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On positionality of trigger strategies Nash equilibria in SCAR

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ABSTRACT

We study here the positionality of *trigger strategies* Nash equilibria $\overline{\sigma}$ for the N-player SCAR games $\Gamma_N(G|s_0, \gamma, \varepsilon)$ (with $N \geq 3$). Our study is exhaustive with respect to types of graphs G, initial states s_0 and values of N, γ, ε. We conclude that in the majority of cases, profiles $\overline{\sigma}$ are nonpositional. Whenever $\overline{\sigma}$ are positional a key role is played by paths and the ε, γ values (especially whether $\varepsilon > 0$ or not). A crucial concept in our analysis is the *state cop number*, which is first introduced in the current paper.

1. Introduction

In [3] we introduced the game of *selfish cops and adversarial robber* (SCAR) denoted by $\Gamma_N(G|s_0, \gamma, \varepsilon)$. This can be seen as an N-player variant of the classic, two-player cops and robber (CR) game [4,6] where each of the $N-1$ cops is controlled by a different player (whereas in CR a single player controls all cops). G denotes the graph of the game, s_0 the starting *state* or *position* and γ, ε are game parameters. In that paper we prove (among other results) that $\Gamma_N(G|s_0, \gamma, \varepsilon)$ has a *Nash equilibrium* (NE) in *trigger strategies* which is generally *nonpositional* (i.e., some player's next move depends on the past).

Starting from repeated games and then extending the idea to other dynamic/multistage games (including stochastic games) the construction of Nash equilibria by means of trigger strategies has been used widely in the literature (see for example [5] on repeated games and [7] on stochastic games). Hence our construction in [3] is in line with this approach regarding graphical, or more generally stochastic games. In these games *positionality* is an important property of strategies and strategy profiles. As noted above, trigger strategies are in general (and by construction) nonpositional. However, choosing appropriately G, s_0, N, γ and ε we can easily construct examples of the SCAR $\Gamma_N(G|s_0, \gamma, \varepsilon)$ games where the trigger strategies profiles are actually *positional* (i.e., each player's next move depends only on the current state). The generalization of this simple observation leads to the analysis we present in the current paper, as part of the effort for a better understanding of the equilibrium structure of the NEa of SCAR. Hence, we herewith aim to a *complete* analysis of the conditions on the form of G, s_0 and the values of N, γ, ε under which, a trigger strategies NE of $\Gamma_N(G|s_0, \gamma, \varepsilon)$ is positional.

To this end, we also employ a graph classification based on a novel concept, the *state cop number*, which is a refinement of the classic cop number [1].

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2. Preliminaries

All graphs considered here are undirected, finite, connected and simple. $N(u)$ denotes the open neighborhood of vertex u, $d(u, v)$ the distance between u and v and $|A|$ the cardinality of set A. In classic CR, $c(G)$ denotes the *cop number of graph* G, defined as *the minimum number of cops needed to ensure capture.*[1]

In this paper we focus on the *auxiliary games* $\Gamma_N^n(G|s_0, \gamma, \varepsilon)$, which we use in [3] to construct the trigger strategies NE profiles of the SCAR game $\Gamma_N(G|s_0, \gamma, \varepsilon)$. The main difference between game Γ_N and games Γ_N^n is that, whereas the former is an N-player, mixed motive game (at least as far as the cops are concerned), the latter are two-player zero-sum games. The full story can be found in [3]. Here we present basic notation and facts.

For each $n \in \{1, ..., N\}$, $\Gamma_N^n(G|s_0, \gamma, \varepsilon)$ is a zero-sum two-player *stochastic game* [2] played on the graph $G = (V, E)$; $\gamma \in (0, 1)$ and $\varepsilon \in \left[0, \frac{1}{N-1}\right]$ are *game parameters*. Player P_n controls the n-th token, i.e., cop C_n for $n \in \{1, ..., N-1\}$, or the robber R for $n = N$; player P_{-n} controls the rest tokens.

A *state* s of the game has the form $s = \left(x^1, ..., x^N, n\right)$ where $x^i \in V$ is the location of the i-th token and $n \in \{1, ..., N\}$ is the token moving next. S denotes the set of all states and can be partitioned as:

1. $S = S_c \cup S_{nc} \cup \{\tau\}$ where, S_c is the set of *capture states,* in which at least one C_n is on the same vertex as R (i.e., $x^n = x^N$), S_{nc} the set of *noncapture states* where no C_n is on the same vertex as R and τ the terminal state that finally occurs in case of capture; or
2. $S = \cup_{n=1}^N S^n \cup \{\tau\}$ where $S^n := \left\{s : s = \left(x^1, ..., x^N, n\right)\right\}$ is the set of states in which the n-th token has the move.

The game begins at some *initial state* s_0, which also specifies the first token to move. In each turn a single token is moved from its current vertex to a neighboring one, always following the sequence $..., C_1, C_2, ..., C_{N-1}, R, C_1, ...$. The game lasts an infinite number of turns but is effectively over as soon as capture occurs, if it does, since right after the system moves to state τ and stays there ad infinitum.

Regarding the players' *payoffs* and since $\Gamma_N^n(G|s_0, \gamma, \varepsilon)$ is a zero-sum game, it suffices to specify P_n's payoff. If the *capture time* (i.e., the time that a capture state $s_t \in S_c$ occurs) is t, then:

1. in the game $\Gamma_N^N(G|s_0, \gamma)$, P_N's payoff is $-\gamma^t$[2];
2. in the game $\Gamma_N^n(G|s_0, \gamma, \varepsilon)$ ($n \in \{1, ..., N-1\}$), P_n's payoff is

$$\frac{(1-\varepsilon)}{K}\gamma^t \quad : \text{when } K \text{ cops } (K \in \{1, ..., N-2\}) \textit{ including } C_n \text{ are in the same vertex as } R;$$

$$\frac{\varepsilon}{N-K-1}\gamma^t \quad : \text{when } K \text{ cops } (K \in \{1, ..., N-2\}) \textit{ but no } C_n \text{ are in the same vertex as } R;$$

$$\frac{\gamma^t}{N-1} \quad : \text{when all } N-1 \text{ cops are in the same vertex as } R.$$

In case capture never takes place, P_n's payoff is zero. P_{-n}'s payoff is always the negative of P_n's.

$\Gamma_N^N(G|s_0, \gamma)$ is essentially the *modified* CR game introduced in [3], the basic difference with classic CR being that time is counted in turns instead of rounds, and a single token, cop or robber moves in each turn. Hence P_{-N} aims to effect capture as soon as possible and P_N aims to delay it as much as possible. Here the players' optimal strategies depend on G and s_0, but not on γ (neither on ε).

In games $\Gamma_N^n(G|s_0, \gamma, \varepsilon)$ however, where C_n plays against the robber and $N-2$ "robber-friendly" cops, the players' optimal strategies depend also on the values of γ, ε. If $\varepsilon > 0$, then P_{-n}'s best outcome is evasion of the robber and from then on, depending on γ, ε and the respective capture times he may prefer a capture by one of his tokens, or a *joint* capture involving also C_n, or a late as possible *pure* C_n capture. The difference in case $\varepsilon = 0$ is that P_{-n} is indifferent between letting the robber evade and capturing him by one of his tokens, since in both cases he gets his best outcome, i.e., a zero payoff.

We will shortly return to these optimal strategies. But first let us introduce a few additional notions.

A *history* $h = (s_0, s_1, ...)$ is a finite or infinite sequence of states. Let H_f denote the set of *finite* length histories $(s_0, ..., s)$, H_f^n those where token n moves, i.e., $s \in S^n$ and H_{fnc}^n those where $s \in S^n \cap S_{nc}$.

A *pure* (or deterministic) strategy for the m-th token is a function σ^m which maps finite histories to next moves. That is, $\forall h = (s_0, s_1, ..., s_t) \in H_f$, $\sigma^m(h) = v$ specifies that: if the game started at s_0 and passed through $s_1, ..., s_t$, then next the m-th token should move to vertex v.

[1] It is not hard to see that $c(G) = k$ if and only if k is the minimum number such that, k cops can win from any position in k-cops CR, whether it is cops' turn to move or the robber's.

[2] Note that variable ε is not included as an argument here since it is not relevant in this game.

A strategy is *positional* (or *stationary Markovian*) if it depends only on the current state s_t, but not on previous states or current time t, i.e.,

$$\forall \left(h_1 = (s_0, s_1, ..., s_{t_1}), h_2 = (s_0, s'_1 ..., s'_{t_2})\right) \in H_f : s_{t_1} = s'_{t_2} = s, \sigma^m (h_1) = \sigma^m (h_2) = \sigma^m (s) .$$

A *strategy profile* (or, simply a profile) is a tuple $\sigma = \left(\sigma^1, ..., \sigma^N\right)$. A profile is *positional* if it consists solely of positional strategies. Otherwise it is called *nonpositional*.

Standard results [2] yield that the players in games $\Gamma_N^n (G|s_0, \gamma, \varepsilon)$ have optimal, *pure positional* strategies for all n, s_0 and γ, ε. This is the only kind of strategies we consider then for these games. Furthermore, since P_n controls only the n-th token, his strategy consists of a single function σ^n; a P_{-n} strategy though is a "vector" function $\sigma^{-n} = \left(\sigma^m\right)_{m\in\{1,...,N\}\backslash n}$ with one strategy for each of P_{-n}'s tokens; and both players' strategies together yield profile $\sigma = \left(\sigma^1, ..., \sigma^N\right) = \left(\sigma^n, \sigma^{-n}\right)$.

Since the form of the games is deterministic, if moreover the players use pure strategies, the games *evolve deterministically*. That is, given an initial state s and a pure strategy profile σ, the tuple (s, σ) leads in a deterministic manner to either capture or evasion of the robber.

Keeping the above in mind, in the sequel, we will need the following definitions.

$T (s, \sigma)$ denotes the *capture time* (finite or not) starting from state s under pure profile σ.

$\widehat{\sigma}^n$ denotes a pure positional *optimal strategy for the* n-th *token in modified CR game* $\Gamma_N^N (G|s_0, \gamma)$, and $\widehat{\Sigma}^n$ the set of all $\widehat{\sigma}^n$'s. Strategies $\widehat{\sigma}^n$ will be called *CR-optimal*.

Note that: for *any* initial state s and CR-optimal profile $\widehat{\sigma} = (\widehat{\sigma}^1, ..., \widehat{\sigma}^N)$ such that $(s, \widehat{\sigma})$ leads to capture, *it is always the same cop effecting capture and at the same time* $T (s, \widehat{\sigma})$. This follows from the facts: (i) in each turn only one token moves and (ii) under CR-optimal play, R does *never* run into a cop.[3] Let $\widehat{C}(s)$ then denote the *cop effecting capture under every CR-optimal profile* $\widehat{\sigma}$, *when the game starts at* s, and $\widehat{T}(s) := T (s, \widehat{\sigma})$ the respective *number of moves*. If $\widehat{C}(s) = C_m$ for some $m \in \{1, ..., N-1\}$, then at times we denote this by $\widehat{C}_m(s)$.

Finally, for all $n, m \in \{1, ..., N\}$, let ϕ_m^n be an *optimal, pure positional strategy for the* n*-th token in game* $\Gamma_N^m (G|s_0, \gamma, \varepsilon)$ (where dependence on γ, ε has been suppressed); let Φ_m^n be the respective set.

We now turn to SCAR games $\Gamma_N (G|s_0, \gamma, \varepsilon)$ and the so-called *trigger strategies* profiles $\overline{\sigma}$, where

$$\overline{\sigma} := \left(\overline{\sigma}^1, ..., \overline{\sigma}^N\right)$$

and each trigger strategy $\overline{\sigma}^n$ is composed from strategies ϕ_m^n as follows:

For all $h = (s_0, ..., s) \in H_f$

$$\overline{\sigma}^n(h) := \begin{cases} \phi_n^n(s) & \text{as long as every player } m \in \{1, ..., N\} \backslash n \text{ follows } \phi_m^m; \\ \phi_m^n(s) & \text{as soon as some player } m \in \{1, ..., N\} \backslash n \text{ deviates from } \phi_m^m. \end{cases}$$

In general, $\overline{\sigma}^n$ is nonpositional by construction (and so is $\overline{\sigma}$ then) since it takes into account the players' past behavior. However, if $\phi_n^n (s) = \phi_m^n (s)$ for all m and "relevant" states s, $\overline{\sigma}^n$ becomes positional.

To better understand the meaning of this latter condition, consider the following. Roughly speaking, cop C_n's optimal strategy ϕ_n^n in the game Γ_N^n (where he plays against a "coalition" of the remaining players) must be also optimal (i) in every game Γ_N^m with $m \in \{1, ..., N-1\} \backslash n$ (where he and the remaining players ally against cop C_m) and (ii) in the CR game Γ_N^N (where he and the remaining cops chase the robber). In other words: $\overline{\sigma}$ is positional iff, for every m and n, there exists a CR-optimal strategy $\widehat{\sigma}^n$ of token n which is optimal in every Γ_N^m. This can be stated formally as follows.

Condition 2.1. Let $\overline{\sigma} = (\overline{\sigma}^1, ..., \overline{\sigma}^N)$ be a trigger strategies profile in $\Gamma_N(G|s_0, \gamma, \varepsilon)$ with $\overline{\sigma}^n$ consisting of $\left(\phi_m^n\right)_{m=1}^N$. Then $\overline{\sigma}$ is positional (resp. nonpositional) iff **A1** (resp. **A2**) holds:

$$\mathbf{A1} : \forall n, m \in \{1, ..., N\}, \exists \widehat{\sigma}^n \in \widehat{\Sigma}^n : \forall h = (s_0, ..., s) \in H_{fnc}^n, \phi_m^n(s) = \widehat{\sigma}^n(s), \tag{1}$$

$$\mathbf{A2} : \exists n, m \in \{1, ..., N\} : \forall \widehat{\sigma}^n \in \widehat{\Sigma}^n, \exists h = (s_0, ..., s) \in H_{fnc}^n : \phi_m^n(s) \neq \widehat{\sigma}^n(s). \tag{2}$$

The goal of this paper is to explore the form of graphs G and initial states s_0 and the values of parameters N, γ and ε, under which, each of the mutually exclusive conditions **A1** or **A2** holds. Given Γ_N^N is the modified CR, $\Phi_N^n = \widehat{\Sigma}^n$ and **A1** holds always for $m = N$. Hence we will be examining the remaining cases.

[3] Contrary to the classic CR, where capture under optimal play occurs in the minimum number of *rounds* and it can possibly be effected by *different* cops, in modified CR capture under optimal play (i.e., for every $\widehat{\sigma} \in \widehat{\Sigma} = \times_{n\in\{1,...,N\}} \widehat{\Sigma}^n$) occurs in the minimum number of *moves* and thus always by the *same* cop.

3. Analysis

The study is divided as follows. In Section 3.1 we study the case $|V|=2, \varepsilon \geq 0$ and in Section 3.2 the case $|V|>2, \varepsilon>0$; Section 3.3 concerns the case $|V|>2, \varepsilon=0$ and is further divided in Section 3.3.1, where $c(G) \leq N-1$, and in Section 3.3.3, where $c(G)>N-1$. Section 3.3.2 consists of an interlude on state cop number. To avoid trivialities we only consider initial states $s_0 \in S_{nc}$.

3.1. Case $|V|=2$ (the path $\mathcal{P}_2$) $\varepsilon \geq 0$

Proposition 3.1. *Let $G=(V,E)$ with $|V|=2$. Then $\Gamma_3(G|s_0,\gamma,\varepsilon)$ has a (unique) positional trigger strategies profile $\overline{\sigma}$ iff $\varepsilon \in \left[0,\frac{1}{2}\right)$ and $\gamma \in \left[\sqrt{\frac{\varepsilon}{1-\varepsilon}},\frac{1}{2-2\varepsilon}\right]$.*

Proof. The set of noncapture states is

$$S_{nc}=\{(1,1,2,1),(1,1,2,2),(1,1,2,3),(2,2,1,1),(2,2,1,2),(2,2,1,3)\}.$$

Due to symmetry we only consider initial states $s_0=(1,1,2,n)$, $n \in \{1,2,3\}$; then each $(1,1,2,n)$ has two possible successors (e.g., the successors of $(1,1,2,1)$ are $(1,1,2,1)$ and $(1,1,2,2)$) i.e., given s_0, each token has two positional strategies. Thus, in this case Condition **A1** becomes:

$$\forall n,m \in \{1,2,3\}, \exists \widehat{\sigma}^n \in \widehat{\Sigma}^n : \phi_m^n(1,1,2,n)=\widehat{\sigma}^n(1,1,2,n). \tag{3}$$

We examine under which conditions (3) holds and hence $\overline{\sigma}=(\overline{\sigma}^1,\overline{\sigma}^2,\overline{\sigma}^3)$ is positional.

To begin with, the (unique) CR-optimal strategies are: for the cops, $\widehat{\sigma}^1(1,1,2,1)=\widehat{\sigma}^2(1,1,2,2)=2$ (immediate capture), and for the robber $\widehat{\sigma}^3(1,1,2,3)=2$ (stay in place).

I. In game $\Gamma_3^1(G|s_0,\gamma,\varepsilon)$ (P_1 controls C_1, P_{-1} controls C_2 and R) we have the following.

For token C_1, the unique optimal strategy $\phi_1^1 \in \Phi_1^1$ prescribes immediate capture at $s=(1,1,2,1)$, i.e., $\phi_1^1(s)=2$. Indeed, if C_1 captures say at time t, then P_1's payoff is $(1-\varepsilon)\gamma^t$. Otherwise, depending on the values of γ,ε, P_{-1} will optimally play so that, either C_2 effects capture in the next move, and then P_1's payoff will be $\varepsilon\gamma^{t+1}$, or C_2 stays put and R runs into both C_1, C_2 in the next move, and P_1's payoff will be $\frac{1}{2}\gamma^{t+2}$. But $(1-\varepsilon)\gamma^t>\max\left(\varepsilon\gamma^{t+1},\frac{1}{2}\gamma^{t+2}\right)$ for all $(\gamma,\varepsilon) \in (0,1)\times[0,\frac{1}{2}]$. Hence,

$$\forall(\gamma,\varepsilon) \in (0,1)\times[0,\frac{1}{2}], \phi_1^1(1,1,2,1)=2=\widehat{\sigma}^1(1,1,2,1). \tag{4}$$

For token C_2, moving at time t from state $s=(1,1,2,2)$ we have the following possibilities.

1. C_2 captures at t; P_{-1}'s loss is $\varepsilon\gamma^t$;
2. C_2 stays put at t, R runs into both C_1, C_2 at $t+1$; P_{-1}'s loss is $\frac{1}{2}\gamma^{t+1}$;
3. C_2 and R stay put and C_1 captures at $t+2$; P_{-1}'s loss is $(1-\varepsilon)\gamma^{t+2}$.

For a positional $\overline{\sigma}^2$, P_{-1} must not prefer (2) or (3) to (1), i.e.,

$$\varepsilon\gamma^t \leq \min\left(\frac{1}{2}\gamma^{t+1},(1-\varepsilon)\gamma^{t+2}\right) \Rightarrow \gamma \geq \max\left(2\varepsilon,\sqrt{\frac{\varepsilon}{1-\varepsilon}}\right) \tag{5}$$

Given $\gamma<1$, from $\gamma \geq 2\varepsilon$ we get $\varepsilon<\frac{1}{2}$. Therefore,

$$\varepsilon \in \left[0,\frac{1}{2}\right) \text{ and } \gamma \in \left[\sqrt{\frac{\varepsilon}{1-\varepsilon}},1\right) \tag{6}$$

guarantee the existence of a ϕ_1^2 such that:

$$\phi_1^2(s)=\widehat{\sigma}^2(s).$$

For token R, moving at time t from state $s=(1,1,2,3)$ we have the following possibilities.

1. R stays put at t and C_1 effects capture at $t+1$; P_{-1}'s loss is $(1-\varepsilon)\gamma^{+1}$.
2. R runs into both C_1, C_2 at t; P_{-1}'s loss is $\frac{1}{2}\gamma^t$.

For a positional $\overline{\sigma}^3$, P_{-1} must not prefer (2) to (1):

$$(1-\varepsilon)\gamma^{t+1} \le \frac{1}{2}\gamma^t \Rightarrow \gamma \le \frac{1}{2-2\varepsilon}. \tag{7}$$

II. In game $\Gamma_3^2(G|s_0,\gamma,\varepsilon)$ (P_2 controls C_2, P_{-2} controls C_1 and R) we have the following.

Regarding token C_2: moving at t from $s=(1,1,2,2)$, the unique optimal strategy prescribes immediate capture, i.e., $\phi_2^2(s)=2=\widehat{\sigma}^2(s)$, for all $(\gamma,\varepsilon)\in(0,1)\times[0,\frac{1}{2}]$. Indeed, P_2's payoff in this case is $(1-\varepsilon)\gamma^t$. Otherwise, optimally R stays in place at $t+1$ and at $t+2$ C_1 captures with P_2's payoff being $\varepsilon\gamma^{t+2}<(1-\varepsilon)\gamma^t$.

Regarding token C_1:

1. if C_1 captures at t, then P_{-2}'s loss is $\varepsilon\gamma^t$;
2. otherwise (optimally) C_2 captures at $t+1$ and P_{-2}'s loss is $(1-\varepsilon)\gamma^{t+1}$.

For a positional $\overline{\sigma}^1$ it must be:

$$\varepsilon\gamma^t \le (1-\varepsilon)\gamma^{t+1} \Rightarrow \gamma \ge \frac{\varepsilon}{1-\varepsilon}. \tag{8}$$

Regarding the robber token R we have the following possibilities.

1. R stays put at time t and C_1 effects capture at $t+1$; P_{-2}'s loss is $\varepsilon\gamma^{t+1}$;
2. R and C_1 stay put and C_2 captures at time $t+2$; P_{-2}'s loss is $(1-\varepsilon)\gamma^{t+2}$.
3. R runs into both C_1, C_2 at t; P_{-2}'s loss is $\frac{1}{2}\gamma^t$.

For a positional $\overline{\sigma}^3$ (1) must be at least as good for P_{-2} as (2) and (3). By (8) we have,

$$\gamma \ge \frac{\varepsilon}{1-\varepsilon} \Rightarrow \varepsilon\gamma^{t+1} \le (1-\varepsilon)\gamma^{t+2}$$

and P_{-2} does not prefer (2) to (1). For (1) to be at least as good as (3) it must be $\varepsilon\gamma^{t+1}\le\frac{1}{2}\gamma^t$. If $\varepsilon=0$, this holds always. If $\varepsilon>0$ it must be

$$\gamma \le \frac{1}{2\varepsilon}. \tag{9}$$

For a positional profile $\overline{\sigma}=\left(\overline{\sigma}^1,\overline{\sigma}^2,\overline{\sigma}^3\right)$, (6)-(9) must all hold; this yields the required result. □

Proposition 3.2. *Let $G=(V,E)$ with $|V|=2$; let $N>3$ and $\varepsilon>0$. Then every trigger strategies profile $\overline{\sigma}$ of $\Gamma_N(G|s_0,\gamma,\varepsilon)$ is nonpositional, for all $\gamma\in(0,1)$.*

Proof. Consider game $\Gamma_N^1(G|s_0,\gamma,\varepsilon)$ (P_1 controls C_1, P_{-1} controls $C_2, C_3, ..., C_{N-1}$ and R). Let $s=(1,...,1,2,2)$ and consider C_2's optimal move at time t from s. If C_2 captures R, then P_{-1}'s loss is $\varepsilon\gamma^t$; if C_2 stays put and C_3 captures at $t+1$, P_{-1}'s loss is $\varepsilon\gamma^{t+1}<\varepsilon\gamma^t$ (since $\varepsilon>0$). So for the unique $\phi_1^2,\widehat{\sigma}^2$ it is $\phi_1^2(s)=1\neq 2=\widehat{\sigma}^2(s)$. We conclude that $\overline{\sigma}^2$ and hence $\overline{\sigma}$ is always nonpositional. □

Proposition 3.3. *Let $G=(V,E)$ with $|V|=2$; let $N>3$ and $\varepsilon=0$. The following hold.*

1. *$\Gamma_N(G|s_0,\gamma,\varepsilon)$ has at least one nonpositional trigger strategies profile $\overline{\sigma}$, for all $\gamma\in(0,1)$.*
2. *$\Gamma_N(G|s_0,\gamma,\varepsilon)$ has at least one positional trigger strategies profile $\overline{\sigma}$ iff $\gamma\in\left(0,\frac{1}{N-1}\right]$.*

Proof. All $s\in S_{nc}$ are of the form $s=(v_1,...,v_1,v_2,n)$ with $v_1\neq v_2$ (no cop is in the same vertex as the robber). At any such state s the unique CR-optimal strategies are: for cop C_n, $\widehat{\sigma}^n(v_1,...,v_1,v_2,n)=v_2$ (immediate capture), for the robber R, $\widehat{\sigma}^N(v_1,...,v_1,v_2,N)=v_2$ (stay in place).

1. Consider game $\Gamma_N^1(G|s_0,\gamma,\varepsilon)$ and token C_2 having the next move at state $s=(v_1,...,v_1,v_2,2)$. Moving to v_2 effects a capture which gives P_{-1} his minimum loss of 0; but so does staying in place, provided some other C_k ($k\in\{3,...,N-1\}$) effects the capture. Thus there exists ϕ_1^2: $\phi_1^2(s)=v_1\neq v_2=\widehat{\sigma}^2(s)$. Hence there exists $\overline{\sigma}^2$ and thus $\overline{\sigma}$ which is nonpositional.

2. Consider game $\Gamma_N^m(G|s_0,\gamma,\varepsilon)$ ($m\in\{1,2,...,N-1\}$).

For C_m: immediate capture (i.e., moving to v_2) results to a payoff of $(1-\varepsilon)\gamma^t=\gamma^t>0$ for P_m. If C_m does not capture immediately, then optimally P_{-m} effects a C_n ($n\neq m$) capture any time before C_m resulting to a payoff of $0<\gamma^t$ for P_m. So P_m prefers immediate capture and thus for the only ϕ_m^m it is $\phi_m^m(v_1,...,v_1,v_2,m)=v_2=\widehat{\sigma}^m(v_1,...,v_1,v_2,m)$.

For C_n ($n \in \{1, ..., N-1\} \backslash m$): moving to v_2 effects a capture which gives P_{-m} his minimum loss of 0. Thus there exists $\phi_m^n : \phi_m^n (v_1, ..., v_1, v_2, n) = v_2 = \widehat{\sigma}^n (v_1, ..., v_1, v_2, n)$.

For R: If $m \in \{2, ..., N-1\}$, then R optimally stays put and capture is effected by any C_n with $n < m$, resulting to a loss of 0 for P_{-m}. For $m = 1$ we have the following. If R stays put at time t, C_1 captures at $t+1$ and P_{-m}'s loss is $(1-\varepsilon)\gamma^{t+1} = \gamma^{t+1}$. If R runs into (all) cops, P_{-m}'s loss is $\frac{1}{N-1}\gamma^t$. Thus if $\gamma^{t+1} \leq \frac{\gamma^t}{N-1} \Leftrightarrow \gamma \leq \frac{1}{N-1}$, there exists ϕ_m^N : $\phi_m^N (v_1, ..., v_1, v_2, N) = v_2 = \widehat{\sigma}^N (v_1, ..., v_1, v_2, N)$. Hence, **A1** holds and a positional profile $\overline{\sigma}$ exists iff $\gamma \in \left(0, \frac{1}{N-1}\right]$. □

3.2. Case $|V| \geq 3$ and $\varepsilon > 0$

From this point on, unless otherwise specified, we consider $N \geq 3$.

Proposition 3.4. *Let $G = (V, E)$ with $|V| \geq 3$; let $\varepsilon > 0$. Then every trigger strategies profile $\overline{\sigma}$ of $\Gamma_N(G|s_0, \gamma, \varepsilon)$ is nonpositional, for all $\gamma \in (0, 1)$.*

Proof. Given $|V| \geq 3$, from any s_0 it is possible to reach at some t a state $s = (v_1, v_2, ..., v_N, 2) \in S_{nc}$ (so C_2 moves next) with $v_1 \neq v_2$ and $v_N \in N(v_2)$. Now, at s, every CR-optimal $\widehat{\sigma}^2$ dictates immediate capture by C_2, i.e., $\forall \widehat{\sigma}^2 \in \widehat{\Sigma}^2$, $\widehat{\sigma}^2(s) = v_N$. In game $\Gamma_N^1(G|s_0, \gamma, \varepsilon)$ though at state s, if C_2 captures immediately R, P_{-1}'s loss is $\varepsilon\gamma^t$. If P_{-1} keeps $C_2, ..., C_{N-1}$ in place and lets R move to v_2 at $t+N-2$, then, if $k \in \{1, ..., N-2\}$ is the number of P_{-1}'s cop tokens located at v_2 including C_2, P_{-1}'s loss is $\frac{\varepsilon}{N-k-1}\gamma^{t+N-2} < \varepsilon\gamma^t$. Hence, at s, P_{-1} prefers to defer capture and *never* capture with C_2. Thus,

$$\forall \phi_1^2 \in \Phi_1^{\cdot 2}, \forall \widehat{\sigma}^2 \in \widehat{\Sigma}^2, \exists h = (s_0, ..., s) \in H_{fnc}^n : \phi_1^2(s) \neq v_N = \widehat{\sigma}^2(s).$$

Consequently *every* $\overline{\sigma}^2$ and corresponding profile $\overline{\sigma}$ is nonpositional. □

3.3. Case $|V| \geq 3$ and $\varepsilon = 0$

Proposition 3.5. *Let $G = (V, E)$ with $|V| \geq 3$ and $\varepsilon = 0$. Then in every game $\Gamma_N(G|s_0, \gamma, \varepsilon)$ there exists always a nonpositional trigger strategies profile $\overline{\sigma}$, for all $\gamma \in (0, 1)$.*

Proof. Given $|V| \geq 3$, from any starting s_0 it is possible to reach at some t state $s = (v_1, v_2, ..., v_N, N) \in S_{nc}$ (so R has the next move) such that $v_1 \neq v_2$ and $v_N \in N(v_2)$. Now, under no CR-optimal $\widehat{\sigma}^N$ R ever moves to v_2. In game $\Gamma_N^1(G|s_0, \gamma, \varepsilon)$ though P_{-1} *can*, under optimal play move R to v_2 since then he has a minimum loss of $\frac{\varepsilon}{N-k-1}\gamma^t = 0$, where k is the number of P_{-1}'s cop tokens located at v_2, including C_2. Thus,

$$\exists \phi_1^N : \forall \widehat{\sigma}^N \in \widehat{\Sigma}^N, \forall s_0 \in S_{nc}, \exists h = (s_0, ..., s) \in H_{fnc}^N : \phi_1^N(s) \neq \widehat{\sigma}^N(s);$$

i.e., there always exists a nonpositional $\overline{\sigma}^N$ and a corresponding nonpositional $\overline{\sigma}$. □

3.3.1. Case: $c(G) \leq N-1$

In this part of the paper we connect positionality of $\overline{\sigma}$ to the cop number $c(G)$ of graph G. First we examine the case where G is a path (hence $c(G) = 1$) with $|V| \geq 3$.[4]

Proposition 3.6. *Let G be a path with $|V| \geq 3$; let $\varepsilon = 0$. Then $\Gamma_N(G|s_0, \gamma, \varepsilon)$ has a positional trigger strategies profile $\overline{\sigma}$ iff (i) s_0 is such that all cops are to one side of the robber, and (ii) $\gamma \in \left(0, \frac{1}{N-1}\right]$.*

Proof. Let S'_{nc} denote the set of states where the robber is between some cops and $S''_{nc} = S_{nc} \backslash S'_{nc}$ the set of states where all cops are to one side of the robber. In Part I we show that in every game $\Gamma_N(G|s_0, \gamma, \varepsilon)$ with $s_0 \in S'_{nc}$ every $\overline{\sigma}$ is nonpositional. In part II then we show that, if $s_0 \in S''_{nc}$, then a positional $\overline{\sigma}$ exists iff $\gamma \in \left(0, \frac{1}{N-1}\right]$. The combination of Parts I and II yields the result sought.

I. Initial states $s_0 \in S'_{nc}$. From any such s_0 we can always reach, at say time t, a state $\widetilde{s}$ in which R has the move, some cops are immediately to his left and the rest are immediately to his right. Now, every CR-optimal robber strategy $\widehat{\sigma}^N$ at state $\widetilde{s}$ dictates that the robber stays in place. In game $\Gamma_N^1(G|s_0, \gamma, \varepsilon)$ on the contrary, every optimal strategy ϕ_1^N at $\widetilde{s}$ dictates

[4] In a sense this proposition can be seen as an extension of Proposition 3.3, part 2, to paths with $|V| > 2$.

that the robber moves into the vertex not occupied by C_1, because this yields a minimum loss of $\varepsilon\gamma^t = 0$ for P_{-n}, whereas otherwise C_1 optimally captures right after and P_{-n}'s loss is $(1-\varepsilon)\gamma^{t+1} = \gamma^{t+1} > 0$. Thus,

$$\forall \phi_1^N \in \Phi_1^N, \forall \widehat{\sigma}^N \in \widehat{\Sigma}^N, \forall s_0 \in S'_{nc}, \exists h = (s_0, ..., \widetilde{s}) \in H^N_{fnc} : \phi_1^N(s) \neq \widehat{\sigma}^N(s).$$

Hence in this case, every strategy $\overline{\sigma}^N$ and corresponding profile $\overline{\sigma}$ is nonpositional.

II. Initial states $s_0 \in S''_{nc}$. Let $S''_{ncl} \subset S''_{nc}$ be the states where all cops are to the left of R. A similar argument holds for the symmetric case. For every $s_0 \in S''_{ncl}$ let $S''_{ncl}(s_0)$ be the set of states that can occur starting from s_0, i.e.,

$$S''_{ncl}(s_0) := \left\{ s : \exists h = (s_0, ..., s) \in H_{fnc} \right\}.$$

Observe that, for all $s_0 \in S''_{ncl}$, $S''_{ncl}(s_0) = S''_{ncl}$. Then existence of a positional $\overline{\sigma}$ implies:

$$\forall m, n \in \{1, ..., N\}, \exists \phi_m^n, \widehat{\sigma}^n : \forall s \in S''_{ncl} \cap S^n, \phi_m^n(s) = \widehat{\sigma}^n(s). \tag{10}$$

Given G is a path, for any $s \in S''_{ncl}$ and under every CR-optimal profile $\widehat{\sigma}$, the cop $\widehat{C}(s)$ that captures is the one that is “closer” to R, taking also into account whose turn is to move and he does so at time $\widehat{T}(s)$. Let $\widehat{\sigma}_* = (\widehat{\sigma}_*^1, ..., \widehat{\sigma}_*^N)$ be the CR-optimal profile where, C_n ($n \in \{1, 2, ..., N-1\}$) always moves towards R, and R moves away from the cops, reaches the end of the path and waits there until capture. Instead of (10) we show the following which is equivalent:

$$\forall m, n \in \{1, ..., N\}, \exists \phi_m^n : \forall s \in S''_{ncl} \cap S^n, \phi_m^n(s) = \widehat{\sigma}_*^n(s). \tag{11}$$

Now fix an $m \in \{1, 2, ..., N-1\}$ for the rest of the proof and consider game $\Gamma_N^m(G|s_0, \gamma, \varepsilon)$. We partition S''_{ncl} into two mutually disjoint sets S_A, S_B defined below and examine each case separately.

$$S_A := \left\{ s \in S''_{ncl} : \text{under (every) } \widehat{\sigma}, \text{cop } C_n \ (n \neq m) \text{ captures (i.e., } \widehat{C}(s) = C_n) \text{ at } \widehat{T}(s) \right\},$$
$$S_B := \left\{ s \in S''_{ncl} : \text{under (every) } \widehat{\sigma}, \text{cop } C_m \text{ captures (i.e., } \widehat{C}(s) = C_m) \text{ at } \widehat{T}(s) \right\}.$$

Case II.A: $s \in S_A$. For any $s \in S_A$, if P_{-m} uses the chosen CR-optimal (cop and robber) strategies $\widehat{\sigma}_*^n$ ($n \in \{1, ..., N\} \backslash m$) he can force a C_n capture at time $\widehat{T}(s)$ for any strategy of P_m and get his minimum loss of $\varepsilon\gamma^{\widehat{T}(s)} = 0$. Given this strategy of P_{-m}, P_m cannot affect the outcome. Thus *any* strategy is optimal for him and so is the CR-optimal strategy $\widehat{\sigma}_*^m$. Hence

$$\forall n \in \{1, ..., N\} \ \exists \phi_m^n : \forall s \in S_A \cap S^n, \phi_m^n(s) = \widehat{\sigma}_*^n(s). \tag{12}$$

Case II.B: $s \in S_B$. Assume P_m uses $\widehat{\sigma}_*^m$ for C_m and consider the options of P_{-m}. It can be seen that, depending on the state s, there exist only two possibilities, which partition further S_B as follows:

1. States $s \in S_{B_1}$: P_m can force a *pure* C_m capture, for *any* strategy of P_{-m}, and
2. States $s \in S_{B_2}$: P_{-m} can effect a *joint capture*, i.e., one involving C_m and some P_{-m} cop tokens.

If $s \in S_{B_1}$, then under optimal play (in $\Gamma_N^m(G|s_0, \gamma, \varepsilon)$) C_m chases R till the end of the path and captures him at time $\widehat{T}(s)$; this describes the optimal strategies for C_m and R. The remaining tokens C_n cannot affect the outcome. Thus, any strategy is optimal for them and so is the chosen $\widehat{\sigma}_*^n$. Hence

$$\forall n \in \{1, ..., N\} \ \exists \phi_m^n : \forall s \in S_{B_1} \cap S^n, \phi_m^n(s) = \widehat{\sigma}_*^n(s). \tag{13}$$

Let now $s \in S_{B_2}$. First note that the only optimal strategies for P_m in this case are strategies $\widehat{\sigma}^m \in \widehat{\Sigma}^m$. Indeed, and for any strategy σ^{-m} of P_{-m}, if P_m uses any $\widehat{\sigma}^m$ for C_m the outcome is either a pure C_m capture, at the fastest possible time, or a joint capture, at the fastest possible time. Any other strategy of P_m leads to suboptimal for him outcomes, even to pure C_n ($n \neq m$) capture.

Assuming P_m uses $\widehat{\sigma}_*^m$ for C_m, then P_{-m} can effect a joint capture at some time t. First note that a joint capture can only happen after a move by R and only if he deviates from $\widehat{\sigma}^N$, and second that this can only be at a time $t < \widehat{T}(s)$.[5] Moreover it is clear that P_{-m} always prefers the joint capture which: (i) happens at the latest possible time, call it $\widetilde{T}(s)$ and (ii) involves the largest possible number of his cops, call it $\widetilde{K}(s)$. Now note that both these maximum values are achieved by following $\widehat{\sigma}_*^{-m}$ until $\widetilde{T}(s) - 1$, at which time the robber is at the path end and $\widetilde{K}(s) + 1$ cops are next to him and letting R fall on the cops at $\widetilde{T}(s)$; call this strategy $\widetilde{\sigma}^{-m}$. In this case P_{-m}'s loss is

[5] The latter is a consequence of the following: In a *pure* C_m capture under every $\widehat{\sigma}$, only the moves of C_m and R are relevant (i.e., the remaining cops cannot affect the outcome). Thus if P_{-m} could effect a joint capture at $t > \widehat{T}(s)$, given C_m follows $\widehat{\sigma}^m$, he would be able to do so only due to moves of R. But if R alone could achieve capture later than $\widehat{T}(s)$, when C_m uses $\widehat{\sigma}^m$, then $\widehat{T}(s)$ would not be the optimal CR time, which is a contradiction.

$$\frac{1-\varepsilon}{\widetilde{K}(s)+1}\gamma^{\widetilde{T}(s)}=\frac{1}{\widetilde{K}(s)+1}\gamma^{\widetilde{T}(s)}.$$

Alternatively, P_{-m} can choose to stick to $\widehat{\sigma}_*^{-m}$ until the end and let C_m capture at $\widehat{T}(s)$. Since at $\widehat{T}(s)$ (resp. at $\widetilde{T}(s)$) C_m (resp. R) has the move, we have $\widehat{T}(s)=\widetilde{T}(s)+m$. In this case P_{-m}'s loss is

$$(1-\varepsilon)\gamma^{\widehat{T}(s)}=\gamma^{\widehat{T}(s)}.$$

Then $\widehat{\sigma}_*^{-m}$ is optimal for P_{-m} iff

$$\gamma^{\widehat{T}(s)}\leq\frac{1}{\widetilde{K}(s)+1}\gamma^{\widetilde{T}(s)}\Leftrightarrow\gamma^{m}\leq\frac{1}{\widetilde{K}(s)+1}\Leftrightarrow\gamma\leq\left(\frac{1}{\widetilde{K}(s)+1}\right)^{1/m}. \tag{14}$$

For a positional $\overline{\sigma}$ to exist (14) must hold for all $s\in S_{B_2}$ and thus also for the minimum of $\left(\frac{1}{\widetilde{K}(s)+1}\right)^{1/m}$. This quantity is increasing in m and decreasing in $\widetilde{K}(s)$ and thus takes its minimum for $m=1$ and $\widetilde{K}(s)=N-2$, i.e., when $C_m=C_1$ and at $\widetilde{T}(s)$ all cops are next to the robber. Then (14) becomes

$$\gamma\leq\frac{1}{N-1}. \tag{15}$$

Hence, under *and only under (15)* we have

$$\forall n\in\{1,\dots,N\}\ \exists\phi_m^n:\forall s\in S_{B_2}\cap S^n,\ \phi_m^n(s)=\widehat{\sigma}_*^n(s). \tag{16}$$

Given $S_B=S_{B_1}\cup S_{B_2}$, $S''_{ncl}=S_A\cup S_B$ and combining (12), (13) and (16) yields that a positional trigger strategies profile $\overline{\sigma}$ exists iff (15) holds. □

Remark 3.7. We now move on to graphs other than paths. It is not hard to see that, for any such graph and pair of noncapture states s_0, s, there exists a history h which starts at s_0 and ends at s. I.e.,

$$\forall s_0, s\in S_{nc},\ \exists h=(s_0,\dots,s)\in H_{fnc}$$

This means that, *for every* s_0, the set of endstates of *all* finite noncapture histories starting at s_0 is exactly S_{nc}. Thus Conditions **A1**, **A2** for positional and nonpositional respectively profiles reduce to:

Condition 3.8. Let $\overline{\sigma}=(\overline{\sigma}^1,\dots,\overline{\sigma}^N)$ be a trigger strategies profile in $\Gamma_N(G|s_0,\gamma,\varepsilon)$, with $\overline{\sigma}^n$ consisting of $(\phi_m^n)_{m=1}^N$. Then $\overline{\sigma}$ is positional (resp. nonpositional) iff **B1** (resp. **B2**) holds:

$$\mathbf{B1}:\forall n,m\in\{1,\dots,N\},\ \exists\widehat{\sigma}^n\in\widehat{\Sigma}^n:\forall s\in S^n\cap S_{nc},\ \phi_m^n(s)=\widehat{\sigma}^n(s), \tag{17}$$

$$\mathbf{B2}:\exists n,m\in\{1,\dots,N\}:\forall\widehat{\sigma}^n\in\widehat{\Sigma}^n,\ \exists s\in S^n\cap S_{nc}:\phi_m^n(s)\neq\widehat{\sigma}^n(s). \tag{18}$$

The next proposition settles the issue for all rest graphs G with $c(G)\leq N-1$.

Proposition 3.9. *Consider $\Gamma_N(G|s_0,\gamma,\varepsilon)$ where $G=(V,E)$ is not a path, $c(G)\leq N-1$ and $\varepsilon=0$. Then every trigger strategies profile $\overline{\sigma}$ is nonpositional for all $\gamma\in(0,1)$.*

Proof. Given G is not a path, it is $|V|\geq 3$ and, G contains, or is equal to either the clique $\mathcal{K}_3$ with $E=\{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_1\}\}$, or the star $\mathcal{S}_3$ $E=\{\{v_1,v_2\},\{v_1,v_3\},\{v_1,v_4\}\}$. By Remark 3.7 then we have that, for any initial state s_0, the state $\widetilde{s}$ where, some of the cops are on vertex v_2, the rest are on v_3, the robber is on v_1 and it is the robber's turn to move can always occur. Moreover, $c(G)\leq N-1$ means that, starting from $\widetilde{s}$, capture occurs under CR-optimal play by $\widehat{C}(\widetilde{s})$; let $\widehat{C}(\widetilde{s})=C_m$.

Given Proposition 3.5, to show the current claim suffices to show there exist no positional profiles $\overline{\sigma}$. Assume (towards contradiction) there exists positional profile $\overline{\sigma}$. Now consider game $\Gamma_N^m(G|s_0,\gamma,\varepsilon)$ and the case where state $\widetilde{s}$ has been reached. Then it must be that, there exist optimal ϕ_m^n such that, Condition **B1** is satisfied for that particular m and $s=\widetilde{s}$. However, if a CR-optimal strategy $\widehat{\sigma}^n$ is used for every $n\in\{1,\dots,N\}$, then starting from $\widetilde{s}$, profile $\widehat{\sigma}$ leads to capture by C_m at $\widehat{T}(\widetilde{s})$, in which case P_{-m}'s loss will be $\gamma^{\widehat{T}(\widetilde{s})}>0$. But we know that, starting from $\widetilde{s}$, P_{-m} can achieve his minimum loss of 0 by moving R into whichever of v_2 or v_3 does not contain C_m. Hence using $\widehat{\sigma}^n$ for every $n\in\{1,\dots,N\}$ is suboptimal for P_{-m} when starting from $\widetilde{s}$. Thus, there exists no positional $\overline{\sigma}$. □

3.3.2. Interlude: state cop number

In this section we present the *state s cop number* $c_N(G|s)$ (or simply $c(G|s)$ when N is implied by the context) which we will need to proceed with our analysis. All the notions presented here concern exclusively *capturability* and thus they depend only on the graph G and the state s to which they refer (and thus on the number of players, their locations and whose turn is to move). Payoffs and initial states play no role. This motivates us to define a *CR-pregame*[6] $\breve{\Gamma}_N(G)$ consisting of a graph G, $N-1$ cop tokens and one robber token, where move and capture rules are the same as in SCAR $\Gamma_N(G|s_0, \gamma, \varepsilon)$, but no payoffs or initial state are specified. These notions apply also to the games $\Gamma_N^m(G|s_0, \gamma, \varepsilon)$, including modified CR and to any other game sharing this basic structure, since, for any initial state s_0 and strategy profile σ, the infinite history produced is the same in all of them.

First we define *s-guaranteed capture profiles of k-th order*, i.e. k-cops profiles which, when used in $\breve{\Gamma}_N(G)$, guarantee capture from state s, no matter how the rest $N-k$ players (including R) play.

Definition 3.10. A k-cops profile $\sigma = \left(\sigma^{i_1}, ..., \sigma^{i_k}\right)$ $(k \in \{1, ..., N-1\})$ in $\breve{\Gamma}_N(G)$ is called *s-guaranteed capture profile* (s-gcp) *of k-th order* iff, given s has been reached,[7] the profile (σ, σ') leads to capture for all strategy profiles $\sigma' = \left(\sigma^{j_1}, ..., \sigma^{j_{N-k}}\right)$ with $\left\{j_1, ..., j_{N-k}\right\} = \{1, ..., N\} \setminus \{i_1, ..., i_k\}$ of the rest players.

Note that the definition of an s-gcp σ implies guaranteed capture by *some* (one or more) cops, but *not necessarily by one of the cops involved in* σ.

And now we define the state cop number $c(G|s)$ in $\breve{\Gamma}_N(G)$.

Definition 3.11. Consider the pregame $\breve{\Gamma}_N(G)$ $(N \geq 2)$ and $s \in S_{nc}$.

1. If a k-th order $(k \in \{1, ..., N-1\})$ s-gcp exists, then the *state s cop number in* $\breve{\Gamma}_N(G)$ is denoted by $c(G|s)$ and defined to be the minimum k for which such a k-th order s-gcp exists.
2. Otherwise $c(G|s) = \infty$.

Example 3.12. Consider $\breve{\Gamma}_3(G)$ on graph G depicted in both sides of Fig. 1; note that $c(G) = 2$. Suppose the current state is $s_1 = (3, 7, 5, 3)$ as depicted in the left side of Fig. 1 with R having the move. It is clear that, starting from s_1, C_1 can always ensure capture by going towards R and thus either effecting capture himself, or forcing R to run into C_2. Hence $c(G|s_1) = 1$.[8] From $s_2 = (6, 5, 3, 3)$ however depicted on the right side of Fig. 1 with R having again the move, both cops are needed to ensure capture and thus $c(G|s_2) = 2$.

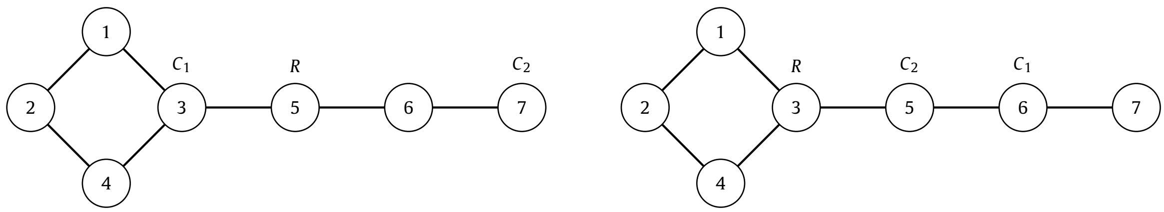

Fig. 1. The state on the left part of the figure is $s_1 = (3, 7, 5, 3)$ and the state on the right part is $s_2 = (6, 5, 3, 3)$.

Finally we present the following theorem which establishes the connection between $c(G|s)$ and the classic cop number $c(G)$.

Theorem 3.13. *Consider pregame* $\breve{\Gamma}_N(G)$ *with* $N \geq 2$. *Then*

$$c(G) = K \leq N-1 \Leftrightarrow \max_{s \in S_{nc}} c(G|s) = K \leq N-1 \tag{19}$$

$$c(G) > N-1 \Leftrightarrow \max_{s \in S_{nc}} c(G|s) = \infty. \tag{20}$$

[6] A similar situation occurs with *extensive game forms with perfect information*, that is, structures of extensive games where players' preferences are not specified [5, p. 90].

[7] Note that the definition holds also in case state s does not occur under profile (σ, σ'), for some game $\Gamma(G|s_0)$.

[8] Note however that C_1, R can also move so that C_2 cannot effect capture.

Proof. The complete proof will be given in a forthcoming paper. Here we prove only (20) which is what we will use in this paper. Assume $c(G) > N-1$. This means there exists s' in $\breve{\Gamma}_N(G)$ where the $N-1$ available cops do not suffice to ensure capture. Then by definition of $c(G|s)$ it is $c(G|s') = \infty$ and thus $\max_{s\in S_{nc}} c(G|s) = \infty$. Conversely, $\max_{s\in S_{nc}} c(G|s) = \infty$ implies there exists $s' \in S^N$ such that $c(G|s') = \infty$ and hence $N-1$ cops do not suffice to ensure capture in $\breve{\Gamma}_N(G)$ from state s'. It is fairly straightforward then to see that $c(G) > N-1$. □

3.3.3. Case: $c(G) > N-1$

For any given N, we define $\mathcal{G}(N)$ by

$$\mathcal{G}(N) := \{G : c(G) > N-1\}, \tag{21}$$

i.e., the set of graphs examined in this section[9] or, equivalently by (20), the set

$$\mathcal{G}(N) := \{G : \exists s \text{ with } c(G|s) = \infty\}. \tag{22}$$

Elements of $\mathcal{G}(3)$ include Dodecahedron and the Petersen graph (both of which have $c(G) = 3$) as well as any other graph resulting by *bridging* either of these to another graph.[10] The following propositions involve sets of graphs that form a partition of $\mathcal{G}(N)$. Moreover, we sometimes simply write $\mathcal{G}$ rather than $\mathcal{G}(N)$ and likewise for its constituents.

Given N, consider the following subsets of $\mathcal{G}$:

1. $\mathcal{G}_1$ consists of those graphs in $\mathcal{G}$ where there exists a state starting from which, cooperation of two or more cops, up to $N-1$, is necessary and sufficient to ensure capture in $\breve{\Gamma}_N(G)$. I.e.,

$$\mathcal{G}_1 := \{G \in \mathcal{G} : \exists s \text{ with } c(G|s) = k \in \{2, ..., N-1\}\}. \tag{23}$$

2. $\mathcal{G}_1'$ is the complement of $\mathcal{G}_1$ (with respect to $\mathcal{G}$). Here, at every noncapture state, either the robber evades under CR-optimal play, or there exists a single cop who can ensure capture. I.e.,

$$\mathcal{G}_1' := \mathcal{G}\backslash\mathcal{G}_1 = \{G \in \mathcal{G} : \forall s \in S_{nc}, c(G|s) \in \{1, \infty\}\}. \tag{24}$$

3. Finally, $\mathcal{G}_2$ consists of graphs such that, when it is the robber's turn to move, he can always evade. I.e.,

$$\mathcal{G}_2 := \{G \in \mathcal{G} : \forall s \in S^N \cap S_{nc} \text{ it is } c(G|s) = \infty\}. \tag{25}$$

As we will shortly see, $\mathcal{G}_2$ is a subset of $\mathcal{G}_1'$.

Graphs in $\mathcal{G}_1(3)$ are typically graphs of $\mathcal{G}(3)$ containing as subgraphs cycles of length $l \geq 4$. Some graphs in $\mathcal{G}_1'(3)$ are those resulting by bridging Dodecahedron or Petersen with paths or trees. Some graphs in $\mathcal{G}_2(3)$ are Dodecahedron and Petersen themselves.

The next proposition concerns $\mathcal{G}_1$. Note that holds, not only for $\varepsilon = 0$ but for every $\varepsilon \in \left[0, \frac{1}{2}\right]$.[11]

Proposition 3.14. *Consider $\Gamma_N(G|s_0, \gamma, \varepsilon)$ with $G \in \mathcal{G}_1$ and $\varepsilon \in \left[0, \frac{1}{2}\right]$. Then every profile $\overline{\sigma}$ is nonpositional for all $\gamma \in (0, 1)$.*

Proof. Let $s \in S_{nc}$ with $c(G|s) = k \in \{2, ..., N-1\}$; let $\widehat{C}(s) = C_m$. Consider game $\Gamma_N^m(G|s, \gamma, \varepsilon)$. Given $c(G|s) \geq 2$, P_{-m} can enforce robber evasion in $\Gamma_N^m(G|s, \gamma, \varepsilon)$. Hence it cannot be $\phi_m^n(s_t) = \widehat{\sigma}^n(s_t)$ for all $n \in \{1, ..., N-1\}\backslash m$ and all states $s_t \in S^n$ following s, because this allows P_m to capture R and is suboptimal play for P_{-m}. Hence, there exists at least one $n \in \{1, ..., N-1\}\backslash m$ and at least one $s_t \in S^n$ following s such that, $\phi_m^n(s_t) \neq \widehat{\sigma}^n(s_t)$, $\forall \widehat{\sigma}^n \in \widehat{\Sigma}^n$, leading to the result sought. □

In the following lemma we show that $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$ and hence $\mathcal{G}_2 \subset \mathcal{G}_1'$.

Lemma 3.15. $\mathcal{G}_1(N) \cap \mathcal{G}_2(N) = \emptyset$.

Proof. Given N, assume on the contrary $\mathcal{G}_1 \cap \mathcal{G}_2 \neq \emptyset$ and let $G \in \mathcal{G}_1 \cap \mathcal{G}_2$. $G \in \mathcal{G}_1$ means there exists state s such that $c(G|s) = k \in \{2, ..., N-1\}$. This in its turn means that, at s, there exists no cop who (a) is located next to R and (b) plays before R (because otherwise it would have been $c(G|s) = 1$). This again means that, starting from s and irrespective of the moves of the cops playing before R, R will have the chance to move i.e., the game will reach a state $s' \in S^N \cap S_{nc}$.

[9] It is a well known result [1] that, for every $k \in \mathbb{N}$, there exists a graph G with $c(G) > k$.

[10] Recall that a "bridge" is an edge whose deletion results to a disconnected graph. Then our claim follows from the (easy to see) fact that, if a graph G with $c(G) = k$ is bridged to another graph, the resulting graph H will have $c(H) \geq k$.

[11] However, for $\varepsilon > 0$ the claim has been already shown in Proposition 3.4.

Now by assumption G belongs also to $\mathcal{G}_2$ and thus $c(G|s') = \infty$ (i.e., starting from s' R evades under CR-optimal robber play). But then (from the previous argument) we have that, under CR-optimal robber play R evades also from s and thus $c(G|s) = \infty$, which however contradicts the assumption $c(G|s) = k \in \{2, ..., N-1\}$. Hence there exists no such graph G and thus $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$. □

Furthermore $\mathcal{G}_1'$ clearly contains graphs that do not belong in $\mathcal{G}_2$ and thus we have the following.

Corollary 3.16. $\mathcal{G}_2(N) \subset \mathcal{G}_1'(N)$.

Remark 3.17. An important fact is the following: $c(G|s) = \infty$ for all $s \in S^N \cap S_{nc}$ implies that, under CR-optimal robber play, the *only* states that can lead to capture in any $G \in \mathcal{G}_2$ are those where *a cop is next to R and moves before him.*[12]

Next we identify one more set of graphs and respective games where positional profiles $\overline{\sigma}$ exist.

Proposition 3.18. *Consider $\Gamma_N(G|s_0, \gamma, \varepsilon)$ with $G \in \mathcal{G}_2$ and $\varepsilon = 0$. Then there exists a positional profile $\overline{\sigma}$ for all $\gamma \in (0, 1)$.*

Proof. Fix an $m \in \{1, ..., N-1\}$ for the rest of the proof and consider $\Gamma_N^m(G|s_0, \gamma, \varepsilon)$. We partition S_{nc} into three mutually disjoint sets S_A, S_B and S_C defined below and examine each case separately.

$$S_A := \{s \in S_{nc} : \text{under (every)}\, \widehat{\sigma} \text{ the robber evades}\},$$
$$S_B := \left\{s \in S_{nc} : \text{under (every)}\, \widehat{\sigma}, \text{cop } C_n\ (n \neq m) \text{ captures (i.e., } \widehat{C}(s) = C_n) \text{ at } \widehat{T}(s)\right\},$$
$$S_C := \left\{s \in S_{nc} : \text{under (every)}\, \widehat{\sigma}, \text{cop } C_m \text{ captures (i.e., } \widehat{C}(s) = C_m) \text{ at } \widehat{T}(s)\right\}.$$

<u>**Case A:** $s \in S_A$</u>. For any $s \in S_A$, if P_{-m} uses any CR-optimal cop and robber strategies $\widehat{\sigma}^n$ $(n \in \{1, ..., N\} \backslash m)$ he can force evasion of R for any strategy of P_m and get his minimum loss of 0. Given P_{-m}'s strategy, P_m cannot affect the outcome. Thus *any* strategy is optimal for him and so is any CR-optimal strategy $\widehat{\sigma}^m$. Hence

$$\forall n \in \{1, ..., N\}\ \exists \phi_m^n, \widehat{\sigma}^n : \forall s \in S_A \cap S^n, \phi_m^n(s) = \widehat{\sigma}^n(s). \tag{26}$$

<u>**Case B:** $s \in S_B$</u>. By Remark 3.17, these are states such that: (i) a cop C_n $(n \neq m)$ is next the robber and moves before him and (ii) if there exists another cop C_k who is also next the robber, then C_k moves after C_n. Now, for any $s \in S_B$, if P_{-m} uses any CR-optimal strategies $\widehat{\sigma}^n$ $(n \in \{1, ..., N\} \backslash m)$ C_n captures at time $\widehat{T}(s)$ for any strategy of P_m and P_{-m} gets his minimum loss of $\varepsilon \gamma^{\widehat{T}(s)} = 0$. Given P_{-m}'s strategy, P_m cannot affect the outcome. Thus any strategy is optimal for him and so is any CR-optimal strategy $\widehat{\sigma}^m$. Hence

$$\forall n \in \{1, ..., N\}\ \exists \phi_m^n, \widehat{\sigma}^n : \forall s \in S_B \cap S^n, \phi_m^n(s) = \widehat{\sigma}^n(s). \tag{27}$$

<u>**Case C:** $s \in S_C$</u>. By Remark 3.17, these are states such that: (i) cop C_m is next to the robber and moves before him and (ii) if there exists another cop C_n who is also next the robber, then C_n moves after C_m. Now, for any $s \in S_C$, if P_m uses any CR-optimal cop strategy $\widehat{\sigma}^m$, then C_m captures at time $\widehat{T}(s)$ for any strategy of P_{-m} and P_m gets his maximum gain of $(1-\varepsilon)\gamma^{\widehat{T}(s)} = \gamma^{\widehat{T}(s)}$. Given P_m's strategy, P_{-m} cannot affect the outcome. Thus any strategy is optimal for him and so is any CR-optimal strategies $\widehat{\sigma}^n$ $(n \in \{1, ..., N\} \backslash m)$. Hence

$$\forall n \in \{1, ..., N\}\ \exists \phi_m^n, \widehat{\sigma}^n : \forall s \in S_C \cap S^n, \phi_m^n(s) = \widehat{\sigma}^n(s). \tag{28}$$

Combining (26)-(28) and $S_A \cup S_B \cup S_C = S_{nc}$ we get the result sought. □

Given N, let $\mathcal{G}_2'$ be the complementary of $\mathcal{G}_2$ in $\mathcal{G}_1'$, i.e.,

$$\mathcal{G}_2' := \mathcal{G}_1' \backslash \mathcal{G}_2 = \{G \in \mathcal{G}_1' : \exists s \in S^N \text{ with } c(G|s) = 1\}. \tag{29}$$

Summarizing to this point we have: (a) partitions $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_1'$ and $\mathcal{G}_1' = \mathcal{G}_2 \cup \mathcal{G}_2'$ and (b) $\mathcal{G}_2'$ is the last class of graphs remaining to examine.

In search of positional profiles $\overline{\sigma}$ within $\mathcal{G}_2'$, a process of trial and error (coupled with some intuition) led us to the following (and the last) partition of $\mathcal{G}_2'$. $\mathcal{G}_3$ is the subset of $\mathcal{G}_2'$ with graphs satisfying the property that: at *every* state s with $c(G|s) = 1$, cop $\widehat{C}_m(s)$ $(m \in \{1, ..., N-1\})$ can always *effect* capture using $\widehat{\sigma}^m$, *no matter what the rest players do*; $\mathcal{G}_3'$ is the complement of $\mathcal{G}_3$ with respect to $\mathcal{G}_2'$. Formally, given N and $\widehat{C}_m(s)$ $(m \in \{1, ..., N-1\})$ for all $s \in S_{nc}$, define:

[12] Note that this type of states is a trivial case of states s with $c(G|s) = 1$ existing in every graph.

$$\mathcal{G}_3 := \left\{G \in \mathcal{G}_2' : \forall \left(s \text{ with } c(G|s) = 1 \text{ and } \sigma^{-m}\right) \text{ we have: } (s, \widehat{\sigma}^m, \sigma^{-m}) \to \widehat{C}_m(s) \text{ capture}\right\}, \tag{30}$$

$$\mathcal{G}_3' := \left\{G \in \mathcal{G}_2' : \exists \left(s \text{ with } c(G|s) = 1 \text{ and } \sigma^{-m}\right) \text{ such that: } (s, \widehat{\sigma}^m, \sigma^{-m}) \nrightarrow \widehat{C}_m(s) \text{ capture}\right\}; \tag{31}$$

where $\to$ (resp. $\nrightarrow$) means "leads to" (resp. "does not lead to").

The next proposition concerns graphs belonging to $\mathcal{G}_3'$.[13]

Proposition 3.19. *Consider $\Gamma_N(G|s_0, \gamma, \varepsilon)$ with $G \in \mathcal{G}_3'$ and $\varepsilon = 0$. Then every profile $\overline{\sigma}$ is nonpositional for all $\gamma \in (0, 1)$.*

Proof. Let $G \in \mathcal{G}_3'$, s satisfying the conditions in (31) and $\widehat{C}_m(s)$, $\widehat{T}(s)$ as known. Consider game $\Gamma_N^m(G|s, \gamma, \varepsilon)$. Starting from s, under every profile $\widehat{\sigma}$, $\widehat{C}_m(s)$ captures at $\widehat{T}(s)$ and P_{-m}'s loss is $(1-\varepsilon)\gamma^{\widehat{T}(s)} = \gamma^{\widehat{T}(s)}$. Now, if σ^{-m} is such that $(s, \widehat{\sigma}^m, \sigma^{-m})$ does not lead to $\widehat{C}_m(s)$ capture, then, using σ^{-m} P_{-m} can force, either evasion of R, or capture by C_n with $n \neq m$, achieving in both cases a minimum loss of $0 < \gamma^{\widehat{T}(s)}$. Hence, there exists at least one $n \in \{1, ..., N\}\backslash m$ and state $s_t \in S^n$ following s where $\phi_m^n(s_t) \neq \widehat{\sigma}^n(s_t)$, $\forall \widehat{\sigma}^n \in \widehat{\Sigma}^n$ and thus, every profile $\overline{\sigma}$ is nonpositional (and for all $\gamma \in (0, 1)$). □

It remains to examine graphs in $\mathcal{G}_3$. As a first step, we aim to elucidate the form of such graphs. The following lemma makes clearer the distinction between sets $\mathcal{G}_3$ and $\mathcal{G}_3'$.

Lemma 3.20. *The following hold for any N:*

$$\textit{If } G \in \mathcal{G}_3 \textit{ then: } \forall \left(s \in S^N \textit{ with } c(G|s) = 1 \textit{ and } \sigma^{-m}\right) \textit{ we have: } (s, \widehat{\sigma}^m, \sigma^{-m}) \to \widehat{C}_m(s) \textit{ capture.} \tag{32}$$

$$\textit{If } G \in \mathcal{G}_3' \textit{ then: } \exists \left(s \in S^N \textit{ with } c(G|s) = 1 \textit{ and } \sigma^{-m}\right) \textit{ such that: } (s, \widehat{\sigma}^m, \sigma^{-m}) \nrightarrow \widehat{C}_m(s) \textit{ capture.} \tag{33}$$

Proof. Clearly (30) implies (32). For (33), let s, σ^{-m} satisfy the conditions in (31). If $s \in S^N$ then (33) holds. If $s \notin S^N$, then starting from s (and for any strategy of the cops) the game will certainly reach a state $s' \in S^N$ (i.e., the robber will move at least once) by the following argument.

If the robber does not move at least once, then he is captured before he can move. This means that: at s a cop (a) is next to R and (b) moves before R. But then by definition this cop is $\widehat{C}_m(s)$, which contradicts condition (31).

So, let s' be the *first* state in S^N that occurs starting from s, under profile $(\widehat{\sigma}^m, \sigma^{-m})$, where σ^{-m} is the same as in (31). Then s' clearly satisfies (33) for this same σ^{-m}. □

Remark 3.21. Thus we see that for every $G \in \mathcal{G}_3$, each state $s \in S^N \cap S_{nc}$ is such that, either (i) $c(G|s) = \infty$, or (ii) $c(G|s) = 1$ and s satisfies condition (32).

Next we show that, for every state s satisfying (32), there is a state $\widetilde{s}$ which satisfies the exact same conditions as s and differs from s only in that the capturing cop C_m is next to the robber.

Lemma 3.22. *Let $s = (x^1, ..., x^m, ..., x^{N-1}, u, N) \in S^N$ satisfying (32) for some $\widehat{C}_m(s)$ and $x^m \notin N(u)$. Then there exists $\widetilde{s} = (x^1, ..., \widetilde{x}^m, ..., x^{N-1}, u, N) \in S^N$ such that (i) $\widetilde{x}^m \in N(u)$, (ii) $c(G|\widetilde{s}) = 1$, (iii) $\widehat{C}(\widetilde{s}) = C_m$ and (iv) for every $\widetilde{\sigma}^{-m}$, $(\widetilde{s}, \widehat{\sigma}^m, \widetilde{\sigma}^{-m})$ leads to $\widehat{C}_m(\widetilde{s})$ capture.*

Proof. Let $\acute{\sigma}^{-m}$ be the profile where all players besides C_m stay put. Then, starting from s and under $(\widehat{\sigma}^m, \acute{\sigma}^{-m})$, there will come a time where C_m reaches R's neighborhood (because otherwise R could stay put indefinitely and evade) and it is R's turn to move. Let the respective state be $\widetilde{s}$. First note that $\widetilde{s}$ has the required form; and $\widetilde{x}^m \in N(u)$ i.e., condition (i) holds. Now, given $(s, \widehat{\sigma}^m, \sigma^{-m})$ leads to $\widehat{C}_m(s)$ capture for *every* σ^{-m}, this must be also true for *any* profile σ^{-m} that copies $\acute{\sigma}^{-m}$ until $\widetilde{s}$ occurs and follows *any* profile $\widetilde{\sigma}^{-m}$ thereafter. Thus, $(\widetilde{s}, \widehat{\sigma}^m, \widetilde{\sigma}^{-m})$ leads to $\widehat{C}_m(s)$ capture for *every* $\widetilde{\sigma}^{-m}$ and condition (iv) also holds. Conditions (ii)-(iii) follow immediately from (iv). □

The following lemma reveals an important fact for graphs (as those in $\mathcal{G}_3$) satisfying (32).

Lemma 3.23. *Let $G \in \mathcal{G}_3$. If $s = (x^1, ..., x^m, ..., x^{N-1}, u, N) \in S^N$ satisfies (32), then:*

1. *$|N(u)| = 1$ (i.e., u is a leaf) and*
2. *if $\{v\} = N(u)$, then for all $s' = (y^1, ..., y^{N-1}, v, N) \in S^N$ it is $c(G|s') = \infty$.*

[13] One graph in $\mathcal{G}_3'$ is a graph where the Petersen graph is bridged to a path. Then a state satisfying the condition in (31) is the one where R is "crammed" between C_1 and C_2 on the path and it is R's turn to move. Under CR-optimal play R stays put in the first move and C_1 captures right after. However, R can run into C_2 straight ahead.

Proof. **1.** Assume towards contradiction $|N(u)| \geq 2$. We distinguish the following cases.

1.A. s is such that $x^m \in N(u)$. Then $N(u)$ contains at least one more vertex z. There are two possibilities.

(i) z is occupied by some cop $C_i \neq \widehat{C}_m(s)$. But this violates the requirement that $(s, \widehat{\sigma}^m, \sigma^{-m})$ leads to $\widehat{C}_m(s)$ capture, for every σ^{-m} since R can run into C_i on his first move.

(ii) z is not occupied by any cop. Moving some cop $C_i \neq \widehat{C}_m(s)$ to vertex z we create state $s'' \in S^N$ which differs from s only in C_i's new location. Now we have (a) $c(G|s'') = 1$ and (b) letting $\widehat{C}(s'') = C_n$ (where C_n may be C_m or C_i) then (given $G \in \mathcal{G}_3$) we must have that $(s'', \widehat{\sigma}^n, \sigma^{-n})$ leads to C_n capture for every σ^{-n}. But (b) is violated since R can run into whichever cop is not C_n (either C_m or C_i).

Hence, if s is such that $x^m \in N(u)$ then $|N(u)| = 1$ and u is a leaf.

1.B. s is such that $x^m \notin N(u)$. Then there exists some $\widetilde{s}$ of the type described in Lemma 3.22 and we return to Case A.

We conclude that if s satisfies (32) then u is always a leaf.

2. Assume on the contrary that v, the neighbor of u, is such that there exists $s' = (y^1, ..., y^{N-1}, v, N) \in S^N$ with $c(G|s') \neq \infty$; then, since $G \in \mathcal{G}_3$, it must be $c(G|s') = 1$; therefore, s' also satisfies (32). From Part 1 then we have that v is a leaf; given u is also a leaf, it follows that G is the path $\mathcal{P}_2$; but this contradicts $G \in \mathcal{G}_3$. □

Corollary 3.24. *Let $G \in \mathcal{G}_3$. Then u is not a leaf iff*

$$\forall s = (x^1, ..., x^{N-1}, u, N) \in S^N \text{ it is } c(G|s) = \infty. \tag{34}$$

Proof. The left to right implication follows from Lemma 3.23; the reverse from the fact that if u is a leaf, then there always exist states s with $c(G|s) = 1$ e.g. those where a cop occupies u's neighbor. □

The last two results lead directly to the following characterization of the form of graphs in $\mathcal{G}_3$.

Corollary 3.25. *Let $G \in \mathcal{G}_3$ and $u \in V(G)$. Then (i) either u is a leaf, in which case its neighbor satisfies the condition in Part 2 of Lemma 3.23 (ii) or u is not a leaf and then condition (34) holds.*

Remark 3.26. In other words, Corollary 3.25 means that, $\mathcal{G}_3$ consists of the graphs obtained by attaching to each graph $G \in \mathcal{G}_2$ an arbitrary (but positive) number of leaves. Some graphs in $\mathcal{G}_3$ then are graphs like Petersen, or Dodecahedron to which some leaves have been attached to.

The following proposition concludes this study.

Proposition 3.27. *Consider $\Gamma_N(G|s_0, \gamma, \varepsilon)$ with $G \in \mathcal{G}_3$ and $\varepsilon = 0$. Then there exists a positional profile $\overline{\sigma}$ iff $\gamma \in (0, \frac{1}{N-1}]$.*

Proof. Take first any vertex u which is not a leaf. Then condition (34) holds. But (34) is the same as the defining condition of $\mathcal{G}_2$. Hence, the same analysis as that in Proposition 3.18 leads to the conclusion that: whenever the robber occupies a vertex that is not a leaf, every $\widehat{\sigma}^n \in \widehat{\Sigma}^n$ is optimal for each token $n \in \{1, ..., N\}$ and in every game $\Gamma_N^m(G|s_0, \gamma, \varepsilon)$ for $m \in \{1, ..., N\}$ and $\gamma \in (0, 1)$.

Thus, we only need to examine cases where the robber occupies an arbitrary leaf of the graph. Fix an $m \in \{1, ..., N-1\}$ for the rest of the proof and consider $\Gamma_N^m(G|s_0, \gamma, \varepsilon)$. For the arbitrary leaf u of G, let S_{ncu} denote the set of all states $s \in S_{nc}$ where the robber occupies u. We now partition S_{ncu} as follows and consider for each set of states players' P_{-m} and P_m optimal strategies.

$$S_A := \{s \in S_{ncu} : \text{under (every)}\, \widehat{\sigma} \text{ the robber evades}\}$$
$$S_B := \left\{s \in S_{ncu} : \text{under (every)}\, \widehat{\sigma}, \text{cop } C_m \text{ captures (i.e., } \widehat{C}\,(s) = C_m) \text{ at } \widehat{T}(s)\right\}$$
$$S_C := \left\{s \in S_{ncu} : \text{under (every)}\, \widehat{\sigma}, \text{cop } C_n\ (n \neq m) \text{ captures (i.e., } \widehat{C}\,(s) = C_n) \text{ at } \widehat{T}(s)\right\}.$$

Case A: $s \in S_A$. For any $s \in S_A$, if P_{-m} uses any CR-optimal (cop and robber) strategies $\widehat{\sigma}^n$ $(n \in \{1, ..., N\} \backslash m)$ he can force evasion of R for any strategy of P_m and get his minimum loss of 0. Given P_{-m}'s strategy, P_m cannot affect the outcome. Thus *any* strategy is optimal for him and so is any CR-optimal strategy $\widehat{\sigma}^m$. Hence

$$\forall n \in \{1, ..., N\}\ \exists \phi_m^n, \widehat{\sigma}^n : \forall s \in S_A \cap S^n, \phi_m^n(s) = \widehat{\sigma}^n(s). \tag{35}$$

Let us pause here and consider the form of states $s = (x^1, ..., x^{N-1}, u, n)$ (i.e., R occupies u) where capture can occur under CR-optimal play. Let v denote u's (unique) neighbor in G. Then it is not hard to see that, under CR-optimal play, a capture can occur iff there exists at least one cop C_i, who is (i) either already at v, i.e., $x^i = v$, or (ii) he can cover v in case R moves there, i.e., either (ii.a) $d(x^i, v) = 1$, or (ii.b) $d(x^i, v) = 2$ and C_i moves before R. In any other case R can evade using $\widehat{\sigma}^N$. Furthermore note that, in all such states s it is $c(G|s) = 1$ and $\widehat{C}_m(s)$ is the cop that is "closer" to R, taking into account also whose turn is to move.[14]

Case B: $s \in S_B$. Assume P_m uses $\widehat{\sigma}^m$ for C_m and consider the options of P_{-m}. It can be seen that, depending on the state s, there exist only two possibilities, which partition further S_B as follows:

1. States $s \in S_{B_1} : C_m$ effects a pure capture, for any strategy of P_{-m}, and
2. States $s \in S_{B_2} : P_{-m}$ can effect a *joint capture*, i.e., one involving C_m and some P_{-m} cop tokens.[15]

If $s \in S_{B_1}$, then under optimal play (in $\Gamma_N^m(G|s_0, \gamma, \varepsilon)$) C_m goes straight towards R, while R stays put and C_m captures at time $\widehat{T}(s)$. This describes the optimal strategies for C_m and R. The remaining tokens C_n cannot affect the outcome. Thus, any strategy is optimal for them and so is any $\widehat{\sigma}^n$. Hence

$$\forall n \in \{1, ..., N\}\ \exists \phi_m^n, \widehat{\sigma}^n : \forall s \in S_{B_1} \cap S^n, \phi_m^n(s) = \widehat{\sigma}^n(s). \tag{36}$$

If $s \in S_{B_2}$, then the exact same analysis as in case II.B, part 2 of Proposition 3.6 leads to the conclusion that, whereas for P_m using $\widehat{\sigma}^m$ is always optimal in $\Gamma_N^m(G|s_0, \gamma, \varepsilon)$, P_{-m}, uses optimally $\widehat{\sigma}^{-m}$ in $\Gamma_N^m(G|s_0, \gamma, \varepsilon)$ for all $m \in \{1, ..., N-1\}$ and states s iff

$$\gamma \le \frac{1}{N-1}. \tag{37}$$

Thus we have that iff (37) holds, then

$$\forall n \in \{1, ..., N\}\ \exists \phi_m^n, \widehat{\sigma}^n : \forall s \in S_{B_2} \cap S^n, \phi_m^n(s) = \widehat{\sigma}^n(s). \tag{38}$$

Case C: $s \in S_C$. In this case P_{-m} employing CR-optimal strategies $\widehat{\sigma}^n$ for his cop and robber tokens results to capture by C_n and his best outcome, i.e., a loss of 0. Similarly P_m loses anyhow so he may as well employ $\widehat{\sigma}^m$ for his cop token C_m. Hence,

$$\forall n \in \{1, ..., N\}\ \exists \phi_m^n, \widehat{\sigma}^n : \forall s \in S_C \cap S^n, \phi_m^n(s) = \widehat{\sigma}^n(s). \tag{39}$$

Given $S_A \cup S_B \cup S_C = S_{ncv}$ for every leaf v of G and relations (35), (36), (38) and (39), we have that, iff (37) holds, then

$$\forall n, m \in \{1, ..., N\}\ \exists \phi_m^n, \widehat{\sigma}^n : \forall s \in S_{ncu} \cap S^n, \phi_m^n(s) = \widehat{\sigma}^n(s),$$

which completes the proof. □

4. Conclusion

Our aim was to identify the cases of SCAR games $\Gamma_N(G|s_0, \gamma, \varepsilon)$ [3] where, the generally nonpositional trigger strategies Nash equilibria $\overline{\sigma}$ are in fact positional. The current, exhaustive study regarding the form and the values of G, s_0 and N, γ, ε respectively showed that positional $\overline{\sigma}$ profiles exist in exceptional cases, as reflected in Table 1.

Table 1
Cases of positional $\overline{\sigma}$.

ε	G	N	s_0	γ
$\varepsilon \in (0, \frac{1}{N-1}]$	G is path $\mathcal{P}_2$	$N = 3$	$s_0 \in S_{nc}$	$\left[\sqrt{\frac{\varepsilon}{1-\varepsilon}}, \frac{1}{2-2\varepsilon}\right]$
$\varepsilon = 0$	G is path $\mathcal{P}_2$	$N > 3$	$s_0 \in S_{nc}$	$\left(0, \frac{1}{N-1}\right]$
$\varepsilon = 0$	G is path $\mathcal{P}_n$ $(n \ge 2)$	$N \ge 3$	s_0 s.t. cops on one side of the robber	$\left(0, \frac{1}{N-1}\right]$
$\varepsilon = 0$	$G \in \mathcal{G}_2$	$N \ge 3$	$s_0 \in S_{nc}$	$(0, 1)$
$\varepsilon = 0$	$G \in \mathcal{G}_3$	$N \ge 3$	$s_0 \in S_{nc}$	$\left(0, \frac{1}{N-1}\right]$

[14] That is, (i) either $\widehat{C}_m(s)$ is already at v and for any other cop C_j that might be also at v, $\widehat{C}_m(s)$ moves before C_j, or (ii) $\widehat{C}_m(s)$ is at distance 1 from v and for any other cop C_j that might also be at distance 1 from v, $\widehat{C}_m(s)$ moves before C_j, or (iii) $\widehat{C}_m(s)$ is at distance 2 from v, no other cop's distance from v is less than 2, and for any other cop C_j that might also be at distance 2 from v, $\widehat{C}_m(s)$ moves before C_j and $\widehat{C}_m(s)$ moves before R.

[15] Note that, the expression $(s, \widehat{\sigma}^m, \sigma^{-m})$ leads to $\widehat{C}_m(s)$ capture in (30) does not exclude the possibility of a joint capture, which however involves $\widehat{C}_m(s)$ as well.

Clearly paths and the values of ε and γ play a major role. If $\varepsilon > 0$ where cops in the SCAR game $\Gamma_N(G|s_0, \gamma, \varepsilon)$ have an incentive to cooperate, a positional $\overline{\sigma}$ exists only in the exceptional (if not trivial) case where two cops chase the robber on the path $\mathcal{P}_2$ and only for these values of γ. In all remaining cases $\overline{\sigma}$ are nonpositional. Also note that, games played on graphs in the class $\mathcal{G}_2$ are the *only* ones where a path is not involved, since those in $\mathcal{G}_3$ can be seen as graphs in $\mathcal{G}_2$ connected to $\mathcal{P}_2$.

In addition to the above results, in the current study we have introduced the *state cop number* $c(G|s)$. This was crucial in our analysis for the following reasons. First, purely graph theoretical notions do not suffice because they do not take into account the number of players. For example, if G is the Petersen graph, then: (i) if $N = 3$, G belongs to $\mathcal{G}_2(N)$ and all $\Gamma_N(G|s_0, \gamma, \varepsilon)$ possess positional profiles $\overline{\sigma}$ (see Proposition 3.18), whereas (ii) if $N = 4$, all $\Gamma_N(G|s_0, \gamma, \varepsilon)$ possess only nonpositional profiles $\overline{\sigma}$ (see Proposition 3.9). Second, the classical cop number $c(G)$ is not sufficient for our analysis. For example, the study of $\mathcal{G}(N)$ class in Section 3.3.3, required a finer subdivision into six subclasses defined in terms of $c(G|s)$.

The current research can be extended in several directions. One such direction is the complete presentation of the state cop number concept, the formal establishment of its connection with the classical cop number $c(G)$ (Theorem 3.13) and the elaboration of its application in the analysis of CR and related problems.

Another future research direction is a refined, subgame perfect equilibrium (SPE) analysis of the SCAR game $\Gamma_N(G|s_0)$, which will extend the NE analysis of [3]. To this end, several elements of the current paper will prove especially useful. First is the use of the state cop number $c(G|s)$. Finding SPEa requires the analysis of optimal play in every subgame of $\Gamma_N(G|s_0)$; given the nature of the game, this boils down to analysis of all games $\Gamma_N(G|s)$, for every s that can be reached from s_0, on the respective states digraph. And $c(G|s)$ gives important information for these games. Finally, the subdivision of class $\mathcal{G}(\mathcal{N})$ and the analysis of zero-sum games Γ_N^m will also be useful.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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