, pick any $v \in [a \wedge x, a \vee y]$ and define $z_v = (v \vee x) \wedge y$. By distributivity, we also have $z_v = (v \wedge y) \vee x$. Now, $x \leq (v \wedge y) \vee x = z_v = (v \vee x) \wedge y \leq y$. So $z_v \in [x,y]$. Also, $z_v \wedge a = [(v \vee x) \wedge y] \wedge a = (v \vee x) \wedge (y \wedge a) = (v \wedge y \wedge a) \vee (x \wedge y \wedge a)$. Since $v \wedge y \wedge a \leq v$ and $x \wedge y \wedge a = x \wedge a \leq v$, it follows that $z_v \wedge a \leq v$. Similarly we can show that $z_v \vee a \geq v$. In short, $z_v \wedge a \leq v \leq z_v \vee a$ and so $v \in az_v$. Hence $z_v \in [x,y]$ and $v \in az_v$, which implies that $v \in a[x,y]$. So $[x \wedge a, y \vee a] \subseteq a[x,y]$

Hence we conclude that $[x \wedge a, y \vee a] = a[x, y]$ and the proof of (i) is complete.

(ii) (For this proof we are indebted to V.G. Kaburlasos.) Let us first prove that $[a, b] \cdot [x, y] \subseteq [a \wedge x, b \vee y]$. Take any $u \in [a, b] \cdot [x, y] = \cup_{a \le z \le b} z[x, y] = \cup_{a \le z \le b} [z \wedge x, z \vee y]$ (by part (i) of the Proposition). Hence there is some $z_1 \in [a, b]$ such that $z_1 \wedge x \le u \le z_1 \vee y$. But, $a \wedge x \le z_1 \wedge x$, since $a \le z_1$. Similarly, $z_1 \vee y \le b \vee y$, since $z_1 \le b$. Hence, $a \wedge x \le z_1 \wedge x \le u \le z_1 \vee y \le b \vee y$, i.e. $u \in [a \wedge x, b \vee y]$, and so $[a, b] \cdot [x, y] \subseteq [a \wedge x, b \vee y]$.

Next we prove that $[a \wedge x, b \vee y] \subseteq [a, b] \cdot [x, y]$. Take $v \in [a \wedge x, b \vee y]$; then $a \wedge x \le v \le b \vee y$. Now take $z_1 = (v \vee x) \wedge y = (v \wedge y) \vee x$ and $z_2 = (v \vee a) \wedge b = (v \wedge b) \vee a$. It is easy to check that $z_1 \in [x, y]$ and $z_2 \in [a, b]$. Also, $z_1 \wedge z_2 = [(v \vee x) \wedge y] \wedge [(v \vee a) \wedge b] = [v \vee (a \wedge x)] \wedge [b \wedge y] = [v \wedge (b \wedge y)] \vee [a \wedge x] \le v$ (since $v \wedge (b \wedge y) \le v$ and $a \wedge x \le v$). In short, we have shown that $z_1 \wedge z_2 \le v$. Similarly we show that $v \le z_1 \vee z_2$, i.e. we have $v \in z_1 z_2 \subseteq [x, y] \cdot [a, b]$. In short, $[a \wedge x, b \vee y] \subseteq [a, b] \cdot [x, y]$. This, together with $[a, b] \cdot [x, y] \subseteq [a \wedge x, b \vee y]$, implies that $[a, b] \cdot [x, y] = [a \wedge x, b \vee y]$.■

We now are ready to establish the basic properties of the join hyperoperation.

Proposition 6 *The following properties hold for any $a, b \in \mathbf{L}$.*

(i) Idempotence: $aa = a$.

(ii) Commutativity: $ab = ba$.

(iii) Associativity: $(ab)c = a(bc)$.

(iv) Reproduction: $a\mathbf{L} = \mathbf{L}$.

(v) There is no scalar *in $\mathbf{L}$, i.e. there is no a such that for all $x \in \mathbf{L}$ we have $|ax| = 1$.*

Proof. (i) is immediate: $aa = [a \wedge a, a \vee a] = [a, a] = a$. The proof of (ii) is obvious. The proof of (iii) uses Proposition 5: we have $(ab)c = [a \wedge b, a \vee b]c = [a \wedge b \wedge c, a \vee b \vee c]$; similarly $a(bc) = a[b \wedge c, b \vee c] = [a \wedge b \wedge c, a \vee b \vee c]$. For (iv), take any $a \in \mathbf{L}$. We have $a\mathbf{L} = \cup_{x \in \mathbf{L}} ax \supseteq \cup_{x \in \mathbf{L}} x = \mathbf{L}$. On the other hand, $a\mathbf{L}$ is clearly a subset of $\mathbf{L}$. So $a\mathbf{L} = \mathbf{L}$. Finally, for (v), take any $a \in \mathbf{L}$ and any $x \in \mathbf{L}$, $x \neq a$; then $a, x \in ax$ and we have $|ax| \ge 2$.■

Conclusion 7 $(\mathbf{L}, \cdot)$ *is a hypergroup; each of its elements is idempotent; none of its elements is scalar.*

Proposition 8 *For every $a, b \in \mathbf{L}$ we have that $(a \cdot b, \le)$ is a convex sub-lattice of $\mathbf{L}$.*

Proof. This follows immediately from the fact that $a \cdot b$ is an interval. ■

Proposition 9 *For every $a, b \in \mathbf{L}$ we have that $(a \cdot b, \cdot)$ is a sub-hypergroup of $\mathbf{L}$.*

Proof. Choose any $a, b \in \mathbf{L}$ and keep them fixed for the rest of the proof. We need to prove that for all $x, y \in a \cdot b$ we have: (i) $x \cdot y \subseteq a \cdot b$ and (ii) $x \cdot (a \cdot b) = a \cdot b$. Choose any $x, y \in a \cdot b$, then $a \wedge b \le x, y \le a \vee b$. Hence it follows that $a \wedge b \le x \wedge y \le x \vee y \le a \vee b$ which means $x \cdot y \subseteq a \cdot b$ and (i) is proved. Regarding (ii), we have $x \cdot (a \cdot b) = [x \wedge a \wedge b, x \vee a \vee b]$ (from Proposition 5); since $a \wedge b \le x \le a \vee b$, we have $[x \wedge a \wedge b, x \vee a \vee b] = [a \wedge b, a \vee b]$ and so $x \cdot (a \cdot b) = (a \cdot b)$. ■

Remark. When $(\mathbf{L}, \le)$ is not distributive, it is possible that the join hyperoperation is not associative. For example, consider the lattice $(\mathbf{L}, \le)$ of Figure 1. Clearly $(\mathbf{L}, \le)$ is not distributive; in fact it is not even modular. We can compute $a(fe)$ and $(af)e$ explicitly. It turns out that $a(fe) = \{a, b, c, e, f\}$, $(af)e = \{a, b, c, d, e, f\}$. So we see that associativity of join can fail in a non-modular lattice.

Figure 1 to be placed here

Remark. We have proved that distributivity of $(\mathbf{L}, \leq)$ is sufficient for associativity of the hyperbox join. It is an open question whether modularity is also sufficient.
Remark. Finally, neither distributivity nor modularity of $(\mathbf{L}, \leq)$ are *necessary* for the hyperbox join to be associative. Consider for instance the lattice $\mathbf{N}_5$ depicted in Figure 2. It can be checked directly that for all $x, y, z \in \mathbf{N}_5$ we have $(xy)z = x(yz)$ (we performed this computation using the computer algebra package *Maple*).

Figure 2 to be placed here

An alternative proof of the associativity of join on $\mathbf{N}_5$ is also presented. This requires the following auxiliary Propositions 10, 11, 12.

Proposition 10 *For* any *lattice* $(\mathbf{L}, \leq)$ *(i.e. even a non-distributive lattice) and for any* $x, y, z \in \mathbf{L}$*, we have:* $x \wedge y \wedge z, x \vee y \vee z \in x(yz) \cap (xy)z$ *and so* $x(yz) \sim (xy)z$*.*

Proof. $x \cdot (yz) = x \cdot\ [y \wedge z, y \vee z]$ and so both $x \wedge y \wedge z$ and $x \vee y \vee z$ belong to $x \cdot\ [y \wedge z, y \vee z]$; similarly for $(xy) \cdot z$.■

Proposition 11 *For* any *lattice* $(\mathbf{L}, \leq)$ *(i.e. even a non-distributive lattice) and any* $x, y, z \in \mathbf{L}$*, the quantitites* $\wedge(x(yz))$*,* $\wedge((xy)z)$*,* $\vee(x(yz))$*,* $\vee((xy)z)$ *exist and in fact we have:*

(i) $\wedge(x(yz)) = \wedge((xy)z) = x \wedge y \wedge z$;

(ii) $\vee(x(yz)) = \vee((xy)z) = x \vee y \vee z$.

Proof. We only prove (i), since (ii) can be proved dually. Take any $u \in x(yz) = \cup_{w \in yz} xw$. Then there is some w_u such that $x \wedge w_u \leq u$ and $y \wedge z \leq w_u$, from which follows $x \wedge y \wedge z \leq u$. In short: $u \in x(yz) \Rightarrow x \wedge y \wedge z \leq u$. Since, $x \wedge y \wedge z \in x(yz)$, it follows that $\wedge(x(yz)) = x \wedge y \wedge z$. Similarly we can prove that $u \in (xy)z) \Rightarrow x \wedge y \wedge z \leq u$ and, since, $x \wedge y \wedge z \in (xy)z)$, it follows that $\wedge((xy)z)) = x \wedge y \wedge z$.
■

Proposition 12 *For all* $x, y, z \in \mathbf{N}_5$*, both* $x(yz)$ *and* $(xy)z$ *are closed intervals.*

Proof. This can be checked by exhaustive computation (we did this with the computer program *Maple*). ■

Conclusion 13 *We see that, since* $x(yz)$ *and* $(xy)z$ *are closed intervals and they have identical infimum and supremum, they must be the same interval. Hence we have proved that join is associative in the nonmodular lattice* $\mathbf{N}_5$*.*

Remark. Propositions 10 and 11 are additionally useful as the basis for developing a theory of the hyperbox join on *non-distributive lattices*; in this case we see that even if associativity fails, weak associativity holds (hence we can build a theory of *weakly* join spaces, along the lines of [30, 31, 32, 34, 35]).

2.2 Compatibility of Join with the Lattice Order

We will now demonstrate that the order relation $\leq$ (on elements of $\mathbf{L}$), produces a new order $\precsim$ (on *intervals* of elements of $\mathbf{L}$), *which is compatible with the join operation* and hence we will conclude that $(\mathbf{L}, \leq, \cdot)$ is a *partially-ordered join space.* To this end we will define *two* preorders on $\wp(\mathbf{L})$ and then we will show that, when restricted on the class of intervals of $\mathbf{L}$, these preorders are identical and, in fact, are an order.

Definition 14 *We define the $\precsim$ relation on pairs (A, B) where $A, B \in \wp(\mathbf{L})$, as follows*

$$A \precsim B \Leftrightarrow \begin{cases} \forall a_1 \in A \quad \exists b_1 \in B: \quad a_1 \leq b_1 \\ \forall b_2 \in B \quad \exists a_2 \in A: \quad a_2 \leq b_2. \end{cases}$$

Proposition 15 *(i) $\precsim$ is a pre-order on $\wp(\mathbf{L})$; (ii) $\precsim$ is an order on $I(\mathbf{L})$.*

Proof. (i) Obviously, $A \precsim A$ for all $A \in \wp(\mathbf{L})$. Now take any $A, B, C \in \wp(\mathbf{L})$ such that $A \precsim B$ and $B \precsim C$. Since $A \precsim B$, it follows that for every $a \in A$, there is some $b_a \in B$ such that $a \leq b_a$; but since $B \precsim C$ there is also some $c \in C$ such that $b_a \leq c$. Hence, for every $a \in A$, there is some $c \in C$ such that $a \leq c$. We prove similarly that for every $c \in C$ there is some $a \in A$ such that $a \leq c$ and so we have proved that $\precsim$ is a preorder on $\wp(\mathbf{L})$.

(ii) In case $A = [a_1, a_2]$, $B = [b_1, b_2]$, such that $A \precsim B$ and $B \precsim A$, since $A \precsim B$, it follows for a_2 that there is some $b \in [b_1, b_2]$ such that $a_2 \leq b \leq b_2$. It also follows for b_1 that there is some $a \in [a_1, a_2]$ such that $a_1 \leq a \leq b_1$. Using $B \precsim A$ we prove similarly that $b_1 \leq a_1$ and $b_2 \leq a_2$. In short, we have obtained that $a_1 = b_1$ and $a_2 = b_2$, i.e. $A = B$. Hence $\precsim$ is an order on $I(\mathbf{L})$.■

Definition 16 *We define the $\preccurlyeq$ relation on pairs (A, B) where $A, B \in \wp(\mathbf{L})$, as follows*

$$A \preccurlyeq B \Leftrightarrow \begin{cases} \exists \overline{b} \in B \text{ such that } \quad \forall a \in A: \quad a \leq \overline{b} \\ \exists \overline{a} \in A \text{ such that } \quad \forall b \in B: \quad \overline{a} \leq b. \end{cases}$$

Proposition 17 *(i) $\preccurlyeq$ is a pre-order on $\wp(\mathbf{L})$; (ii) $\preccurlyeq$ is an order on $I(\mathbf{L})$.*

Proof. This is proved similarly to Proposition 15. ■

Proposition 18 *The $\precsim$ and $\preccurlyeq$ relations are identical on $I(\mathbf{L})$, i.e. for all $A, B \in I(\mathbf{L})$ we have*

$$A \precsim B \Longleftrightarrow A \preccurlyeq B.$$

Proof. Let us take $A = [a_1, a_2]$ and $B = [b_1, b_2]$. First let us show that $A \precsim B \Rightarrow A \preccurlyeq B$. If $A \precsim B$, then for a_2 there is some $\widehat{b} \in [b_1, b_2]$ such that $a_2 \leq \widehat{b} \leq b_2$. Then, for any $a \in [a_1, a_2]$ we have $a \leq a_2 \leq b_2$. Similarly we show that for any $b \in [b_1, b_2]$ we have $a_1 \leq b$. Hence $A \preccurlyeq B$. It can be shown immediately that $A \preccurlyeq B \Rightarrow A \precsim B$. ■

Proposition 19 *$(I(\mathbf{L}), \precsim)$ is a lattice; in particular, for any $A, B \in I(\mathbf{L})$ where $A = [a_1,\ a_2]$ and $B = [\ b_1,\ b_2]$, we have*

$$\inf(A, B) = [a_1 \wedge b_1, a_2 \wedge b_2], \qquad \sup(A, B) = [a_1 \vee b_1, a_2 \vee b_2] \tag{3}$$

Proof. This is can be proved easily. By Proposition 18 the relations $\precsim$ and $\preccurlyeq$ are equivalent on $I(\mathbf{L})$. But for A, B intervals, $A \preccurlyeq B$ is equivalent to $a_1 \leq b_1$ and $a_2 \leq b_2$. Now take any two intervals $[a_1, a_2]$, $[b_1, b_2]$; we certainly have $[a_1 \wedge b_1, a_2 \wedge b_2] \preccurlyeq [a_1, a_2]$ and $[a_1 \wedge b_1, a_2 \wedge b_2] \preccurlyeq [b_1, b_2]$. Furthermore, suppose there is some interval $[c_1, c_2]$ which satisfies

$$[a_1 \wedge b_1, a_2 \wedge b_2] \preccurlyeq [c_1, c_2] \preccurlyeq [a_1, a_2], [b_1, b_2].$$

It follows that $a_1 \wedge b_1 \leq c_1 \leq a_1, b_1$ which implies $c_1 = a_1 \wedge b_1$; and similarly $a_2 \wedge b_2 \leq c_2 \leq a_2, b_2$ which implies $c_2 = a_2 \wedge b_2$. In short, $\inf(A, B) = [a_1 \wedge b_1, a_2 \wedge b_2]$. It can be shown similarly that $\sup(A, B) = [a_1 \vee b_1, a_2 \vee b_2]$.$\blacksquare$

Hence we have obtained (by two alternative definitions) an order $\precsim$ on $I(\mathbf{L})$. We are now ready to state and prove the compatibility of join with the $\precsim$ order. To this end, let us first give precise definitions of *ordered hypergroup* and *ordered join space*.

Definition 20 $(\mathbf{L}, \leq, \cdot)$ *is called a* strictly ordered hypergroup (respectively, strictly ordered join space) *iff:*

(i) $(\mathbf{L}, \leq)$ *is a lattice,*

(ii) $(\mathbf{L}, \cdot)$ *is a hypergroup (respectively, join space),*

(iii) for all $x, y \in \mathbf{L}$ *we have that* xy *is an interval,*

(iv) for all $a, x, y \in \mathbf{L}$ *such that* $x \leq y$ *we have* $a \cdot x \precsim a \cdot y$.

The above definition follows [27]. We are now ready to prove that $(\mathbf{L}, \leq, \cdot)$ is an ordered hypergroup.

Proposition 21 *For all* $a, b, x, y \in \mathbf{L}$ *we have: (i)* $x \leq y \Rightarrow ax \precsim ay$, *(ii)* $a \leq b$ *and* $x \leq y \Rightarrow ax \precsim by$.

Proof. (i) Since $ax = [a \wedge x, a \vee x]$, $ay = [a \wedge y, a \vee y]$, it follows by Proposition 19 that $\inf(ax, ay)$ $= [(a \wedge x) \wedge (a \wedge y), (a \vee x) \wedge (a \vee y)]$. However, since $x \leq y$ we have $(a \wedge x) \wedge (a \wedge y) = (a \wedge x)$ and $(a \vee x) \wedge (a \vee y) = (a \vee x)$. Hence $\inf(ax, ay) = [a \wedge x, a \vee x] = ax$, i.e. $ax \precsim ay$.

(ii) From (i) and $x \leq y$ we have $ax \precsim ay$; from (i) and $a \leq b$ we have $ay \precsim by$; these two inequalities imply that $ax \precsim by$. $\blacksquare$

Conclusion 22 $(\mathbf{L}, \leq, \cdot)$ *is a strictly ordered hypergroup.*

2.3 Distributivity Properties

In this section we explore the distribution: (a) of $\vee, \wedge$ on join; (b) of join on $\vee, \wedge$.

Proposition 23 *For all* $a, b, c \in \mathbf{L}$ *we have:*

(i) $(ab) \vee c = (a \vee c) \cdot (b \vee c)$.

(ii) $(ab) \wedge c = (a \wedge c) \cdot (b \wedge c)$.

Proof. (i) $(ab) \vee c = [(a \wedge b) \vee c, a \vee b \vee c]$ from Proposition 2; $(a \vee c)(b \vee c) = [(a \vee c) \wedge (b \vee c), (a \vee c) \vee (b \vee c)]$. Clearly $(a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c$ (by distributivity) and $(a \vee c) \vee (b \vee c) = a \vee b \vee c$; (ii) is proved dually and we are done.$\blacksquare$

Proposition 24 *For all* $a, b, c \in \mathbf{L}$ *we have:*

(i) $a(b \vee c) = ab \vee ac.$

(ii) $a(b \wedge c) = ab \wedge ac.$

Proof. Take any $z \in a(b \vee c)$ and define $x = (a \vee b) \wedge z$, $y = (a \vee c) \wedge z$. Since $a(b \vee c) = [a \wedge (b \vee c), a \vee b \vee c]$ it follows that $z \leq a \vee b \vee c$. Also $x \vee y = [(a \vee b) \wedge z] \vee [(a \vee c) \wedge z] = (a \vee b \vee c) \wedge z = z$. In short, $z = x \vee y$. On the other hand, since $x = (a \vee b) \wedge z$ it follows that $x \leq a \vee b$. And since $z \geq a \wedge (b \vee c)$ it follows that $z \geq a \wedge b$; so $x = (a \vee b) \wedge z \geq a \wedge b$. In short, $x \in ab$. Similarly we can show that $y \in ac$. In conclusion, we have shown that for every $z \in a(b \vee c)$ there exist $x \in ab$, $y \in ac$ such that $z = x \vee y$. Hence $a(b \vee c) \subseteq (ab) \vee (ac)$.

Take any $u \in (ab) \vee (ac)$. Then there is some $x \in ab$ and some $y \in ac$ such that $u = x \vee y$. So $u = x \vee y \geq (a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$; similarly $u \leq a \vee (b \vee c)$. So $u \in a(b \vee c)$ which means $(ab) \vee (ac) \subseteq a(b \vee c)$.

Hence $(ab) \vee (ac) = a(b \vee c)$ and the proof of (i) is complete; (ii) is proved dually. ■

In anticipation of the results of Section 2.3 we also define *lattice-ordered hypergroup* and *lattice-ordered join space*.

Definition 25 $(\mathbf{L}, \leq, \cdot)$ *is called a* strictly lattice-ordered hypergroup (respectively, join space) *iff:*

(i) $(\mathbf{L}, \leq)$ *is a lattice,*

(ii) $(\mathbf{L}, \cdot)$ *is a hypergroup (respectively, join space),*

(iii) for all $x, y \in \mathbf{L}$ *we have that* xy *is an interval,*

(iv) for all $a, x, y \in \mathbf{L}$ *we have:* $a \cdot (x \vee y) = (ax) \vee (ay)$ *and* $a \cdot (x \wedge y) = (ax) \wedge (ay)$.

Conclusion 26 *From Definition 3, Proposition 24 and Conclusion 7 follows that* $(\mathbf{L}, \leq, \cdot)$ *is a strictly lattice-ordered hypergroup.*

Definition 27 *Given a strictly lattice-ordered hypergroup* $(\mathbf{H}, \cdot)$*, every convex subhypergroup of* $\mathbf{H}$ *is called* lattice-ordered hyperideal*.*

Remark. The above definition is analogous to the one presented in [26].

Conclusion 28 *From Propositions 8 and 9 follows that for every* $a, b \in \mathbf{L}$*,* $a \cdot b$ *is a strictly lattice-ordered hyperideal.*

3 The Extension Hyperoperation

3.1 Definitions and Basic Properties

We will now introduce the *extension* hyperoperation which is derived, in the manner of Prenowitz [22], from the join hyperoperation.

Definition 29 *The* extension *hyperoperation is denoted by* a/b *(*$a, b \in \mathbf{L}$*) and is defined by:*

$$a/b \doteq \{x : x \wedge b \leq a \leq x \vee b\} = \{x : a \in xb\}.$$

It is seen immediately that for any $a, b \in \mathbf{L}$ we have $a \in a/b$. In addition, the extension hyperoperation enjoys the *join* property [22].To prove this, we first need an auxiliary proposition.

Proposition 30 *In any lattice* $\mathbf{L}$ *(i.e. not necessarily distributive) we have*

$$\left.\begin{array}{c} x \wedge w \leq y \vee z \\ y \wedge z \leq x \vee w \end{array}\right\} \Leftrightarrow xw \sim yz.$$

Proof. (i) We first prove the $\Rightarrow$ direction. Assume that $x \wedge w \leq y \vee z$ and $y \wedge z \leq x \vee w$. It follows immediately that

$$x \wedge w \leq (x \wedge w) \vee (y \wedge z) \leq x \vee w, \qquad y \wedge z \leq (x \wedge w) \vee (y \wedge z) \leq y \vee z. \tag{4}$$

We similarly obtain

$$x \wedge w \leq (x \vee w) \wedge (y \vee z) \leq x \vee w, \qquad y \wedge z \leq (x \vee w) \wedge (y \vee z) \leq y \vee z. \tag{5}$$

It also follows form the hypothesis that

$$(x \wedge w) \vee (y \wedge z) \leq (x \vee w) \wedge (y \vee z). \tag{6}$$

Defining $a = (x \wedge w) \vee (y \wedge z)$ and $b = (x \vee w) \wedge (y \vee z)$, we see from (6) that $[a, b]$ is well defined; furthermore, from (4), (5) we see that

$$[a, b] \subseteq (xw) \cap (yz) \Rightarrow xw \sim yz$$

and the first part of the proof is completed.

(ii) It is very easy to prove the $\Leftarrow$ direction. If $xw \sim yz$, then there is some $u \in xw \cap yz$ such that

$$\left.\begin{array}{c} x \wedge w \leq u \leq x \vee w \\ y \wedge z \leq u \leq y \vee z \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x \wedge w \leq u \leq y \vee z \\ y \wedge z \leq u \leq x \vee w. \end{array}\right.$$

This completes the proof of (ii) and of the proposition.$\blacksquare$

We can now prove that $/$ and $\cdot$ enjoy the join property.

Proposition 31 *For all* $a, b, c, d \in \mathbf{L}$ *we have:* $(a/b) \sim (c/d) \Rightarrow ad \sim bc$.

Proof. If $(a/b) \sim (c/d)$, then there is some $u \in (a/b) \cap (c/d)$ such that $u \wedge b \leq a \leq u \vee b$ and $u \wedge d \leq c \leq u \vee d$. From $a \leq u \vee b$ follows that $a \wedge d \leq (u \vee b) \wedge d = (u \wedge d) \vee (b \wedge d) \leq c \vee (b \wedge d) \leq c \vee b$. From $c \leq u \vee d$ follows that $c \wedge b \leq (u \vee d) \wedge b = (u \wedge b) \vee (d \wedge b) \leq a \vee (d \wedge b) \leq a \vee d$. But, from Proposition 30 we have $a \wedge d \leq c \vee b$ and $c \wedge b \leq a \vee d \Rightarrow bc \sim ad$ and the proof is complete.$\blacksquare$

Conclusion 32 $(\mathbf{L}, \leq, \cdot)$ *is a strictly lattice-ordered* join space.

The following proposition gives conditions under which the converse of the join property holds.

Proposition 33 *For all* $a, b, c, d \in \mathbf{L}$ *such that* $a \leq d$ *and* $b \leq c$, *we have:* $ad \sim bc \Rightarrow (a/c) \sim (b/d)$.

Proof. Choose some $x \in ad \cap bc$, then $a \leq x \leq d$ and $b \leq x \leq c$. Take $y = a \wedge b$. Then $y \wedge c \leq a$. Also, $a \leq x \leq d$ and $b \leq x \leq c$ imply that $a \leq x \leq d \wedge c \leq c \leq y \vee c$. It follows that $y \vee c \geq a$. So $y \wedge c \leq a \leq y \vee c \Rightarrow y \in (a/c)$. Similarly we prove that $y \in (b/d)$.$\blacksquare$

Remark. Because of the commutativity of join, there are several more combinations implicit in the above condition, For instance, if $a \leq d$ and $b \leq c$, we have $ad \sim bc \Rightarrow (a/b) \sim (c/d)$.

3.2 Additional Properties

In this section we present various properties of the extension hyperoperation. The definition of the extension hyperoperation does not give an explicit description of the elements which belong in a/b. The next proposition is a step towards a more complete description of such elements, for the case when a, b are comparable.

Proposition 34 *For all $a, b \in \mathbf{L}$ we have:*

(i) If $b \leq a$, then $a/b = \{x : b \vee x = a \vee x\}$.

(ii) If $a \leq b$, then $a/b = \{x : b \wedge x = a \wedge x\}$.

Proof. (i) Take some $x \in a/b$; then $a \leq b \vee x \Rightarrow a \vee x \leq b \vee x$; also $b \leq a \Rightarrow b \vee x \leq a \vee x$. So we have $b \vee x = a \vee x$. Hence $a/b \subseteq \{x : b \vee x = a \vee x\}$. On the other hand, if $b \vee x = a \vee x$ then it follows that $a \leq a \vee x = b \vee x$; in addition, also $b \leq a \Rightarrow b \wedge x \leq a \wedge x \leq a$. In short, $b \vee x = a \vee x$ implies that $b \wedge x \leq a \leq b \vee x$. In short, $a/b \supseteq \{x : b \vee x = a \vee x\}$. It follows that $a/b = \{x : b \vee x = a \vee x\}$. (ii) is proved dually.■

Corollary 35 *It can be easily verified that in every lattice $(\mathbf{L}, \leq)$ we have $a/a = \mathbf{L}$.*

Corollary 36 *It can be easily verified that the following identities hold in every lattice $(\mathbf{L}, \leq)$ with a minimum element 0 and a maximum element 1: $a/0 = [a, 1]$, $a/1 = [0, a]$, $0/a = \{x : x \wedge a = 0\}$, $1/a = \{x : x \vee a = 1\}$.*

Additional properties are described by the following propositions.

Proposition 37 *For all $a, b \in \mathbf{L}$, $(a/b, \leq)$ is a convex sublattice of $(\mathbf{L}, \leq)$.*

Proof. Take $x, y \in a/b$ and keep them fixed for the rest of the proof. First, note that we have

$$\left.\begin{array}{l} x \wedge b \leq a \leq x \vee b \\ y \wedge b \leq a \leq y \vee b \end{array}\right\} \Rightarrow (x \wedge b) \wedge (y \wedge b) \leq a \leq (x \vee b) \wedge (y \vee b) \Rightarrow$$

$$(x \wedge y) \wedge b \leq a \leq (x \wedge y) \vee b \Rightarrow x \wedge y \in (a/b).$$

We can prove dually that $x \vee y \in (a/b)$. Hence a/b is a sublattice.

Next take any $z \in xy$. Since $(x \vee y) \wedge b \leq a$ and $z \leq x \vee y$, it follows that $z \wedge b \leq a$. Since $a \leq (x \wedge y) \vee b$ and $x \wedge y \leq z$, it follows that $a \leq z \vee b$. In short $z \in a/b$ and $xy \subseteq a/b$. Now, if we also have $x \leq y$, then $[x, y] = [x \wedge y, x \vee y] = xy \subseteq a/b$; so a/b is a *convex* sublattice.■

Proposition 38 *a/b is a subhypergroup for all $a, b \in \mathbf{L}$.*

Proof. Take any $a, b \in \mathbf{L}$. We have to show that: (i) for all $x, y \in a/b$ we have $xy \subseteq a/b$, and (ii) for all x we have $x(a/b) = a/b$.

As for (i), it has been shown in the course of proving the previous proposition.

For proving (ii), any $x \in a/b$ and keep it fixed for the rest of the proof. First, note that for all $y \in a/b$ we have, by (i), that $xy \subseteq a/b$, and so $x(a/b) = \cup_{y \in a/b} xy \subseteq a/b$. So $x(a/b) \subseteq a/b$. Second, note that for all $z \in a/b$ we have $z \in xz$ and so $a/b = \cup_{z \in a/b} z \subseteq \cup_{z \in a/b} xz = x(a/b)$; so $a/b \subseteq x(a/b)$. Hence we have $a/b = x(a/b)$.■

Conclusion 39 *From Propositions 37 and 38 follows that a/b is a lattice-ordered hyperideal for all $a, b \in \mathbf{L}$.*

Proposition 40 *For all $a, b \in \mathbf{L}$ we have $ab \cap (a/b) = a$.*

Proof. Clearly $a \in ab \cap (a/b)$, so $ab \cap (a/b)$ is not empty. Take any $x \in ab \cap a/b$.

$$\left.\begin{array}{c} a \wedge b \leq x \leq a \vee b \\ x \wedge b \leq a \leq x \vee b \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \wedge b \leq x \wedge b \\ x \wedge b \leq a \wedge b \end{array}\right\} \Rightarrow a \wedge b = x \wedge b.$$

Dually we get $a \vee b = x \vee b$. In a distributive lattice, $a \vee b = x \vee b$ and $a \wedge b = x \wedge b$ imply $a = x$ [1].■

Proposition 41 *For all $a, b \in \mathbf{L}$, $(b/a) = (a/b) \Rightarrow a = b$.*

Proof. $(b/a) = (a/b) \Rightarrow (b/a) \cap ab = (a/b) \cap ab$. But from the previous proposition $(b/a) \cap ab = b$ and $(a/b) \cap ab = a$.■

Now suppose that $\mathbf{L}$ is equipped with a positive valuation[1] $v(\cdot)$ and an associated metric $d(\cdot, \cdot)$.

Proposition 42 *For all $a, b, c \in \mathbf{L}$, we have: $(c/b) \subseteq (a/b) \Rightarrow d(a, b) \leq d(c, b)$.*

Proof. $(c/b) \subseteq (a/b) \Rightarrow c \in a/b$. Then: $b \wedge c \leq a \Rightarrow b \wedge c \leq b \wedge a \Rightarrow v(b \wedge c) \leq v(b \wedge a)$. Similarly: $b \vee c \geq a \Rightarrow b \vee c \geq b \vee a \Rightarrow v(b \vee c) \geq v(b \vee a)$. From $v(b \wedge c) \leq v(b \wedge a)$ and $v(b \vee c) \geq v(b \vee a)$ follows that $v(b \vee c) - v(b \wedge c) \geq v(b \vee a) - v(b \wedge a)$ which is equivalent to $d(c, b) = d(b, c) \geq d(b, a) = d(a, b)$.■

3.3 Conditions for a/b to be a Closed Interval

In this section we give a more complete description of a/b. In particular we show that, under certain conditions, a/b is a closed interval and we characterize its endpoints. The key requirement is that $\mathbf{L}$ is not only distributive, but *general distributive*. The definition of this important property is given below. (In the following definition, $\{x_s\}_{s \in S}$ is a set and S is an appropriate index set, such that for all $s \in S$ we have $x_s \in \mathbf{L}$.)

Definition 43 *(i)A lattice $\mathbf{L}$ is called general $\wedge$ -distributive iff: $\forall a \in \mathbf{L}, \forall \{x_s\}_{s \in S}$ we have*

$$a \wedge (\vee_{s \in S} x_s) = \vee_{s \in S}(a \wedge x_s).$$

(ii) A lattice $\mathbf{L}$ is called general $\vee$-distributive iff: $\forall a \in \mathbf{L}, \forall \{x_s\}_{s \in S}$ we have

$$a \vee (\wedge_{s \in S} x_s) = \wedge_{s \in S}(a \vee x_s).$$

(iii) A lattice is called general distributive iff it is general $\wedge$-distributive and general $\vee$-distributive.

Let us now present two propositions related to general distributivity.

Proposition 44 *If a lattice $\mathbf{L}$ is complete then*

(i) $\mathbf{L}$ is general $\wedge$-distributive iff $\{x : b \wedge x \leq a\}$ is a closed interval for all $a, b \in \mathbf{L}$.

(ii) $\mathbf{L}$ is general $\vee$-distributive iff $\{x : b \vee x \geq a\}$ is a closed interval for all $a, b \in \mathbf{L}$.

[1] For instance, in a modular lattice of finite length, the height function is always a positive valuation.

Proof. (i) is proved in p. 128 of [1]; (ii) can be proved dually.■

Remark. Birkhoff uses the term "Brouwerian lattice" for a lattice **L** with the property of Prop.44.(i), i.e. a lattice **L** such that $\{x : b \wedge x \leq a\}$ contains a greatest element; he denotes this element by $a : b$ and calls it the *relative pseudocomplement of* b *in* a. We call this element briefly *rpc* of b in a. Similarly, the *dual relative pseudocomplement of* b *in* a (denoted by *drpc* of b in a), is the least element of the set $\{x : b \vee x \geq a\}$.

Proposition 45 *If a lattice* **L** *is complete and* a, b *are any elements of* **L***, then*

$$\left.\begin{array}{l} \{x : b \wedge x \leq a\} \textit{ is a closed interval} \\ \{x : b \vee x \geq a\} \textit{ is a closed interval} \end{array}\right\} \Rightarrow a/b \textit{ is a closed interval.}$$

Proof. By assumption we have that: $\{x : b \wedge x \leq a\} = [p_1, p_2]$; $\{x : b \vee x \geq a\} = [q_1, q_2]$.We immediately recognize that $p_1 = 0$, since 0 certainly belongs to $\{x : b \wedge x \leq a\}$; dually we have $q_2 = 1$. Next, note that $[0, p_2] \cap [q_1, 1] \neq \emptyset$, since a belongs to both of them. It follows that $q_1 \leq a \leq p_2$. In other words $[0, p_2] \cap [q_1, 1] = [q_1, p_2]$. It follows immediately that any x which belongs to a/b also belongs to $[0, p_2]$ and to $[q_1, 1]$ and so to $[q_1, p_2]$. In short: $a/b \subseteq [q_1, p_2]$. On the other hand, take any $x \in [q_1, p_2]$. We have

$$q_1 \leq x \leq p_2 \Rightarrow \left\{ \begin{array}{c} b \vee q_1 \leq b \vee x \\ b \wedge x \leq b \wedge p_2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} a \leq b \vee x \\ b \wedge x \leq a \end{array} \right\} \Rightarrow x \in a/b.$$

So $a/b \supseteq [q_1, p_2]$. Hence $a/b = [q_1, p_2]$ and the proof is complete. It is worth noting that the lower (respectively upper) bound of the closed interval a/b is the drpc (respectively rpc) of b with respect to a.■

The next proposition gives the required sufficient condition for a/b to be a closed interval.

Proposition 46 *If a complete lattice* **L** *is general distributive, then* a/b *is a closed interval.*

Proof. The proof can be obtained directly; but it can also be obtained from the preceding propositions. First, by general distributivity and Proposition 44, we have that $\{x : b \wedge x \leq a\} = [0, p]$ and $\{x : b \vee x \geq a\} = [q, 1]$. From this and Proposition 45, we see that $a/b = [q, p]$ and the proof is complete.■

Corollary 47 *If* **L** *is finite then* a/b *is an interval for any* $a, b \in$ **L**.

Remark. In a non-modular lattice a/b is not necessarily a closed interval. Consider for example the non-modular lattice $\mathbf{N}_5$. Referring to Figure 2 we see that $b/c = \{a, b, d\}$ which is not an interval.

Remark. In a lattice which is not general distributive, a/b is not necessarily a closed interval. In the following counterexample (adapted from [1])we see a lattice which is complete and distributive, but *not* general distributive and in which a/b is not always a closed interval.

Counterexample. The lattice of all closed subsets of [0,1], with order provided by set theoretic inclusion, is both complete and distributive (see [1]). Denote the *inf* by $\overline{\wedge}$ (actually it is just set theoretic inclusion, i.e. $A \,\overline{\wedge}\, B = A \cap B$) and the *sup* by $\underline{\vee}$ ($A \,\underline{\vee}\, B$ is the intersection of all closed sets containing $A \cup B$). Now take $A = \emptyset$, $B = \{1\}$ and $X_n = [0, 1 - 1/n]$, $n = 1, 2, ...$. Consider $A/B = \{X : \{1\} \,\overline{\wedge}\, X \subseteq \emptyset \subseteq \{1\} \,\underline{\vee}\, X\}$. Clearly $X_n \in A/B$ for all n. Suppose that A/B is a closed interval of subsets of $[0, 1]$, i.e. $A/B = [\underline{X}, \overline{X}]$. We must have $X_1 \subseteq X_2 \subseteq X_3 \subseteq ... \subseteq \overline{X}$ and so $\cup_{n=1}^{\infty}[0, 1 - 1/n] \subseteq \overline{X}$ which implies $[0, 1) \subseteq \overline{X}$, i.e. $[0, 1] = \overline{X}$. But $B \,\overline{\wedge}\, \overline{X} = B \cap \overline{X} = 1 \not\subseteq \emptyset$. Hence $\overline{X} \notin A/B$, i.e. A/B is not a closed interval of sets.

In light of the above remarks, the following question arises: is general distributivity a *necessary* condition for a/b to be a closed interval? We have not found an answer to this question.

A case where a/b is an interval, with particular importance for applications, is the following.

Proposition 48 *If* $\mathbf{L}$ *is Boolean then* $a/b = ab'$.

Proof. Take any $x \in a/b$. Then $a \leq x \vee b$ and we have $0 \leq a \wedge b' \leq (x \vee b) \wedge b' = (x \wedge b') \vee (b \wedge b') = x \wedge b'$. In short, we have $0 \leq a \wedge b' \leq x \wedge b'$. But then, $0 = 0 \wedge x' \leq (a \wedge b') \wedge x' \leq (x \wedge b') \wedge x' = 0$. It follows that $a \wedge b' \wedge x' = 0$ and then $x' \leq (a \wedge b')'$, or equivalently, $x \geq a \wedge b'$. Starting from $x \wedge b \leq a$, we proceed dually and obtain $x \leq a \vee b'$. In short, $a \wedge b' \leq x \leq a \vee b' \Rightarrow x \in ab'$and so $a/b \subseteq ab'$.

Take any $x \in ab'$. Then $a \wedge b' \leq x \Rightarrow (a \wedge b') \vee b \leq x \vee b \Rightarrow (a \vee b) \wedge (b' \vee b) \leq x \vee b \Rightarrow a \vee b \leq x \vee b$; from which follows $a \leq x \vee b$. Starting from $x \leq a \vee b'$ we proceed dually and obtain $x \wedge b \leq a$. In short, we have showed that $x \in ab' \Rightarrow x \wedge b \leq a \leq x \vee b \Rightarrow x \in a/b$ and so $ab' \subseteq a/b$.

From the previous two steps it is clear that $a/b = ab'$.■

3.4 Join Properties

The first proposition of this section summarizes properties of the join and extension hyperoperation which depend on the join property but on the order relationship $\leq$ (i.e. they remain valid for a general join space). These properties are presented in [22] and are listed here for completeness; the proofs can be found in [22].

Proposition 49 *For any* $A, B, C, D \in \wp(\mathbf{L})$ *and any* $x \in \mathbf{L}$*, we have:*

(i) $A \subseteq B, C \subseteq D \Rightarrow (AC \subseteq BD, A/C \subseteq B/D)$.

(ii) $B \subseteq C \Rightarrow (AB \subseteq AC, A/B \subseteq A/C, B/A \subseteq C/A)$.

(iii) $x \in A \Rightarrow (x \in AB, x \in A/B)$.

(iv) $AB \sim C \Leftrightarrow A \sim C/B \Leftrightarrow B \sim C/A$.

(v) $A \sim B/C \Leftrightarrow AC \sim B$.

(vi) $A/(BC) = (A/B)/C$.

(vii) $A/(B/C) \subseteq (AC)/B$.

(viii) $B \subseteq A/(A/B)$.

(ix) $(A/B)(C/D) = (AC)/(BD)$.

(x) $(A/B)/(C/D) = (AD)/(BC)$.

(xi) $A(B/C) \subseteq (AB)/C$.

Proof. As already remarked, all of the above properties are proved in [22].■

The validity of the next proposition depends on the underlying lattice order.

Proposition 50 *For any* $a, b, c \in \mathbf{L}$*, we have:*

(i) $(a/b)a = a/b$,

(ii) $(a/c)(b/c) = (ab)/c$,

(iii) $(a/b)(a/c) = a/(bc)$.

Proof. (i) Since $a \in a/b$, then for every $x \in a/b$ we have (by Proposition 38) $xa \subseteq a/b$ and so $(a/b)a \subseteq (a/b)$. On the other hand, $(a/b)a = \cup_{x \in a/b} xa \supseteq \cup_{x \in a/b} x = (a/b)$. Hence $(a/b)a = a/b$.

(ii) and (iii) follow from Proposition 49.(ix), taking $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{c\}$, and using $cc = c$.■

3.5 Ordering Properties

The first proposition of this section describes the compatibility of extension with the $\leq$ order.

Proposition 51 *For all $a, b, c \in \mathbf{L}$ we have: (i) $a \leq b \Rightarrow (a/c) \precsim (b/c)$,(ii) $a \leq b \Rightarrow (c/b) \precsim (c/a)$.*

Proof. (i) Take any $x \in a/c$, so $x \wedge c \leq a \leq x \vee c$. Define $y = b \vee x$. Clearly $x \leq y$. Also, $y \vee c = (b \vee x) \vee c \geq b$. Finally, $y \wedge c = (b \vee x) \wedge c = (b \wedge c) \vee (x \wedge c)$, but $b \wedge c \leq b$ and $x \wedge c \leq a \leq b$, so $y \wedge c \leq b$. In short, for all $x \in a/c$, exists $y \in b/c$ such that $x \leq y$.

On the other hand, take any $y \in b/c$ and define $x = a \wedge y \leq y$. Then $x \wedge c = a \wedge y \wedge c \leq a$. Also, $x \vee c = (a \wedge y) \vee c = (a \vee c) \wedge (y \vee c) \geq b \geq a$. In short, for all $y \in b/c$, exists $x \in a/c$ such that $x \leq y$. It follows that $(a/c) \precsim (b/c)$.

(ii) Take any $x \in c/b$. Define $y = x \vee c$; then $x \leq y$ and $c \leq y$. Clearly $c \leq y \vee a$. Also, $a \wedge y = a \wedge (x \vee c) = (a \wedge x) \vee (a \wedge c)$. Since $a \wedge x \leq b \wedge x \leq c$, it follows that $a \wedge y \leq c$. So $y \in c/a$. In short, for every $x \in c/b$, exists $y \in c/a$ such that $x \leq y$. It follows that $(c/b) \precsim (c/a)$.

On the other hand, take any $y \in c/a$, so $y \wedge a \leq c \leq y \vee a$. Define $y = x \wedge c$; then $x \leq y$ and $x \leq c$. Clearly $x \wedge b \leq c$. Also, $b \vee x = b \vee (y \wedge c) = (b \vee y) \wedge (b \vee c)$. Since $b \vee y \geq a \vee y \geq c$, it follows that $b \vee x \geq c$. So $x \in c/b$. In short, for all $y \in c/a$, exists $x \in c/b$ such that $x \leq y$.■

Remark. Notice that the above proposition holds even when a/c, b/c etc. are not intervals; however, in this case $\precsim$ must be understood as a *preorder*, rather than an order.

The next proposition relates the join hyperoperation to set theoretic inclusion; it is well known that this is an (alternative to $\leq$) order on $\wp(\mathbf{L})$.

Proposition 52 *For all $a, b, x \in \mathbf{L}$ we have: $x \in (a/b) \Longleftrightarrow xa \subseteq xb$.*

Proof. $x \in a/b \Rightarrow x \wedge b \leq a \leq x \vee b$. Now, $x \wedge b \leq a \Rightarrow x \wedge b \leq x \wedge a$. Also, $a \leq x \vee b$ and $x \leq x \vee b$ imply that $x \vee a \leq x \vee b$. Hence $x \wedge b \leq x \wedge a \leq x \vee a \leq x \vee b \Rightarrow xa \subseteq xb$. On the other hand, $xa \subseteq xb$ is equivalent to $x \wedge b \leq x \wedge a \leq x \vee a \leq x \vee b$ which implies $x \wedge b \leq x \wedge a \leq a \leq x \vee a \leq x \vee b$, i.e. $x \in a/b$.■

3.6 Distributivity Properties

Proposition 53 *For all $a, b, c \in \mathbf{L}$ we have: (i) $(a/b) \vee c \subseteq (a \vee c)/b$, (ii) $(a/b) \wedge c \subseteq (a \wedge c)/b$.*

Proof. (i) Take any $u \in (a/b) \vee c$; then $u = x \vee c$, where $x \in a/b$. So $a \vee c \leq (x \vee b) \vee c = u \vee b$. On the other hand, $a \vee c \geq (x \wedge b) \vee c = (x \vee c) \wedge (b \vee c) = u \wedge (b \vee c) \geq u \wedge b$. Hence $u \in (a \vee c)/b$ and so $(a/b) \vee c \subseteq (a \vee c)/b$.

(ii) Take any $u \in (a/b) \wedge c$; then $u = x \wedge c$, where $x \in a/b$. So $a \wedge c \geq (x \wedge b) \wedge c = u \wedge b$. On the other hand, $a \wedge c \leq (x \vee b) \wedge c = (x \wedge c) \vee (b \wedge c) = u \vee (b \wedge c) \leq u \vee b$. Hence $u \in (a \wedge c)/b$ and so $(a/b) \vee c \subseteq (a \vee c)/b$. ■

Remark. It is worth remarking that the inclusions in the above proposition can be proper; for instance consider Figure 3, where $(a/b) \vee c = \{e, d, q, c\}$ and $(a \vee c)/b = \{j, i, q, c, e, d\}$.

Figure 3 to be placed here

The next two propositions relate join and extension to set theoretic inclusion.

Proposition 54 *For all $a, b, c \in \mathbf{L}$ we have:*

(i) $a/(b \vee c) \precsim (a/b) \vee (a/c)$.

(ii) $a/(b \wedge c) \precsim (a/b) \wedge (a/c)$.

(iii) $(b \vee c)/a \supseteq (b/a) \vee (c/a)$.

(iv) $(b \wedge c)/a \supseteq (b/a) \wedge (c/a)$.

Proof. (i) From Proposition 51 we know that for all x, y, z such that $x \leq y$, we have $z/y \precsim z/x$. Let us put $z = a$, $x = b$, $y = b \vee c$. Then we have $a/(b \vee c) \precsim a/b$. Similarly we get $a/(b \vee c) \precsim a/c$.

Now take any $z \in a/(b \vee c)$. Since $a/(b \vee c) \precsim a/b$, there exists $x \in a/b$ such that $z \leq x$; since $a/(b \vee c) \precsim a/c$, there exists $y \in a/c$ such that $z \leq y$. Define $u = x \vee y$; clearly $z \leq u$ and $u \in (a/b) \vee (a/c)$. In short, for any $z \in a/(b \vee c)$ there exists $u \in (a/b) \vee (a/c)$ such that $z \leq u$.

Next, take any $u \in (a/b) \vee (a/c)$, i.e. $u = x \vee y$, where $x \in a/b$, $y \in a/c$. Since $a/(b \vee c) \precsim a/b$, there exists $z_1 \in a/(b \vee c)$ such that $z_1 \leq x$; since $a/(b \vee c) \precsim a/c$, there exists $z_2 \in a/(b \vee c)$ such that $z_2 \leq y$. By Proposition 37 $a/(b \vee c)$ is a sublatttice of $\mathbf{L}$, so $z = z_1 \wedge z_2$ is in $a/(b \vee c)$ and, clearly, $z \leq x \vee y = u$. In short, for any $u \in (a/b) \vee (a/c)$ there exists $z \in a/(b \vee c)$ such that $z \leq u$. This, together with the previous paragraph shows that $a/(b \vee c) \precsim (a/b) \vee (a/c)$. (ii) is proved dually.

(iii) Take any $u \in (b/a) \vee (c/a)$; then there exist $x \in b/a$, $y \in c/a$ such that $u = x \vee y$. We have $x \wedge a \leq b$, $y \wedge a \leq c$ and so $(x \wedge a) \vee (y \wedge a) = (x \vee y) \wedge a \leq b \vee c$; it is also clear that $(x \vee y) \vee a \geq b \vee c$. In short, recalling that $u = x \vee y$, we have shown that $u \in (b \vee c)/a$; hence $(b/a) \vee (c/a) \subseteq (b \vee c)/a$ and we have proved (iii); (iv) is proved dually. ■

Remark. Regarding parts (i) and (ii) of Proposition 54, note that they hold true even when a/b, b/c etc. are not intervals; in this case, however, $\precsim$ is a *preorder*, rather than an order.

Proposition 55 *For all* $a, b \in \mathbf{L}$ *we have: (i)* $a/b \subseteq a/(a \vee b)$*, (ii)* $a/b \subseteq a/(a \wedge b)$.

Proof. (i) Take any $x \in a/b$. Then $a \leq x \vee b \leq x \vee (a \vee b)$. Also, since $a \geq x \wedge b$ and $a \geq x \wedge a$, it follows that $a \geq (x \wedge b) \vee (x \wedge a) = x \wedge (a \vee b)$. So $x \in a/(a \vee b)$ and so $a/b \subseteq a/(a \vee b)$. (ii) is proved dually.■

Remark. The set inclusion in the above proposition can be proper, as can be seen by the following example. Take $\mathbf{L}$ to be R^2 and the order relation on elements of R^2 to be as follows: for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in R^2$ we have $a \leq b \Leftrightarrow [a_1 \leq b_1$ and $a_2 \leq b_2$]. It can be checked easily that $(\mathbf{L}, \leq)$ is a distributive lattice. Now take $a = (1, 2)$, $b = (2, 1)$; then we have $a \vee b = (2, 2)$ and (see Figure 4) $a/(a \vee b) = \{(x_1, x_2) : x_1 \leq 1\}$, $a/b = \{(x_1, x_2) : x_1 \leq 1, x_2 \geq 2\}$. So we see that the inclusion is in this case proper.

Figure 4 to be placed here

4 Conclusion

Motivated from machine learning applications, we have introduced the "hyperbox" join hyperoperation (and the associated extension hyperoperation) in a distributive lattice. The resulting algebraic structure is a lattice ordered join space. We have studied the basic properties of the hyperoperations. In the future we want to study the properties of a general lattice ordered join space (i.e. not related to the hyperbox join). Secondly, we want to study the properties of the collection of lattice intervals generated by the hyperbox join hyperoperation in a distributive lattice; in particular we are interested in the approximation of arbitrary subsets of the lattice by join and extension hyperoperations. Finally, we want to study the application of the hyperbox join in a lattice of intervals of fuzzy membership functions, in connection to Corsini's remarks in [10], [11].

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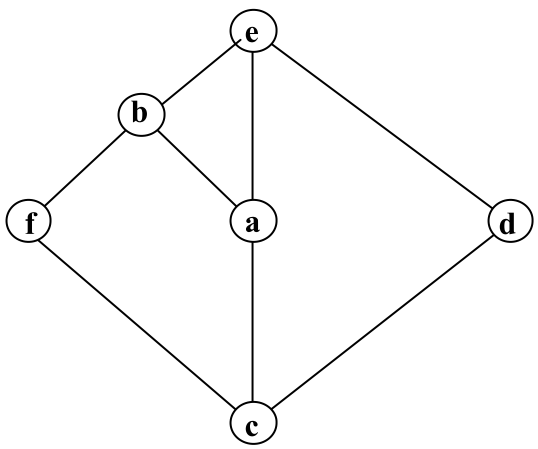

Figure 1

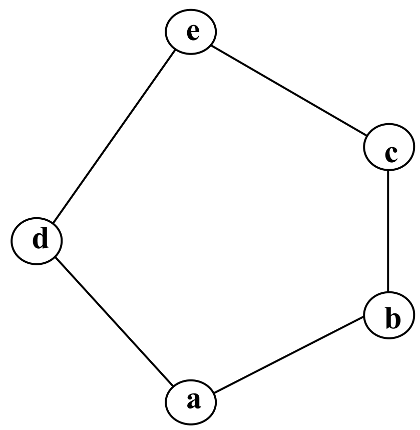

Figure 2

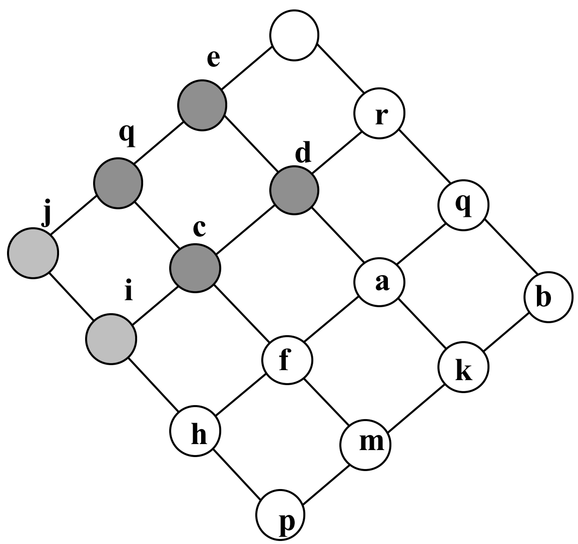

Figure 3

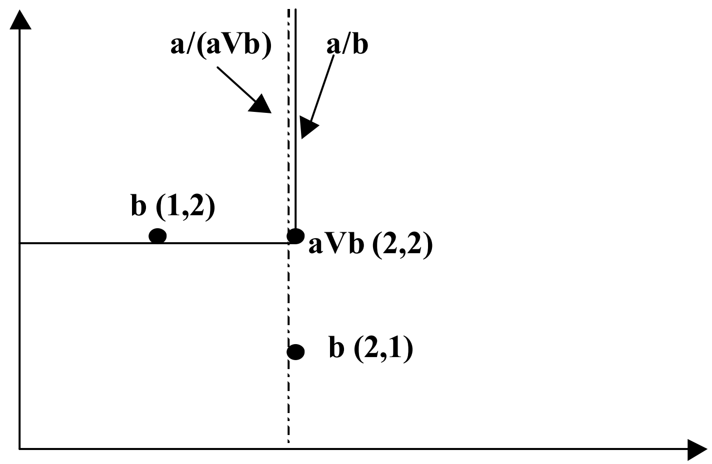

Figure 4